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Infinitely many weak solutions for $p(x)$-Laplacian-like problems with sign-changing potential

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Abstract. This study is concerned with the $p(x)$-Laplacian-like problems and arising from capillarity phenomena of the following type

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, and the primitive of the nonlinearity f of super-p^+ growth near infinity in u and is also allowed to be sign-changing. Based on a direct sum decomposition of a space $W_0^{1,p(x)}(\Omega)$, we establish the existence of infinitely many solutions via variational methods for the above equation. Furthermore, our assumptions are suitable and different from those studied previously.

Keywords: $p(x)$-Laplacian-like, variational method, multiple solutions, sign-changing potential.

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1 Introduction and main results

The present study is concerned with the existence of infinitely many nontrivial solutions for the nonlinear eigenvalue problems involving the $p(x)$-Laplacian-like operators, originated from a capillary phenomena,

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition and the primitive of the nonlinearity f is allowed to be sign-changing.

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems.

Recently, problem (P) has begun to receive more and more attention, see, for example, [2,7,8,11–13,15]. Let us recall some known results on problem (P). When the the primitive F of f oscillates at infinity, Shokooh and Neirameh [12] showed the existence of infinitely many weak solutions for this problem by using Ricceri's variational principle. For the case of f is p^+-superlinear at infinity, Zhou [15] and Ge [7] both obtained the existence of nontrivial solution of problem (P) for every parameter $\lambda > 0$, under suitable conditions on f. Rodrigues in [11], by using Fountain Theorem, established the existence of sequence of high energy solutions for problem (P), by assuming the following assumptions:

(h_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $t \to f(x,t)$ is continuous for a.e. $x \in \Omega$, and $x \to f(x,t)$ is Lebesgue measurable for all $t \in \mathbb{R}$;

(h_2) There exists a positive constant C such that

$$|f(x,t)| \le C(1 + |t|^{r(x)-1}),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$, where $r \in C_+(\overline{\Omega})$ such that $1 < p^- \le p^+ < r^- \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \ge N$;

$(h_3)'$ there exist $M > 0$, $\mu > p^+$ such that for $|t| \ge M$ and a.e. $x \in \Omega$,

$$0 < \mu F(x,t) \le t f(x,t),$$

where $F(x,t) = \int_0^t f(x,s)ds$;

(h_4) $f(x,-t) = -f(x,t)$, for all $(x,t) \in \Omega \times \mathbb{R}$.

Specifically, the author established the following theorem in [11].

Theorem 1.1 ([11, Theorem 4.7]). *Suppose that (h_1), (h_2), $(h_3)'$ and (h_4) hold. Then the problem (P) has an unbounded sequence of weak solutions for every $0 < \lambda < \frac{2r^+}{p^+}$.*

Observe that condition $(h_3)'$ plays an important role for showing that any Palais–Smale sequence is bounded in the work. However, there are some functions which do not satisfy condition $(h_3)'$, for example,

$$f(x,u) = |u|^{p^+-2}u \ln(1 + |u|).$$

In the present paper, we shall prove the same result as in [11] for problem (P) under more general assumptions on the nonlinearity, which unifies and significantly improves the result of [11]. The underlying idea for proving our main result is motivated by the argument used in [10]. In order to state the main result of this paper, we need the following assumptions:

(h_3) $\lim\limits_{|t|\to\infty} \frac{|F(x,t)|}{|t|^{p^+}} = +\infty$ uniformly in x, and there exists $r_0 > 0$ such that

$$F(x,t) \geq 0, \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ |t| \geq r_0;$$

(h_5) $\mathcal{F}(x,t) := \frac{1}{p^+} f(x,t)t - F(x,t) \geq 0$, and there exist $c_0 > 0$ and $\sigma \in C_+(\Omega)$ with $\sigma^- > \max\{1, \frac{N}{p^-}\}$ such that

$$|F(x,t)|^{\sigma(x)} \leq c_0 |t|^{p^-\sigma(x)} \mathcal{F}(x,t), \qquad \forall (x,t) \in \Omega \times \mathbb{R},\ |t| \geq r_0;$$

(h_6) there exist $\mu > p^+$ and $\theta > 0$ such that

$$\mu F(x,t) \leq t f(x,t) + \theta |t|^{p^-}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

We are now in the position to state our main results.

Theorem 1.2. *Suppose that (h_1)–(h_5) hold. Then for each $\lambda \in \left(0, \frac{2r^+}{p^+}\right)$, problem (P) possesses infinitely many nontrivial solutions.*

Theorem 1.3. *Suppose that (h_1)–(h_4) hold. and (h_6) hold. Then for each $\lambda \in \left(0, \frac{2r^+}{p^+}\right)$, problem (P) possesses infinitely many nontrivial solutions.*

Remark 1.4. It is easy to see that (h_3) and (h_5) are weaker than $(h_3)'$. In particular, $F(x,t)$ is allowed to be sign-changing in Theorems 1.2 and Theorems 1.3. The role of $(h_3)'$ is to ensure the boundedness of the Palais–Smale sequences of the energy functional, it is also significant to construct the variational framework. This is very crucial in applying the critical point theory. However, there are many functions which are superlinear at infinity, but do not satisfy the condition $(h_3)'$ for any $\mu > p^+$. For example, set $f(x,t) = p^+|t|^{p^+-2}t\ln(1+t^2)$, then $F(x,t) = |t|^{p^+}\ln(1+t^2) - \frac{2|t|^{p^+}t}{1+t^2}$. It is easy to check that $f(x,t)$ satisfy assumptions (h_3) and (h_5).

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, the proof of the main results is given.

2 Preliminaries

In order to discuss problem (P), we need some facts on space $W_0^{1,p(x)}(\Omega)$ which are called variable exponent Sobolev space. For this reason, we will recall some properties involving the variable exponent Lebesgue–Sobolev spaces, which can be found in [3–6,9] and references therein.

Throughout this paper, we always assume $p(x) > 1$, $p \in C(\overline{\Omega})$. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we will denote

$$h^- = \min_{x\in\overline{\Omega}} h(x), \qquad h^+ = \max_{x\in\overline{\Omega}} h(x)$$

and denote by $h_1 \ll h_2$ the fact that $\inf_{x\in\Omega}(h_2(x) - h_1(x)) > 0$.

For $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$, and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

We recall that spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{L^{p(x)}(\Omega)} |v|_{L^{q(x)}(\Omega)}, \qquad u \in L^{p(x)}(\Omega), \quad v \in L^{q(x)}(\Omega) \tag{2.1}$$

holds. Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u|^{p(x)} dx,$$

then the following relations hold

$$|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1), \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}. \tag{2.4}$$

Next, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Moreover, we have the following.

Proposition 2.1 ([6]).

(1) The Poincaré inequality in $W_0^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant C such that

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

(2) If $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$.

By (1) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

Proposition 2.2 ([4]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$ almost every where in Ω. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+},$$

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then $\left| |u|^p \right|_{q(x)} = |u|_{pq(x)}^p$.

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} \right) dx, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

We know that $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. If we denote $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([11]). *Set $E = W_0^{1,p(x)}(\Omega)$, A is as above, then*

(1) $A : E \to E^*$ is a convex, bounded and strictly monotone operator;

(2) $A : E \to E^*$ is a mapping of type $(S)_+$, i.e., $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$, implies $u_n \to u$ in E;

(3) $A : E \to E^*$ is a homeomorphism.

3 Variational setting and proof of the main results

For each $u \in E$, we define

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x,u)dx. \tag{3.1}$$

Then we have the following lemma.

Lemma 3.1. *If assumptions (h_1)–(h_2) hold, then $\varphi \in C^1(E, \mathbb{R})$ and*

$$\langle \varphi_\lambda'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx - \lambda \int_\Omega f(x,u)v dx \tag{3.2}$$

for all $u, v \in E$. Moreover, $\psi' : E \to E^*$ is weakly continuous, where $\psi(u) = \int_\Omega F(x,u)dx$.

Proof. To prove $\varphi_\lambda \in C^1(E, \mathbb{R})$ and (3.2), we only need to show that $\psi \in C^1(E, \mathbb{R})$ and

$$\langle \psi'(u), v\rangle = \int_\Omega f(x,u)v dx, \qquad \forall u, v \in E.$$

On the one hand, for any $u, v \in E$ and $0 < |t| < 1$, by condition (h_2), we obtain

$$\begin{aligned} |f(x, u+tv)v| &\le C(1 + |u+tv|^{r(x)-1})|v| \\ &\le C(|v| + 2^{r^+-1}|u|^{r(x)-1}|v| + 2^{r^+-1}|v|^{r(x)}). \end{aligned}$$

Note that $1 < p(x) < r(x) < p^*(x)$, the Hölder inequality implies that

$$|v| + 2^{r^+-1}|u|^{r(x)-1}|v| + 2^{r^+-1}|v|^{r(x)} \in L^1(\Omega).$$

Consequently, by the mean value theorem and the Lebesgue dominated convergence theorem, there exists $0 < \lambda < 1$ such that

$$\begin{aligned}\langle \psi'(u), v\rangle &= \lim_{t\to 0}\int_\Omega \frac{F(x,u+tv)-F(x,u)}{t}dx\\ &= \lim_{t\to 0}\int_\Omega f(x,u+\lambda tv)vdx\\ &= \int_\Omega f(x,u)vdx,\end{aligned}$$

for all $u, v \in E$. Hence ψ is Gateaux differentiable.

It remains to prove that ψ' is weakly continuous. Assume that $u_n \rightharpoonup u$ in E. By Proposition 2.1, we conclude that $u_n \to u$ in $L^{r(x)}(\Omega)$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$. Recalling

$$\begin{aligned}\|\psi'(u_n)-\psi'(u)\|_{E_*} &= \sup_{\|v\|\le 1}|\langle \psi'(u_n)-\psi'(u), v\rangle|\\ &\le \sup_{\|v\|\le 1}\int_\Omega |f(x,u_n)-f(x,u)||v|dx.\end{aligned}$$

Set $\alpha := \lim_{n\to+\infty}\sup_{\|v\|\le 1}\int_\Omega |f(x,u_n)-f(x,u)||v|dx$. We claim that $\alpha = 0$. Suppose, by contradiction, that $\alpha > 0$. Hence, there exists a sequence $\{\phi_n\} \subseteq E$ and $\|\phi_n\| = 1$ such that $|\int_\Omega |f(x,u_n)-f(x,u)||\phi_n|dx| > \frac{\alpha}{2}$ for enough large n. By (h_2), one has

$$\begin{aligned}|(f(x,u_n)-f(x,u))\phi_n| &\le C(1+|u_n|^{r(x)-1})|\phi_n| + C(1+|u|^{r(x)-1})|\phi_n|\\ &\le C(2|\phi_n| + |u_n|^{r(x)-1}|\phi_n| + |u|^{r(x)-1}|\phi_n|).\end{aligned}$$

Using again Hölder inequality, we get $2|\phi_n| + |u_n|^{r(x)-1}|\phi_n| + |u|^{r(x)-1}|\phi_n| \in L^1(\Omega)$. In view of [14, Lemma A.1], there exist $w_1 \in L^1(\Omega)$ and $\xi_1, w_2 \in L^{r(x)}(\Omega)$ such that

$$\max\{|u_n(x)|, |u(x)|\} \le |\xi_1(x)| \text{ and } |\phi_n(x)| \le \min\{|w_1(x)|, |w_2(x)|\}.$$

Therefore, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty}\int_\Omega |f(x,u_n)-f(x,u)||\phi_n|dx = 0,$$

which contradicts with $\alpha > 0$. Hence, $\|\psi'(u_n)-\psi'(u)\|_{E_*} \to 0$ as $n \to +\infty$. The proof is completed. □

Definition 3.2. We say that $\varphi_\lambda \in C^1(E, \mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\} \subset E$ satisfying

$$\varphi_\lambda(u_n) \to c \quad \text{and} \quad \|\varphi_\lambda'(u_n)\|_{E^*}(1+\|u_n\|) \to 0 \tag{3.3}$$

contains a convergent subsequence.

Now, we present the following theorem which will play a crucial role in the proof of Main Theorems.

Let X be a reflexive and separable Banach space. It is well known that there exist $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

(i) $\langle e_n^*, e_m\rangle = \delta_{n,m}$, where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ for $n \neq m$;

(ii) $X = \overline{\text{span}\{e_n : n \in N\}}$ and $X^* = \overline{\text{span}\{e_n^* : n \in N\}}$.

Let $X_i = \mathbb{R}e_i$, then $X = \overline{\oplus_{i\geq 1} X_i}$. Now, we define

$$Y_n = \oplus_{i=1}^{n} X_i \text{ and } Z_n = \overline{\oplus_{i\geq n} X_i}. \tag{3.4}$$

Then we have the following Fountain Theorem.

Lemma 3.3 ([1, 14]). *Assume that $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$ and I is even. If for each sufficiently large $n \in N$, there exist $\rho_n > \delta_n > 0$ such that the following conditions hold:*

(A_1) $b_n := \inf\{I(u) : u \in Z_n, \|u\| = \delta_n\} \to +\infty$ *as* $n \to +\infty$;

(A_2) $a_n := \inf\{I(u) : u \in Y_n, \|u\| = \rho_n\} \leq 0$.

Then the functional I has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_n\} \subset X$ such that $I'(u_n) = 0$ and $I(u_n) \to +\infty$ as $n \to +\infty$.

Lemma 3.4. *Assume that (h_2), (h_3) and (h_5) hold. Then any $(C)_c$ sequence is bounded.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence. To complete our goals, arguing by contradiction, suppose that $\|u_n\| \to \infty$, as $n \to \infty$. Observe that for n large,

$$\begin{aligned}
c+1 &\geq \varphi_\lambda(u_n) - \frac{1}{p^+}\langle \varphi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{p^+}\int_\Omega \left(|\nabla u_n|^{p(x)} + \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)dx + \frac{\lambda}{p^+}\int_\Omega f(x,u_n)u_n dx \\
&\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx + \lambda\int_\Omega \mathcal{F}(x,u_n)dx \\
&\geq \lambda\int_\Omega \mathcal{F}(x,u_n)dx.
\end{aligned} \tag{3.5}$$

Since $\|u_n\| > 1$ for n large, using (3.3) we have

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^{p^-}} = \lim_{n\to\infty}\frac{\varphi_\lambda(u_n)}{\|u_n\|^{p^-}} \\
&\geq \frac{1}{p^+}\frac{\int_\Omega\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx}{\|u_n\|^{p^-}} - \lambda\int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^-}}dx \\
&\geq \frac{2}{p^+} - \lambda\int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^-}}dx,
\end{aligned}$$

which implies that

$$\frac{2}{p^+\lambda} \leq \limsup_{n\to\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx. \tag{3.6}$$

For $0 \leq \alpha < \beta$, let $\Omega_n(\alpha,\beta) = \{x \in \Omega : \alpha \leq |u_n(x)| < \beta\}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $|v_n|_{r(x)} \leq C_0\|v_n\| = C_0$ for some $C_0 > 0$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in E and

$$v_n \to v \quad \text{in } L^{s(x)}(\Omega), \qquad 1 \leq s(x) < p^*(x) \quad \text{and} \quad v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \tag{3.7}$$

Now, we consider two possible cases: $v = 0$ or $v \neq 0$.

(1) If $v = 0$, then we have that $v_n \to 0$ in $L^{s(x)}(\Omega)$ and $v_n(x) \to 0$ a.e. on Ω. Hence, it follows from (h_2) that

$$\int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}} dx \leq \frac{C(r_0 + r_0^{\bar{r}})\text{meas}(\Omega)}{\|u_n\|^{p^-}} \to 0 \quad \text{as } n \to +\infty, \tag{3.8}$$

where $\bar{r} = r^+$ if $r_0 \geq 1$, $\bar{r} = r^-$ if $r_0 < 1$.

Set $\sigma'(x) = \frac{\sigma(x)}{\sigma(x)-1}$. Since $\sigma^- > \max\{1, \frac{N}{p^-}\}$ one sees that $1 < p^-\sigma'(x) < p^*(x)$. So, $v_n \to 0$ in $L^{p^-\sigma'(x)}(\Omega)$ as $n \to +\infty$. Hence, we deduce from Proposition 2.2, (h_5), (3.5) and (3.7) that

$$\begin{aligned}
&\int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^{p^-}} |v_n|^{p^-} dx \\
&\leq 2 \left| \frac{|F(x,u_n)|}{|u_n|^{p^-}} \right|_{L^{\sigma(x)}(\Omega_n(r_0,+\infty))} \left| |v_n|^{p^-} \right|_{L^{\sigma'(x)}(\Omega_n(r_0,+\infty))} \\
&\leq 2 \max\left\{ \left(\int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|^{\sigma(x)}}{|u_n|^{(p^-)\sigma(x)}} dx \right)^{\frac{1}{\sigma^+}}, \left(\int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|^{\sigma(x)}}{|u_n|^{(p^-)\sigma(x)}} dx \right)^{\frac{1}{\sigma^-}} \right\} \\
&\quad \times \max\left\{ \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^-}}, \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^+}} \right\} \\
&\leq 2 \max\left\{ \left(\int_{\Omega_n(r_0,+\infty)} \mathcal{F}(x,u_n) dx \right)^{\frac{1}{\sigma^+}}, \left(\int_{\Omega_n(r_0,+\infty)} \mathcal{F}(x,u_n) dx \right)^{\frac{1}{\sigma^-}} \right\} \\
&\quad \times \max\left\{ \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^-}}, \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^+}} \right\} \\
&\leq 2 \max\left\{ \left(\frac{c_0}{\lambda}(c+1) \right)^{\frac{1}{\sigma^+}}, \left(\frac{c_0}{\lambda}(c+1) \right)^{\frac{1}{\sigma^-}} \right\} \\
&\quad \times \max\left\{ \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^-}}, \left(\int_{\Omega_n(r_0,+\infty)} |v_n|^{p^-\sigma'(x)} dx \right)^{\frac{1}{(\sigma')^+}} \right\} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), we get

$$\begin{aligned}
\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}} dx &= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}} dx + \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}} dx \\
&= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}} dx + \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^{p^-}} |v_n|^{p^-} dx \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned} \tag{3.10}$$

which contradicts (3.6).

(2) If $v \neq 0$, set $\Omega_{\neq} := \{x \in \Omega : v(x) \neq 0\}$, then $\text{meas}(\Omega_{\neq}) > 0$. For a.e. $x \in \Omega_{\neq}$, we have $\lim_{n\to\infty} |u_n(x)| = +\infty$. Hence, $\Omega_{\neq} \subset \Omega_n(r_0, \infty)$ for large $n \in N$. As the proof of (3.8), we also obtain that

$$\int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^{p^+}} dx \leq \frac{C(r_0 + r_0^{\bar{r}})\text{meas}(\Omega)}{\|u_n\|^{p^+}} \to 0, \quad \text{as } n \to +\infty. \tag{3.11}$$

It follows from (h_2), (h_3), (3.11) and Fatou's Lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{\varphi(u_n)}{\|u_n\|^{p^+}} \\
&\le \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} \right. \\
&\qquad \left. - \lambda \int_{\Omega_n(0,r_0)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \qquad (3.12) \\
&\le \lim_{n\to\infty} \left[\frac{1}{p^-} \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + 1 + |\nabla u_n|^{p(x)} \right) dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \lim_{n\to\infty} \left[\frac{2}{p^-} \frac{\int_\Omega |\nabla u_n|^{p(x)} dx}{\|u_n\|^{p^+}} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&\le \limsup_{n\to\infty} \left[\frac{2}{p^-} - \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^{p^+}} dx \right] \\
&= \frac{2}{p^-} - \liminf_{n\to\infty} \lambda \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{|u_n|^{p^+}} |v_n|^{p^+} dx \\
&= \frac{2}{p^-} - \liminf_{n\to\infty} \lambda \int_\Omega \frac{F(x,u_n)}{|u_n|^{p^+}} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^{p^+} dx \\
&\le \frac{2}{p^-} - \lambda \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^{p^+}} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^{p^+} dx \\
&\to -\infty, \quad \text{as } n \to \infty,
\end{aligned}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. The proof is accomplished. □

Lemma 3.5. *Suppose that (h_2), (h_3) and (h_5) hold. Then any $(C)_c$-sequence of φ has a convergent subsequence in E.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence. In view of the Lemma 3.4, the sequence $\{u_n\}$ is bounded in E. Then, up to a subsequence we have $u_n \rightharpoonup u$ in E. By Proposition 2.2, it follows that

$$u_n \to u \text{ in } L^{r(x)}(\Omega),$$
$$\{u_n\} \text{ is bounded in } L^{r(x)}(\Omega).$$

It is easy to compute directly that

$$\begin{aligned}
&\int_\Omega |f(x,u_n)-f(x,u)||u_n-u|dx \\
&\quad\le \int_\Omega (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
&\quad\le \int_\Omega [C(1+|u_n|^{r(x)-1})+C(1+|u|^{r(x)-1})]|u_n-u|dx \\
&\quad\le 2C\int_\Omega |u_n-u|dx + C\int_\Omega |u_n|^{r(x)-1}|u_n-u|dx \\
&\qquad + \int_\Omega |u|^{r(x)-1}|u_n-u|dx \\
&\quad\le 2C|u_n-u|_1 + 2C\big||u_n|^{r(x)-1}\big|_{r'(x)}|u_n-u|_{r(x)} \\
&\qquad + 2C\big||u|^{r(x)-1}\big|_{r'(x)}|u_n-u|_{r(x)} \\
&\quad\le 2C|u_n-u|_1 + 2C\max\Big\{|u_n|_{r(x)}^{r^+-1},|u_n|_{r(x)}^{r^--1}\Big\}|u_n-u|_{r(x)} \\
&\qquad + 2C\max\Big\{|u|_{r(x)}^{r^+-1},|u|_{r(x)}^{r^--1}\Big\}|u_n-u|_{r(x)} \\
&\quad\to 0, \quad \text{as } n\to\infty,
\end{aligned} \tag{3.13}$$

where $\frac{1}{r(x)}+\frac{1}{r'(x)}=1$. Noting that

$$\begin{aligned}
\langle A(u_n)-A(u),u_n-u\rangle &= \langle \varphi'(u_n)-\varphi'(u),u_n-u\rangle \\
&\quad + \int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx.
\end{aligned} \tag{3.14}$$

Moreover, by (3.3), one infers

$$\lim_{n\to\infty}\langle \varphi'(u_n)-\varphi'(u),u_n-u\rangle = 0. \tag{3.15}$$

Finally, the combination of (3.13)–(3.15) implies

$$\lim_{n\to\infty}\langle A(u_n)-A(u),u_n-u\rangle = 0. \tag{3.16}$$

Since A is of type $(S)_+$ by Lemma 2.3, we obtain $u_n\to u$ in E. The proof is complete. □

Lemma 3.6. *Suppose that (h_2), (h_3) and (h_6) hold. Then any $(C)_c$-sequence of φ has a convergent subsequence in E.*

Proof. Similar to the proof of Lemma 3.5, we only prove that $\{u_n\}$ is bounded in E. Suppose by contradiction that $\|u_n\|\to\infty$ as $n\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$ and $|v_n|_{r(x)}\le C_0\|v_n\|=C_0$ for some $C_0>0$. Going if necessary to a subsequence, we may assume that $v_n\rightharpoonup v$ in E,

$$v_n\to v \quad \text{in } L^{r(x)}(\Omega), \qquad 1\le r(x)<p^*(x) \quad \text{and} \quad v_n(x)\to v(x) \quad \text{a.e. on } \Omega. \tag{3.17}$$

By (3.1), (3.2) and (h_6), one has

$$\begin{aligned}
c+1 &\ge \varphi_\lambda(u_n)-\frac{1}{\mu}\langle \varphi_\lambda'(u_n),u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{\mu}\int_\Omega \left(|\nabla u_n|^{p(x)}+\frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)dx + \frac{\lambda}{\mu}\int_\Omega f(x,u_n)u_n dx
\end{aligned}$$

$$\geq \left(\frac{1}{p^+}-\frac{1}{\mu}\right)\int_\Omega\left(|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx-\frac{\lambda\theta}{\mu}\int_\Omega |u_n|^{p^-}dx$$
$$\geq \frac{\mu-p^+}{p^+\mu}\|u_n\|^{p^-}-\frac{\lambda\theta}{\mu}|u_n|_{p^-}^{p^-},$$

for $n\in N$, which implies

$$1\leq \frac{\lambda\theta p^+}{\mu-p^+}\limsup_{n\to\infty}|v_n|_{p^-}^{p^-}. \tag{3.18}$$

In view of (3.17), $v_n\to v$ in $L^{p^-}(\Omega)$. Hence, we deduce from (3.18) that $v\neq 0$. By a similar reasoning as in the proof of Lemma 3.4 step (2), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest proof is the same as that in Lemma 3.5. □

Proof of Theorem 1.2. Let $X=E$, Y_n and Z_n be defined by (3.4). Obviously, $\varphi_\lambda(u)=\varphi_\lambda(-u)$ by (h_4), and Lemma 3.5 implies that φ_λ satisfies the $(C)_c$ condition for any $\lambda>0$. Hence, to prove Theorem 1.2, it remains to verify the conditions (A_1) and (A_2) in Lemma 3.3.

Verification of (A_1). Set $\beta_n:=\sup_{u\in Z_n,\,\|u\|=1}|u|_{r(x)}$, where $p^+<r^-\leq r(x)<p^*(x)$ and $n\in N$. We claim that $\beta_n\to 0$ as $n\to\infty$. Indeed, it is obvious that $\beta_n\geq\beta_{n+1}\geq 0$. so $\beta_n\to\beta\geq 0$ as $n\to\infty$. For each $n=1,2,\dots$, taking $u_n\in Z_n$, $\|u_n\|=1$ such that $0\leq\beta_n-|u_n|_{r(x)}\leq\frac{1}{n}$. As E is reflexive, $\{u_n\}$ has a weakly convergent subsequence, without loss of generality, suppose $u_n\rightharpoonup u$ in E. By definition of Z_n, one knows that $u=0$. Proposition 2.3 implies that $u_n\to 0$ in $L^{r(x)}(\Omega)$. Thus we have proved that $\beta=0$.

By the above definition of β_n, for $u\in Z_n$ with $\|u\|>1$, we have

$$|u|_{r(x)}\leq\beta_n\|u\|. \tag{3.19}$$

Moreover, we consider the real function $k:\mathbb{R}\to\mathbb{R}$, defined by

$$k(t)=\frac{1}{p^+}t^{p^-}-\lambda C\beta_n^{r^-}t^{r^+}.$$

Choosing $\delta_n=(2Cr^+\beta_n^{r^-})^{\frac{1}{p^--r^+}}$ for $n\in N$, it is clear that

$$\begin{aligned}k(\delta_n)&=\frac{1}{p^+}\delta_n^{p^-}-\lambda C\beta_n^{r^-}\delta_n^{r^+}\\&=(2Cr^+\beta_n^{r^-})^{\frac{p^-}{p^--r^+}}\left[\frac{1}{p^+}-\frac{\lambda}{2r^+}\right].\end{aligned} \tag{3.20}$$

Therefore, since $r^->p^+$, $\lambda<\frac{2r^+}{p^+}$ and $\beta_n\to 0$ as $n\to+\infty$, we obtain that

$$\delta_n\to+\infty,\qquad k(\delta_n)\to+\infty,\quad \text{as } n\to+\infty. \tag{3.21}$$

It follows from (h_2) that

$$F(x,t)\leq C(|t|+|t|^{r(x)})\leq 2C(1+|t|^{r(x)})$$

for all $(x,t)\in\Omega\times\mathbb{R}$. Then, for any $u\in Z_n$, assume that $\|u\|=\delta_n$. It follows from (h_2), (3.19), (3.20) and (3.21) that

$$
\begin{aligned}
\varphi_\lambda(u) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u)dx \\
&\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\text{meas}(\Omega) - 2\lambda C\int_\Omega |u|^{r(x)}dx \\
&\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\text{meas}(\Omega) - 2\lambda C\max\left\{|u|^{r^+}_{r(x)}, |u|^{r^-}_{r(x)}\right\} \\
&\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\text{meas}(\Omega) - 2\lambda C\max\left\{\beta_n^{r^+}\|u\|^{r^+}, \beta_n^{r^-}\|u\|^{r^-}\right\} \\
&\geq \frac{2}{p^+}\|u\|^{p^-} - 2\lambda C\text{meas}(\Omega) - 2\lambda C\beta_n^{r^-}\|u\|^{r^+} \\
&= 2k(\delta_n) - 2\lambda C\text{meas}(\Omega) \to +\infty, \quad \text{as } n \to +\infty.
\end{aligned}
\tag{3.22}
$$

This gives relation (A_1).

Verification of (A_2). Assume that (A_2) of Lemma 3.3 does not hold for some given n. Then there exists a sequence $\{u_k\} \subset Y_n$ such that

$$\|u_k\| \to +\infty \quad \text{as } k \to +\infty \quad \text{and} \quad \varphi_\lambda(u_k) \geq 0. \tag{3.23}$$

Let $w_k = \frac{u_k}{\|u_k\|}$. Then it is obvious that $\|w_k\| = 1$. Since $\dim Y_n < +\infty$, there exists $w \in Y_n \setminus \{0\}$ such that up to a subsequence, $\|w_k - w\| \to 0$ and $w_k(x) \to w(x)$ a.e. $x \in \Omega$ as $k \to +\infty$.

If $w(x) \neq 0$, then $|u_k(x)| \to +\infty$ as $k \to +\infty$. By virtue of (h_3), we get $\lim_{k\to+\infty}\frac{F(x,u_k(x))}{\|u_k\|^{p^+}} = \lim_{k\to+\infty}\frac{F(x,u_k(x))}{|u_k(x)|^{p^+}}|w_k(x)|^{p^+} = +\infty$, for all $x \in \Omega_0 := \{x \in \Omega : w(x) \neq 0\}$. The estimate in Lemma 3.4 implies that

$$\int_{\Omega_0}\frac{F(x,u_k)}{\|u_k\|^{p^+}}dx \to +\infty \text{ as } k \to +\infty.$$

Note that, $\Omega_0 \subset \Omega_n(r_0,\infty)$ for large $n \in N$. Therefore, we have

$$
\begin{aligned}
\varphi_\lambda(u_k) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_k|^{p(x)} + \sqrt{1+|\nabla u_k|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u_k)dx \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_k|^{p(x)} + \sqrt{1+|\nabla u_k|^{2p(x)}}\right)dx \\
&\quad - \lambda\int_{\Omega_k(0,r_0)} F(x,u_k)dx - \lambda\int_{\Omega_k(r_0,+\infty)} F(x,u_k)dx \\
&\leq \frac{1}{p^-}\|u_k\|^{p^+} + C\int_{\Omega_k(0,r_0)}(r_0 + r_0^r)dx - \int_{\Omega_k(r_0,+\infty)} F(x,u_k)dx \\
&\leq \frac{1}{p^-}\|u_k\|^{p^+} + C(r_0 + r_0^r)\text{meas}(\Omega) - \int_{\Omega_k(r_0,+\infty)\cap\Omega_0} F(x,u_k)dx \\
&\leq \|u_k\|^{p^+}\left(\frac{1}{p^-} + \frac{C(r_0+r_0^r)\text{meas}(\Omega)}{\|u_k\|^{p^+}} - \int_{\Omega_k(r_0,+\infty)\cap\Omega_0}\frac{F(x,u_k)}{\|u_k\|^{p^+}}dx\right) \\
&\to -\infty, \text{ as } k \to +\infty,
\end{aligned}
$$

which is contradiction to (3.23). This gives relation (A_2). Hence, all conditions of Lemma 3.3 are satisfied. Namely, for each $\lambda \in (0, \frac{2r^+}{p^+})$, problem (P) possesses infinitely many nontrivial solutions sequence $\{u_n\}$ such that $\varphi_\lambda(u_n) \to +\infty$ as $n \to +\infty$. □

Proof of Theorem 1.3. Let $X = E$, Y_n and Z_n be defined by (3.4). We know that φ_λ satisfies the $(C)_c$ condition from Lemma 3.6 and $\varphi_\lambda(u) = \varphi_\lambda(-u)$. The rest proof is the same as that of Theorem 1.2. □

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