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Regularity criteria for the 3D tropical climate model in Morrey–Campanato space

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Abstract. In this paper we investigate the regularity criterion for the local-in-time smooth solution to the three-dimensional (3D) tropical climate model in the Morrey–Campanato space. It is shown that if u satisfies

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}}{\ln(\|u(t)\|_{L^2}+e)}dt<\infty \quad \text{with } 0<r<1,$$

then the smooth solution (u,v,θ) can be extended past time T.

Keywords: tropical climate model, regularity criterion, Morrey–Campanato space.

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1 Introduction

In this paper, we consider the cauchy problem for the following 3D tropical climate model introduced by Frierson, Majda and Panluis in [1]:

$$\begin{cases}\partial_t u+u\cdot\nabla u-\Delta u+\nabla p+\operatorname{div}(v\otimes v)=0,\\ \partial_t v+u\cdot\nabla v-\Delta v+\nabla\theta+v\cdot\nabla u=0,\\ \partial_t\theta+u\cdot\nabla\theta-\Delta\theta+\operatorname{div}v=0,\\ \nabla\cdot u=0,\\ u(x,0)=u_0(x),v(x,0)=v_0(x),\theta(x,0)=\theta_0(x),\end{cases}\tag{1.1}$$

where $x\in\mathbb{R}^3$, $t>0$ and $u=\big(u_1(x,t),u_2(x,t),u_3(x,t)\big)$ is the barotropic mode, $v=\big(v_1(x,t),v_2(x,t),v_3(x,t)\big)$ is the first baroclinic mode of vector velocity, $\theta=\theta(x,t)$ is a scalar function denoting the temperature and $p=p(x,t)$ is the scalar pressure, respectively. u_0, v_0, θ_0 are the prescribed initial data with $\nabla\cdot u_0=0$.

By performing a Galerkin truncation to the hydrostatic Boussinesq equations, the original system derived in [1] has no viscous terms in $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$, in other words, there

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without any Laplacian terms in system (1.1). Recently, for the 2D case, Li and Titi [2] obtained the global well-posedness of strong solutions for the system (1.1) while without diffusivity in the temperature equations. Later, inspired by [2], Wan [5] proved the global well-posedness with the small data to the 2D tropical climate model without thermal diffusion. The global well-posedness with the small data by using the spectral analysis for a viscous tropical climate model with only a damp term have been proved by Wan and Ma [3]. In [4], Ye established the global regularity of a tropical climate with the very weak dissipation barotropic by utilizing the "weakly nonlinear" energy estimate approach and maximal $L_t^q L_x^p$ regularity for heat kernel. Subsequently, Yu and Yang [7] established a new blowup criterion for smooth solution to the 2D generalized tropical climate model. More global regularity for the tropical climate model with fractional dissipation have been established (see, for example, [6, 8–10] and references therein). It should be pointed out that for the system (1.1), Wang et al. [11] first showed the following regularity criteria involving ∇u:

$$\nabla u \in L^q(0,T;L^p(\mathbb{R}^3)), \qquad \frac{3}{p}+\frac{2}{q}\le 2, \quad 2<p\le 3. \tag{1.2}$$

Here it is worth particularly mentioning that system (1.1) and the magnetohydrodynamic (MHD) equations are very similar in terms of the structure of the equation. Obviously, when $\theta =$ constant, the system (1.1) reduces to the 3D MHD-type equations (here we regard velocity v as magnetic b). It is well known that the question of global regularity for 3D incompressible MHD equations has been one of the most outstanding open problems in applied analysis, as well as that for the 3D tropical climate model (1.1). It is an interesting topic of finding sufficient conditions for local smooth solutions such that they can be extended smoothly past T in mathematical fluid mechanics. For MHD equations, Zhou et al. [13] obtained some known regularity criteria of weak solutions in the multiplier space $\dot{X}_r$, provided that one of the following conditions hold:

$$u \in L^{\frac{2}{1-r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0\le r<1 \tag{1.3}$$

or

$$\nabla u \in L^{\frac{2}{2-r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0\le r\le 1. \tag{1.4}$$

Chen et al. [12] established logarithmically improved regularity criteria in terms of the velocity field or on the gradient of velocity field in terms of the critical Morrey–Campanato spaces. More precisely, they proved the following regularity condition

$$\int_0^T \frac{\|u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}}{1+\ln(e+\|u(t)\|_{L^\infty})}dt<\infty \quad \text{with } 0<r<1, \tag{1.5}$$

or

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}}{1+\ln(e+\|u(t)\|_{L^\infty})}dt<\infty \quad \text{with } 0<r\le 1. \tag{1.6}$$

More regularity conditions of the incompressible fluid equations, see [14–19] and so forth.

The purpose of this paper is to improve and extend some known regularity criterion for the 3D tropical climate model (1.1) in the Morrey–Campanato space $\dot{\mathcal{M}}_{2,3/r}$ (see Definition 2.2 in Section 2). It is a natural way to extend the space widely and improve the previous results [11]. Meanwhile, our results extend and generalize the recent works [12, 13] respectively on the regularity criteria for the three-dimensional MHD equations.

Now we state our result as follows.

Theorem 1.1. *Assume that* $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$. *Let* (u, v, θ) *be a local smooth solution to the system* (1.1) *on some interval* $[0, T)$. *If additionally,*

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}}{\ln(\|u(t)\|_{L^2} + e)} dt < \infty \tag{1.7}$$

for some $0 < r < 1$, *then the solution can be extended smoothly past* T.

Remark 1.2. Due to $\frac{\|\nabla u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}}{\ln(\|u(t)\|_{L^2}+e)} \leq \|\nabla u(t)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}$, it is easy to get regularity condition $\nabla u \in L^{\frac{2}{r}}(0, T, \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))$.

Remark 1.3. We are unable to obtain regularity condition $\nabla u \in L^{\frac{2}{2-r}}\big(0, T, \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)\big)$, the main difficulty comes from the term $(\text{without} \nabla \cdot v = 0)$ $I_6 = -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} v_j \partial_k u_j \partial_k \partial_i v_i dx$.

Remark 1.4. Since the critical Morrey–Campanato space $\dot{\mathcal{M}}_{2,3/r}$ is much wider than the Lebesgue space $L^{\frac{3}{r}}$ hence our result extend the recent results given by Wang et al. [11]. Moreover, these can be regarded as an generalize of previous results [12,13] in some sense.

Note that $\dot{B}^{\frac{3}{p}-r}_{p,\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ for $0 < r < \frac{3}{2}$ with $p < \frac{3}{r}$, we obtain a corresponding regularity criterion. Here $\dot{B}^s_{p,q}$ denote the homogenous Besov spaces.

Corollary 1.5. *Assume that* $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$. *Let* (u, v, θ) *be a local smooth solution to the system* (1.1) *on some interval* $[0, T)$. *If additionally,*

$$\nabla u \in L^{\frac{2}{r}}\big(0, T, \dot{B}^{\frac{3}{p}-r}_{p,\infty}(\mathbb{R}^3)\big) \tag{1.8}$$

for some $0 < r < \frac{3}{2}$ *with* $p < \frac{3}{r}$, *then the solution can be extended smoothly past* T.

2 Preliminaries

Now, we recall the definition and some properties of the spaces to be used later. These spaces play an important role in studying the regularity of solutions to partial differential equations, see e.g. [22,23] and the references therein.

Definition 2.1. For $0 \leq r < 3/2$, the space $\dot{X}_r(\mathbb{R}^3)$ is defined as the space of functions $f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty.$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2} u\|_{L^2}$.

We have the following homogeneity properties: For all $x_0 \in \mathbb{R}^3$,

$$\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}$$
$$\|f(\lambda \cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \lambda > 0.$$

Also we have the imbedding

$$L^{1/3}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \quad \text{for } 0 \leq r < \frac{3}{2}.$$

Now we recall the definition of the Morrey–Campanato spaces.

Definition 2.2. For $1 < p \le q \le +\infty$, the Morrey–Campanato space $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3) = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^3) : \|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^3}\sup_{R>0} R^{3/q-3/p}\|f\|_{L^p(B(x,R))} < \infty \Big\}. \tag{2.1}$$

It is easy to check the equality

$$\|f(\lambda\cdot)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0.$$

For $2 < p \le 3/r$ and $0 < r < 3/2$ we have the following embeddings:

$$L^{1/3}(\mathbb{R}^3) \hookrightarrow L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3).$$

The relation

$$L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3)$$

is shown as follows

$$\begin{aligned}
\|f\|_{\dot{\mathcal{M}}_{p,\frac{3}{r}}} &\le \sup_E |E|^{\frac{r}{3}-\frac{1}{2}} \Big(\int_E |f(y)|^p dy \Big)^{1/p} \quad (f \in L^{3/r,\infty}(\mathbb{R}^3)) \\
&= \Big(\sup_E |E|^{\frac{pr}{3}-1} \int_E |f(y)|^p dy \Big)^{1/p} \\
&\cong \Big(\sup_{R>0} R|\{x \in \mathbb{R}^3 : |f(y)|^p > R\}|^{pr/3} \Big)^{1/p} \\
&= \sup_{R>0} R|\{x \in \mathbb{R}^p : |f(y)| > R\}|^{r/3} \\
&\cong \|f\|_{L^{3/r,\infty}}.
\end{aligned}$$

For $0 < r < 1$, we use the fact that

$$L^2 \cap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r.$$

Thus we can replace the space $\dot{X}_r$ by the pointwise multipliers from Besov space $\dot{B}^r_{2,1}$ to L^2. Then we have the following lemmas.

Lemma 2.3 ([21]). *For $0 \le r < 3/2$, the space $\dot{Z}_r(\mathbb{R}^3)$ is defined as the space of functions $f(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\le 1} \|fg\|_{L^2} < \infty.$$

Then $f \in \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ if and only if $f \in \dot{Z}_r(\mathbb{R}^3)$ with equivalence of norms.

Additionally, for $2 < p \le \frac{3}{r}$ and $0 \le r < \frac{3}{2}$, we have the following inclusions:

$$\dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).$$

The relation

$$\dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$$

is shown as follows: Let $f \in \dot{X}_r(\mathbb{R}^3)$, $0 < R \leq 1$, $x_0 \in \mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$\begin{aligned} R^{r-\frac{3}{2}}\Big(\int_{|x-x_0|\leq R}|f(x)|^2dx\Big)^{1/2} &= R^r\Big(\int_{|y-\frac{x_0}{R}|\leq 1}|f(Ry)|^2dy\Big)^{1/2} \\ &\leq R^r\Big(\int_{y\in\mathbb{R}^3}|f(Ry)\phi(y)|^2dy\Big)^{1/2} \\ &\leq R^r\|f(R.)\|_{\dot{X}_r}\|\phi\|_{H^r} \\ &\leq \|f\|_{\dot{X}_r}\|\phi\|_{H^r} \\ &\leq C\|f\|_{\dot{X}_r}. \end{aligned}$$

Lemma 2.4 ([24]). *For $0 < r < 1$, we have*

$$\|f\|_{\dot{B}^r_{2,1}} \leq C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r, \tag{2.2}$$

where C only depends on r.

Lemma 2.5 ([20]). *For $s > 0$ and $1 < p < \infty$. If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then we have a basic estimate*

$$\|[J^s, f]g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|J^{s-1}g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{2.3}$$

with $p_2, p_3 \in (1, \infty)$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, where $J^s = (I - \Delta)^{\frac{s}{2}}$, $[J^s, f]g = J^s(fg) - fJ^s(g)$.

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof is based on the establishment of a priori estimates under condition (1.7).

Multiplying the first equation of system (1.1) by u, the second equation of system (1.1) by v, and the third equation of system (1.1) by θ, respectively. Integrating over $\mathbb{R}^3$, then we adding them together, it yields

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 = 0. \tag{3.1}$$

Applying Gronwall's inequality to (3.1), we get the fundamental energy estimate

$$\|(u, v, \theta)(t)\|_{L^2}^2 + 2\int_0^t \|(\nabla u, \nabla v, \nabla\theta)(s)\|_{L^2}^2 ds = \|(u_0, v_0, \theta_0)\|_{L^2}^2. \tag{3.2}$$

Next, we are going to derive the estimates for ∇u, ∇v and $\nabla\theta$. Multiplying the first equation of system (1.1) by $-\Delta u$, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla u\cdot\Delta u dx + \int_{\mathbb{R}^3}\mathrm{div}(v\otimes v)\cdot\Delta u dx. \tag{3.3}$$

Similarly, multiplying the second and third of system (1.1) by $-\Delta v$ and $-\Delta\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla v\cdot\Delta v dx + \int_{\mathbb{R}^3}\nabla\theta\cdot\Delta v dx + \int_{\mathbb{R}^3} v\cdot\nabla u\cdot\Delta v dx \tag{3.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2=\int_{\mathbb{R}^3}u\cdot\nabla\theta\cdot\Delta\theta dx+\int_{\mathbb{R}^3}\mathrm{div}v\cdot\Delta\theta dx. \tag{3.5}$$

Adding up (3.3)–(3.5), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla\theta\|_2^2)+\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\Delta\theta\|_2^2\\
&\quad=\int_{\mathbb{R}^3}u\cdot\nabla u\cdot\Delta u dx+\int_{\mathbb{R}^3}\mathrm{div}(v\otimes v)\cdot\Delta u dx+\int_{\mathbb{R}^3}u\cdot\nabla v\cdot\Delta v dx\\
&\qquad+\int_{\mathbb{R}^3}v\cdot\nabla u\cdot\Delta v dx+\int_{\mathbb{R}^3}\nabla\theta\cdot\Delta v dx+\int_{\mathbb{R}^3}u\cdot\nabla\theta\cdot\Delta\theta dx+\int_{\mathbb{R}^3}\mathrm{div}v\cdot\Delta\theta dx\\
&\quad=-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_j\partial_i u_j\partial_k u_i dxdx-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k v_j\partial_i v_j\partial_k u_i dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k v_j\partial_i u_j\partial_k v_i\\
&\qquad-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_j\partial_i v_j\partial_k v_i dx-\sum_{i,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_k\theta\partial_i\theta dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}v_j\partial_k u_j\partial_k\partial_i v_i dx\\
&\quad=\sum_{i=1}^{6}I_i,
\end{aligned} \tag{3.6}$$

where we use integration by parts, the fact that $\nabla\cdot u=0$.

To estimate I_1, we apply Hölder's inequality, Young's inequality and (2.2), we get

$$\begin{aligned}
I_1&=\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i u_j\partial_i\partial_k u_j dx\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\cdot\nabla u\|_{L^2}\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{L^2}\|\nabla u\|_{\dot{B}^r_{2,1}}\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1-r}\|\nabla^2 u\|_{L^2}^{r}\\
&=C\Big(\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2\Big)^{\frac{2-r}{2}}\|\nabla^2 u\|_{L^2}^r\\
&\le\frac{1}{4}\|\nabla^2 u\|_{L^2}^2+C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.7}$$

Similarly, for the term I_2, I_3 and I_4, we have

$$\begin{aligned}
I_2+I_3+I_4&\le C\|\nabla v\|_{L^2}\|\nabla u\cdot\nabla v\|_{L^2}\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{\dot{B}^r_{2,1}}\|\nabla v\|_{L^2}\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{L^2}^{1-r}\|\nabla^2 v\|_{L^2}^{r}\|\nabla v\|_{L^2}\\
&\le\frac{1}{4}\|\nabla^2 v\|_{L^2}^2+C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{3.8}$$

For I_5, we get

$$\begin{aligned}
I_5&=-\sum_{i,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_k\theta\partial_i\theta dx\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{\dot{B}^r_{2,1}}\\
&\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2}^{1-r}\|\nabla^2\theta\|_{L^2}^{r}\\
&\le\frac{1}{4}\|\nabla^2\theta\|_{L^2}^2+C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{3.9}$$

Finally, for I_6, by Hölder's inequality, Young's inequality and (2.2), we get

$$\begin{aligned} I_6 &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} v_j\partial_k u_j\partial_k\partial_i v_i dx \\ &\le C\|\nabla^2 v\|_{L^2}\|v\cdot\nabla u\|_{L^2} \\ &\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla^2 v\|_{L^2}\|v\|_{\dot{B}^r_{2,1}} \\ &\le C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla^2 v\|_{L^2}\|v\|_{L^2}^{1-r}\|\nabla v\|_{L^2}^{r} \\ &\le \frac{1}{4}\|\nabla^2 v\|_{L^2}^2 + C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^2\|v\|_{L^2}^{2(1-r)}\|\nabla v\|_{L^2}^{2r} \\ &\le \frac{1}{4}\|\nabla^2 v\|_{L^2}^2 + \frac{1}{4}\|v\|_{L^2}^2 + C\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}\|\nabla v\|_{L^2}^2. \end{aligned} \tag{3.10}$$

Inserting the above estimates (3.7)–(3.10) into (3.6), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|\nabla\theta(t)\|_2^2) + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\Delta\theta\|_2^2 \\ &\quad\le \frac{1}{2}\|v\|_2^2 + C(\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}} + \|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}})(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\ &\quad\le C(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|u_0\|_{L^2} + 1 + e) \\ &\qquad\times\left(C\frac{\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}} + \|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}}{\ln(\|u\|_{L^2}+e)}\right) \\ &\qquad\times\ln(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|u_0\|_{L^2} + 1 + e). \end{aligned} \tag{3.11}$$

Due to Gronwall's inequality, it follows from (3.11) that

$$\begin{aligned} &\ln(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|u_0\|_{L^2} + 1 + e) \\ &\quad\le \ln(\|\nabla u_0\|_{L^2}^2 + \|\nabla v_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + \|u_0\|_{L^2} + 1 + e) \\ &\qquad\times\exp C\int_0^T \frac{\|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}} + \|\nabla u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{r}}}{\ln(\|u\|_{L^2}+e)}ds, \end{aligned} \tag{3.12}$$

which implies that

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 dt \le C, \tag{3.13}$$

where we used the following Sobolev's inequality:

$$\nabla u \in L^{\frac{2}{r}}(0,T,\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)) \subset L^{\frac{2}{2-r}}(0,T,\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)).$$

Thus, the above inequality (3.13) implies

$$u, v, \theta \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)). \tag{3.14}$$

Applying Δ to the equations $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$, multiplying the resulting equations by Δu, Δv and $\Delta\theta$ respectively, adding them up and using the incompressible conditions

$\nabla \cdot u = 0$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\Delta\theta\|_2^2) + \|\nabla\Delta u\|_2^2 + \|\nabla\Delta v\|_2^2 + \|\nabla\Delta\theta\|_2^2 \\
&\quad = \int_{\mathbb{R}^3} \Delta(u\cdot\nabla u)\cdot\Delta u dx + \int_{\mathbb{R}^3} \Delta(u\cdot\nabla v)\cdot\Delta v dx + \int_{\mathbb{R}^3} \Delta(u\cdot\nabla\theta)\cdot\Delta\theta dx \\
&\qquad + \int_{\mathbb{R}^3} \Delta(v\cdot\nabla v)\cdot\Delta u dx + \int_{\mathbb{R}^3} \Delta(\nabla\cdot v)v\cdot\Delta u dx + \int_{\mathbb{R}^3} \Delta(v\cdot\nabla u)\cdot\Delta v dx dx \\
&\quad = \sum_{i=1}^{6} J_i.
\end{aligned} \tag{3.15}$$

Now, by using the Kato–Ponce commutator estimate (i.e. (2.3) when $p_1 = p_4 = 3$, $p_2 = p_3 = 6$) to estimate each term on the right hand side of (3.15) separately, we get

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^3} \Delta(u\cdot\nabla u)\cdot\Delta u dx \\
&= \int_{\mathbb{R}^3} (\triangle(u\cdot\nabla u) - u\nabla\Delta u)\cdot\Delta u dx \\
&\le C\|\Delta(u\cdot\nabla u) - u\nabla\Delta u\|_{L^2}\|\Delta u\|_{L^2} \\
&\le C\|\nabla u\|_{L^3}\|\Delta u\|_{L^6}\|\Delta u\|_{L^2} \\
&\le C\|\nabla u\|_{L^3}\|\nabla\Delta u\|_{L^2}\|\Delta u\|_{L^2} \\
&\le \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^2\|\Delta u\|_{L^2}^2.
\end{aligned} \tag{3.16}$$

Similarly, for the term J_2, J_3, we have

$$J_2 \le \frac{1}{8}(\|\nabla\Delta u\|_{L^2}^2 + \|\nabla\Delta v\|_{L^2}^2) + C(\|\nabla u\|_{L^3}^2 + \|\nabla v\|_{L^3}^2)\|\Delta v\|_{L^2}^2 \tag{3.17}$$

and

$$J_3 \le \frac{1}{8}(\|\nabla\Delta u\|_{L^2}^2 + \|\nabla\Delta\theta\|_{L^2}^2) + C(\|\nabla u\|_{L^3}^2 + \|\nabla\theta\|_{L^3}^2)\|\Delta\theta\|_{L^2}^2. \tag{3.18}$$

For the term J_4, J_5 and J_6 can be bounded as

$$\begin{aligned}
J_4 &= \int_{\mathbb{R}^3} \triangle(v\cdot\nabla v)\cdot\Delta u dx \\
&\le \|v\|_{L^\infty}\|\triangle v\|_{L^2}\|\nabla\Delta u\|_{L^2} + \|\nabla v\|_{L^6}\|\nabla v\|_{L^3}\|\nabla\Delta u\|_{L^2} \\
&\le C\|v\|_{L^\infty}^2\|\Delta v\|_{L^2}^2 + \frac{1}{16}\|\nabla\Delta u\|_{L^2}^2 + C\|\Delta v\|_{L^2}\|\nabla v\|_{L^3}\|\nabla\Delta u\|_{L^2} \\
&\le C(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^3}^2)\|\Delta v\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2.
\end{aligned} \tag{3.19}$$

Similarly, the last two terms J_5 and J_6 can be bounded by

$$J_5 \le C(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^3}^2)\|\Delta v\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2 \tag{3.20}$$

and

$$J_6 \le C(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^3}^2)\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta v\|_{L^2}^2. \tag{3.21}$$

Combining the above estimates (3.16)–(3.21) and (3.15), we deduce

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta u(t)\|_2^2+\|\Delta v(t)\|_2^2+\|\Delta\theta(t)\|_2^2)+\frac{1}{4}(\|\nabla\Delta u\|_2^2+\|\nabla\Delta v\|_2^2+\|\nabla\Delta\theta\|_2^2)\\
&\quad\le C(\|v\|_{L^\infty}^2+\|\nabla u\|_{L^3}^2+\|\nabla v\|_{L^3}^2+\|\nabla\theta\|_{L^3}^2)(\|\Delta u\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)\\
&\quad\le C(\|\Delta v\|_{L^2}^2+\|\nabla u\|_{L^3}^2+\|\nabla v\|_{L^3}^2+\|\nabla\theta\|_{L^3}^2)(\|\Delta u\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2).
\end{aligned}\tag{3.22}$$

Due to Gronwall's inequality and (3.14), we conclude that

$$u,v,\theta\in L^\infty(0,T;H^2(\mathbb{R}^3))\cap L^2(0,T;H^3(\mathbb{R}^3)).$$

This completes the proof of Theorem 1.1. □

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