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On the Oscillation of Third Order Half-linear Neutral Type Difference Equations

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Abstract

In this paper, the authors study the oscillatory properties of third order quasilinear neutral difference equation of the form

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-\delta}))^\alpha) + q_n x_{n-\tau}^\alpha = 0, \quad n \geq 0, \qquad (E)$$

where $\alpha > 0, q_n \geq 0, 0 \leq p_n \leq p < \infty$. By using Riccati transformation we establish some new sufficient conditions which ensure that every solution of equation (E) is either oscillatory or converges to zero. These results improve some known results in the literature. Examples are provided to illustrate the main results.

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1 Introduction

Consider a neutral type difference equation of the form

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-\delta}))^\alpha) + q_n x_{n-\tau}^\alpha = 0, n \in \mathbb{N}, \qquad (1.1)$$

where δ and τ are nonnegative integers, $\{a_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$ for all $n_0 \in \mathbb{N} = \{1, 2, ...\}$, $\{p_n\}$ is a bounded nonnegative real sequence, $\{q_n\}$ is a nonnegative real sequence, and α is a ratio of odd positive integers.

Let $\theta = \max(\delta, \tau)$. By *solution* of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq 1 - \theta$ and satisfies equation (1.1) for all $n \in \mathbb{N}$. A nontrivial solution of equation (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. The equation (1.1) is said to be *almost oscillatory* if all its solutions are either oscillatory or tend to zero as $n \to \infty$.

The oscillation theory of difference equations and their applications have been receiving intensive attention in the last few decades, see for example [1, 5, 11] and the references cited therein. Especially the study of oscillatory behavior of second order equations of various types occupied a great deal of interest. However, the study of third order difference equations has received considerably less attention even though such equations have wide applications in the fields such as economics, mathematical biology and many other areas of mathematics.

In [7], the authors considered the equation

$$\Delta(c_n\Delta(d_n\Delta x_n)) + q_n f(x_{n-\sigma+1}) = 0 \tag{1.2}$$

and studied oscillatory and asymptotic behavior of solutions of equation (1.2) subject to the conditions

$$\Delta c_n \geq 0, \sum_{n=n_0}^{\infty} \frac{1}{c_n} = \sum_{n=n_0}^{\infty} \frac{1}{d_n} = \infty. \tag{1.3}$$

In [2], the authors classified the nonoscillatory solutions of equation (1.2) into different classes and established conditions concerning the existence of solutions in these classes.

In [6], the authors considered the equation

$$\Delta(c_n(\Delta^2 x_n)^\alpha) + q_n f(x_{\sigma(n)}) = 0 \tag{1.4}$$

where $\sigma(n) < n$ and α is a quotient of odd positive integers, and studied the oscillatory behavior of equation (1.4) under the condition $\sum\limits_{n=n_0}^{\infty} \frac{1}{c_n^{\frac{1}{\alpha}}} < \infty$.

In [14], the authors studied the oscillatory and asymptotic behavior of solutions of the equation

$$\Delta(c_n\Delta(d_n(\Delta x_n)^\alpha)) + q_n f(x_{n-\sigma}) = 0 \tag{1.5}$$

under the conditions $\sum\limits_{n=n_0}^{\infty} \frac{1}{c_n^\alpha} = \infty$ and $\sum\limits_{n=n_0}^{\infty} \frac{1}{d_n} = \infty$.

In [23], the authors considered the following equation

$$\Delta(c_n\Delta(d_n\Delta(x_n + p_n x_{n-k}))) + q_n f(x_{n-m}) = 0 \tag{1.6}$$

and established criteria for the oscillation of all solutions of equation (1.6) under the condition (1.3).

In [15] the authors considered the third order equation of the form

$$\Delta(c_n(\Delta(d_n\Delta(x_n + p_n x_{n-\tau})))^\alpha) + q_n f(x_{n-\sigma}) = 0 \tag{1.7}$$

and established conditions for the oscillation of all solutions of equation (1.7) under the condition (1.3) without assuming $\Delta c_n \geq 0$. For further results concerning the oscillatory and asymptotic behavior of third order difference equation one can refer to [2, 13, 16–22] and the references cited there in.

From the review of literature it is found that most of the results for the oscillation of third order neutral type difference equations are obtained under the assumption $-1 < p_n < 1$. So it is interesting to study the oscillatory behavior of equation (1.1) under the condition $0 \leq p_n \leq p < \infty$. To the best of our knowledge, there are no results regarding the oscillation of equation(1.1) under the assumption $p_n \geq 1$. Therefore the purpose of this paper is to present some new oscillatory and asymptotic criteria for equation (1.1). We establish criteria for the equation (1.1) to be almost oscillatory.

The paper is organized as follows. In Section 2, we present the main results and in Section 3, we provide some examples to illustrate the main results.

2 Oscillation Results

In this section, we establish some new oscillation criteria for the equation (1.1). We begin with some useful lemmas, which will be used later. We set $z_n = x_n + p_n x_{n-\delta}$, and we may deal only with the positive solutions of equation (1.1) since the proof for the opposite case is similar. We also introduce a usual convention, namely for any sequence $\{f_k\}$ and any $m \in \mathbb{N}$ we put $\sum\limits_{k=m}^{m-1} f_k = 0$ and $\prod\limits_{k=m}^{m-1} f_k = 1$.

Lemma 2.1. *Assume that* $\alpha \geq 1$, $x_1, x_2 \in [0, \infty)$. *Then*

$$x_1^\alpha + x_2^\alpha \geq \frac{1}{2^{\alpha-1}}(x_1 + x_2)^\alpha. \tag{2.1}$$

Proof. The proof can be found in [8, pp. 292] and also in [9, Remark 2.1].

Lemma 2.2. *Assume that* $0 < \alpha \leq 1$, $x_1, x_2 \in [0, \infty)$. *Then*

$$x_1^\alpha + x_2^\alpha \geq (x_1 + x_2)^\alpha. \tag{2.2}$$

Proof. Assume that $x_1 = 0$ or $x_2 = 0$. Then we have (2.2). Assume that $x_1 > 0, x_2 > 0$. Define $f(x_1, x_2) = x_1^\alpha + x_2^\alpha - (x_1 + x_2)^\alpha$. Fix x_1. Then

$$\begin{aligned} \frac{df(x_1, x_2)}{dx_2} &= \alpha x_2^{\alpha-1} - \alpha(x_1 + x_2)^{\alpha-1} \\ &= \alpha[x_2^{\alpha-1} - (x_1 + x_2)^{\alpha-1}] \geq 0, \text{ since } 0 < \alpha \leq 1. \end{aligned}$$

Thus, f is nondecreasing with respect to x_2, which yields $f(x_1, x_2) \geq 0$. This completes the proof.

Lemma 2.3. *Let* $\{f_n\}$ *and* $\{g_n\}$ *be real sequences, and suppose there exists a* $\sigma > 0$ *and a sequence* $\{h_n\}$ *such that* $f_n = h_n + g_n h_{n-\sigma}$ *holds for all* $n \geq n_0 \in \mathbb{N}$. *Suppose that* $\lim\limits_{n\to\infty} f_n$ *exists and* $\lim\limits_{n\to\infty} \inf g_n > -1$. *Then* $\lim\limits_{n\to\infty} \sup h_n > 0$ *implies* $\lim\limits_{n\to\infty} f_n > 0$.

Proof. The proof can be modeled similar to that of Lemma 3 of [10], and hence the details are omitted.

Lemma 2.4. *Assume that* $\{x_n\}$ *is a positive solution of equation* (1.1) *and* $\lim\limits_{n\to\infty} x_n \neq 0$. *If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a_{s-\delta}} \sum_{t=s}^{\infty} Q_t \right)^{\frac{1}{\alpha}} = \infty \tag{2.3}$$

where

$$Q_n = \min\{q_n, q_{n-\delta}\}, \tag{2.4}$$

then

$$z_n > 0, \Delta z_n > 0, \Delta^2 z_n > 0, \Delta(a_n(\Delta^2 z_n)^\alpha) \leq 0 \tag{2.5}$$

for $n \geq n_1 \in \mathbb{N}$, *where* n_1 *is sufficiently large.*

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1). We may deal only with the case $\alpha \geq 1$, since the case $0 < \alpha \leq 1$ is similar. From equation(1.1), we see that $z_n \geq x_n > 0$ and

$$\Delta(a_n(\Delta^2 z_n)^\alpha) = -q_n x_{n-\tau}^\alpha \leq 0. \tag{2.6}$$

Then, $\{(a_n(\Delta^2 z_n)^\alpha)\}$ is nonincreasing and eventually of one sign. Therefore $\{\Delta^2 z_n\}$ is also of one sign and so we have two possibilities: $\Delta^2 z_n > 0$ or $\Delta^2 z_n < 0$ for all $n \geq n_1 \in \mathbb{N}$. We claim that $\Delta^2 z_n > 0$. If not, then there exists a constant $M > 0$ such that

$$(a_n(\Delta^2 z_n)^\alpha) \leq -M < 0.$$

Summing the above inequality from n_1 to $n-1$, we obtain

$$\Delta z_n \leq \Delta z_{n_1} - M^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}.$$

Therefore, $\lim_{n\to\infty} \Delta z_n = -\infty$. Then, from $\Delta^2 z_n < 0$ and $\Delta z_n < 0$, we have $\lim_{n\to\infty} z_n = -\infty$. This contradiction proves that $\Delta^2 z_n > 0$.

Next, we prove that $\Delta z_n > 0$. Otherwise, we assume that $\Delta z_n \leq 0$. From equation (1.1), we have

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha) + q_n x_{n-\tau}^\alpha + p^\alpha q_{n-\delta} x_{n-\tau-\delta}^\alpha = 0$$

and then using Lemma 2.1, we obtain

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha) + \frac{Q_n}{2^{\alpha-1}} z_{n-\tau}^\alpha \leq 0. \tag{2.7}$$

Summing the last inequality from n to ∞, we obtain

$$a_n(\Delta^2 z_n)^\alpha + p^\alpha (a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha) \geq \frac{1}{2^{\alpha-1}} \sum_{s=n}^{\infty} Q_s z_{s-\tau}^\alpha.$$

In view of (2.6), we see that

$$a_n(\Delta^2 z_n)^\alpha \leq a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha.$$

Thus

$$(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha) \geq \frac{1}{2^{\alpha-1}(1+p^\alpha)} \sum_{s=n}^{\infty} Q_s z_{s-\tau}^\alpha.$$

Since $\lim_{n\to\infty} x_n \neq 0$, from Lemma 2.3, $\lim_{n\to\infty} z_n = L > 0$, and $z_{n-\tau}^\alpha \geq L^\alpha$. Then we obtain

$$\Delta^2 z_{n-\delta} \geq L \left(\frac{1}{2^{\alpha-1}(1+p^\alpha)} \right)^{\frac{1}{\alpha}} \left(\frac{1}{a_{s-\delta}} \sum_{s=n}^{\infty} Q_s \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from n to ∞, we have

$$-\Delta z_{n-\delta} \geq L \left(\frac{1}{2^{\alpha-1}(1+p^\alpha)} \right)^{\frac{1}{\alpha}} \sum_{s=n}^{\infty} \left(\frac{1}{a_{s-\delta}} \sum_{t=s}^{\infty} Q_t \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality again from n_1 to ∞, we have

$$z_{n_1-\delta} \geq L \left(\frac{1}{2^{\alpha-1}(1+p^\alpha)} \right)^{\frac{1}{\alpha}} \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \left(\frac{1}{a_{s-\delta}} \sum_{t=s}^{\infty} Q_t \right)^{\frac{1}{\alpha}},$$

which contradicts (2.3). Thus $\Delta z_n > 0$. The proof is now complete.

Lemma 2.5. *Assume that $\{z_n\}$ satisfies (2.5) for $n \geq n_1 \in \mathbb{N}$. Then*

$$\Delta z_n \geq \left(a_n^{\frac{1}{\alpha}} \Delta^2 z_n \right) \beta_1(n, n_1), \tag{2.8}$$

and

$$z_n \geq \left(a_n^{\frac{1}{\alpha}} \Delta^2 z_n \right) \beta_2(n, n_1), \tag{2.9}$$

where

$$\beta_1(n, n_1) = \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}, \quad \beta_2(n, n_1) = \sum_{s=n_1}^{n-1} \frac{(n-1-s)}{a_s^{\frac{1}{\alpha}}}.$$

Proof. Since $\Delta(a_n(\Delta^2 z_n)^\alpha) \le 0$, we have $a_n(\Delta^2 z_n)^\alpha$ is nonincreasing. Then we obtain,

$$\begin{aligned}\Delta z_n \ge \Delta z_n - \Delta z_{n_1} &= \sum_{s=n_1}^{n-1} \frac{(a_s(\Delta^2 z_s)^\alpha)^{\frac{1}{\alpha}}}{a_s^{\frac{1}{\alpha}}} \\ &\ge \left(a_n^{\frac{1}{\alpha}} \Delta^2 z_n\right) \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}.\end{aligned}$$

Similarly, we have

$$z_n \ge (a_n^{\frac{1}{\alpha}} \Delta^2 z_n) \sum_{s=n_1}^{n-1} \sum_{t=n_1}^{s-1} \frac{1}{a_t^{\frac{1}{\alpha}}}.$$

Since

$$\sum_{s=n_1}^{n-1} \sum_{t=n_1}^{s-1} \frac{1}{a_t^{\frac{1}{\alpha}}} = \sum_{t=n_1}^{n-1} \sum_{s=t+1}^{n-1} \frac{1}{a_t^{\frac{1}{\alpha}}} = \sum_{t=n_1}^{n-1} \frac{(n-1-t)}{a_t^{\frac{1}{\alpha}}},$$

and therefore

$$z_n \ge \left(a_n^{\frac{1}{\alpha}} \Delta^2 z_n\right) \beta_2(n, n_1).$$

Lemma 2.6. *Let $\alpha > 0$. If $f_n > 0$ and $\Delta f_n > 0$ for all $n \ge n_0 \in \mathbb{N}$, then*

$$\Delta f_n^\alpha \ge \alpha f_n^{\alpha-1} \Delta f_n \text{ if } \quad \alpha \ge 1,$$

and

$$\Delta f_n^\alpha \ge \alpha f_{n+1}^{\alpha-1} \Delta f_n \text{ if } \quad 0 < \alpha \le 1$$

for all $n \ge n_0$.

Proof. By Mean value theorem, we have for $n \ge n_0$

$$\Delta f_n^\alpha = f_{n+1}^\alpha - f_n^\alpha = \alpha t^{\alpha-1} \Delta f_n$$

where $f_n < t < f_{n+1}$. The result follows by taking $t > f_n$ when $\alpha \ge 1$ and $t < f_{n+1}$ when $0 < \alpha \le 1$.

Next, we state and prove the main theorems.

Theorem 2.1. *Let $\alpha \ge 1$. Assume that (2.3) holds and $\tau \ge \delta$. Further, assume that there exists a positive nondecreasing sequence $\{\rho_n\}$, such that for any $n_1 \in \mathbb{N}$, there exists an integer $n_2 > n_1$, with*

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta \rho_s)^{\alpha+1}}{(\rho_s \beta_1(s-\tau, n_1))^\alpha}\right] = \infty. \tag{2.10}$$

Then equation (1.1) is almost oscillatory.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1) which does not tend to zero as $n \to \infty$. From the proof of Lemma 2.4, we obtain (2.5) and (2.7). Define

$$w_n = \rho_n \frac{a_n(\Delta^2 z_n)^\alpha}{z_{n-\tau}^\alpha}. \tag{2.11}$$

Then $w_n > 0$ due to Lemma 2.4. From (2.11) and Lemma 2.6, we have

$$
\begin{aligned}
\Delta w_n &= \Delta\rho_n \frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha}{z^\alpha_{n+1-\tau}} + \rho_n \Delta\left(\frac{a_n(\Delta^2 z_n)^\alpha}{z^\alpha_{n-\tau}}\right)\\
&= \Delta\rho_n \frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha}{z^\alpha_{n+1-\tau}} + \rho_n \frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{z^\alpha_{n-\tau}} - \rho_n \frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha}{z^\alpha_{n+1-\tau} z^\alpha_{n-\tau}} \Delta z^\alpha_{n-\tau}\\
&\le \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \frac{\rho_n \Delta(a_n(\Delta^2 z_n)^\alpha)}{z^\alpha_{n-\tau}} - \frac{\alpha\rho_n a_{n+1}(\Delta^2 z_{n+1})^\alpha}{z^\alpha_{n+1-\tau} z^\alpha_{n-\tau}} z^{\alpha-1}_{n-\tau} \Delta z_{n-\tau}.
\end{aligned}
\tag{2.12}
$$

From (2.5) and (2.8), we have

$$
\Delta z_{n-\tau} \ge (a^{\frac{1}{\alpha}}_{n-\tau} \Delta^2 z_{n-\tau})\beta_1(n-\tau, n_1) \ge (a^{\frac{1}{\alpha}}_{n+1} \Delta^2 z_{n+1})\beta_1(n-\tau, n_1).
$$

It follows from (2.11) and (2.12) that

$$
\Delta w_n \le \frac{\rho_n \Delta(a_n(\Delta^2 z_n)^\alpha)}{z^\alpha_{n-\tau}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n \beta_1(n-\tau, n_1)}{\rho^{\frac{1}{\alpha}+1}_{n+1}} w^{\frac{\alpha+1}{\alpha}}_{n+1}. \tag{2.13}
$$

Similarly, define another function v_n by

$$
v_n = \rho_n \frac{(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z^\alpha_{n-\tau}}. \tag{2.14}
$$

Then $v_n > 0$ due to Lemma 2.4. From (2.14) and Lemma 2.6, we have

$$
\begin{aligned}
\Delta v_n &= \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} + \rho_n \Delta\left(\frac{a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha}{z^\alpha_{n-\tau}}\right)\\
&= \rho_n \frac{\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z^\alpha_{n-\tau}} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\rho_n a_{n+1-\delta}(\Delta^2 z_{n+1-\delta})^\alpha}{z^\alpha_{n+1-\tau} z^\alpha_{n-\tau}} \Delta z^\alpha_{n-\tau}\\
&\le \rho_n \frac{\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z^\alpha_{n-\tau}} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \alpha \frac{\rho_n a_{n+1-\delta}(\Delta^2 z_{n+1-\delta})^\alpha}{z^\alpha_{n+1-\tau} z_{n-\tau}} \Delta z_{n-\tau}.
\end{aligned}
\tag{2.15}
$$

From (2.5) and (2.8) and $\tau \ge \delta$, we have

$$
\Delta z_{n-\tau} \ge (a^{\frac{1}{\alpha}}_{n-\tau} \Delta^2 z_{n-\tau})\beta_1(n-\tau, n_1) \ge (a^{\frac{1}{\alpha}}_{n-\delta} \Delta^2 z_{n-\delta})\beta_1(n-\tau, n_1).
$$

Then from (2.15), we have

$$
\Delta v_n \le \rho_n \frac{\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z^\alpha_{n-\tau}} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha\rho_n \beta_1(n-\tau, n_1) v^{\frac{(\alpha+1)}{\alpha}}_{n+1}}{\rho^{1+\frac{1}{\alpha}}_{n+1}}. \tag{2.16}
$$

From (2.13) and (2.16), we obtain

$$
\begin{aligned}
\Delta w_n + p^\alpha \Delta v_n \le\;& \frac{\rho_n \left[\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)\right]}{z^\alpha_{n-\tau}}\\
&+ \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n \beta_1(n-\tau, n_1)}{\rho^{1+\frac{1}{\alpha}}_{n+1}} w^{\frac{\alpha+1}{\alpha}}_{n+1}\\
&+ p^\alpha \left[\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha\rho_n \beta_1(n-\tau, n_1) v^{\frac{(\alpha+1)}{\alpha}}_{n+1}}{\rho^{1+\frac{1}{\alpha}}_{n+1}}\right].
\end{aligned}
\tag{2.17}
$$

From (2.7) and (2.17), we have

$$\begin{aligned}\Delta w_n + p^\alpha \Delta v_n &\leq -\rho_n \frac{Q_n}{2^{\alpha-1}} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha \rho_n \beta_1(n-\tau, n_1)}{\rho_{n+1}^{1+\frac{1}{\alpha}}} w_{n+1}^{\frac{\alpha+1}{\alpha}} \\ &+ p^\alpha \left[\frac{\Delta \rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha \rho_n \beta_1(n-\tau, n_1) v_{n+1}^{\frac{(\alpha+1)}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}}} \right]. \end{aligned} \tag{2.18}$$

Using (2.18) and the inequality

$$Bu - Au^{\frac{(\alpha+1)}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0 \tag{2.19}$$

we have

$$\Delta w_n + p^\alpha \Delta v_n \leq -\rho_n \frac{Q_n}{2^{\alpha-1}} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta \rho_n)^{\alpha+1}}{(\rho_n \beta_1(n-\tau, n_1))^\alpha} + \frac{(\Delta \rho_n)^{\alpha+1}}{(\rho_n \beta_1(n-\tau, n_1))^\alpha} \frac{p^\alpha}{(\alpha+1)^{\alpha+1}}.$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} \left[\rho_s \frac{Q_s}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} (1+p^\alpha) \frac{(\Delta \rho_s)^{\alpha+1}}{(\rho_s \beta_1(s-\tau, n_1))^\alpha} \right] \leq w_{n_2} + p^\alpha v_{n_2}.$$

Taking lim sup in the above inequality, we obtain a contradiction with (2.10). The proof is complete.

By using the inequality in Lemma 2.2, we obtain the following result.

Theorem 2.2. *Let $0 < \alpha \leq 1$. Assume that (2.3) holds and $\tau \geq \delta$. Further, assume that there exists a positive nondecreasing sequence $\{\rho_n\}$, such that for any $n_1 \in \mathbb{N}$, there exists an integer $n_2 > n_1$, with*

$$\limsup_{n\to\infty} \sum_{s=n_2}^{n-1} \left[\rho_s Q_s - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta \rho_s)^{\alpha+1}}{(\rho_s \beta_1(s-\tau, n_1))^\alpha} \right] = \infty.$$

Then equation (1.1) is almost oscillatory.

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted.

Theorem 2.3. *Let $\alpha \geq 1$. Assume that (2.3) holds and $\tau \geq \delta$. Further, assume that there exists a positive nondecreasing sequence $\{\rho_n\}$, such that for any $n_1 \in \mathbb{N}$, there exists an integer $n_2 > n_1$, with*

$$\limsup_{n\to\infty} \sum_{s=n_2}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{4\alpha} \frac{(\Delta \rho_s)^2}{\rho_s (\beta_2(s-\tau, n_1))^{\alpha-1} \beta_1(s-\tau, n_1)} \right] = \infty. \tag{2.20}$$

Then equation (1.1) is almost oscillatory.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1), which does not tend to zero asymptotically. By the proof of Lemma 2.4, we have (2.5) and (2.7). Then from Lemma 2.5, we obtain (2.8) and (2.9).

Define w_n and v_n by (2.11) and (2.14) respectively. Proceeding as in the proof of Theorem 2.1, we obtain (2.12) and (2.15). It follows from (2.12) that

$$\Delta w_n \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} + \frac{\rho_n \Delta(a_n (\Delta^2 z_n)^\alpha)}{z_{n-\tau}^\alpha} - \frac{\alpha \rho_n (\rho_{n+1} a_{n+1} (\Delta^2 z_{n+1})^\alpha)^2 z_{n-\tau}^{\alpha-1} \Delta z_{n-\tau}}{\rho_{n+1} z_{n+1-\tau}^{2\alpha} (\rho_{n+1} a_{n+1} (\Delta^2 z_{n+1})^\alpha)}. \tag{2.21}$$

In view of (2.5),(2.8) and (2.9), we see that

$$\frac{z_{n-\tau}^{\alpha-1}\Delta z_{n-\tau}}{a_{n+1}(\Delta^2 z_{n+1})^\alpha} \geq \frac{z_{n-\tau}^{\alpha-1}\Delta z_{n-\tau}}{a_n(\Delta^2 z_n)^\alpha} \geq (\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1). \tag{2.22}$$

Substituting (2.22) in (2.21), we have

$$\Delta w_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} + \frac{\rho_n\Delta(a_n(\Delta^2 z_n)^\alpha)}{z_{n-\tau}^\alpha} - \frac{\alpha\rho_n}{\rho_{n+1}^2}(\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1)w_{n+1}^2. \tag{2.23}$$

On the other hand, from (2.15), we have

$$\Delta v_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} + \frac{\rho_n\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z_{n-\tau}^\alpha} - \frac{\alpha\rho_n(\rho_{n+1}a_{n+1-\delta}(\Delta^2 z_{n+1-\delta})^\alpha)^2 z_{n-\tau}^{\alpha-1}\Delta z_{n-\tau}}{\rho_{n+1}^2 z_{n+1-\tau}^{2\alpha}a_{n+1-\delta}(\Delta^2 z_{n+1-\delta})^\alpha}. \tag{2.24}$$

By (2.5),(2.8),(2.9) and $\tau > \delta$, we see that

$$\frac{z_{n-\tau}^{\alpha-1}\Delta z_{n-\tau}}{a_{n+1-\delta}(\Delta^2 z_{n+1-\delta})^\alpha} \geq \frac{z_{n-\tau}^{\alpha-1}\Delta z_{n-\tau}}{a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha} \geq (\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1). \tag{2.25}$$

Substituting (2.25) into (2.24), we obtain

$$\Delta v_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} + \frac{\rho_n\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z_{n-\tau}^\alpha} - \frac{\alpha\rho_n}{\rho_{n+1}^2}(\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1)v_{n+1}^2. \tag{2.26}$$

Using (2.23) and (2.26), we have

$$\begin{aligned}\Delta w_n + p^\alpha\Delta v_n &\leq \frac{\rho_n\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z_{n-\tau}^\alpha} \\ &+ \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\alpha\rho_n}{\rho_{n+1}^2}(\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1)w_{n+1}^2 \\ &+ p^\alpha\left[\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1} - \frac{\alpha\rho_n}{\rho_{n+1}^2}(\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1)v_{n+1}^2\right]. \end{aligned} \tag{2.27}$$

Applying (2.7), and the inequality $Bu - Au^2 \leq \frac{B^2}{4A}$, $A > 0$ in (2.27), we have

$$\Delta w_n + p^\alpha\Delta v_n \leq -\rho_n\frac{Q_n}{2^{\alpha-1}} + \frac{(1+p^\alpha)}{4\alpha\rho_n}\frac{(\Delta\rho_n)^2}{(\beta_2(n-\tau,n_1))^{\alpha-1}\beta_1(n-\tau,n_1)}. \tag{2.28}$$

Summing (2.28) from $n_2(n_2 \geq n_1)$ to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1}\left[\rho_s\frac{Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{4\alpha\rho_s}\frac{(\Delta\rho_s)^2}{(\beta_2(s-\tau,n_1))^{\alpha-1}\beta_1(s-\tau,n_1)}\right] \leq w_{n_2} + p^\alpha v_{n_2}.$$

Taking lim sup in the above inequality, we obtain a contradiction with (2.20). The proof is complete.

From Lemma 2.2, similar to the proof of Theorem 2.3, we obtain the following result.

Theorem 2.4. *Let $0 < \alpha \leq 1$. Assume that (2.3) holds and $\tau \geq \delta$. Further more, assume that there exists a positive nondecreasing sequence $\{\rho_n\}$, such that for any $n_1 \in \mathbb{N}$, there exists an integer $n_2 > n_1$, with*

$$\lim_{n\to\infty}\sup\sum_{s=n_2}^{n-1}\left[\rho_s Q_s - \frac{(1+p^\alpha)}{4\alpha}\frac{(\Delta\rho_s)^2}{\rho_s(\beta_2(s-\tau,n_1))^{\alpha-1}\beta_1(s-\tau,n_1)}\right] = \infty.$$

Then equation (1.1) is almost oscillatory.

Next we establish some criteria for the oscillation of equation (1.1) for the case when $\tau \leq \delta$.

Theorem 2.5. *Assume that* (2.3) *holds,* $\alpha \geq 1$ *and* $\tau \leq \delta$*. Further, assume that there exists a positive nondecreasing sequence* $\{\rho_n\}$*, such that for any* $n_1 \in \mathbb{N}$*, there exists an integer* $n_2 > n_1$*, with*

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s(\beta_1(s-\delta, n_1))^\alpha} \right] = \infty. \tag{2.29}$$

Then equation (1.1) *is almost oscillatory.*

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1), which does not tend to zero as $n \to \infty$. From the proof of Lemma 2.4, we obtain (2.5) and (2.7). Hence by Lemma 2.5, we have (2.8). Define

$$w_n = \rho_n \frac{a_n(\Delta^2 z_n)^\alpha}{z_{n-\delta}^\alpha}. \tag{2.30}$$

Then $w_n > 0$. From (2.30) and Lemma 2.6 we have

$$\begin{aligned} \Delta w_n &= \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \rho_n \Delta\left(\frac{a_n(\Delta^2 z_n)^\alpha}{z_{n-\delta}^\alpha}\right) \\ &\leq \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \rho_n \frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{z_{n-\delta}^\alpha} - \frac{\alpha\rho_n a_{n+1}(\Delta^2 z_{n+1})^\alpha}{z_{n+1-\delta}^\alpha z_{n-\delta}^\alpha} z_{n-\delta}^{\alpha-1} \Delta z_{n-\delta.} \end{aligned} \tag{2.31}$$

By (2.5) and (2.8), we have

$$\Delta z_{n-\delta} \geq (a_{n-\delta}^{\frac{1}{\alpha}}(\Delta^2 z_{n-\delta}))\beta_1(n-\delta, n_1) \geq (a_{n+1}^{\frac{1}{\alpha}} \Delta^2 z_{n+1})\beta_1(n-\delta, n_1).$$

It follows from (2.31) and (2.30) that

$$\Delta w_n \leq \frac{\rho_n \Delta(a_n(\Delta^2 z_n)^\alpha)}{z_{n-\delta}^\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta, n_1) w_{n+1}^{\frac{(\alpha+1)}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}. \tag{2.32}$$

Similarly, define another function v_n by

$$v_n = \rho_n \frac{a_{n-\delta(\Delta^2 z_{n-\delta})^\alpha}}{z_{n-\delta}^\alpha}. \tag{2.33}$$

Then $v_n > 0$,

$$\Delta v_n \leq \rho_n \frac{\Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)}{z_{n-\delta}^\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta, n_1) v_{n+1}^{\frac{(\alpha+1)}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}. \tag{2.34}$$

From (2.32) and (2.34), we have

$$\begin{aligned} \Delta w_n + p^\alpha \Delta v_n \leq\ & \frac{\rho_n\left[\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-\delta}(\Delta^2 z_{n-\delta})^\alpha)\right]}{z_{n-\delta}^\alpha} \\ &+ \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta, n_1)}{\rho_{n+1}^{1+\frac{1}{\alpha}}} w_{n+1}^{\frac{\alpha+1}{\alpha}} \\ &+ p^\alpha \left[\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta, n_1) v_{n+1}^{\frac{(\alpha+1)}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right]. \end{aligned} \tag{2.35}$$

By (2.5),(2.7),(2.35) and $\tau \leq \delta$, we obtain

$$\begin{aligned}\Delta w_n + p^\alpha \Delta v_n &\leq -\rho_n \frac{Q_n}{2^{\alpha-1}} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta,n_1)}{\rho_{n+1}^{1+\frac{1}{\alpha}}} w_{n+1}^{\frac{\alpha+1}{\alpha}} \\ &+ p^\alpha \left[\frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \frac{\alpha\rho_n\beta_1(n-\delta,n_1) v_{n+1}^{\frac{(\alpha+1)}{\alpha}}}{\rho_{n+1}^{1+\frac{1}{\alpha}}}\right]. \qquad (2.36)\end{aligned}$$

From (2.36) and the inequality (2.19), we have

$$\Delta w_n + p^\alpha \Delta v_n \leq -\rho_n \frac{Q_n}{2^{\alpha-1}} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_n)^{\alpha+1}}{(\rho_n\beta_1(n-\delta,n_1))^\alpha} + \frac{(\Delta\rho_n)^{\alpha+1}}{(\rho_n\beta_1(n-\delta,n_1))^\alpha} \frac{p^\alpha}{(\alpha+1)^{\alpha+1}}.$$

Summing the last inequality from $n_2 (n_2 \geq n_1)$ to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} \left[\rho_s \frac{Q_s}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} (1+p^\alpha) \frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s\beta_1(s-\delta,n_1))^\alpha}\right] \leq w_{n_2} + p^\alpha v_{n_2}.$$

Taking lim sup in the above inequality, we obtain a contradiction with (2.29).The proof is complete.

From Lemma 2.2, similar to the proof of Theorem 2.5, we obtain the following result.

Theorem 2.6. *Assume that* (2.3) *holds,* $0 < \alpha \leq 1$ *and* $\tau \leq \delta$. *Further, assume that there exists a positive nondecreasing function* $\{\rho_n\}$, *such that for any* $n_1 \in \mathbb{N}$, *there exists an integer* $n_2 > n_1$, *with*

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\rho_s Q_s - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s\beta_1(s-\delta,n_1))^\alpha}\right] = \infty.$$

Then equation (1.1) *is almost oscillatory.*

Using the method of proof adapted in Theorem 2.3, we obtain the following result.

Theorem 2.7. *Assume that* (2.3) *holds,* $\alpha \geq 1$ *and* $\tau \leq \delta$. *Further, assume that there exists a positive nondecreasing sequence* $\{\rho_n\}$, *such that for any* $n_1 \in \mathbb{N}$, *there exists an integer* $n_2 > n_1$, *with*

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{4\alpha} \frac{(\Delta\rho_s)^2}{\rho_s(\beta_2(s-\delta,n_1))^{\alpha-1}\beta_1(s-\delta,n_1)}\right] = \infty.$$

Then equation (1.1) *is almost oscillatory.*

From Lemma 2.2 and Theorem 2.7, similar to the proof of Theorem 2.3 we establish the following result.

Theorem 2.8. *Assume* (2.3) *holds,* $0 < \alpha \leq 1$ *and* $\tau \leq \delta$. *Further, assume that there exists a positive nondecreasing sequence* $\{\rho_n\}$, *such that for any* $n_1 \in \mathbb{N}$, *there exists an integer* $n_2 > n_1$, *with*

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\rho_s Q_s - \frac{(1+p^\alpha)}{4\alpha} \frac{(\Delta\rho_s)^2}{\rho_s(\beta_2(s-\delta,n_1))^{\alpha-1}\beta_1(s-\delta,n_1)}\right] = \infty.$$

Then equation (1.1) *is almost oscillatory.*

Remark 1: From Theorems 2.1 - 2.8, one can derive several oscillation criteria for the equation (1.1) by choosing specific sequence for $\{\rho_n\}$.

3 Examples

In this section, we present three examples to illustrate the main results.

Example 3.1. Consider the third order half-linear neutral difference equation

$$\Delta(n(\Delta^2(x_n + p\, x_{n-1}))^3) + \frac{\lambda}{n^6} x_{n-2}^3 = 0, \quad n \geq 1 \quad . \tag{3.1}$$

Here $a_n = n, p_n = p > 0, \tau = 2, \delta = 1, \alpha = 3, q_n = \frac{\lambda}{n^6}, \lambda > 0$. Then $Q_n = q_n = \frac{\lambda}{n^6}$ and $\beta_1(n,1) = \sum_{s=1}^{n-1} \frac{1}{s^{\frac{1}{3}}} \geq (n-1)^{\frac{2}{3}}$, for n sufficiently large. It is easy to see that (2.3) holds. Set $\rho_n = n^5$. We obtain

$$\begin{aligned} \lim_{n\to\infty} \quad \sup \quad & \sum_{s=4}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s \beta_1(s-\tau, n_1))^\alpha} \right] \\ \geq \quad & \lim_{n\to\infty} \sup \sum_{s=4}^{n-1} \left(\frac{\lambda}{4s} - \frac{5^4(1+p^3)}{4^4} \frac{s}{(s-3)^2} \right) = \infty \end{aligned}$$

if $\lambda > \frac{5^4(1+p^3)}{4^3}$. Hence by Theorem 2.1, equation (3.1) is almost oscillatory when $\lambda > \frac{5^4(1+p^3)}{4^3}$.

Example 3.2. Consider the third order half - linear difference equation

$$\Delta\left(\frac{1}{n^3}\,(\Delta^2(x_n + p\, x_{n-1}))^3\right) + \frac{\lambda}{n^6}\, x_{n-1}^3 = 0, \quad n \geq 1. \tag{3.2}$$

Here $a_n = \frac{1}{n^3},\ p_n = p > 0,\ \tau = \delta = 1,\ \alpha = 3$ and $q_n = \dfrac{\lambda}{n^6},\ \lambda > 0$. Then $Q_n = q_n = \dfrac{\lambda}{n^6}$,

$$\beta_1(n,1) = \frac{n(n-1)}{2} \text{ and } \beta_2(n,1) = \frac{1}{6}\, n(n-1)(n-2)$$

for n sufficiently large. It is easy to see that (2.3) holds. Set $\rho_n = n^5$. We obtain

$$\begin{aligned} & \lim_{n\to\infty} \sup \sum_{s=4}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{4\alpha} \frac{(\Delta\rho_s)^2}{\rho_s\ (\beta_2(s-\delta, n_1))^{\alpha-1} \beta_1(s-\delta, n_1)} \right] \\ & = \lim_{n\to\infty} \sup \sum_{s=4}^{n-1} \left(\frac{\lambda}{4s} - \frac{150(1+p^3)n^3}{(n-1)^3(n-2)^3(n-3)^2} \right) = \infty, \end{aligned}$$

if $\lambda > 0$. Hence, by Theorem 2.7, equation (3.2) is almost oscillatory when $\lambda > 0$.

Example 3.3. Consider the third order difference equation of the form

$$\Delta^3(x_n + \frac{1}{3} x_{n-2}) + \frac{\lambda}{n^2} x_{n-1} = 0, \quad n \geq 1. \tag{3.3}$$

Here $a_n = 1, p_n = \frac{1}{3}, \tau = 1, \delta = 2, \alpha = 1, q_n = \frac{\lambda}{n^2}, \lambda > 0$. Then $Q_n = q_n = \frac{\lambda}{n^2}$, $\beta_1(n,1) = n-1$. It is easy to see that (2.3) holds. Set $\rho_n = n$.

$$\begin{aligned} \lim_{n\to\infty} \quad \sup \quad & \sum_{s=3}^{n-1} \left[\frac{\rho_s Q_s}{2^{\alpha-1}} - \frac{(1+p^\alpha)}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s \beta_1(s-\tau, n_1))^\alpha} \right] \\ = \quad & \lim_{n\to\infty} \sup \sum_{s=3}^{n-1} \left(\frac{\lambda}{s} - \frac{1}{3} \frac{1}{s(s-2)} \right) = \infty, \end{aligned}$$

if $\lambda > 0$. Hence by Theorem 2.5, equation (3.3) is almost oscillatory. However one cannot derive this conclusion from Theorem 3.1 of [15] since condition (h_4) of Theorem 3.1 of [15] is not satisfied.

4 Conclusion

In this paper, we have established some new oscillation theorems for the equation (1.1) for the case $0 \leq p_n \leq p < \infty$, and τ and δ are nonnegative integers. If τ is nonnegative and δ is negative then the condition $\tau \geq \delta$ in Theorems 2.1 to 2.4 and if τ is negative and δ is nonnegative then the condition $\delta \geq \tau$ in Theorems 2.5 to 2.8 is satisfied and hence our results can be extended to these cases and the details are left to the reader. The reader can refer [3,9,12,24] for oscillation results of higher order neutral difference equations with different ranges of the neutral coefficient. It would be interesting to study equation (1.1) under the cases when $p_n < -1$ or $\lim_{n\to\infty} p_n = \infty$ or $\{p_n\}$ is an oscillatory sequence.

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