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Transmission Robin problem for singular $p(x)$-Laplacian equation in a cone

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Abstract. We study the behavior near the boundary angular or conical point of weak solutions to the transmission Robin problem for an elliptic quasi-linear second-order equation with the variable $p(x)$-Laplacian.

Keywords: $p(x)$-Laplacian, angular and conical points.

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1 Introduction

The aim of our article is the investigation of the behavior of the weak solutions to the transmission Robin problem for quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian in a neighborhood of an angular or a conical boundary point of the bounded cone. The case for the constant p-Laplacian was investigated in our monograph [4]. The transmission problems appear frequently in various areas of physics and engineering. For instance, one of the important problems of the electrodynamics of solid media is the research of electromagnetic processes in ferromagnetic with various dielectric constants. This type of problems appears in solid mechanics if a body consists of composite materials as well. Let us mention also vibrating folded membranes, composite plates, folded plates, junctions in elastic multi-structures etc.

In this article we obtain estimates of the weak solutions to the elliptic transmission problem for the variable $p(x)$-Laplacian near singularities on the boundary (conical boundary point or edge). The same problems for $p(x) = p = const$ were studied in our monograph [4].

Boundary value problems for elliptic second order equations with a non-standard growth in function spaces with variable exponents actively studied in recent years. We refer to [8] for an overview and the recent paper [1, 9–11] and reference therein. Differential equations with variable exponents-growth conditions arise from the nonlinear elasticity theory, electrorheological fluids, etc. There are many essential differences between the variable exponent problems and the constant exponent problems. In the variable exponent problems, many

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singular phenomena occurred and many special questions were raised. V. Zhikov [12, 13] has gave examples of the Lavrentiev phenomenon for the variational problems with variable exponent.

Most of the works devoted to the quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian refers to the Dirichlet problem in smooth bounded domains (see [8]). We know only a few articles studying the Robin problem for such equations, but in these works a domain is smooth and lower order terms depend only on (x, u) and do not depend on $|\nabla u|$. Our articles [2, 3] is deduced to the Robin problem **in a cone** for such equations with **a singular $p(x)$-power gradient lower order term**. Here we describe qualitatively the behavior of the weak solution near a conical point, namely we derive the sharp estimate of the type $|u(x)| = O(|x|^{\varkappa})$ for the weak solution modulus (for the solution decrease rate) of our transmission problem near a conical boundary point. We establish the comparison principle for weak solutions as well. We shall use calculations and some results of our previous article [2].

We introduce the following notations: let $\mathbb{C}$ be an open cone in $\mathbb{R}^n$, $n \geq 2$, with the vertex at the origin $\mathcal{O}$ and the angular opening of cone $\omega_0 \in (0, \pi)$. Let B_r be an open ball with radius r centered at $\mathcal{O}$. We use the following standard notations:

- S^{n-1}: a unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- $(r, \omega), \omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$:

$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2 \sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1; \end{aligned}$$

- Ω: a domain on the unit sphere S^{n-1} with the smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathbb{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathbb{C} \cap S^{n-1}$;
- $G_0^d \equiv \mathbb{C} \cap B_d = \{(r, \omega) \mid 0 < r < d;\ \omega \in \Omega\}$;
- $\Gamma_0^d \equiv \partial\mathbb{C} \cap B_d = \{(r, \omega) \mid 0 < r < d;\ \omega \in \partial\Omega\}$;
- $\Omega_d = \overline{G_0^d} \cap \{|x| = d\}$;

We assume that $G_0^d = G_+^d \cup G_-^d$ is divided into two subdomains

$$G_+^d := \{(r, \omega) : 0 < r < d,\ \omega \in \Omega_+\} \quad \text{and} \quad G_-^d : \{(r, \omega) : 0 < r < d,\ \omega \in \Omega_-\}$$

by a $\Sigma_0^d := G_0^d \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0^d}$,

$$\Omega_+ = \Omega \cap \{x_n > 0\},\ \Omega_- = \Omega \cap \{x_n < 0\} \Longrightarrow \Omega = \Omega_+ \cup \Omega_-;$$

$$\sigma_0 = \overline{\Sigma_0^d} \cap \Omega_d;\ \partial_\pm\Omega = \overline{\Omega_\pm} \cap \partial\mathcal{C};\ \partial\Omega_\pm = \partial_\pm\Omega \cup \sigma_0;$$

- $\Gamma_0^d = \Gamma_+^d \cup \Gamma_-^d, \quad \Gamma_\pm^d := \{(r, \omega) : 0 < r < d,\ \omega \in \Omega_\pm\}$;

- $u(x) = \begin{cases} u_+(x), & x \in G_+^d, \\ u_-(x), & x \in G_-^d; \end{cases} \quad f(x) = \begin{cases} f_+(x), & x \in G_+^d, \\ f_-(x), & x \in G_-^d \end{cases}$ etc.;
- $[u]_{\Sigma_0^d}$ denotes the saltus of the function $u(x)$ on crossing Σ_0^d, i.e.

$$[u]_{\Sigma_0^d} = u_+(x)\Big|_{\Sigma_0^d} - u_-(x)\Big|_{\Sigma_0^d},$$

where $u_\pm(x)\big|_{\Sigma_0^d} = \lim_{G_\pm^d \ni y \to x \in \Sigma_0^d} u_\pm(y)$;

- $n_i = \cos(\overrightarrow{n}, x_i)$, $i = 1, 2$, where $\overrightarrow{n}$ denotes the unit outward vector with respect to G_+^d (or G_0^d) normal to Σ_0^d (respectively $\partial G_0^d \setminus \mathcal{O}$).

We use the standard function spaces:

- $C^k(\overline{G_\pm})$ with the norm $|u_\pm|_{k,G_\pm}$;
- the Lebesgue space $L_p(G_\pm), p \geq 1$ with the norm $\|u_\pm\|_{p,G_\pm}$;
- the Sobolev space $W^{k,p}(G_\pm)$ with the norm $\|u_\pm\|_{p,k;G_\pm}$

and introduce their direct sums

- $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_+}) \dotplus C^k(\overline{G_-})$ with the norm $|u|_{k,G} = |u_+|_{k,G_+} + |u_-|_{k,G_-}$;
- $\mathbf{L}_p(G) = L_p(G_+) \dotplus L_p(G_-)$ with the norm

$$\|u\|_{\mathbf{L}_p(G)} = \left(\int_{G_+} |u_+|^p dx\right)^{\frac{1}{p}} + \left(\int_{G_-} |u_-|^p dx\right)^{\frac{1}{p}};$$

- $\mathbf{W}^{k,p}(G) = W^{k,p}(G_+) \dotplus W^{k,p}(G_-)$ with the norm

$$\|u\|_{p,k;G} = \left(\int_{G_+} \sum_{|\beta|=0}^{k} |D^\beta u_+|^p dx\right)^{\frac{1}{p}} + \left(\int_{G_-} \sum_{|\beta|=0}^{k} |D^\beta u_-|^p dx\right)^{\frac{1}{p}}.$$

We investigate the behavior in a neighborhood of the origin $\mathcal{O}$ of solutions to the transmission Robin problem:

$$\begin{cases} -\triangle_{p(x)} u + a(x) u|u|^{p(x)-1} + b(u, \nabla u) = f(x), & x \in G_0^{d_0}, \\ [u]_{\Sigma_0^{d_0}} = 0, \\ \left[|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \overrightarrow{n}}\right]_{\Sigma_0^{d_0}} + \frac{\beta}{|x|^{p(x)-1}} u|u|^{p(x)-2} = h(x, u), & x \in \Sigma_0^{d_0}, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \overrightarrow{n}} + \frac{\gamma}{|x|^{p(x)-1}} u|u|^{p(x)-2} = g(x, u), & x \in \Gamma_0^{d_0}, \end{cases} \tag{TRQL}$$

where $0 < d_0 \ll 1$ (d_0 is fixed) and

$$\triangle_{p(x)} u \equiv \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right). \tag{1.1}$$

We will work under the following assumptions:

(i) $1 < p_- \leq p(x) \leq p_+ = p(0) < n; \forall x \in \overline{G_0^{d_0}}$;

(ii) the Lipschitz condition: $p(x) \in C^{0,1}(\overline{G_0^{d_0}}) \implies 0 \le p_+ - p(x) \le L|x|,\ \forall x \in \overline{G_0^{d_0}}$; where L is the Lipschitz constant for $p(x)$;

(iii) $\beta = const > 0,\ \gamma = const > 0$, such that

$$\begin{cases} \beta \ge 2 \quad \text{and} \quad \gamma \ge \frac{1}{2}\beta, & \text{if} \quad p_+ \ge 2; \\ 1 \le \gamma \le \frac{1}{2}\beta, & \text{if} \quad 1 < p_+ \le 2; \end{cases} \tag{1.2}$$

(iv) the function $b(u,\xi)$ is differentiable with respect to the u, ξ variables in $\mathfrak{M} = \mathbb{R} \times \mathbb{R}^n$ and satisfy in $\mathfrak{M}$ the following inequalities:

$(\mathbf{iv})_\mathbf{a}$ $$|b(u,\xi)| \le \delta |u|^{-1}|\xi|^{p(x)} + b_0|u|^{p(x)-1}, \quad \begin{cases} 0 \le \delta < \mu, & \text{if } \mu > 0; \\ \delta \ge 0, & \text{if } \mu = 0; \end{cases}$$

$(\mathbf{iv})_\mathbf{b}$ $$b(u,\xi)\,\mathrm{sign}\, u \ge \nu |u|^{-1}|\xi|^{p(x)} - b_0|u|^{p(x)-1}, \quad \nu > 0; \qquad \text{if } \mu = 0;$$

$(\mathbf{iv})_\mathbf{c}$ $$\sqrt{\sum_{i=1}^{n} \left| \frac{\partial b(u,\xi)}{\partial \xi_i} \right|^2} \le b_1 |u|^{-1}|\xi|^{p(x)-1}; \quad \frac{\partial b(u,\xi)}{\partial u} \ge b_2|u|^{-2}|\xi|^{p(x)};$$

$$b_0 \ge 0,\ b_1 \ge 0,\ b_2 \ge 0;$$

(v) $0 \le a_0 \le a(x) \le const \cdot |x|^{-p(x)}$;

$|f(x)| \le f_0|x|^{\beta(x)}, \quad f_0 \ge 0, \quad \beta(x) \ge \frac{p_+-1}{p_+-1+\mu}\left(p(x)-1\right)\lambda - p(x);\ \forall x \in \overline{G_0^{d_0}};\ 0 \le \mu < 1$ and λ is the least positive eigenvalue of problem $(NEVP)$ (see below);

(vv) $|h(x,u)| \le h_0|x|^{1+\beta(x)};\ \forall u \in \mathbf{L}_\infty;\ \frac{\partial h(x,u)}{\partial u} \le 0,\ h(x,0) \equiv 0,\ x \in \Sigma_0^{d_0}$;
$|g(x,u)| \le g_0|x|^{1+\beta(x)};\ \forall u \in \mathbf{L}_\infty;\ \frac{\partial g(x,u)}{\partial u} \le 0,\ g(x,0) \equiv 0,\ x \in \Gamma_0^{d_0}$;

(vvv) the spherical region $\Omega \subset S^{n-1}$ is invariant with respect to rotations in S^{n-2}.

We consider the functions class

$$\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0}) = \left\{ u \;\middle|\; u(x) \in L_\infty(G_0^{d_0}) \text{ and } \int_{G_0^{d_0}} \left\langle |x|^{-p(x)}|u|^{p(x)} + |u|^{-1}|\nabla u|^{p(x)} \right\rangle dx < \infty \right\}$$

which was introduced in [5]. It is obvious that $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0}) \subset \mathbf{W}^{1,p(x)}(G_0^{d_0})$.

Definition 1.1. The function u is called a weak bounded solution of problem (*TRQL*) provided that $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0})$ and satisfies the integral identity

$$\begin{aligned} Q(u,\eta) :\equiv & \int_{G_0^{d_0}} \left\langle |\nabla u|^{p(x)-2} u_{x_i}\eta_{x_i} + a(x)u|u|^{p(x)-1}\eta(x) + b\left(u, \nabla u\right)\eta(x) \right\rangle dx \\ & + \gamma \int_{\Gamma_0^{d_0}} r^{1-p(x)} u|u|^{p(x)-2}\eta(x)dS + \beta \int_{\Sigma_0^{d_0}} r^{1-p(x)} u|u|^{p(x)-2}\eta(x)dS \\ & - \int_{\Omega_{d_0}} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial r}\eta(x)d\Omega_d - \int_{\Gamma_0^{d_0}} g(x,u)\eta(x)dS - \int_{\Sigma_0^{d_0}} h(x,u)\eta(x)dS \\ = & \int_{G_0^{d_0}} f(x)\eta(x)dx \end{aligned} \tag{II}$$

for all $\eta(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0})$.

Proposition 1.2. *Above assumptions* **(i)–(vv)** *ensure the existence of integrals over $G_0^{d_0}, \Sigma_0^{d_0}$ and $\Gamma_0^{d_0}$ in (II). Therefore, the Definition 1.1 is correct.*

Proof. **1)** We use well known the Hölder inequality with $p(x), p'(x) : \frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and inequality

$$\||\nabla u|^{p(x)-1}\|_{p'(x)} \le \begin{cases} \|\nabla u\|_{p(x)}^{\frac{p_-}{p_+}(p_- -1)}, & \text{if } \|\nabla u\|_{p(x)} \le 1; \\ \|\nabla u\|_{p(x)}^{\frac{p_+}{p_-}(p_+ -1)}, & \text{if } \|\nabla u\|_{p(x)} \ge 1; \\ 1, & \text{if } \|\nabla u\|_{p(x)} \ge 1; \end{cases}$$

we have

$$\int_{G_0^{d_0}} |\nabla u|^{p(x)-2} u_{x_i} \eta_{x_i} dx \le \int_{G_0^{d_0}} |\nabla u|^{p(x)-1} |\nabla \eta| dx \le 2\|\nabla \eta\|_{p(x)} \cdot \||\nabla u|^{p(x)-1}\|_{p'(x)} < \infty, \quad \forall u(x), \eta(x) \in \mathfrak{N}_{-1,\infty}^{1,p(x)}(G_0^{d_0});$$

2) By assumption **(v)**,

$$\int_{G_0^{d_0}} a(x) u|u|^{p(x)-1} \eta(x) dx \le const \cdot \|\eta\|_{\mathbf{L}_\infty} \cdot \int_{G_0^{d_0}} |x|^{-p(x)} |u|^{p(x)} dx < \infty, \quad \forall u(x), \eta(x) \in \mathfrak{N}_{-1,\infty}^{1,p(x)}(G_0^{d_0});$$

3) By assumption $\mathbf{(iv)_a}$,

$$\int_{G_0^{d_0}} b\,(u, \nabla u)\, \eta(x) dx \le \delta \|\eta\|_{\mathbf{L}_\infty} \int_{G_0^{d_0}} |u|^{-1} |\nabla u|^{p(x)} dx + b_0 \int_{G_0^{d_0}} |u(x)|^{p(x)-1} |\eta(x)| dx < \infty, \quad \forall u(x), \eta(x) \in \mathfrak{N}_{-1,\infty}^{1,p(x)}(G_0^{d_0});$$

because of

$$\int_{G_0^{d_0}} |u(x)|^{p(x)-1} |\eta(x)| dx \le \|\eta\|_{\mathbf{L}_\infty} \cdot \left(\text{meas}\, G_0^{d_0}\right) \cdot \begin{cases} 1, & \text{if } |u| \le 1; \\ \|u\|_{\mathbf{L}_\infty}^{p_+ -1}, & \text{if } |u| > 1. \end{cases}$$

4)

$$\int_{\Gamma_0^{d_0} \cup \Sigma_0^{d_0}} r^{1-p(x)} u|u|^{p(x)-2} \eta(x) dS \le \|\eta\|_{\mathbf{L}_\infty} \cdot \|u\|_{\mathbf{L}_\infty}^{p_+ -1} \int_{\Gamma_0^{d_0} \cup \Sigma_0^{d_0}} r^{1-p(x)} dS$$
$$\le const \cdot \|\eta\|_{\mathbf{L}_\infty} \cdot \|u\|_{\mathbf{L}_\infty}^{p_+ -1} \int_0^{d_0} r^{n-p_+-1} dr = \frac{const}{n - p_+} \|\eta\|_{\mathbf{L}_\infty} \cdot \|u\|_{\mathbf{L}_\infty}^{p_+ -1} \cdot d_0^{n-p_+}.$$

5) By assumption **(v)** and (1.3) (see below),

$$\int_{G_0^{d_0}} f(x) \eta(x) dx$$
$$\le f_0 \|\eta\|_{\mathbf{L}_\infty} \int_{G_0^{d_0}} |x|^{\beta(x)} dx = f_0 \|\eta\|_{\mathbf{L}_\infty} (\text{meas}\, \Omega_{d_0}) \int_0^{d_0} r^{\varkappa(p(x)-1)-p(x)+n-1} dr$$
$$\le f_0 \|\eta\|_{\mathbf{L}_\infty} (\text{meas}\, \Omega_{d_0}) \int_0^{d_0} r^{\varkappa(p_- -1)-p_+ +n-1} dr = \frac{f_0 \|\eta\|_{\mathbf{L}_\infty} (\text{meas}\, \Omega_{d_0})}{\varkappa(p_- -1) - p_+ + n} d_0^{\varkappa(p_- -1)-p_+ +n}.$$

6) By assumption **(vv)**,

$$\int_{\Gamma_0^{d_0}} g(x,u)\eta(x)dS \le g_0\|\eta\|_{\mathbf{L}_\infty} \int_{\Gamma_0^{d_0}} r^{1+\beta(x)}dS = g_0\|\eta\|_{\mathbf{L}_\infty} (\mathrm{meas}\,\partial\Omega_{d_0}) \int_0^{d_0} r^{\varkappa(p(x)-1)-p(x)+n}dr$$
$$\le g_0\|\eta\|_{\mathbf{L}_\infty} (\mathrm{meas}\,\partial\Omega_{d_0}) \int_0^{d_0} r^{\varkappa(p_- -1)-p_+ +n}dr = \frac{g_0\|\eta\|_{\mathbf{L}_\infty} (\mathrm{meas}\,\partial\Omega_{d_0})}{\varkappa(p_- -1)-p_+ +n+1} d_0^{\varkappa(p_- -1)-p_+ +n+1}.$$

Similarly we verify the existence and the finiteness of $\int_{\Sigma_0^{d_0}} h(x,u)\eta(x)dS$.

7) Finally, the existence and the finiteness of $\int_{\Omega_{d_0}} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial r}\eta(x)d\Omega_d$ follows from the equality (*II*). □

The main result is the following statement.

Theorem 1.3. *Let u be a weak bounded solution of problem (TRQL), $M_0 = \sup_{x\in G_0^{d_0}} |u(x)|$[*] and let λ be the least positive eigenvalue of problem (NEVP) (see Section 2). Suppose that* **(i)–(vvv)** *hold. Then there exist $\widetilde{d} \in (0,d_0)$ and a constant $C_0 > 0$ depending only on $\lambda, d_0, M_0, p_+, p_-, L, n, (\mu-\delta), \nu, b_0, f_0$ and such that*

$$|u(x)| \le C_0|x|^{\varkappa}, \quad \varkappa = \frac{p_+ -1}{p_+ -1+\mu}\lambda; \quad \forall x \in G_0^{\widetilde{d}}. \tag{1.3}$$

2 Nonlinear eigenvalue problem

Let $\overrightarrow{\nu}$ be the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$ and $\overrightarrow{\tau}$ be the exterior with respect to Ω_+ normal to σ_0 (lying in the tangent to Ω plane). To prove the main result we shall consider the nonlinear eigenvalue problem for $\psi(\omega) \in \mathbf{C}^2(\Omega)\cap\mathbf{C}^1(\overline{\Omega})$, where

$$\psi(\omega) = \begin{cases} \psi_+(\omega), & \omega \in \overline{\Omega_+}, \\ \psi_-(\omega), & \omega \in \overline{\Omega_-}, \end{cases}$$

$$\begin{cases} -\mathrm{div}_\omega \left((\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\nabla_\omega\psi \right) \\ \qquad = \lambda\left(\lambda(p_+-1)+n-p_+\right)(\lambda^2\psi^2+|\nabla_\omega\psi|^2)^{(p_+-2)/2}\psi, \quad \omega\in\Omega, \\ [\psi]_{\sigma_0} = 0; \left[(\lambda^2\psi^2+|\nabla_\omega\psi|^2)^{(p_+-2)/2}\frac{\partial\psi}{\partial\overrightarrow{\tau}} \right]_{\sigma_0} + \beta\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\cdot\psi|\psi|^{p_+-2}\Big|_{\sigma_0} = 0, \\ (\lambda^2\psi^2+|\nabla_\omega\psi|^2)^{(p_+-2)/2}\frac{\partial\psi}{\partial\overrightarrow{\nu}} + \gamma\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\cdot\psi|\psi|^{p_+-2} = 0, \quad \omega\in\partial\Omega, \end{cases} \tag{NEVP}$$

where $|\nabla_\omega\psi|$ denotes the projection of the vector $\nabla\psi$ onto the tangent plane to the unit sphere at the point ω:

$$\nabla_\omega\psi = \left\{ \frac{1}{\sqrt{q_1}}\frac{\partial\psi}{\partial\omega_1}, \ldots, \frac{1}{\sqrt{q_{n-1}}}\frac{\partial\psi}{\partial\omega_{n-1}} \right\},$$

$$|\nabla_\omega\psi|^2 = \sum_{i=1}^{n-1}\frac{1}{q_i}\left(\frac{\partial\psi}{\partial\omega_i}\right)^2, \quad q_1 = 1, \quad q_i = (\sin\omega_1\cdots\sin\omega_{i-1})^2, \quad i\ge 2.$$

Proposition 2.1.

$$\lambda\left(\lambda(p_+-1)+n-p_+\right) > 0 \implies \lambda > 0. \tag{2.1}$$

[*]see Section 3, Theorem 3.3

Proof. We multiply the (*NEVP*) equation by $\psi(\omega)$ and integrate over Ω:

$$\int_\Omega \left\langle -\psi(\omega)\mathrm{div}_\omega \left((\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\nabla_\omega\psi \right) \right\rangle d\Omega$$
$$= \lambda\left(\lambda(p_+-1)+n-p_+\right)\int_\Omega (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\psi^2(\omega)d\Omega.$$

Integrating by parts the left integral, we obtain

$$\begin{aligned}
&\int_\Omega \left\langle -\psi(\omega)\mathrm{div}_\omega \left((\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\nabla_\omega\psi \right) \right\rangle d\Omega \\
&= \int_\Omega (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}|\nabla_\omega\psi|^2 d\Omega - \int_{\sigma_0}\left[(\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\frac{\partial\psi}{\partial\overrightarrow{\nu}}\right]\psi(\omega)d\sigma \\
&\quad - \int_{\partial\Omega}(\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\frac{\partial\psi}{\partial\overrightarrow{\nu}}\psi(\omega)d\sigma \\
&= \int_\Omega (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}|\nabla_\omega\psi|^2 d\Omega + \beta\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\cdot\int_{\sigma_0}|\psi|^p d\sigma \\
&\quad + \gamma\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\cdot\int_{\partial\Omega}|\psi|^p d\sigma.
\end{aligned}$$

From the above obtained equalities we derive

$$\begin{aligned}
&\lambda\left(\lambda(p_+-1)+n-p_+\right)\int_\Omega (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\psi^2(\omega)d\Omega \\
&\quad = \int_\Omega (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}|\nabla_\omega\psi|^2 d\Omega + \beta\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\int_{\sigma_0}|\psi|^p d\sigma \\
&\quad + \gamma\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\int_{\partial\Omega}|\psi|^p d\sigma. \qquad (2.2)
\end{aligned}$$

Because of $\beta > 0,\ \gamma > 0$ and $\psi(\omega) \not\equiv 0$, the last equality means that inequality (2.1) is true. Since $p_+ \in (1, n)$ we have also $\lambda > 0$. □

Proposition 2.2. *Let $y(\omega) = \frac{\nabla_\omega\psi(\omega)}{\psi(\omega)}$. There exists a constant $Y_0 = c(p_+, \lambda, n)$ that satisfies the inequality*

$$|y(\omega)| \le Y_0, \quad \forall\omega\in\Omega. \qquad (2.3)$$

Proof. From (2.2) it follows that

$$\begin{aligned}
&\int_\Omega |\psi(\omega)|^{p_+}(\lambda^2 + y^2(\omega))^{\frac{p_+-2}{2}}y^2(\omega)d\Omega \\
&\quad \le \lambda\left\langle\lambda(p_+-1)+n-p_+\right\rangle\int_\Omega |\psi(\omega)|^{p_+}(\lambda^2 + y^2(\omega))^{\frac{p_+-2}{2}}d\Omega \\
&\quad \Longrightarrow \int_\Omega |\psi(\omega)|^{p_+}(\lambda^2 + y^2(\omega))^{\frac{p_+-2}{2}}\left\{y^2(\omega) - \lambda\left\langle\lambda(p_+-1)+n-p_+\right\rangle\right\}d\Omega \le 0.
\end{aligned}$$

Hence it is clear the desired estimate (2.3)

$$|y(\omega)| \le \sqrt{\lambda\left\langle\lambda(p_+-1)+n-p_+\right\rangle} \equiv Y_0. \qquad \square$$

If we rename $\omega = \omega_1, \omega' = (\omega_2, \dots, \omega_{n-1})$ then, by assumption **(vvv)**, we can see that $\psi(\omega_1, \omega')$ do not depend on ω'. Therefore our problem (*NEVP*) is equivalent the following nondivergent form:

$$\begin{cases} \left(\lambda^2\psi^2 + (p_+ - 1)\psi'^2\right)\psi''(\omega) + (n-2)\cot\omega\left(\lambda^2\psi^2 + \psi'^2\right)\psi'(\omega) \\ \qquad +\lambda\left(\lambda(2p_+ - 3) + n - p_+\right)\psi'^2\psi(\omega) \\ \qquad +\lambda^3\left(\lambda(p_+ - 1) + n - p_+\right)\psi^3(\omega) = 0, \quad \omega \in \Omega = \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right) \setminus \{0\}; \\ \psi_+(0) = \psi_-(0) = \psi(0); \\ \left[\lambda^2\psi^2 + \psi'^2\right)^{(p_+-2)/2}\psi'(\omega)\Big]_{\omega=0} + \beta\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\psi(0)|\psi(0)|^{p_+-2} = 0; \\ \pm(\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi'(\omega) + \gamma\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\psi|\psi|^{p_+-2} = 0, \quad \omega = \pm\omega_0/2. \end{cases} \tag{OEVP}$$

2.1 Properties of the (*OEVP*) eigenvalue and corresponding eigenfunction

First of all, we note that $\psi_-(-\omega) = \psi_+(\omega)$. Note that any two eigenfunctions are scalar multiples of each other if they solve problem for the same λ. Therefore, without loss of generality we can assume that

$$\psi_+\left(\frac{\omega_0}{2}\right) = \psi_-\left(-\frac{\omega_0}{2}\right) = 1. \tag{2.4}$$

Lemma 2.3.

1) *$y_-(\omega) > 0,\ \omega \in \overline{\Omega_-},\ y_+(\omega) < 0,\ \omega \in \overline{\Omega_+}$. Moreover, if $n = 2$ then $y'_\pm(\omega) < 0,\ \omega \in \Omega_\pm$ and therefore $y_\pm(\omega)$ are decreasing functions.*

2) *There exists a constant $\psi_0 = c(p_+, n, \lambda, \gamma, \beta, \mu, \omega_0)$ that satisfies the inequality*

$$1 \le \psi(\omega) \le \psi_0, \quad \omega \in \overline{\Omega}. \tag{2.5}$$

Proof. Let us consider the $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$. From (*OEVP*) we obtain the problem

$$\begin{cases} \left((p_+ - 1)y^2 + \lambda^2\right)y'(\omega) + (n-2)\cot\omega\left(y^2 + \lambda^2\right)y(\omega) + (p_+ - 1)y^4 \\ \qquad +\lambda\left(2\lambda(p_+ - 1) + n - p_+\right)y^2 + \lambda^3\left(\lambda(p_+ - 1) + n - p_+\right) = 0, & \omega \in \Omega, \\ \left[y\left\langle\lambda^2 + y^2\right\rangle^{\frac{p_+-2}{2}}\right]_{\omega=0} = -\beta\left(\dfrac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1}, \\ y_+\left\langle\lambda^2 + y_+^2\right\rangle^{\frac{p_+-2}{2}} = -\gamma\left(\dfrac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1}, & \omega = \dfrac{\omega_0}{2}, \\ y_-\left\langle\lambda^2 + y_-^2\right\rangle^{\frac{p_+-2}{2}} = +\gamma\left(\dfrac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1}, & \omega = -\dfrac{\omega_0}{2}. \end{cases} \tag{CP}$$

At first, we note that $y_-(-\omega) = -y_+(\omega)$ and therefore $y'_-(-\omega) = y'_+(\omega)$. From this and from the conjunction condition (*CP*) we get

$$y_+(0)\left\langle\lambda^2 + y_+^2(0)\right\rangle^{\frac{p_+-2}{2}} - y_-(0)\left\langle\lambda^2 + y_-^2(0)\right\rangle^{\frac{p_+-2}{2}} = -\beta\left(\frac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1},$$

$$\Longrightarrow \begin{cases} y_+(0)\left\langle\lambda^2 + y_+^2(0)\right\rangle^{\frac{p_+-2}{2}} = -\dfrac{\beta}{2}\left(\dfrac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1}, \\ y_-(0)\left\langle\lambda^2 + y_-^2(0)\right\rangle^{\frac{p_+-2}{2}} = \dfrac{\beta}{2}\left(\dfrac{p_+ - 1 + \mu}{p_+ - 1}\right)^{p_+-1}. \end{cases} \tag{2.6}$$

Thus we have:

$$y_+(0) < 0,\ y_+\left(\frac{\omega_0}{2}\right) < 0; \quad y_-(0) > 0,\ y_-\left(-\frac{\omega_0}{2}\right) > 0. \tag{2.7}$$

Now we consider two cases: $n = 2$ and $n > 2$.

$$\boxed{n = 2.}$$

From the equation (CP) it follows that $y'_\pm(\omega) < 0,\ \omega \in \Omega$ and therefore $y_\pm(\omega)$ are decreasing functions. Hence it follows that

$$\begin{aligned} y_+\left(\frac{\omega_0}{2}\right) \le y_+(\omega) \le y_+(0) < 0, \qquad & \omega \in \overline{\Omega_+}; \\ 0 < y_-(0) \le y_-(\omega) \le y_-\left(-\frac{\omega_0}{2}\right), \qquad & \omega \in \overline{\Omega_-}. \end{aligned} \tag{2.8}$$

Remark 2.4. The assumption (1.2) guaranties inequalities (2.8).

Now, we shall estimate $y_\pm\left(\pm\frac{\omega_0}{2}\right)$ and $y_\pm(0)$ from boundary condition and (2.6). For this we consider the equation

$$|t|\left(\lambda^2 + t^2\right)^{\frac{p_+-2}{2}} = a, \tag{2.9}$$

where a is a given positive number. We assert that this equation has a bounded solution. In fact:

- for $p_+ = 2$ a solution is $|t| = a$;
- for $p_+ > 2$ we have $|t|^{p_+-1} \le |t|\left(\lambda^2 + t^2\right)^{\frac{p_+-2}{2}} = a, \implies |t| \le a^{\frac{1}{p_+-1}}$;
- if $1 < p_+ < 2$ then the equation cannot have a unbounded solution, because of

  $$\lim_{|t|\to+\infty} \frac{|t|}{\left(\lambda^2 + t^2\right)^{\frac{2-p_+}{2}}} = \lim_{|t|\to+\infty} |t|^{p_+-1} = +\infty,$$

  which contradicts (2.9).

Hence it follows that there exist such constants $c(p_+, \lambda, \gamma, \mu)$ and $c(p_+, \lambda, \beta, \mu)$ that

$$\left|y_\pm\left(\pm\frac{\omega_0}{2}\right)\right| \le c(p_+, \lambda, \gamma, \mu); \quad |y_\pm(0)| \le c(p_+, \lambda, \beta, \mu). \tag{2.10}$$

By the definition of $y(\omega)$ and (2.4), we derive from (2.8) and (2.10)

$$1 \le \psi_+(\omega) = \exp\left(-\int_\omega^{\frac{\omega_0}{2}} y_+(\xi)d\xi\right) \le c(p_+, \lambda, \gamma, \mu, \omega_0) = \psi_0, \quad \omega \in [0, \omega_0/2]; \tag{2.11}$$

$$1 \le \psi_-(\omega) = \exp\left(\int_{-\frac{\omega_0}{2}}^{\omega} y_-(\xi)d\xi\right) \le c(p_+, \lambda, \gamma, \mu, \omega_0) = \psi_0, \quad \omega \in [-\omega_0/2, 0]. \tag{2.12}$$

$$\boxed{n > 2.}$$

Let us prove that $y_+(\omega) < 0$ for all $\omega \in [0, \omega_0/2]$. It is true on the ends of this segment (see (2.7)). Let us assume that $y_+(\omega) \ge 0$ in some interval $[\omega_1, \omega_2] \subset [0, \omega_{0/2}]$ and $y_+(\omega_1) = y_+(\omega_2) = 0$ (by the continuity of $y_+(\omega)$ at least two such points can be found). Therefore $y'_+(\omega_1) > 0$, but it impossible by differential equation: in those points, where the function becomes zero, itâĂŹs first derivative is strongly negative. The same we obtain if in one point

$y_+(\omega_1)=0$, where the function (curve) touches abscissa axis. Now it is clear that $y_-(\omega)>0$ for all $\omega\in[-\omega_0/2,0]$. Hence, by the definition of $y(\omega)$ and (2.4), we derive from (2.3) the following:

$$1\le\psi_+(\omega)=\exp\left(-\int_\omega^{\frac{\omega_0}{2}}y_+(\xi)d\xi\right)\le\exp\left(\frac{\omega_0}{2}Y_0\right)=\psi_0,\quad\omega\in[0,\omega_0/2];\tag{2.13}$$

$$1\le\psi_-(\omega)=\exp\left(\int_{-\frac{\omega_0}{2}}^{\omega}y_-(\xi)d\xi\right)\le\exp\left(\frac{\omega_0}{2}Y_0\right)=\psi_0,\quad\omega\in[-\omega_0/2,0].\tag{2.14}$$

□

Proposition 2.5. *Let* $\left|y_\pm\left(\pm\frac{\omega_0}{2}\right)\right|=|y_0|$. *If assumptions* **(i)** *and* (1.2) *satisfy then*

$$\begin{aligned}\left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\right)^{p(x)-p(0)}&\le1,\quad\forall x\in\Gamma_0^d,\\ \left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_\pm^2(0)}\right)^{p(x)-p(0)}&\le1,\quad\forall x\in\Sigma_0^d,\end{aligned}\tag{2.15}$$

where $\varkappa$ *is defined by* (1.3).

Proof. For the proof of the first inequality of (2.15) we refer to the proof of Proposition 2.1 of [2]. The proof of the second inequality of (2.15) is analogous. □

Proposition 2.6.

$$\frac{y'(\omega)}{y^2+\lambda^2}\ge-\frac{(p_+-1)(y^2+\lambda^2)+(n-p_+)\lambda}{(p_+-1)y^2+\lambda^2},\quad\omega\in\Omega_d.\tag{2.16}$$

Proof. Since

$$\begin{aligned}&(p_+-1)y^4+\lambda\left(2\lambda(p_+-1)+n-p_+\right)y^2+\lambda^3\left(\lambda(p_+-1)+n-p_+\right)\\&\qquad=(p_+-1)(y^2+\lambda^2)\left(y^2+\lambda^2+\frac{n-p_+}{p_+-1}\lambda\right),\end{aligned}$$

the (*CP*) equation can be rewritten as follows

$$\frac{y_+'(\omega)}{y_+^2+\lambda^2}=-\frac{(n-2)\cot\omega}{(p_+-1)y_+^2+\lambda^2}\cdot y_+(\omega)-\frac{(p_+-1)(y_+^2+\lambda^2)+(n-p_+)\lambda}{(p_+-1)y_+^2+\lambda^2},\quad\omega\in\left(0,\frac{\omega_0}{2}\right).$$

$$\frac{y_-'(\omega)}{y_-^2+\lambda^2}=-\frac{(n-2)\cot\omega}{(p_+-1)y_-^2+\lambda^2}\cdot y_-(\omega)-\frac{(p_+-1)(y_-^2+\lambda^2)+(n-p_+)\lambda}{(p_+-1)y_-^2+\lambda^2},\quad\omega\in\left(-\frac{\omega_0}{2},0\right).$$

Now, by Lemma 2.3, $y_+(\omega)<0,\ \omega\in\Omega_+;\ y_-(\omega)>0,\ \omega\in\Omega_-$, hence from this it follows the desired inequality (2.16). □

3 Maximum principle

In this section we derive L_∞-*a priori* estimate of the weak bounded solution to problem (*TRQL*). For this we shall consider the problem in a domain $G\subset\mathbb{R}^n$, $n\ge2$ being a bounded domain with the boundary Γ and that is divided into two subdomains G_+ and G_- by $\Sigma_0=G\cap\{x_n=0\}$.

We suppose that Γ is a smooth surface everywhere except at the origin $\mathcal{O} \in \Gamma$, and near the point $\mathcal{O}$ it is a conical surface whose vertex is $\mathcal{O}$, namely $G \cap B_{d_0} = G_0^{d_0}$. Thus, $G_0^{d_0} \subset G$.

We shall derive L_∞-*a priori* estimate of the weak bounded solution to the transmission Robin problem:

$$\begin{cases} -\triangle_{p(x)} u + a(x)u|u|^{p(x)-1} + b(u, \nabla u) = f(x), & x \in G, \\ [u]_{\Sigma_0} = 0, \\ \left[|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \frac{\beta}{|x|^{p(x)-1}} u|u|^{p(x)-2} = h(x,u), & x \in \Sigma_0, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \vec{n}} + \frac{\gamma}{|x|^{p(x)-1}} u|u|^{p(x)-2} = g(x,u), & x \in \Gamma. \end{cases} \tag{TRPr}$$

Definition 3.1. The function u is called a weak bounded solution of problem (*TRPr*) provided that $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G)$ and satisfies the integral identity

$$\begin{aligned} &\int_G \left\langle |\nabla u|^{p(x)-2} u_{x_i} \eta_{x_i} + a(x)u|u|^{p(x)-1}\eta(x) + b\left(u, \nabla u\right)\eta(x) \right\rangle dx \\ &\qquad + \gamma \int_\Gamma r^{1-p(x)} u|u|^{p(x)-2}\eta(x)dS + \beta \int_\Sigma r^{1-p(x)} u|u|^{p(x)-2}\eta(x)dS \\ &= \int_\Gamma g(x,u)\eta(x)dS + \int_{\Sigma_0} h(x,u)\eta(x)dS + \int_G f(x)\eta(x)dx \end{aligned} \tag{3.1}$$

for all $\eta(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G)$.

At first we formulate well-known lemmas.

Lemma 3.2 (see [7, Lemma 2.1] and [6, Lemma 1.60])**.** *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1, & x \geq 0, \\ -e^{-\varkappa x} + 1, & x \leq 0, \end{cases}$$

where $\varkappa > 0$. Let a, b be positive constants, $m > 1$. If $\varkappa > (2b/a) + m$, then we have

$$a\eta'(x) - b\eta(x) \geq \frac{a}{2} e^{\varkappa x}, \qquad \forall x \geq 0, \tag{3.2}$$

$$\eta(x) \geq \left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \geq 0. \tag{3.3}$$

Moreover, there exist a $d \geq 0$ and an $M > 0$ such that

$$\eta(x) \leq M\left[\eta\left(\frac{x}{m}\right)\right]^m \quad \text{and} \quad \eta'(x) \leq M\left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \geq d; \tag{3.4}$$

$$|\eta(x)| \geq x, \quad \forall x \in \mathbb{R}. \tag{3.5}$$

Theorem 3.3. *Let $u(x)$ be a weak solution of* (*TRQL*)*. If assumptions* **(i)–(vv)** *hold in $\overline{G}$, then there exists a constant $M_0 > 0$ depending only on* meas G*, n, $p_\pm$, s, μ, f_0, g_0, a_0, β_0, γ and such that $\|u\|_{L_\infty(G)} \leq M_0$.*

Proof. Let us define the set $A(k) = \{x \in \overline{G} : |u(x)| > k\}$ and let $\chi_{A(k)}$ be the characteristic function of the set $A(k)$. Note that $A(k+d) \subseteq A(k)$ for all $d > 0$. Putting $\eta((|u| - k)_+)\chi_{A(k)} \operatorname{sign} u$

as the test function in (3.1), where η is defined by Lemma 3.2 and $k \geq k_0$ (without loss of generality we can assume $k_0 \geq 1$), we obtain the inequality

$$\begin{aligned}
&\int_{A(k)} \left\{ |\nabla u|^{p(x)} \eta'((|u|-k)_+) + \langle a(x)|u|^{p(x)} + b(u,\nabla u)\operatorname{sign} u \rangle \eta((|u|-k)_+) \right\} dx \\
&\quad + \gamma \cdot \int_{\Gamma \cap A(k)} \left(\frac{|u|}{r}\right)^{p(x)-1} \eta((|u|-k)_+) ds + \beta \cdot \int_{\Sigma_0 \cap A(k)} \left(\frac{|u|}{r}\right)^{p(x)-1} \eta((|u|-k)_+) ds \\
&\leq \int_{\Gamma \cap A(k)} g(x,u)\operatorname{sign} u \cdot \eta((|u|-k)_+) dS \\
&\quad + \int_{\Sigma_0 \cap A(k)} h(x,u)\operatorname{sign} u \cdot \eta((|u|-k)_+) dS + \int_{A(k)} |f(x)| \eta((|u|-k)_+)\, dx. \qquad (3.6)
\end{aligned}$$

Now, we use the equality $g(x,u) - g(x,0) = \int_0^1 \frac{d}{d\tau} g(x,\tau u) d\tau$; hence we obtain

$$g(x,u)\operatorname{sign} u = g(x,0)\operatorname{sign} u + |u| \cdot \int_0^1 \frac{\partial g(x,\tau u)}{\partial(\tau u)} d\tau \leq 0, \quad \text{by assumption } \textbf{(vv)};$$

Similarly,

$$h(x,u)\operatorname{sign} u = h(x,0)\operatorname{sign} u + |u| \cdot \int_0^1 \frac{\partial h(x,\tau u)}{\partial(\tau u)} d\tau \leq 0.$$

From this and from (3.6), with regard to $a(x) \geq a_0 > 0$ and $\beta > 0$, $\gamma > 0$ (see assumptions **(iii), (v)**), it follows that

$$\begin{aligned}
&\int_{A(k)} \left\{ |\nabla u|^{p(x)} \eta'((|u|-k)_+) + \langle a_0|u|^{p(x)} + b(u,\nabla u)\operatorname{sign} u \rangle \eta((|u|-k)_+) \right\} dx \\
&\qquad \leq \int_{A(k)} |f(x)| \eta((|u|-k)_+)\, dx. \qquad (3.7)
\end{aligned}$$

Now we estimate from below $a_0|u|^{p(x)} + b(u,\nabla u)\operatorname{sign} u$ on $A(k)$. Because of $|u|_{A(k)} > k_0$, for $\mu > 0$, by assumption $(\mathbf{iv})_\mathbf{a}$, we obtain

$$\left\langle a_0|u|^{p(x)} + b(u,\nabla u)\operatorname{sign} u \right\rangle \Big|_{A(k)} \geq (a_0 - b_0 k_0^{-1})|u|^{p(x)} - \mu k_0^{-1} |\nabla u|^{p(x)}; \qquad (3.8)$$

for $\mu = 0$, by assumption $(\mathbf{iv})_\mathbf{b}$, we obtain

$$b(u,\nabla u)\operatorname{sign} u \Big|_{A(k)} \geq -\mu k_0^{-1} |\nabla u|^{p(x)} - b_0 k_0^{-1} |u|^{p(x)},$$

therefore in this case we obtain again (3.8).

As a result from (3.7)–(3.8) we get the inequality

$$\begin{aligned}
&\int_{A(k)} \left\{ |\nabla u|^{p(x)} \langle \eta'((|u|-k)_+) - \mu k_0^{-1} \eta((|u|-k)_+) \rangle + \widetilde{a}_0 |u|^{p(x)} \eta((|u|-k)_+) \right\} dx \\
&\qquad \leq \int_{A(k)} |f(x)| \eta((|u|-k)_+)\, dx; \qquad \text{where} \quad \widetilde{a}_0 = a_0 - b_0 k_0^{-1} > 0, \qquad (3.9)
\end{aligned}$$

if we choose

$$k_0 > \frac{b_0}{a_0}. \qquad (3.10)$$

The inequality (3.9) coincides with the inequality (1.14) of [3]. The further proof is analogous to the proof in [3]. It is necessary to inequality (1.36) of [3] to add the inequality (3.10). □

4 Comparison principle

In G_0^d we consider the second order quasi-linear degenerate operator T of the form

$$\begin{aligned} T(u,\eta) \equiv & \int_{G_0^d} \Big\langle \mathcal{A}_i(x,u_x)\eta_{x_i} + b(x,u,u_x)\eta(x)\Big\rangle dx + \int_{\Gamma_0^d} \frac{\gamma(\omega)}{r^{p(x)-1}} u|u|^{p(x)-2}\eta(x)ds \\ & + \int_{\Sigma_0^d} \frac{\beta(\omega)}{r^{p(x)-1}} u|u|^{p(x)-2}\eta(x)ds - \int_{\Omega_d} \mathcal{A}_i(x,u_x)\cos(r,x_i)\eta(x)d\Omega_d \\ & - \int_{\Sigma_0^d} h(x,u)\eta(x)ds - \int_{\Gamma^d} g(x,u)\eta(x)ds \end{aligned} \tag{4.1}$$

for $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ and for all non-negative $\eta(x)$ belonging to $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ under the following assumptions:

$\mathcal{A}_i(x,\xi), b(x,u,\xi)$ are Caratheodory functions, continuously differentiable with respect to the u,ξ variables in $\mathfrak{M} = \overline{G} \times \mathbb{R} \times \mathbb{R}^n$ and satisfy in $\mathfrak{M}$ the following inequalities:

(i) $\frac{\partial \mathcal{A}_i(x,\xi)}{\partial \xi_j}\zeta_i\zeta_j \geq \varkappa_p |\xi|^{p(x)-2}\zeta^2,\ \forall \zeta \in \mathbb{R}^n \setminus \{0\};\ \varkappa_p > 0;\ p(x) \geq p_- > 1;$

(ii) $\sqrt{\sum_{i=1}^{n}\left|\frac{\partial b(x,u,\xi)}{\partial \xi_i}\right|^2} \leq b_1|u|^{-1}|\xi|^{p(x)-1};\quad \frac{\partial b(x,u,\xi)}{\partial u} \geq b_2|u|^{-2}|\xi|^{p(x)};\quad b_1 \geq 0,\ b_2 \geq 0;$

(iii) $\frac{\partial g(x,u)}{\partial u} \leq 0,\ \frac{\partial h(x,u)}{\partial u} \leq 0,\quad \gamma(\omega) > 0,\ \beta(\omega) > 0.$

Proposition 4.1. *Let T satisfy assumptions (i)–(iii) and functions $u, w \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ satisfy the inequality*

$$T(u,\eta) \leq T(w,\eta) \tag{4.2}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$. Assume also that the inequality

$$u(x) \leq w(x) \quad \text{on } \Omega_d \tag{4.3}$$

holds. Then $u(x) \leq w(x)$ in G_0^d.

Proof. Let us prove this proposition by the contradiction. Let us designate $z = u - w$ and $u^\tau = \tau u + (1-\tau)w,\ \tau \in [0,1]$. We have

$$\begin{aligned} 0 \geq\ & T(u,\eta) - T(w,\eta) \\ =\ & \int_{G_0^d} \left\langle \eta_{x_i} z_{x_j} \int_0^1 \frac{\partial \mathcal{A}_i(x,u_x^\tau)}{\partial u^\tau_{x_j}} d\tau + \eta z_{x_i} \int_0^1 \frac{\partial b(x,u^\tau,u_x^\tau)}{\partial u^\tau_{x_i}} d\tau + \eta z \int_0^1 \frac{\partial b(x,u^\tau,u_x^\tau)}{\partial u^\tau} d\tau \right\rangle dx \\ & - \int_{\Omega_d} \left(\int_0^1 \frac{\partial \mathcal{A}_i(x,u_x^\tau)}{\partial u^\tau_{x_j}} d\tau \right) \cos(r,x_i) \cdot z_{x_j}\eta(x)d\Omega_d \\ & + \int_{\Gamma_0^d} \frac{\gamma(\omega)}{r^{p(x)-1}} \left(\int_0^1 \frac{\partial (u^\tau|u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau \right) z(x)\eta(x)ds \\ & + \int_{\Sigma_0^d} \frac{\beta(\omega)}{r^{p(x)-1}} \left(\int_0^1 \frac{\partial (u^\tau|u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau \right) z(x)\eta(x)ds \\ & - \int_{\Gamma_0^d} \left(\int_0^1 \frac{\partial g(x,u^\tau)}{\partial u^\tau} d\tau \right) z(x)\eta(x)ds - \int_{\Sigma_0^d} \left(\int_0^1 \frac{\partial h(x,u^\tau)}{\partial u^\tau} d\tau \right) z(x)\eta(x)ds \end{aligned} \tag{4.4}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$.

Let us designate the sets

$$(G_0^d)^+ := \{x \in G_0^d \mid \ u(x) > w(x)\} \subset G_0^d,$$
$$(\Sigma_0^d)^+ := \{x \in \Sigma_0^d \mid \ v(x) > w(x)\} \subset \Sigma_0^d,$$
$$(\Gamma_0^d)^+ := \{x \in \Gamma_0^d \mid \ u(x) > w(x)\} \subset \Gamma_0^d$$

and assume that $(G_0^d)^+ \neq \emptyset$, $(\Gamma_0^d)^+ \neq \emptyset$, $(\Sigma_0^d)^+ \neq \emptyset$. Let $k \geq 1$ be any **odd** number. We choose $\eta = \max\{(u-w)^k, 0\}$ as a test function in the integral inequality (4.4). We have

$$\int_0^1 \frac{\partial(u^\tau|u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau = (p(x)-1)\int_0^1 |u^\tau|^{p(x)-2} d\tau > 0.$$

Then, by assumptions *(i)–(iii)* and $\eta\Big|_{\Omega_d} = 0$, we obtain from (4.4) that

$$\int_{(G_0^d)^+} \left\{ k\varkappa_p z^{k-1}\Big(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau\Big)|\nabla z|^2 + b_2 z^{k+1}\Big(\int_0^1 |u^\tau|^{-2}|\nabla u^\tau|^{p(x)} d\tau\Big)\right\} dx$$
$$\leq b_1 \cdot \int_{(G_0^d)^+} z^k \Big(\int_0^1 |u^\tau|^{-1}|\nabla u^\tau|^{p(x)-1} d\tau\Big)|\nabla z| dx. \tag{4.5}$$

By the Cauchy inequality,

$$b_1 z^k|\nabla z||u^\tau|^{-1}|\nabla u^\tau|^{p(x)-1} = \left(|u^\tau|^{-1} z^{\frac{k+1}{2}}|\nabla u^\tau|^{\frac{p(x)}{2}}\right)\cdot\left(b_1 z^{\frac{k-1}{2}}|\nabla z||\nabla u^\tau|^{\frac{p(x)}{2}-1}\right)$$
$$\leq \frac{\varepsilon}{2}|u^\tau|^{-2} z^{k+1}|\nabla u^\tau|^{p(x)} + \frac{b_1^2}{2\varepsilon} z^{k-1}|\nabla z|^2|\nabla u^\tau|^{p(x)-2},\ \forall\varepsilon > 0.$$

Hence, taking $\varepsilon = 2b_2$, we obtain from (4.5) the inequality

$$\left(k\varkappa_p - \frac{b_1^2}{4b_2}\right)\int_{(G_0^d)^+} z^{k-1}|\nabla z|^2 \left(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau\right) dx \leq 0. \tag{4.6}$$

Choosing the odd number $k \geq \max\big(1; \frac{b_1^2}{2b_2\varkappa_p}\big)$, and taking into account that $z(x) \equiv 0$ on $\partial(G_0^d)^+$, we get from (4.6) that $z(x) \equiv 0$ in $(G_0^d)^+$. We got a contradiction to our definition of the set $(G_0^d)^+$, this completes the proof. □

Remark 4.2. For the $p(x)$-Laplacian assumption (i) is satisfied with

$$\varkappa_p = \begin{cases} 1, & \text{if } \ p(x) \geq 2; \\ p_- - 1, & \text{if } \ 1 < p_- \leq p(x) < 2. \end{cases}$$

4.1 Barrier function and eigenvalue problem (*OEVP*)

We shall study the **barrier** function $w(r,\omega) \not\equiv 0$ as a solution of the auxiliary problem:

$$\begin{cases} -\triangle_{p_+} w = \mu w^{-1}|\nabla w|^{p_+}, & x \in G_0^d, \\ \left[|\nabla w|^{p_+-2}\frac{\partial w}{\partial \vec{n}}\right]_{\Sigma_0^d} + \frac{\beta}{|x|^{p_+-1}} w|w|^{p_+-2} = 0, & x \in \Sigma_0^d, \\ |\nabla w|^{p_+-2}\frac{\partial w}{\partial \vec{n}} + \frac{\gamma}{|x|^{p_+-1}} w|w|^{p_+-2} = 0, & x \in \Gamma_0^d, \\ 0 < d \leq d_0 \ll 1. \end{cases} \tag{\textit{BFP}}$$

By direct calculations, we derive a solution of this problem in the form

$$w = w(r,\omega) = r^{\varkappa}\psi^{\varkappa/\lambda}(\omega), \quad \varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda, \tag{BF}$$

where $(\lambda, \psi(\omega))$ is the solution of the eigenvalue problem (*OEVP*). For this function we calculate (we use our designation $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$):

$$\begin{gathered} \frac{\partial w}{\partial r} = \varkappa r^{\varkappa-1}\psi^{\varkappa/\lambda}(\omega); \quad \frac{\partial w}{\partial \omega} = \frac{\varkappa}{\lambda} r^{\varkappa}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\psi'(\omega); \\ |\nabla w| = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\sqrt{\lambda^2\psi^2(\omega) + \psi'^2(\omega)} = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)}. \end{gathered} \tag{4.7}$$

Proposition 4.3 (see [2, Proposition 3.3])**.** $w \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$.

5 The proof of the main Theorem 1.3

Let $A > 1$, and let $w(r,\omega)$ be the barrier function defined above. By the definition of the operator Q in (*II*), we have

$$\begin{aligned} Q(Aw,\eta) \equiv & \int_{G_0^d} \Big\langle A^{p(x)-1}|\nabla w|^{p(x)-2} w_{x_i}\eta_{x_i} + a(x)A^{p(x)}w^{p(x)}\eta(x) \\ & + b\left(Aw, A\nabla w\right)\eta(x)\Big\rangle dx + \gamma\int_{\Gamma_0^d} A^{p(x)-1}r^{1-p(x)}w^{p(x)-1}\eta(x)dS \\ & + \beta\int_{\Sigma_0^d} A^{p(x)-1}r^{1-p(x)}w^{p(x)-1}\eta(x)dS \\ & - \int_{\Omega_d} A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{\partial w}{\partial r}\eta(x)d\Omega_d - \int_{\Gamma_0^d} g(x,Aw)\eta(x)dS \\ & - \int_{\Sigma_0^d} h(x,Aw)\eta(x)dS \end{aligned} \tag{5.1}$$

for all $d \in (0, d_0)$ and all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$. Integrating by parts and next calculating with regard to the problem (*BFP*) (see [2, Section 4]), from (5.1) it follows that

$$Q(Aw,\eta) = J_{G_0^d} + J_{\Gamma_0^d} + J_{\Sigma_0^d}, \tag{5.2}$$

where

$$\begin{aligned} J_{G_0^d} \equiv & \int_{G_0^d}\Bigg\langle \mu A^{p(x)-1}w^{-1}|\nabla w|^{p(x)} - A^{p(x)-1}|\nabla w|^{p_+-2}w_{x_i}\frac{d|\nabla w|^{p(x)-p_+}}{dx_i} \\ & - \frac{\partial A^{p(x)-1}}{\partial x_i}w_{x_i}|\nabla w|^{p(x)-2} + a(x)A^{p(x)}w^{p(x)} + b\left(Aw, A\nabla w\right)\Bigg\rangle\eta(x)dx; \\ J_{\Gamma_0^d} \equiv & \int_{\Gamma_0^d}\Bigg\{\gamma\left(\frac{Aw}{r}\right)^{p(x)-1}\cdot\Bigg\langle 1 - \left(\frac{r|\nabla w|}{w}\right)^{p(x)-p_+}\Bigg\rangle \\ & - Aw\cdot\int_0^1\frac{\partial g(x,\tau Aw)}{\partial(\tau Aw)}d\tau + g(x,0)\Bigg\}\eta(x)dS; \end{aligned}$$

$$J_{\Sigma_0^d} \equiv \int_{\Sigma_0^d} \left\{ \beta \left(\frac{Aw}{r} \right)^{p(x)-1} \cdot \left\langle 1 - \left(\frac{r|\nabla w_+|}{w} \right)^{p(x)-p_+} \right\rangle \right.$$
$$\left. - Aw \cdot \int_0^1 \frac{\partial h(x, \tau Aw)}{\partial(\tau Aw)} d\tau + h(x, 0) \right\} \eta(x) dS.$$

These integrals we estimate from below. At first, by Proposition 2.5 (see (2.15)) and the assumption **(vv)** , we have inequalities

$$J_{\Gamma_0^d} \geq 0, \quad J_{\Sigma_0^d} \geq 0. \tag{5.3}$$

Therefore from (5.2)–(5.3) it follows

$$Q(Aw, \eta) \geq J_{G_0^d}. \tag{5.4}$$

Because of $a(x) \geq a_0 > 0$ (the assumption **(v)**), for further details of the proof of Theorem 1.3, we refer the reader to [2, Section 4].

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