Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **38**, 1–12; https://doi.org/10.14232/ejqtde.2020.1.38 www.math.u-szeged.hu/ejqtde/

Periodic solutions of relativistic Liénard-type equations

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Received 7 November 2019, appeared 24 June 2020
Communicated by Gabriele Villari

Abstract. In this paper, we prove that the relativistic Liénard-type equation

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right)+f(x)\,\dot{x}+g(x)=0, \qquad p>1,$$

and its special case, relativistic Van der Pol-type equation, have a periodic solution. Our results are inspired by the results obtained by Mawhin and Villari [*Nonlinear Anal.* **160**(2017), 16–24] and extend their results to this more general case.

Keywords: closed orbits, periodic solutions, limit cycles, relativistic Liénard-type equations.

2020 Mathematics Subject Classification: 34C05, 34C15, 34C25, 34C26.

1 Introduction

In 1926, Van der Pol [16] considered the equation

$$\ddot{x}+\mu\left(x^{2}-1\right)\dot{x}+x=0, \qquad \mu\neq 0, \tag{1.1}$$

to obtain the results about relaxation oscillations which are important in physics and engineering problems. In 1928, Liénard [9] gave a more general description of relaxation oscillations for the equation

$$\ddot{x}+f(x)\,\dot{x}+g(x)=0, \tag{1.2}$$

where $g(x)$ is positive when $x>0$ and negative when $x<0$, $f(x)$ is negative for small values of $|x|$ and positive for large values of $|x|$. In point of fact, he takes $g(x)=x$. The more general form was first dealt with by Levinson and Smith [8]. The equations (1.1) and (1.2) are known as Van der Pol and Liénard equations, respectively. Since the appearance of Van der

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Pol and Liénard's fundamental papers, various proofs and generalizations or improvements have appeared in the literature. For example, in 1942, Levinson and Smith [8] obtained the relaxation oscillations for a more general equation

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + g(x) = 0,$$

where $g(x)$ is positive when $x > 0$ and negative when $x < 0$, $f(x, \dot{x})$ is damping coefficient which for large $|x|$ is positive and for small $|\dot{x}|$ and $|x|$ is negative.

In the last ten years, the study of the existence and multiplicity of periodic solutions of second order equations where $\ddot{x}$, with $\dot{x}$ denoting the derivative of x with respect to t, is replaced by a relativistic acceleration $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right)$ has been considered by many authors [2, 12, 13, 15]. To the best of our knowledge, this is the first paper using a generalized relativistic acceleration

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right)$$

to study the following problems. It should be noted that the definition of the generalized relativistic acceleration is given by the generalizations chosen in the numerator and the denominator, and this choice of the denominator will be clear in the following Section 2. It is easy to see that the inverse of the generalized curvature operator

$$\Phi_p(v) = \frac{v\,|v|^{p-2}}{\left(1-|v|^{p}\right)^{\frac{p-1}{p}}}, \qquad v \in (-1, 1),$$

is

$$\Phi_q^{-1}(v) = \frac{v\,|v|^{q-2}}{\left(1+|v|^{q}\right)^{\frac{q-1}{q}}}, \qquad v \in \mathbb{R}, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\left|\Phi_q^{-1}\right| < 1$. In the literature, the authors obtained some results for equations with relativistic acceleration by using various methods [1–3, 6, 10–12].

More recently, Fujimoto and Yamaoka [7] and Pérez-González et al. [13] have obtained the results about the existence and uniqueness of limit cycles of the Liénard-type differential equations of forms

$$\frac{d}{dt}\left(\phi(\dot{x})\right) + f(x)\,\phi(\dot{x}) + g(x) = 0$$

and

$$\frac{d}{dt}\left(\varphi(\dot{x})\right) + f(x)\,\psi(\dot{x}) + g(x) = 0$$

involving the curvature operators, respectively.

The aim of this paper is to obtain new results about the existence and uniqueness of limit cycles for the generalized relativistic Liénard equations of the form

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right) + f(x)\,\dot{x} + g(x) = 0, \qquad p > 1, \tag{1.4}$$

where the continuous functions f and g satisfy some conditions, inspired by Mawhin and Villari [12].

2 Relativistic duffing and Liénard-type equations

We now consider the relativistic Liénard-type equation (1.4), with $xg(x) > 0$ and $g(0) = 0$, so that $(0,0)$ is an equilibrium. Solutions of Eq. (1.4) must of course be such that $|\dot{x}(t)| < 1$ for all $t \in \mathbb{R}$, so that, instead of considering the usual phase plane $\mathbb{R}^2$, one is a priori restricted to the strip $\mathbb{R} \times (-1,1)$. A way to avoid this difficulty is to make a change of variable

$$y = \frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1 - |\dot{x}|^{p}\right)^{\frac{p-1}{p}}}, \qquad p > 1,\ |\dot{x}| < 1,$$

which is equivalent to

$$\dot{x} = \frac{y\,|y|^{q-2}}{\left(1 + |y|^{q}\right)^{\frac{q-1}{q}}}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad y \in \mathbb{R},$$

from (1.3). Then, Eq. (1.4) can be written as a pair of first order equations

$$\dot{x} = \frac{y\,|y|^{q-2}}{\left(1 + |y|^{q}\right)^{\frac{q-1}{q}}}, \qquad \dot{y} = -f(x)\,\frac{y\,|y|^{q-2}}{\left(1 + |y|^{q}\right)^{\frac{q-1}{q}}} - g(x). \tag{2.1}$$

On the other hand, Eq. (1.4) can be rewritten in the form below

$$\frac{d}{dt}\left[\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1 - |\dot{x}|^{p}\right)^{\frac{p-1}{p}}} + F(x)\right] + g(x) = 0,$$

where $F(x) = \int_0^x f(s)ds$. If we make the change of variable

$$y = \frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1 - |\dot{x}|^{p}\right)^{\frac{p-1}{p}}} + F(x), \qquad p > 1, \quad |\dot{x}| < 1,$$

then we have

$$\dot{x} = \frac{(y - F(x))\,|y - F(x)|^{q-2}}{\left(1 + |y - F(x)|^{q}\right)^{\frac{q-1}{q}}}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad (y - F(x)) \in \mathbb{R},$$

from (1.3). Thus, Eq. (1.4) can be written as a pair of first order equations

$$\dot{x} = \frac{(y - F(x))\,|y - F(x)|^{q-2}}{\left(1 + |y - F(x)|^{q}\right)^{\frac{q-1}{q}}}, \qquad \dot{y} = -g(x). \tag{2.2}$$

From this follows immediately the following regularity result.

Lemma 2.1. *If $q > 2$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $g : \mathbb{R} \to \mathbb{R}$ locally Lipschitzian, the Cauchy problem for Eq.* (1.4) *or* (2.1) *or* (2.2) *is locally uniquely solvable.*

Proof. It suffices to notice that F is of class C^1, and apply standard results [5] to system (2.2). □

Note that for $1 < q \leq 2$, the first equation of system (2.2) does not satisfy the locally Lipschitz conditions at the origin, and this case will be discussed below.

We now consider the corresponding Duffing-type equation, for which $f \equiv 0$,

$$\frac{d}{dt}\left(\frac{\dot{x}\,|\dot{x}|^{p-2}}{\left(1-|\dot{x}|^{p}\right)^{\frac{p-1}{p}}}\right)+g\left(x\right)=0, \tag{2.3}$$

and the system (2.1) reduces to

$$\dot{x}=\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}, \qquad \dot{y}=-g\left(x\right). \tag{2.4}$$

We observe that the system (2.4) has the Hamiltonian structure

$$\dot{x}=\frac{\partial H}{\partial y}\left(x,y\right), \qquad \dot{y}=-\frac{\partial H}{\partial x}\left(x,y\right),$$

where the Hamiltonian function $H\left(x,y\right)$ is given by

$$H\left(x,y\right)=\left(1+|y|^{q}\right)^{\frac{1}{q}}-1+G\left(x\right)$$

and the function $G(x)$ is the integral of $g(x)$, $G\left(x\right)=\int_0^x g\left(s\right)ds$. It is well known that the level curves of the function $H\left(x,y\right)$ are its solutions. If we consider the level curve

$$\left(1+|y|^{q}\right)^{\frac{1}{q}}-1+G\left(x\right)=C \tag{2.5}$$

in the dynamical interpretation as motion of a particle, the first term represents its kinetic energy and (2.5) expresses the law of conservation of energy as applied to the particle. Note that the constant 1 from $\left(1+|y|^{q}\right)^{\frac{1}{q}}$ is subtracted in order that, for $|y|$ small, the result $\left(1+|y|^{q}\right)^{\frac{1}{q}}-1$ is close to the classical expression $\frac{y^2}{2}$.

Now, we mention a result given by Rebelo [14].

Theorem A ([14, Theorem 1]). *If the initial value (x_0,y_0) is not an equilibrium, that is, that $\nabla H\left(x_0,y_0\right)\neq(0,0)$, the Cauchy problem for Eq. (2.3) or (2.4) is locally uniquely solvable.*

We observe that in virtue of this result for system (2.4) Lemma 2.1 holds also for $1<q\leq 2$ if the initial value is not the origin.

It is easy to see that the origin $(0,0)$ of our (x,y)-phase plane is a global center for the system (2.4) if and only if $G(x)\to+\infty$ as $|x|\to\infty$ as in the classical case. The time rate of change of H along a solution trajectory is given by

$$\begin{aligned}\frac{\partial H}{\partial t}\left(x,y\right)&=\frac{\partial H}{\partial x}\left(x,y\right)\frac{dx}{dt}+\frac{\partial H}{\partial y}\left(x,y\right)\frac{dy}{dt}\\&=g\left(x\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}-\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}\left(f\left(x\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}+g\left(x\right)\right)\\&=-f\left(x\right)\left(\frac{y\,|y|^{q-2}}{\left(1+|y|^{q}\right)^{\frac{q-1}{q}}}\right)^{2}.\end{aligned}$$

Therefore, at points where $f(x)$ is positive, the trajectories of system (2.1) enter trajectories of system (2.4), while, at points where $f(x)$ is negative, the trajectories of system (2.1) exit trajectories of system (2.4). In virtue of this result, being $f(0) < 0$, the unique equilibrium $(0,0)$ for system (2.1) and system (2.4) as well, is a source. Therefore, for both systems, the Cauchy problem is uniquely solvable in future also for $1 < q \leq 2$ and this completes the result of Lemma 2.1. Moreover, the slope of the trajectories of system (2.1) is given by the following expression, where y' denotes the derivative of y with respect to x,

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -f(x) - g(x)\frac{(1+|y|^q)^{\frac{q-1}{q}}}{y|y|^{q-2}}, \tag{2.6}$$

and the 0-isocline, namely the curve in which $\dot{y} = 0$, is given by

$$\frac{y|y|^{q-2}}{(1+|y|^q)^{\frac{q-1}{q}}} = -\frac{g(x)}{f(x)}.$$

At this point, we need to prove the existence of a winding trajectory for system (2.1) in order to apply the Poincaré–Bendixson theorem [5].

3 The relativistic Van der Pol-type equation

At first, we discuss the relativistic Van der Pol-type equation

$$\frac{d}{dt}\left(\frac{\dot{x}|\dot{x}|^{p-2}}{(1-|\dot{x}|^p)^{\frac{p-1}{p}}}\right) + \mu(x^2-1)\dot{x} + x = 0, \tag{3.1}$$

where $p > 1$ and $\mu \neq 0$, although interesting results, and in particular the existence of limit cycles, can be proved in a similar way for Eq. (1.4). Notice the case where $\mu < 0$ is reduced to the case where $\mu > 0$ by changing t into $-t$, so that we can assume without loss of generality that $\mu > 0$.

For this particular equation, system (2.1) becomes

$$\dot{x} = \frac{y|y|^{q-2}}{(1+|y|^q)^{\frac{q-1}{q}}}, \qquad \dot{y} = -\mu(x^2-1)\frac{y|y|^{q-2}}{(1+|y|^q)^{\frac{q-1}{q}}} - x, \tag{3.2}$$

and the 0-isocline is given by

$$\frac{y|y|^{q-2}}{(1+|y|^q)^{\frac{q-1}{q}}} = -\frac{x}{\mu(x^2-1)}. \tag{3.3}$$

Observe first that for $f(x) = \mu(x^2-1)$, $f(0) = -1 < 0$ and hence the origin of the phase plane is a source.

The 0-isocline in the classical Van der Pol equation is given by

$$y = -\frac{x}{\mu(x^2-1)}. \tag{3.4}$$

Of course, points of (3.3) only correspond to those x for which $-\frac{x}{\mu(x^2-1)} \in (-1,1)$, i.e., as easily shown, to the x belonging to the set

$$(-\infty, -x_2) \cup (-x_1, x_1) \cup (x_2, +\infty),$$

where

$$x_1 = -\frac{1}{2\mu} + \sqrt{\frac{1}{4\mu^2} + 1} \in (0,1), \qquad x_2 = \frac{1}{2\mu} + \sqrt{\frac{1}{4\mu^2} + 1} \in (1, +\infty).$$

Hence, (3.3) can be seen as 'stretching' the restriction of (3.4) to $\mathbb{R} \times (-1,1)$ to $\mathbb{R}^2$ (see Figs. 3.1 and 3.2).

We know that define $\gamma^+(S)$ as the positive semi-trajectory starting from S, and assume that $\gamma^+(S)$ moves around the origin and intersects again the y-axis in the same half-plane of S at a point $R = (0, y_R)$. Clearly, such semi-trajectory is winding if $|y_R| < |y_S|$, unwinding if $|y_R| > |y_S|$, and a cycle if $|y_R| = |y_S|$ [4]. At this point, arguing in the same way as in the classical case considered in [17], we are able to produce a winding trajectory. As the origin is a source, we can apply the Poincaré–Bendixson theorem [5] and get the existence of at least one limit cycle for (3.2).

We assume that Λ_1 is the graph of the function

$$\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}} = -\frac{x}{\mu\,(x^2-1)}$$

for $x \in (-\infty, -x_2)$. In this case, from (1.3), the function

$$y_1(x) = -\frac{x\,|x|^{p-2}}{\left(|\mu\,(x^2-1)|^p - |x|^p\right)^{\frac{p-1}{p}}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.5}$$

is an increasing positive function. Similarly, we define that Λ_2 is the graph of (3.3) for $x \in (x_2, +\infty)$ and so the function $y_2(x)$ given by (3.5) is an increasing negative function. Then, we get

$$\lim_{x\to -x_2^-} y_1(x) = +\infty, \qquad \lim_{x\to x_2^+} y_2(x) = -\infty,$$
$$\lim_{x\to -\infty} y_1(x) = 0, \qquad \lim_{x\to +\infty} y_2(x) = 0.$$

From the assumptions in [17], we can choose a point γ in the curve Λ_1 whose abscissa x_γ is to the left of $-x_2$ and whose ordinate is larger than the values which $y_1(x)$ takes for $x < x_\gamma$. We now define the function

$$G(x,y) = -\mu\left(x^2-1\right)\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}} - x.$$

Since

$$\frac{d}{dy}\left(\frac{y\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{q-1}{q}}}\right) = \frac{(q-1)\,|y|^{q-2}}{\left(1+|y|^q\right)^{\frac{2q-1}{q}}} > 0$$

for $q > 1$ and $y \in \mathbb{R}\setminus\{0\}$, the function $\frac{y|y|^{q-2}}{(1+|y|^q)^{\frac{q-1}{q}}}$ is an increasing function of y and so $\dot{y} = G(x,y)$ is a decreasing function of y for each fixed $x \notin [-1,1]$. The trajectory which passes through the point γ comes from 'infinity' without intersecting the x-axis before reaching the point $\gamma = (x_\gamma, y_1(x_\gamma))$ in the curve Λ_1. Since

$$y'(x) = \frac{\dot{y}}{\dot{x}} = -\mu(x^2-1) - x\frac{\left(1+|y|^q\right)^{\frac{q-1}{q}}}{y\,|y|^{q-2}} \tag{3.6}$$

gives the slope of tangent to the path of (3.2) passing through the point (x, y), the trajectory does not have vertical asymptotes and, being bounded away from the x-axis, it must cross the y-axis. By an analogous argument, we can claim that the trajectory, after entering the $x > 0$ half-plane, either will cross the x-axis on the interval $(0, x_2]$, or will cross the line $x = x_2$. In the latter case, $y(x)$ will decrease after $x = x_2$. Since the inequality

$$|x| + \mu\left(x^2-1\right) > |x| > x_2 > 0 \quad \text{for } |x| > x_2 \tag{3.7}$$

holds, the trajectory does not have a horizontal asymptote and it must eventually cross the x-axis for $x > x_2$. From (3.6), the trajectory must meet the y-axis at some $y < 0$.

Afterwards, as a consequence of (3.7) again, the trajectory cuts the x-axis either on the $-x_2 < x < 0$ segment, or at some $x \leq -x_2$. In the latter case, the trajectory may cut the curve Λ_1, but the ordinate of crossing point must be smaller than $\sup_{x\in(-\infty,x_\gamma)} y_1(x)$. Eventually, the trajectory must remain below the graph Λ_1, and so it is bounded.

Similarly, $y_2(x)$ is bounded to corresponding treatment which starts from a point $\delta \in \Lambda_2$ with abscissa $x_\delta > x_2$. Thus, we have found that starting at $t = 0$ from a point γ (or δ), the state $(x(t), y(t))$ moves for $t > 0$ along a bounded trajectory. The limit set is compact and non-empty. Since the only critical point (the origin) is repulsive, we can conclude that the limit set must be a cycle. Therefore, there exists at least one periodic solution for (3.1).

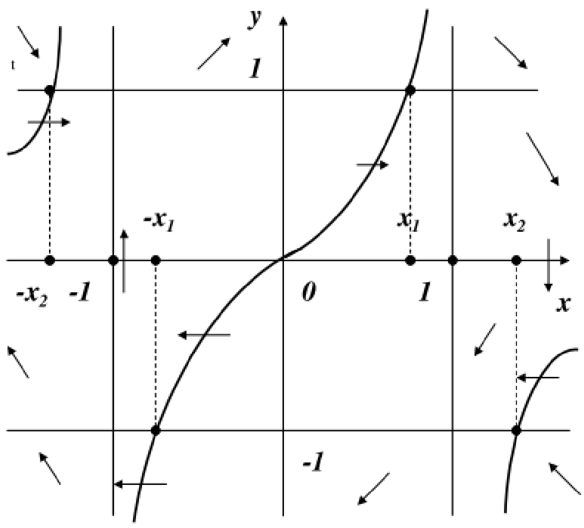

Figure 3.1: Classical Van der Pol equation for $p = 2$. Vertical asymptotes points are -1 and 1.

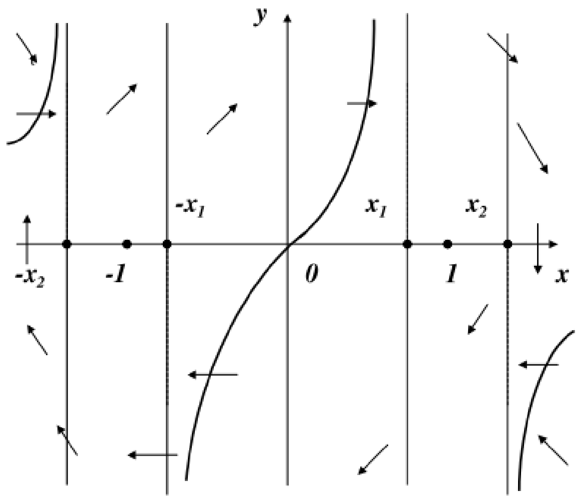

Figure 3.2: Relativistic Van der Pol-type equation for $p > 1$. Vertical asymptotes points are $\pm x_1$ and $\pm x_2$.*

As a result of the above, the following result is given.

Theorem 3.1. *For each $\mu \neq 0$, Eq. (3.1) has a least one nontrivial periodic solution.*

4 The relativistic Liénard-type equation

Following the strategy used in reference [12], we return to system (2.1) and first compare the slope of the relativistic Liénard-type system (2.6) with the slope of the classical Liénard system, namely

$$y'(x) = -f(x) - \frac{g(x)}{y}.$$

Now, we show that a direct comparison of the slopes at the same point (x, y). Since we have $|y|^q < |y|^{\frac{q}{q-1}} + |y|^{q+\frac{q}{q-1}}$ for $q > 1$ and all $y \in \mathbb{R} \setminus \{0\}$, we get

$$1 < \frac{(1 + |y|^q)\,|y|^{\frac{q}{q-1}}}{|y|^q} \quad \text{and so} \quad 1 < \frac{(1 + |y|^q)^{\frac{q-1}{q}}\,|y|}{|y|^{q-1}}.$$

Without loss of generality, we may take $y > 0$. The case when y is negative can similarly be dealt with. It is easy to see that while we have

$$-f(x) - \frac{g(x)}{y} > f(x) - g(x)\frac{(1 + |y|^q)^{\frac{q-1}{q}}}{y\,|y|^{q-2}}$$

when $x > 0$, we have

$$-f(x) - \frac{g(x)}{y} < f(x) - g(x)\frac{(1 + |y|^q)^{\frac{q-1}{q}}}{y\,|y|^{q-2}}$$

*Figures 3.1 and 3.2 are taken from the reference [12, p. 20].

when $x < 0$. Therefore, if $xy > 0$, the trajectories of system (2.1) enter the trajectories of the classical Liénard system

$$\dot{x} = y, \qquad \dot{y} = -f(x)\, y - g(x), \tag{4.1}$$

while if $xy < 0$, the trajectories of system (2.1) exit the trajectories of system (4.1). So, when $xy > 0$, the trajectories of (2.1) are guided by those of (4.1). The question is then the intersection of a positive semitrajectory with the x-axis, because in this way one can prove that trajectories are clockwise and then apply the Poincaré–Bendixson theorem [5].

When $F(x)$ is bounded from below for x positive large enough and bounded from above for x negative large enough, Villari [18] has proved that the condition

$$\limsup_{x\to+\infty} \left(G(x) + F(x)\right) = +\infty \tag{4.2}$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonnegative y intersects the x-axis, and that the condition

$$\limsup_{x\to-\infty} \left(G(x) - F(x)\right) = +\infty$$

is necessary and sufficient in order that a positive semitrajectory starting with a nonpositive y intersects the x-axis. The results are proved in the Liénard plane but hold as well in the phase plane.

More general situations have been considered by Villari and Zanolin in [19], that we shall adapt to the present situation. Likewise in [19], given $f : \mathbb{R} \to \mathbb{R}$ continuous, $F(x) = \int_0^x f(s)ds$, $g : \mathbb{R} \to \mathbb{R}$ continuous, we define $\Gamma_+ : \mathbb{R} \to \mathbb{R}$ by

$$\Gamma_+ = \int_0^x \left(1 + F_+(s)\right)^{-1} g(s)\, ds,$$

where $F_+(x) = \max\{0, F(x)\}$. We also define $G(x) = \int_0^x g(s)ds$.

Theorem 4.1. *Assume that the following conditions hold.*

(1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian, $xg(x) > 0$ for $x \neq 0$, and $f(0) < 0$.

(2) There exists $a > 0$ such that $f(x) > 0$ when $x > a$,

$$\lim_{x\to+\infty} G(x) = K < +\infty, \qquad \lim_{x\to+\infty} F(x) = +\infty.$$

(3) There exists $0 < \gamma < 4$ such that

$$\limsup_{x\to-\infty} \left(\gamma\Gamma_+(x) - F(x)\right) = +\infty.$$

Then Eq. (1.4) has at least a stable limit cycle.

Proof. Notice that Assumption 2 rules the behavior of f and g for $x > 0$ and Assumption 3 for $x < 0$. We first consider the behavior of a trajectory when $x > 0$. Let $K > 0$ be such that $G(x) < K$ for all $x \in \mathbb{R}$, according to the second condition in Assumption 2. We define $H : \mathbb{R}^2 \to \mathbb{R}$

$$H(x,y) := \left(1 + |y|^q\right)^{\frac{1}{q}} - 1 + G(x)$$

and consider the corresponding curve of equation

$$K = \left(1 + |y|^q\right)^{\frac{1}{q}} - 1 + G(x). \tag{4.3}$$

It intersects the y-axis at the point $\left(0, -\left[(K+1)^q - 1\right]^{\frac{1}{q}}\right)$. On the other hand, as $G(x) < K$ for all $x \in \mathbb{R}$ the curve with Eq. (4.3) does not intersect the x-axis. For $a > 0$ given in Assumption 2, the curve with Eq. (4.3) intersects the line $x = a$ at the point of ordinate

$$y = -\beta := -\left\{\left[(K+1) - G(a)\right]^q - 1\right\}^{\frac{1}{q}}.$$

When $G(x) \to K$, this expression tends to 0, as expected. Following an argument that appeared in [4] and [18] and a slope comparison, we observe that the negative semi-trajectory $\gamma^-(P)$ with $P = (a, -\beta)$ does not intersect the x-axis. On the other hand, as its slope is bounded, the semi-trajectory $\gamma^+(P)$ intersects the y-axis, say at point $Q = (0, \overline{y})$ with $\overline{y} < 0$.

We now consider the behavior of a trajectory when $x < 0$. For the classical Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x),$$

we know from [19] that if Assumption 3 holds, then the positive semi-trajectory $\widehat{\gamma}^+(Q)$ starting from some point $Q = (\gamma, -\beta)$ with $\gamma \in (0, 4)$ given in Assumption 3 and $\beta > 0$ intersects the vertical isocline, and therefore the x-axis at some point $R = (\widehat{x}, 0)$. The interesting case is the one where $f(x)$ is eventually negative, which corresponds to the last condition in Assumption 2. Hence, by definition of Γ_+, $G(x)$ must dominate $F(x)$. Using a comparison argument, the positive semi-trajectory $\gamma^+(Q)$ of (2.1) must intersect the x-axis at some point $S = (x, 0)$, with $\widehat{x} < x < 0$. Now, as its slope is bounded, the semi-trajectory $\gamma^*(S)$ must intersect the y-axis at some point $(0, y)$ with $y > 0$ and, in virtue of (4.2), eventually intersects the x-axis at some point $(x, 0)$ with $x > 0$.

Therefore $\gamma(P)$ is winding. The origin being a source because of the last condition in Assumption 1, we apply the Poincaré–Bendixson theorem [5] and obtain the existence of a stable limit cycle. Like in [19], a 'dual' result holds if the conditions for $x > 0$ and $x < 0$ are inverted, whose statement is left to the reader. □

Remark 4.2. It is easy to see that if we take $p = 2$ in our results, then they reduce to that of [12].

Acknowledgments

The author thanks the anonymous referee for his/her valuable suggestions and comments that helped to improve the presentation of this paper.

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