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A note on existence of patterns on surfaces of revolution with nonlinear flux on the boundary

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Abstract. In this note we address the question of existence of non-constant stable stationary solution to the heat equation on surfaces of revolution subject to nonlinear boundary flux involving a positive parameter. Our result is independent of the surface geometry and, in addition, we provide the asymptotic profile of the solutions and some examples where the result applies.

Keywords: patterns, surface of revolution, nonlinear flux, sub-supersolution method, linearized stability.

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1 Introduction

Consider the problem

$$\begin{cases} u_t(x,t) = \Delta_g u(x,t), & (x,t) \in \mathcal{S} \times \mathbb{R}^+ \\ \partial_\nu u(x,t) = \lambda h(u(x,t)), & (x,t) \in \partial\mathcal{S} \times \mathbb{R}^+ \end{cases} \tag{1.1}$$

where $\mathcal{S} \subset \mathcal{M} \subset \mathbb{R}^3$ and $\mathcal{M}$ is a surface of revolution without boundary with metric g; Δ_g stands for the Laplace–Beltrami operator on $\mathcal{M}$; ν is the outer normal to $\partial\mathcal{S}$ with respect to $\mathcal{M}$; λ is a positive parameter and $h(\cdot)$ is a C^2 function such that, for some $\alpha < \beta \in \mathbb{R}$

$$h(\alpha) = h(\beta) = 0, \quad h'(\alpha) < 0 \quad \text{and} \quad h'(\beta) < 0. \tag{1.2}$$

Our concern in this paper is to prove the existence of non-constant stable stationary solutions (herein referred to as *patterns*, for short) to the problem (1.1). By a *stationary solution* of problem (1.1) we mean a solution which does not depend on time. We recall that a stationary solution U_λ of (1.1) is called *stable* (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|u_\lambda(\cdot,t) - U_\lambda\|_{L^\infty(\mathcal{S})} < \epsilon$ for all $t > 0$, whenever $\|u_\lambda(\cdot,0) - U_\lambda\|_{L^\infty(\mathcal{S})} < \delta$, where $\|\cdot\|_{L^\infty(\mathcal{S})}$ stands for the norm of the space $L^\infty(\mathcal{S})$.

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To state our main result, consider a smooth curve C in $\mathbb{R}^3$ parametrized by $(\psi(s), 0, \chi(s))$, $s \in [l_1, l_2] \supset [0,1]$ with $\psi(l_1) = \psi(l_2) = 0$ and the borderless surface of revolution $\mathcal{M}$ generated by C. Then $\mathcal{M}$ is a surface of revolution without boundary parametrized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \qquad (s,\theta) \in [l_1, l_2] \times [0, 2\pi). \tag{1.3}$$

Our problem is considered on $\mathcal{S} \subset \mathcal{M}$ where $\mathcal{S}$ is a surface of revolution with boundary obtained from the restrictions $\psi, \chi|_{[0,1]}$ ($\psi(s), \chi(s)$ are positive for $s \in \{0,1\}$). All details about $\mathcal{S}$ will be discussed in the next section.

Our main result is the following.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$ then (1.1) admits a family of patterns $\{u_\lambda\}_{\lambda > \lambda_0}$. Moreover u_λ is independent of the angular variation θ and $u_\lambda \to \tilde{u}$ as $\lambda \to \infty$ in $C^0([0,1])$ where*

$$\tilde{u}(s) = \frac{\beta - \alpha}{\int_0^1 [1/\psi(t)]\,dt} \int_0^s [1/\psi(t)]\,dt + \alpha, \qquad s \in [0,1]. \tag{1.4}$$

Though quite natural, only recently it has been considered by some authors the question of stability in problems on surfaces of $\mathbb{R}^3$. For instance, about the case with the nonlinear term $h(\cdot)$ acting on $\mathcal{S}$ ($u_t = \Delta_g u + h(u)$ on $\mathcal{S}$) and different boundary conditions (Neumann, Dirichlet, Robin or mixed), we cite [1, 2, 15, 17] and [16] where the problem is posed on $\mathcal{M}$. All these works have a common hypothesis (related to the geometry of $\mathcal{S}$) when the existence of patterns is obtained. Namely, $k'_{g,\mathcal{S}}(s_0) > 0$ at some $s_0 \in (0,1)$ where $k_{g,\mathcal{S}}(s) := \psi'(s)/\psi(s)$ stands for the geodesic curvature of the parallel circles $s =$ constant on $\mathcal{S}$. See also [10, 11] where, even with a non-constant diffusivity term, the surface geometry is related to the existence of patterns.

We also cite the recent article [9] where, a classification result of stable solutions (in a weaker sense) to a problem with nonlinear boundary conditions on a general Riemannian manifold, was obtained with a technique based on a geometric Poincaré-type inequality.

The Theorem 1.1 above shows that, when $h(\cdot)$ satisfying (1.2) is on $\partial\mathcal{S}$ and λ is large enough, the existence of patterns occurs independently of the geometry of $\mathcal{S}$. Below we illustrate two surfaces where Theorem 1.1 applies. For all details, see the examples in Section 3.

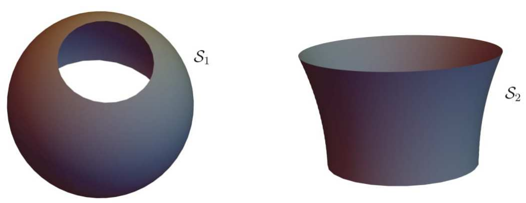

Figure 1.1: Surfaces of revolution where $k'_{g,\mathcal{S}_1}(s) < 0$ and $k'_{g,\mathcal{S}_2}(s) > 0$ for all s.

Still related to this question, it is worth mentioning what is known on the following problems posed in domains of $\mathbb{R}^n$,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \lambda h(u(x,t)), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial_\nu u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

and

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial_\nu u(x,t) = \lambda h(u(x,t)), & (x,t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. It is well known that (1.5) and (1.6) admit patterns for dumbbell-type domain. However, whereas for Ω convex the nonexistence of patterns to (1.5) has long been known [5, 13], until today little has been proved about (1.6) in this case.

Actually, it can be easily proved (see [7], for instance) that if Ω is the n-dimensional ball then (1.6) does not admit patterns. On the other hand, in a computer-assisted work and using bifurcation techniques, the authors in [8] showed strong evidence of existence of patterns to (1.6) when $h(u) = u - u^3$, $\lambda > 2.84083164$ and $\Omega \subset \mathbb{R}^2$ the unit square (i.e. a convex domain). Theorem 1.1 shows that, for surfaces of revolution in $\mathbb{R}^3$ with nonlinear flux on the boundary, the existence of patterns is ensured regardless of the geometry of the domain.

The proof of Theorem 1.1 is made in Section 2 while Section 3 is devoted to presenting some simple examples and remarks. We highlight the adaptation of Theorem 1.1 to a specific class of symmetric Riemannian manifolds.

2 Existence of patterns

2.1 General remarks

Let $\mathcal{M}$ be the surface of revolution parametrized by (1.3). We also assume that $\psi, \chi \in C^2(I)$, $\psi > 0$ in (l_1, l_2) and $(\psi_s)^2 + (\chi_s)^2 = 1$ in $[l_1, l_2]$. Moreover, $\psi_s(l_1) = -\psi_s(l_2) = 1$ and as stated in the Introduction we assume $\psi(l_1) = \psi(l_2) = 0$.

Setting $x^1 = s, x^2 = \theta$ we can conclude that the surface of revolution $\mathcal{M}$ with the above parametrization is a 2-dimensional Riemannian manifold with metric

$$g = ds^2 + \psi^2(s)d\theta^2. \tag{2.1}$$

$\mathcal{M}$ has no boundary and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [4], for instance). The area element on $\mathcal{M}$ is $d\sigma = \psi d\theta ds$ and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2} \partial_\theta u \right).$$

The Laplace–Beltrami operator Δ_g on $\mathcal{M}$ can be expressed as

$$\Delta_g u = u_{ss} + \frac{\psi_s}{\psi} u_s + \frac{1}{\psi^2} u_{\theta\theta}. \tag{2.2}$$

We consider $\mathcal{S} \subset \mathcal{M}$ a surface of revolution with boundary parametrized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \qquad (s, \theta) \in [0, 1] \times [0, 2\pi). \tag{2.3}$$

Hence, $\partial\mathcal{S} = \mathcal{C}_0 \cup \mathcal{C}_1$ where $\mathcal{C}_0$ and $\mathcal{C}_1$ are two circles parametrized by ($\theta \in [0, 2\pi)$)

$$(\psi(0)\cos(\theta), \psi(0)\sin(\theta), \chi(0))$$

and

$$(\psi(1)\cos(\theta), \psi(1)\sin(\theta), \chi(1)),$$

respectively.

For simplicity, we suppose that

$$\chi_s(s) \geq 0, \qquad s \in (0,\epsilon) \cup (1-\epsilon, 1) \tag{2.4}$$

for some $\epsilon > 0$. With this condition it is possible to conclude that $\nu = \frac{\partial}{\partial s}$ on $\mathcal{C}_1$ and $\nu = -\frac{\partial}{\partial s}$ on $\mathcal{C}_0$.

We are interested in solutions of (1.1) that are independent of θ. In fact, it can be proved that these are the only ones that can be stable (the proof is similar to those made in [2, Proposition 5.1], for instance). Thus, based on the above considerations (see (2.2) and (2.4)), we will look for solutions to the following problem (we use $'$ to denote the derivative with respect to s)

$$\begin{cases} \mathcal{L}(u) := u''(s) + \dfrac{\psi'(s)}{\psi(s)} u'(s) = 0, & s \in (0,1) \\ -u'(0) = \lambda h(u(0)), & u'(1) = \lambda h(u(1)). \end{cases} \tag{2.5}$$

We will use the sub-supersolution method in the above problem. We recall that v is a *super-solution* (*sub-solution*) of (2.5) if it satisfies $\mathcal{L}(v) \leq 0$ ($\mathcal{L}(v) \geq 0$), $-v'(0) \geq \lambda h(v(0))$ and $v'(1) \geq \lambda h(v(1))$ ($-v'(0) \leq \lambda h(v(0))$ and $v'(1) \leq \lambda h(v(1))$).

The next result is widely known and a more general version can be found in [14].

Theorem 2.1. *If $\overline{v}$ is a super-solution and $\underline{v}$ is a sub-solution of (2.5) such that $\overline{v} \geq \underline{v}$ then there exists a solution w for the problem (2.5) such that $\overline{v}(s) \geq w(s) \geq \underline{v}(s)$, for all $s \in [0,1]$.*

The classical argument of linearized stability can be applied to the present situation. Let u_λ be a stationary solution of problem (1.1) and μ_1 the principal eigenvalue of the eigenvalue problem for the linearized problem

$$\begin{cases} \Delta_g \phi(x) = \mu \phi(x), & x \in \mathcal{S} \\ \partial_\nu \phi(x) = \lambda h'(u_\lambda(x)) \phi(x), & x \in \partial\mathcal{S}. \end{cases} \tag{2.6}$$

We have the following stability criterion: if $\mu_1 < 0$ then u_λ is stable and if $\mu_1 > 0$ then u_λ is unstable.

It is well known that μ_1 is characterized by Rayleigh variational principle, namely

$$\mu_1 = \sup_{\phi \in H^1(\mathcal{S}),\ \phi \not\equiv 0} J(\phi), \quad \text{where } J(\phi) = \frac{\int_{\mathcal{S}} -|\phi'|^2 + \int_{\partial\mathcal{S}} h'(u_\lambda)\phi^2}{\int_{\mathcal{S}} \phi^2}. \tag{2.7}$$

Finally, we are in position to prove Theorem 1.1.

2.2 Proof of Theorem 1.1

Claim 1. There are $\lambda_0 > 0$ and $l > 0$ such that if $\lambda > \lambda_0$ then $\underline{v_\lambda} = \tilde{u} - l/\lambda$ is a sub-solution and $\overline{v_\lambda} = \tilde{u} + l/\lambda$ is a super-solution of (2.5) where $\tilde{u}$ is the non-constant function given by (1.4).

It is not difficult to see that $\mathcal{L}(\tilde{u}) = 0$. Moreover,

$$\tilde{u}(0) = \alpha \quad \text{and} \quad \tilde{u}(1) = \beta. \tag{2.8}$$

We consider $\lambda_0 = -m/(M\delta)$ and $l = -m/M$ where

- $m = \max\{\tilde{u}'(0), \tilde{u}'(1)\}$;
- $M = \sup_{[\alpha-\delta,\alpha+\delta]\cup[\beta-\delta,\beta+\delta]} h'$ and $\delta > 0$ is so small such that $M < 0$.

Note that $m > 0$ since $\tilde{u}'(s) = \frac{\beta-\alpha}{\int_0^1 1/\psi(t)dt}(1/\psi(s)) > 0$ for any $s \in [0,1]$ and therefore $l > 0$ also.

Hence, $\mathcal{L}(\underline{v_\lambda}) = 0$ and, before analyzing $\underline{v_\lambda}'(s)$ for $s \in \{0,1\}$, we note that if $\lambda > \lambda_0$ then

$$\delta = l/\lambda_0 > l/\lambda.$$

By the Mean Value Theorem

$$h(\tilde{u}(0) - l/\lambda) = h(\tilde{u}(0)) - h'(\tilde{u}(0) - \eta_\lambda^\alpha)(l/\lambda) = -h'(\alpha - \eta_\lambda^\alpha)(l/\lambda)$$

for some $\eta_\lambda^\alpha \in [0, l/\lambda] \subset [0,\delta]$. Analogously

$$h(\tilde{u}(1) - l/\lambda) = h(\tilde{u}(1)) - h'(\tilde{u}(1) - \eta_\lambda^\beta)(l/\lambda) = -h'(\beta - \eta_\lambda^\beta)(l/\lambda)$$

for some $\eta_\lambda^\beta \in [0, l/\lambda] \subset [0,\delta]$.

Now,

$$\begin{aligned} -\underline{v_\lambda}'(0) - \lambda h(\underline{v_\lambda}(0)) &= -\tilde{u}'(0) - \lambda h(\tilde{u}(0) - l/\lambda) \\ &= -\tilde{u}'(0) + h'(\alpha - \eta_\lambda^\alpha)l \\ &\leq 0 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \underline{v_\lambda}'(1) - \lambda h(\underline{v_\lambda}(1)) &= \tilde{u}'(1) - \lambda h(\tilde{u}(1) - l/\lambda) \\ &= \tilde{u}'(1) + h'(\beta - \eta_\lambda^\beta)l \\ &\leq m + Ml = 0. \end{aligned} \tag{2.10}$$

It follows that $\underline{v_\lambda}$ is a sub-solution of (2.5). Similarly (with the same λ_0 and l) we prove that $\overline{v_\lambda}$ is a super-solution of (2.5) and **Claim 1** is proved.

By Theorem 2.1 there are u_λ ($\lambda > \lambda_0$) solutions of (2.5) such that $\underline{v_\lambda}(s) \leq u_\lambda(s) \leq \overline{v_\lambda}(s)$ for all $s \in [0,1]$. We note that u_λ are non-constant functions (for λ large) and

$$u_\lambda \to \tilde{u} \quad \text{as } \lambda \to \infty \text{ in } C^0([0,1]). \tag{2.11}$$

Claim 2: $\{u_\lambda\}_{\lambda>\lambda_0}$ is a family of stable stationary solutions of the problem (1.1).

Indeed, for any $\lambda > \lambda_0$, u_λ is a stationary solution independent of θ of the problem (1.1) and

$$\begin{aligned} u_\lambda(0) &\in [\alpha - l/\lambda, \alpha + l/\lambda] \subset [\alpha - \delta, \alpha + \delta]; \\ u_\lambda(1) &\in [\beta - l/\lambda, \beta + l/\lambda] \subset [\beta - \delta, \beta + \delta]. \end{aligned}$$

Hence, we can conclude that there is $R < 0$ such that for any $\phi \in H^1(\mathcal{S})$ ($\phi \not\equiv 0$),

$$J(\phi) = \frac{\int_{\mathcal{S}} -|\phi'|^2 + \int_{\partial\mathcal{S}} h'(u_\lambda)\phi^2}{\int_{\mathcal{S}} \phi^2} \leq R.$$

By (2.7), $\mu_1 < 0$ and therefore u_λ is stable. **Claim 2** is proved as well as Theorem 1.1.

3 Examples and concluding remarks

Consider $\mathcal{S}_j$ $(j = 1, \ldots, 3)$ surfaces of revolution parametrized by (2.3) where

$(\mathcal{S}_1)$ $\psi_1(s) = (2/\pi)\sin((1/2) + (\pi s)/2)$ and $\chi_1(s) = (2/\pi)\cos((1/2) + (\pi s)/2)$;

$(\mathcal{S}_2)$ $\psi_2(s) = s^2/4 + 1/2$ and $\chi_2(s) = (s/4)\sqrt{4 - s^2} + \arcsin(s/2)$;

$(\mathcal{S}_3)$ $\psi_3(s) = 1$ and $\chi_3(s) = s$

with $s \in [0, 1]$ in all three cases. The surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are plotted in Figure 1.1, while $\mathcal{S}_3$ is a finite straight cylinder. If we suppose that $h(\cdot)$ satisfies (1.2) we can use Theorem 1.1 to conclude that there is $\lambda_0^j > 0$ and a family of patterns $\{u_\lambda^j\}_{\lambda > \lambda_0^j}$ to the problem (1.1) on $\mathcal{S}_j$ $(j = 1, \ldots, 3)$. Moreover, u_λ^j is independent of θ and

$$u_\lambda^j \to \frac{\beta - \alpha}{\int_0^1 [1/\psi_j(t)]\, dt} \int_0^s [1/\psi_j(t)]\, dt + \alpha \quad \text{as } \lambda \to \infty \text{ in } C^0([0,1]).$$

It is important to note that it is not difficult to estimate a value for λ_0^j. For instance, for the problem on the surface $\mathcal{S}_3$ above and $h(u) = -u(u+1)(u-2)$, a direct computation gives us $\lambda_0^3 < 11$.

Remark 3.1. The hypothesis (1.2) is satisfied by notable functions, for instance: the Allen–Cahn and the Peierls–Nabarro nonlinearities, respectively given by $h(u) = u - u^3$ and $h(u) = \sin(\pi u)$.

Remark 3.2. The fact that $\mathcal{S}$ has disconnected boundary is fundamental in the proof of Theorem 1.1. Nothing is known about the same problem with nonlinear flux on the boundary when this boundary is connected (i.e., when $\mathcal{S}$ has one of the poles).

Remark 3.3. It is possible to obtain a similar result if we replace surfaces of revolutions by a specific class of n-dimensional Riemannian manifolds.

Let M_η be a manifold of dimension $n \geq 2$ admitting a pole o whose metric $\tilde{g}$ is given, in polar coordinates around o, by

$$ds^2 = dr^2 + \eta^2(r)d\theta \quad (r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1} \tag{3.1}$$

where r is the geodesic distance of the point $P = (r, \theta)$ to the pole o, $d\theta^2$ is the canonical metric on the unit sphere $\mathbb{S}^{n-1}$ and η is a smooth function in $[0, \infty)$ such that (here, we use $'$ to denote the derivative with respect to r)

$$\eta(0) = \eta''(0) = 0, \qquad \eta'(0) = 1 \quad \text{and} \quad \eta(r) > 0 \quad \text{for } r \in (0, \infty). \tag{3.2}$$

M_η is called *model manifold* or *spherically symmetric manifold* (for more details, see [12]) and we goal is to consider the same diffusion equation with nonlinear flux on the boundary, in $\Lambda := B_1(o) \setminus B_r(o) \subset M_\eta$ $(0 < r < 1)$. It is not difficult to see that (compare (3.1) and (2.1))

$$\Delta_{\tilde{g}} u = u'' + (n-1)\frac{\eta'}{\eta} u_r + \frac{1}{\eta^2}\Delta_{\mathbb{S}^{n-1}}, \tag{3.3}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace–Beltrami operator in $\mathbb{S}^{n-1}$. Thus, if we look only at radial solutions, it is a simple exercise to prove the Theorem 1.1 with Λ instead of $\mathcal{S}$ (see [1–3, 6, 17] where a diffusion problem in Λ also was considered).

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