Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **56**, 1–41; https://doi.org/10.14232/ejqtde.2021.1.56 www.math.u-szeged.hu/ejqtde/

Differentiability of solutions with respect to parameters in a class of neutral differential equations with state-dependent delays

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Received 11 November 2019, appeared 28 July 2021
Communicated by Hans-Otto Walther

Abstract. In this paper we consider a class of nonlinear neutral functional differential equations with state-dependent delays. We study well-posedness and differentiability of the solution with respect to the parameters in a pointwise sense and also in the supremum norm.

Keywords: neutral differential equation, state-dependent delay, differentiability with respect to parameters.

2010 Mathematics Subject Classification: 34K05, 34K40.

1 Introduction

In this paper we consider a neutral functional differential equation with state-dependent delays (SD-NFDEs) of the form

$$\dot{x}(t) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \dot{x}(t - \mu(t)), \dot{x}(t - \rho(t, x_t, \lambda)), \theta\Big), \quad \text{a.e. } t \geq 0. \tag{1.1}$$

Here x_t denotes the solution segment function defined by $x_t(\zeta) = x(t + \zeta)$ for $\zeta \in [-r, 0]$, where $r > 0$ is a fixed finite constant. $\xi \in \Xi$, $\lambda \in \Lambda$ and $\theta \in \Theta$ represent parameters in the formula of τ, ρ and f, respectively. The parameter spaces Ξ, Λ and Θ are finite or infinite dimensional normed linear spaces. The dependence of f on the second argument represent delay terms which are not state-dependent (since we will assume differentiability of the function f with respect to its second argument). Also, the fourth argument of f contains a neutral term with a time-dependent delay. The terms in the third and fifth arguments are delayed terms of x and $\dot{x}$ with explicitly given state-dependent delays. For the simplicity of the presentation only single explicit state-dependent delays in the retarded and neutral terms, and a single time-dependent delay in the neutral term assumed to be present in the equation.

Differential equations with state-dependent delays (SD-DDEs) are studied intensively in the last decades (see, e.g., [3, 9, 11, 23, 27–29, 33, 35, 36, 42] for some recent work and a survey

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for SD-DDEs). SD-NFDEs received much less attention in the literature (see, e.g., [1,2,7,8,12, 13,16,21,22]). Walther in [39–41] studied a class of neutral equations of the form

$$\dot{x}(t) = g(\partial x_t, x_t)$$

under conditions which allow state-dependent delays both in the retarded and in the neutral terms. Here $g: C \times C^1 \supset W \to \mathbb{R}^n$, W is open, and $\partial: C^1 \to C$ denotes the continuous linear operator of differentiation (see Section 2 for definition of the function spaces). Using the state-space of continuously differentiable functions, he proved the existence of continuous semiflows corresponding to certain subsets of initial functions, and a principle of linear stability. For well-posedness results corresponding to a class of NFDEs (without state-dependent delays) of the form $\dot{x}(t) = f(t, x_t, \dot{x}_t)$ we refer to [32]. Here the existence of solutions is proved using a variant of the Krasnoselskii fixed point theorem, and $W^{1,p}$ with finite p is used as the state-space of the solutions. Note that for the case of state-dependent delays in the neutral term the conditions of [32] are not applicable to prove the existence of solutions since $\dot{x}_t$ is assumed to be only an L^p-function.

In this paper we discuss differentiability of solutions with respect to (wrt) parameters, including the initial function. Differentiability of solutions wrt parameters in equations of the form

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau(t, x_t, \xi)), \theta\Big)$$

was first proved in [14,24], and differentiability wrt parameters including the initial time for a slightly more general equations of the form

$$\dot{x}(t) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \theta\Big)$$

was proved later in [17,20]. In [5] and [19], beside of the second order differentiability, the first order differentiability was also proved. Note that in [19] the conditions assumed for the first order differentiability are weaker than the conditions assumed in earlier papers.

In [14] the differentiability of the map $(\varphi, \xi, \theta) \mapsto x_t(\cdot, \varphi, \xi, \theta)$ is proved for a fixed t, where the C-norm is used on the state-space of the solutions, and here φ is the initial function associated to the equation. The key assumption was that the parameters and the continuously differentiable initial function φ satisfy the compatibility condition

$$\dot{\varphi}(0-) = f\Big(0, \varphi(0), \varphi(-\tau(0, \varphi, \xi)), \theta\Big).$$

This condition together with the continuity of f and τ imply that the solution x corresponding to the parameters (φ, ξ, θ) is differentiable wrt the parameters at a fixed parameter value where the compatibility condition is satisfied. Walther in [37,38] proved the existence of C^1-smooth solution semiflow for retarded functional differential equations containing large classes of SD-DDEs restricting the set of initial functions to those which satisfy the compatibility condition.

A different assumption and a different technique was used in [24] to prove differentiabiliy of the map $(\varphi, \xi, \theta) \mapsto x_t(\cdot, \varphi, \xi, \theta)$. It was assumed that (φ, ξ, θ) are parameters such that the corresponding solution x generates a strictly monotone time lag function, more precisely,

$$\operatorname*{ess\,inf}_{t \in [0,\alpha]} \frac{d}{dt}\Big(t - \tau(t, x_t, \xi))\Big) > 0$$

for some $\alpha > 0$. Here $W^{1,p}$ (with finite p) was used as the state-space of the solutions, but the initial functions are assumed to belong to $W^{1,\infty}$. Such monotonicity assumption was also used in [5,17]. In [19,20] the strict monotonicity was relaxed to the assumption that the time lag function is piecewise monotone. Note that in [19] an example is given to demonstrate that in the case when the time lag function is constant on a time interval, the solution may not be differentiable wrt parameters.

Differentiability wrt parameters for "implicit" SD-NFDEs of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x(t-\eta(t)))\Big) = f\Big(t, x_t, x(t-\tau(t, x_t, \xi)), \theta\Big)$$

was discussed in [15], and of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x_t, x(t-\rho(t, x_t, \chi)), \lambda)\Big) = f\Big(t, x_t, x(t-\tau(t, x_t, \xi)), \theta\Big)$$

in [18]. In both manuscripts differentiability results was proved at parameter values where a compatibility condition is satisfied, the C-norm is used on the state-space of the solutions and the $W^{1,\infty}$-norm is used for the initial functions. Note that a similar compatibility condition was used in [30] for NFDEs in order to guarantee the existence of a continuous semiflow on a subset of C^1. Another important assumption used in both papers is that the delay functions η and ρ in the neutral term is bounded below by a positive constant. Similar condition was used in [26,34,39–41].

The structure of the paper is the following: In Section 2 we introduce some notations and preliminary results. In Section 3 we give conditions which imply the well-posedness of the initial value problem (IVP) associated to the "explicit" SD-NFDE (1.1). By a solution of Equation (1.1) we mean an absolutely continuous function x which satisfies (1.1) for a.e. $t \in [0, \alpha]$ for some $\alpha > 0$, and $x(t) = \varphi(t)$ for $t \in [-r, 0]$ with some associated initial function φ. We assume $\varphi \in W^{1,\infty}$ throughout this paper. Then $\dot{x}$ in (1.1) is defined only for a.e. t, so a condition is needed for the measurability of the composite function $\dot{x}(t-\mu(t))$. A simple way to guarantee it is to assume that the function $t - \mu(t)$ is strictly monotone increasing. For the same reason, we will pose conditions which imply that the solution generates a strictly monotone time lag function in the neutral term with state-dependent delay. We show the existence and uniqueness of the solutions in a small neighbourhood P of a fixed parameter $(\bar{\varphi}, \bar{\xi}, \bar{\lambda}, \bar{\theta}) \in \mathcal{M}$, where $\mathcal{M}$ is a special parameter set. In the definition of $\mathcal{M}$ (see details in Section 3 below) we assume that $\varphi \in W^{2,\infty}$, the parameters satisfy a compatibility condition, and a condition which implies that the time lag function $t \mapsto t - \rho(t, x_t, \lambda)$ of the second neutral term is strictly monotone increasing, more precisely,

$$\operatorname*{ess\,inf}_{t\in[0,\alpha]} \frac{d}{dt}\Big(t - \rho(t, x_t, \xi)\Big) > 0$$

for some $\alpha > 0$. We show that $W^{1,\infty}$ and $W^{2,\infty}$ initial functions in P generate $W^{1,\infty}$ and $W^{2,\infty}$ solution segment functions, respectively, and the segment functions are uniformly bounded in the respective norms on $[0, \alpha]$ wrt the parameters from P. The solutions are Lipschitz continuous wrt parameters in the $W^{1,\infty}$-norm (in a restricted sense) with a Lipschitz constant independent of the selection of the parameters from P. Note that the proof uses standard techniques, but it is presented for completeness, and since the uniform estimates mentioned above are important for the proofs of Section 4.

In Section 4 we prove the differentiability of the solutions wrt parameters in a pointwise sense, i.e., differentiability of the function $(\varphi, \xi, \lambda, \theta) \mapsto x(t, \varphi, \xi, \lambda, \theta)$ for any fixed $t \in [0, \alpha]$, and the differentiability of the function $(\varphi, \xi, \lambda, \theta) \mapsto x_t(\cdot, \varphi, \xi, \lambda, \theta)$, where we use the C-norm on the solution segments. In both cases the differentiability result is proved at parameter values which belong to $\mathcal{M} \cap P$, the initial functions are restricted to $W^{2,\infty}$-functions, and we use the $W^{2,\infty}$-norm for the initial functions. See Theorem 4.3 below for the precise formulation of the statement.

2 Notations and preliminaries

A fixed norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. The open ball in a normed linear space $(X, |\cdot|_X)$ around a point x_0 with radius R is denoted by $\mathcal{B}_X(x_0;R)$, i.e., $\mathcal{B}_X(x_0;R) := \{x \in X : |x - x_0|_X < R\}$, and the corresponding closed ball by $\overline{\mathcal{B}}_X(x_0;R)$. The space of bounded linear operators between normed linear spaces X and Y is denoted by $\mathcal{L}(X,Y)$, and the norm on it is by $|\cdot|_{\mathcal{L}(X,Y)}$.

The derivative of a single variable function $v(t)$ wrt t is denoted by $\dot{v}$. Note that all derivatives we use in this paper are Fréchet derivatives. Suppose the function $F(x_1,\dots,x_m)$ takes values in $\mathbb{R}^n$. The partial derivatives of F wrt its first, second, etc. arguments are denoted by D_1F, D_2F, etc. In the case when the argument x_1 of F is real, we simply write $D_1F(x_1,\dots,x_m)$ instead of the more precise notation $D_1F(x_1,\dots,x_m)1$, i.e., here D_1F denotes the vector in $\mathbb{R}^n$ instead of the linear operator $\mathcal{L}(\mathbb{R},\mathbb{R}^n)$. In the case when, let say, $x_2 \in \mathbb{R}^n$, then we identify the linear operator $D_2F(x_1,\dots,x_m) \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ by an $n \times n$ matrix.

The spaces of continuous and continuously differentiable functions from $[-r,0]$ to $\mathbb{R}^n$ are denoted by C and C^1, respectively, where the norms are defined by $|\psi|_C := \max\{|\psi(\zeta)| : \zeta \in [-r,0]\}$ and $|\psi|_{C^1} := \max\{|\psi|_C, |\dot{\psi}|_C\}$. The L^∞-norm of an essentially bounded Lebesgue measurable function $\psi\colon [-r,0] \to \mathbb{R}^n$ is defined by $|\psi|_{L^\infty} := \operatorname{ess\,sup}\{|\psi(\zeta)| : \zeta \in [-r,0]\}$. $W^{1,p}$ ($1 \le p < \infty$) and $W^{1,\infty}$ denote the spaces of absolutely continuous functions $\psi : [-r,0] \to \mathbb{R}^n$ of finite norm $|\psi|_{W^{1,p}} := \left(\int_{-r}^0 |\psi(s)|^p + |\dot{\psi}(s)|^p\,ds\right)^{1/p}$, $1 \le p < \infty$, and $|\psi|_{W^{1,\infty}} := \max\{|\psi|_C, |\dot{\psi}|_{L^\infty}\}$, respectively. We note that $\psi \in W^{1,\infty}$, if and only if ψ is Lipschitz continuous. $W^{2,\infty}$ is the space of continuously differentiable functions from $[-r,0]$ to $\mathbb{R}^n$ with Lipschitz continuous first derivative. The norm on $W^{2,\infty}$ is defined by $|\psi|_{W^{2,\infty}} := \max\{|\psi|_C, |\dot{\psi}|_C, |\ddot{\psi}|_{L^\infty}\}$. For a given $0 < r_0 < r$, the space of continuous functions $\chi\colon [-r,-r_0] \to \mathbb{R}^n$ is denoted by C_{r_0}, and the norm is $|\chi|_{C_{r_0}} := \max\{|\chi(\zeta)| : \zeta \in [-r,-r_0]\}$. Similarly, we use the notation $L^\infty_{r_0} := L^\infty([-r,-r_0],\mathbb{R}^n)$, and $|\chi|_{L^\infty_{r_0}} := \operatorname{ess\,sup}\{|\chi(\zeta)| : \zeta \in [-r,-r_0]\}$.

The following version of the well-known Gronwall's lemma is used in the manuscript.

Lemma 2.1. *Suppose $u\colon [a-r,b] \to [0,\infty)$ is continuous,*

$$u(t) \le A + B\int_a^t |u_s|_C\,ds, \qquad t \in [a,b], \tag{2.1}$$

and

$$|u_a|_C \le A. \tag{2.2}$$

Then

$$u(t) \le |u_t|_C \le Ae^{B(t-a)}, \qquad t \in [a,b]. \tag{2.3}$$

Proof. It is easy to see that (2.1) and (2.2) imply that the function $v(t) := |u_t|_C$ satisfies

$$v(t) \le A + B\int_a^t v(s)\,ds, \qquad t \in [a,b],$$

which yields (2.3). □

The Mean Value Theorem will be used in the following form througout the manuscript.

Lemma 2.2. *Suppose* $\psi \in W^{1,\infty}([a,b],\mathbb{R}^n)$. *Then*

$$|\psi(t_1) - \psi(t_2)| \le |\dot\psi|_{L^\infty([a,b],\mathbb{R}^n)}|t_1 - t_2|, \qquad t_1, t_2 \in [a,b].$$

The following lemma from [4] is a key result to prove Lemma 2.4.

Lemma 2.3 ([4])**.** *Let* $p \in [1,\infty)$, $g \in L^p([-r,\alpha],\mathbb{R}^n)$, $\varepsilon > 0$, *and* $u \in \mathcal{A}(\varepsilon)$, *where*

$$\mathcal{A}(\varepsilon) := \{v \in W^{1,\infty}([0,\alpha],[-r,\alpha]) \,:\, \dot v(s) \ge \varepsilon \text{ for a.e. } s \in [0,\alpha]\}. \tag{2.4}$$

Then

$$\int_0^\alpha |g(u(s))|^p\, ds \le \frac{1}{\varepsilon}\int_{-r}^\alpha |g(s)|^p\, ds.$$

Moreover, if the sequence $u^k \in \mathcal{A}(\varepsilon)$ *is such that* $|u^k - u|_{C([0,\alpha],\mathbb{R})} \to 0$ *as* $k \to \infty$, *then*

$$\lim_{k\to\infty}\int_0^\alpha \Big|g(u^k(s)) - g(u(s))\Big|^p\, ds = 0.$$

Next we recall the following estimate from [17].

Lemma 2.4. *Let* $y \in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$, *and let* $\omega_k \in (0,\infty)$ $(k \in \mathbb{N})$ *be a sequence satisfying* $\omega_k \to 0$ *as* $k \to \infty$. *Let* $\varepsilon > 0$, $\mathcal{A}(\varepsilon)$ *be defined by* (2.4), *and* $p, p^k \in \mathcal{A}(\varepsilon)$ *be such that*

$$|p^k - p|_{C([0,\alpha],\mathbb{R})} \le \omega_k, \qquad k \in \mathbb{N}. \tag{2.5}$$

Then

$$\lim_{k\to\infty}\frac{1}{\omega_k}\int_0^\alpha \Big|y(p^k(s)) - y(p(s)) - \dot y(p(s))(p^k(s) - p(s))\Big|\, ds = 0. \tag{2.6}$$

The following result is a simplified version of Lemma 2.5 from [19].

Lemma 2.5. *Suppose* $g \in L^\infty([c,d],\mathbb{R})$, *and* $u\colon [a,b] \to [c,d]$ *is an absolutely continuous function, and*

$$\operatorname*{ess\,inf}_{s\in[a,b]} \dot u(s) > 0. \tag{2.7}$$

Then the composite function $g \circ u \in L^\infty([a,b],\mathbb{R})$, *and* $|g \circ u|_{L^\infty([a,b],\mathbb{R})} \le |g|_{L^\infty([c,d],\mathbb{R})}$.

3 Well-posedness and continuous dependence on parameters

Consider the SD-NFDE

$$\dot x(t) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \dot x(t - \mu(t)), \dot x(t - \rho(t, x_t, \lambda)), \theta\Big), \quad \text{a.e. } t \in [0,T], \tag{3.1}$$

where $T > 0$ is finite or $T = \infty$, in which case $[0,T]$ denotes the interval $[0,\infty)$.

We associate the initial condition

$$x(t) = \varphi(t), \qquad t \in [-r,0]. \tag{3.2}$$

Next we list our assumptions on the SD-NFDE (3.1) which are used throughout this paper. Let Θ, Ξ and Λ be normed linear spaces with norms $|\cdot|_\Theta$, $|\cdot|_\Xi$ and $|\cdot|_\Lambda$, respectively, and let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \mathbb{R}^n$, $\Omega_4 \subset \mathbb{R}^n$, $\Omega_5 \subset \Theta$, $\Omega_6 \subset \Xi$ and $\Omega_7 \subset \Lambda$ be open subsets of the respective spaces. Let $0 < r_0 < r$ be fixed constants. We assume:

(A1) (i) $f\colon \mathbb{R}\times C\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\Theta\supset[0,T]\times\Omega_1\times\Omega_2\times\Omega_3\times\Omega_4\times\Omega_5\to\mathbb{R}^n$ is locally Lipschitz continuous in the following sense: for every finite $\alpha\in(0,T]$, for every closed subset $M_1\subset\Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_j\subset\Omega_j$ ($j=2,3,4$) of $\mathbb{R}^n$, and closed and bounded subset $M_5\subset\Omega_5$ of Θ, there exists a constant $L_1=L_1(\alpha,M_1,M_2,M_3,M_4,M_5)$ such that

$$\begin{aligned}&|f(t,\psi,u,v,w,\theta)-f(\bar t,\bar\psi,\bar u,\bar v,\bar w,\bar\theta)|\\&\qquad\le L_1\Big(|t-\bar t|+|\psi-\bar\psi|_C+|u-\bar u|+|v-\bar v|+|w-\bar w|+|\theta-\bar\theta|_\Theta\Big),\end{aligned}$$

for $t,\bar t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$, $u,\bar u\in M_2$, $v,\bar v\in M_3$, $w,\bar w\in M_4$ and $\theta,\bar\theta\in M_5$;

(ii) f is differentiable wrt its second, third, fourth, fifth and sixth variables, and the functions

$$\begin{aligned}&\mathbb{R}\times C\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\Theta\supset[0,T]\times\Omega_1\times\Omega_2\times\Omega_3\times\Omega_4\times\Omega_5\to X_j,\\&\quad(t,\psi,u,v,w,\theta)\mapsto D_jf(t,\psi,u,v,w,\theta)\end{aligned}$$

are continuous for $j=2,3,4,5,6$, where $X_2:=\mathcal{L}(C,\mathbb{R}^n)$, $X_3:=X_4:=X_5:=\mathbb{R}^{n\times n}$, and $X_6:=\mathcal{L}(\Theta,\mathbb{R}^n)$;

(A2) (i) $\tau\colon\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_6\to\mathbb{R}$ satisfies

$$0\le\tau(t,\psi,\xi)\le r,\quad\text{for } t\in[0,T],\ \psi\in\Omega_1 \text{ and } \xi\in\Omega_6,$$

and it is locally Lipschitz continuous in the following sense: for every finite $\alpha\in(0,T]$, closed subset $M_1\subset\Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_6\subset\Omega_6$ of Ξ there exists a constant $L_2=L_2(\alpha,M_1,M_6)$ such that

$$|\tau(t,\psi,\xi)-\tau(\bar t,\bar\psi,\bar\xi)|\le L_2\Big(|t-\bar t|+|\psi-\bar\psi|_C+|\xi-\bar\xi|_\Xi\Big)$$

for $t,\bar t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$ and $\xi,\bar\xi\in M_6$;

(ii) τ is differentiable wrt its second and third variables, and the maps

$$\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_6\to Y_j,\quad(t,\psi,\xi)\mapsto D_j\tau(t,\psi,\xi)$$

are continuous for $j=2,3$, where $Y_2:=\mathcal{L}(C,\mathbb{R})$ and $Y_3:=\mathcal{L}(\Xi,\mathbb{R})$;

(A3) $\mu\colon[0,T]\to[r_0,r]$ is a contraction on any finite time interval, i.e., for every finite $\alpha\in(0,T]$ there exists $L_3=L_3(\alpha)<1$ such that

$$|\mu(t)-\mu(\bar t)|\le L_3|t-\bar t|,\qquad t,\bar t\in[0,\alpha];$$

(A4) (i) $\rho\colon\mathbb{R}\times C\times\Lambda\supset[0,T]\times\Omega_1\times\Omega_7\to\mathbb{R}$ satisfies

$$0<r_0\le\rho(t,\psi,\lambda)\le r,\qquad t\in[0,T],\quad\psi\in\Omega_1,\quad\lambda\in\Omega_7,$$

and it is locally Lipschitz continuous in the following sense: for every finite $\alpha\in(0,T]$, closed subset $M_1\subset\Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and bounded and closed subset $M_7\subset\Omega_7$ of Λ there exists $L_4=L_4(\alpha,M_1,M_7)$ such that

$$|\rho(t,\psi,\lambda)-\rho(\bar t,\bar\psi,\bar\lambda)|\le L_4\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|\lambda-\bar\lambda|_\Lambda\Big)$$

for $t,\bar t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$, and $\lambda,\bar\lambda\in M_7$;

(ii) ρ is continuously differentiable, i.e., ρ is differentiable wrt all its variables, and the maps

$$\mathbb{R} \times C \times \Lambda \supset [0, T] \times \Omega_1 \times \Omega_7 \to Z_j, \quad (t, \psi, \lambda) \mapsto D_j\rho(t, \psi, \lambda)$$

are continuous for $j = 1, 2, 3$, where $Z_1 := \mathbb{R}$, $Z_2 := \mathcal{L}(C, \mathbb{R})$ and $Z_3 := \mathcal{L}(\Lambda, \mathbb{R})$.

(iii) The partial derivatives $D_1\rho$ and $D_2\rho$ are uniformly continuous on the sets $[0, \alpha] \times M_1 \times M_7 \subset [0, T] \times \Omega_1 \times \Omega_7$ such that $\alpha > 0$ is finite, M_1 is a closed subset of C which is also a bounded subset of $W^{1,\infty}$, and M_7 is a bounded and closed subset of Λ.

Remark 3.1. It follows from (A4) (i), (ii) that $\rho(t, \psi, \lambda)$ depends only on the restriction of ψ to the interval $[-r, -r_0]$, since if $\psi(\zeta) = \bar{\psi}(\zeta)$ for $\zeta \in [-r, -r_0]$, then $\rho(t, \psi, \lambda) = \rho(t, \bar{\psi}, \lambda)$ and

$$D_j\rho(t, \psi, \lambda) = D_j\rho(t, \bar{\psi}, \lambda), \qquad j = 1, 2, 3. \tag{3.3}$$

Moreover,

$$|D_2\rho(t, \psi, \lambda)h| \le |D_2\rho(t, \psi, \lambda)|_{\mathcal{L}(C,\mathbb{R})} \max_{\zeta \in [-r, -r_0]} |h(\zeta)| \tag{3.4}$$

for $t \in [0, T]$, $\psi \in \Omega_1$, $\lambda \in \Omega_7$ and $h \in C$.

We prove (3.3) for $j = 2$. The proofs for $j = 1$ and $j = 3$ are similar. Let $h \in C$ be a non-zero function, and define the sequence $\eta_k := \frac{1}{k}h$ for $k \in \mathbb{N}$. We have $|\eta_k|_C \to 0$ as $k \to \infty$, and

$$\begin{aligned} \frac{|[D_2\rho(t, \psi, \lambda) - D_2\rho(t, \bar{\psi}, \lambda)]h|}{|h|_C} &= \frac{|[D_2\rho(t, \psi, \lambda) - D_2\rho(t, \bar{\psi}, \lambda)]\eta_k|}{|\eta_k|_C} \\ &\le \frac{|\rho(t, \psi + \eta_k, \lambda) - \rho(t, \psi, \lambda) - D_2\rho(t, \psi, \lambda)\eta_k|}{|\eta_k|_C} \\ &\quad + \frac{|\rho(t, \bar{\psi} + \eta_k, \lambda) - \rho(t, \bar{\psi}, \lambda) - D_2\rho(t, \bar{\psi}, \lambda)\eta_k|}{|\eta_k|_C}. \end{aligned}$$

Since the right-hand side goes to 0 as $k \to \infty$, we get (3.3) with $j = 2$.

To prove (3.4), fix $h \in C$ such that h is nonconstant on $[-r_0, 0]$. Define

$$\bar{h}(\zeta) := \begin{cases} h(\zeta), & -r \le \zeta \le -r_0, \\ h(-r_0), & -r_0 < \zeta \le 0, \end{cases}$$

and the sequence of functions $\chi_k := \frac{1}{k}(h - \bar{h})$. Then $\bar{h}, \chi_k \in C$, $|\chi_k|_C \ne 0$ for all $k \in \mathbb{N}$, and $|\chi_k|_C \to 0$ as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \frac{|\rho(t, \psi + \chi_k, \lambda) - \rho(t, \psi, \lambda) - D_2\rho(t, \psi, \lambda)\chi_k|}{|\chi_k|_C} = 0.$$

On the other hand, since $\chi_k(\zeta) = 0$ for $-r \le \zeta \le -r_0$, we have for $k \in \mathbb{N}$

$$\frac{|\rho(t, \psi + \chi_k, \lambda) - \rho(t, \psi, \lambda) - D_2\rho(t, \psi, \lambda)\chi_k|}{|\chi_k|_C} = \frac{|D_2\rho(t, \psi, \lambda)\chi_k|}{|\chi_k|_C} = \frac{|D_2\rho(t, \psi, \lambda)(h - \bar{h})|}{|h - \bar{h}|_C} = 0.$$

Therefore, $D_2\rho(t, \psi, \lambda)h = D_2\rho(t, \psi, \lambda)\bar{h}$, so

$$|D_2\rho(t, \psi, \lambda)h| = |D_2\rho(t, \psi, \lambda)\bar{h}| \le |D_2\rho(t, \psi, \lambda)|_{\mathcal{L}(C,\mathbb{R})}|\bar{h}|_C = |D_2\rho(t, \psi, \lambda)|_{\mathcal{L}(C,\mathbb{R})} \max_{\zeta \in [-r, -r_0]} |h(\zeta)|,$$

which proves (3.4).

Note that assumptions (A1) and (A2) are very similar to those used in [24] for SD-DDEs. The key assumptions in this paper are that ρ and μ are bounded below by $r_0 > 0$ (see (A3) and (A4) (i)), and $\rho(t,\psi,\lambda)$ depends only on the restriction of ψ to the interval $[-r,-r_0]$. Similar assumption is used for SD-NFDEs in [15], see condition (g1) in [39], [41], and for PDEs with state-dependent delays in [34].

Assumptions (A1), (A2) and (A4) are naturally satisfied for the case when the parameter spaces Ξ, Λ and Θ are finite dimensional. Another typical situation when the assumed Lipschitz continuity conditions can be satisfied is the case when the parameters are functions. For example, we can consider the example when $\Lambda = W^{1,\infty}([0,T],\mathbb{R})$, and ρ has the form

$$\rho(t,\psi,\lambda) = \bar\rho\left(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta, \lambda(t)\right),$$

where $t \in [0,T]$, $\psi \in C$, $\lambda \in \Lambda$. It is easy to formulate natural assumptions on $\bar\rho$, $\nu^1,\ldots,\nu^\ell$ and B which guarantee (A4). Similar specific examples can be given for the parameters θ and ξ, and for the particular form of f and ρ when conditions (A1) and (A2) hold naturally. See Lemma 3.4 in [18] for such related results. We comment that the Arzelà–Ascoli Theorem yields that closed subsets of C which are bounded in $W^{1,\infty}$ are compact in C.

For the rest of the manuscript we will denote the restriction of a function $\psi\colon [-r,0] \to \mathbb{R}^n$ to the interval $[-r,-r_0]$ by

$$\mathcal{P}(\psi) := \psi|_{[-r,-r_0]}.$$

For a function $\psi \in C$ its continuous extension to the interval $(0,\infty)$ by a constant value will be denoted by

$$\widetilde\psi(t) := \begin{cases} \psi(t), & t \in [-r,0], \\ \psi(0), & t > 0. \end{cases} \tag{3.5}$$

Assume $\alpha_1 > 0$ is such that $\alpha_1 \le \min\{r_0, T\}$, and consider the IVP (3.1)–(3.2) on the small time interval $[0,\alpha_1]$. Since $\alpha_1 \le r_0$, it follows

$$x_t(\zeta) = x(t+\zeta) = \varphi(t+\zeta) = \widetilde\varphi(t+\zeta) = \widetilde\varphi_t(\zeta), \qquad t \in [0,\alpha_1],\ \zeta \in [-r,-r_0],$$

therefore (A3) and (A4) (i) yield

$$t - \mu(t) \le 0 \quad \text{and} \quad t - \rho(t,x_t,\lambda) = t - \rho(t,\widetilde\varphi_t,\lambda) \le 0, \qquad t \in [0,\alpha_1]. \tag{3.6}$$

Hence, on $[0,\alpha_1]$, Equation (3.1) is equivalent to the SD-DDE

$$\dot x(t) = f(t, x_t, x(t-\tau(t,x_t,\xi)), \dot\varphi(t-\mu(t)), \dot\varphi(t-\rho(t,\widetilde\varphi_t,\lambda)),\theta), \qquad \text{a.e. } t \in [0,\alpha_1], \tag{3.7}$$

where φ is the initial function from (3.2). It is known (see, e.g., [6, 14]) that the initial function must be Lipschitz continuous in order to guarantee the uniqueness of the solutions of (3.7). But then $\dot\varphi$ is only almost everywhere differentiable, and we need to ensure that $\dot\varphi(t-\mu(t))$ and $\dot\varphi(t-\rho(t,\widetilde\varphi_t,\lambda))$ are both defined for a.e. $t \in [0,\alpha_1]$. An easy way to guarantee it is to use the condition formulated in Lemma 2.5, where it is assumed that the essential infimum of the time derivative of the inner function is positive, therefore the inner function is strictly monotone increasing. Note that an other (more technical) assumption could be to assume piecewise monotonicity of the inner function (see [19] for precise definition and for more details). In this manuscript we will assume conditions which imply that the inner functions in the neutral terms in (3.1) (and the arguments of $\dot{\varphi}$ in (3.7)) are strictly monotone increasing in the sense of (2.7).

Note that (A3) implies $\dot{\mu}(t) \leq L_3(\alpha)$, and so $\frac{d}{dt}(t-\mu(t)) \geq 1 - L_3(\alpha) > 0$ for a.e. $t \in [0,\alpha]$ on any finite interval $[0,\alpha]$. For the reason described above, we will restrict the parameter λ and the initial function φ so that

$$\operatorname*{ess\,inf}_{t\in[0,\alpha_1]} \frac{d}{dt}\Big(t - \rho(t,\widetilde{\varphi}_t,\lambda)\Big) > 0 \tag{3.8}$$

with some $0 < \alpha_1 \leq r_0$.

The parameter space is defined as $\Gamma := W^{1,\infty} \times \Xi \times \Lambda \times \Theta$, and we use the short notation $\gamma = (\varphi,\xi,\lambda,\theta)$ or $\gamma = (\gamma^\varphi,\gamma^\xi,\gamma^\lambda,\gamma^\theta)$ as a parameter vector, and the product norm $|\gamma|_\Gamma := |\varphi|_{W^{1,\infty}} + |\xi|_\Xi + |\lambda|_\Lambda + |\theta|_\Theta$ is used as the norm on Γ. We introduce the set of feasible parameters

$$\Pi := \Big\{(\varphi,\xi,\lambda,\theta) \in \Gamma\colon\ \varphi \in \Omega_1,\ \ \varphi(-\tau(0,\varphi,\xi)) \in \Omega_2,\ \ \theta \in \Omega_5,\ \ \xi \in \Omega_6,\ \ \lambda \in \Omega_7\Big\}.$$

Next define the special parameter set

$$\begin{aligned}\mathcal{M} := \Big\{(\varphi,\xi,\lambda,\theta) \in \Pi \,:\, &\varphi \in W^{2,\infty},\quad D_1\rho(0,\varphi,\lambda) + D_2\rho(0,\varphi,\lambda)\dot{\varphi} < 1,\\ &\dot{\varphi}(-\mu(0)) \in \Omega_3,\quad \dot{\varphi}(-\rho(0,\varphi,\lambda)) \in \Omega_4,\\ &\dot{\varphi}(0-) = f\Big(0,\varphi,\varphi(-\tau(0,\varphi,\xi)),\dot{\varphi}(-\mu(0)),\dot{\varphi}(-\rho(0,\varphi,\lambda)),\theta\Big)\Big\}.\end{aligned}$$

As an example, let $\Lambda = C^1([0,T],\mathbb{R})$, and suppose ρ has the form $\rho(t,\psi,\lambda) = \bar{\rho}(\psi(0),\lambda(t))$ for some function $\bar{\rho} \in C^1(\mathbb{R}^n \times \mathbb{R},\mathbb{R})$. Then for $\varphi \in C^1$ it follows

$$D_1\rho(0,\varphi,\lambda) + D_2\rho(0,\varphi,\lambda)\dot{\varphi} = D_2\bar{\rho}(\varphi(0),\lambda(0))\dot{\lambda}(0) + D_1\bar{\rho}(\varphi(0),\lambda(0))\dot{\varphi}(0).$$

Clearly, if $\dot{\lambda}(0) = 0 = \dot{\varphi}(0)$, then condition $D_1\rho(0,\varphi,\lambda) + D_2\rho(0,\varphi,\lambda)\dot{\varphi} < 1$ holds for any $\bar{\rho}$. It also holds for the special case when $\dot{\lambda}(0) = 0$ and $D_1\bar{\rho}(\varphi(0),\lambda(0))\dot{\varphi}(0) < 1$. It is easy to satisfy the compatibility condition in the case when the parameter θ, or part of it appears in an additive way in the formula of f. Suppose, e.g., $\Theta = C^1([0,T],\mathbb{R}^d) \times C^1([0,T],\mathbb{R}^n)$, $\theta = (\theta_1,\theta_2) \in \Theta$, and f has the form $f(t,\psi,u,v,w,\theta) = \bar{f}(t,\psi,u,v,w,\theta_1(t)) + \theta_2(t)$. If $\theta_2(0) = \dot{\varphi}(0-) - \bar{f}(0,\varphi,\varphi(-\tau(0,\varphi,\xi)),\dot{\varphi}(-\mu(0)),\dot{\varphi}(-\rho(0,\varphi,\lambda)),\theta_1(0))$, then the compatibility condition in $\mathcal{M}$ holds. The above simple examples demonstrate that the conditions in $\mathcal{M}$ can be satisfied for certain classes of f, τ and ρ. We assume in this manuscript that f, τ and ρ are such that the set $\mathcal{M}$ is non-empty.

In the proof of Theorem 3.3 we will need the following result. We show that if a fixed parameter $\bar{\gamma}$ belongs to $\mathcal{M}$, in particular, if $\bar{\varphi} \in W^{2,\infty}$ and $D_1\rho(0,\bar{\varphi},\bar{\lambda}) + D_2\rho(0,\bar{\varphi},\bar{\lambda})\dot{\bar{\varphi}} < 1$, then there exists $\alpha_1 > 0$ such that (3.8) holds for φ and λ close to $\bar{\varphi}$ and $\bar{\lambda}$. Note that the method of the proof is similar to that of Lemma 5.2 from [24].

Lemma 3.2. *Suppose ρ satisfies (A4), and let $\bar{\varphi} \in W^{2,\infty} \cap \Omega_1$ and $\bar{\lambda} \in \Omega_7$ be such that*

$$D_1\rho(0,\bar{\varphi},\bar{\lambda}) + D_2\rho(0,\bar{\varphi},\bar{\lambda})\dot{\bar{\varphi}} < 1.$$

Then there exist finite constants $0 < \alpha_1^* \leq \min\{r_0,T\}$, $0 < \varepsilon^* < 1$ and $\delta^* > 0$ such that $\widetilde{\varphi}_t \in \Omega_1$ for $t \in [0,\alpha_1^*]$, and

$$\frac{d}{dt}\Big(t - \rho(t,\widetilde{\varphi}_t,\lambda)\Big) \geq \varepsilon^*, \quad \text{a.e. } t \in [0,\alpha_1^*], \quad \varphi \in \mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta^*), \quad \lambda \in \mathcal{B}_\Lambda(\bar{\lambda};\delta^*). \tag{3.9}$$

Proof. We note that $\bar{\varphi} \in C^1$ by the assumption $\bar{\varphi} \in W^{2,\infty}$. Let ε_0^* be such that $0 < \varepsilon_0^* < 1/2$ and

$$D_1\rho(0,\bar{\varphi},\bar{\lambda}) + D_2\rho(0,\bar{\varphi},\bar{\lambda})\dot{\bar{\varphi}} < 1 - 2\varepsilon_0^*. \tag{3.10}$$

Let $m_1^* > 0$ be fixed. The openness of Ω_1 and Ω_7, and the assumed continuity of $D_1\rho$ and $D_2\rho$ and Remark 3.1 yield that there exist finite constants $0 < T_1^* \le T$, $\kappa_1^* > 0$, $\varepsilon_1^* > 0$ and $\delta_1^* > 0$ such that $\overline{\mathcal{B}}_C(\bar{\varphi};\kappa_1^*) \subset \Omega_1$, $\overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_1^*) \subset \Omega_7$, and

$$\Big| D_2\rho(t,\psi,\lambda) - D_2\rho(0,\bar{\varphi},\bar{\lambda}) \Big|_{\mathcal{L}(C,\mathbb{R})} < m_1^* \tag{3.11}$$

and

$$\Big| D_1\rho(t,\psi,\lambda) + D_2\rho(t,\psi,\lambda)\dot{\chi} - D_1\rho(0,\bar{\varphi},\bar{\lambda}) - D_2\rho(0,\bar{\varphi},\bar{\lambda})\dot{\bar{\varphi}} \Big| < \varepsilon_0^* \tag{3.12}$$

for $t \in [0,T_1^*]$, $\psi \in \overline{\mathcal{B}}_C(\bar{\varphi};\kappa_1^*)$, $\lambda \in \overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_1^*)$ and $\chi \in C^1$ satisfying $|\mathcal{P}(\dot{\chi}) - \mathcal{P}(\dot{\bar{\varphi}})|_{C_{r_0}} \le \varepsilon_1^*$. Define the constant $m^* := \big|D_2\rho(0,\bar{\varphi},\bar{\lambda})\big|_{\mathcal{L}(C,\mathbb{R})} + m_1^*$. Combining the definition of m^* with (3.11), and (3.10) with (3.12) we get

$$\Big| D_2\rho(t,\psi,\lambda) \Big|_{\mathcal{L}(C,\mathbb{R})} < m^* \tag{3.13}$$

and

$$D_1\rho(t,\psi,\lambda) + D_2\rho(t,\psi,\lambda)\dot{\chi} < 1 - \varepsilon_0^* \tag{3.14}$$

for $t \in [0,T_1^*]$, $\psi \in \overline{\mathcal{B}}_C(\bar{\varphi};\kappa_1^*)$, $\lambda \in \overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_1^*)$ and $\chi \in C^1$ satisfying $|\mathcal{P}(\dot{\chi}) - \mathcal{P}(\dot{\bar{\varphi}})|_{C_{r_0}} \le \varepsilon_1^*$.

Define the constant

$$\alpha_1^* := \begin{cases} \min\left\{ \frac{\kappa_1^*}{|\bar{\varphi}|_{C^1}}, \frac{\varepsilon_1^*}{|\ddot{\bar{\varphi}}|_{L^\infty}}, r_0, T_1^* \right\}, & |\ddot{\bar{\varphi}}|_{L^\infty} \ne 0,\ |\bar{\varphi}|_{C^1} \ne 0, \\ \min\left\{ \frac{\kappa_1^*}{|\bar{\varphi}|_{C^1}}, r_0, T_1^* \right\}, & |\ddot{\bar{\varphi}}|_{L^\infty} = 0,\ |\bar{\varphi}|_{C^1} \ne 0, \\ \min\{r_0, T_1^*\}, & |\bar{\varphi}|_{C^1} = 0, \end{cases}$$

and introduce the extension of $\bar{\varphi} \in C^1$ to $[-r,\infty)$ by

$$\Phi(t) := \begin{cases} \bar{\varphi}(t), & t \in [-r,0], \\ \dot{\bar{\varphi}}(0-)t + \bar{\varphi}(0), & t > 0. \end{cases} \tag{3.15}$$

Then Φ is continuously differentiable on $[-r,\infty)$. Since $\alpha_1^* \le r_0$, we have

$$\widetilde{\varphi}_t(\zeta) = \widetilde{\varphi}(t+\zeta) = \bar{\varphi}(t+\zeta) = \Phi(t+\zeta) = \Phi_t(\zeta), \qquad \zeta \in [-r,-r_0],\ t \in [0,\alpha_1^*], \tag{3.16}$$

so Remark 3.1 yields

$$\rho(t,\widetilde{\varphi}_t,\bar{\lambda}) = \rho(t,\Phi_t,\bar{\lambda}), \qquad t \in [0,\alpha_1^*].$$

The definitions of Φ, $\widetilde{\varphi}$, α_1^* and the Mean Value Theorem imply

$$|\Phi_t - \bar{\varphi}|_C = \max_{-r\le\zeta\le 0} |\Phi(t+\zeta) - \Phi(\zeta)| \le |\bar{\varphi}|_{C^1}\alpha_1^* \le \kappa_1^*, \qquad t \in [0,\alpha_1^*],$$
$$|\widetilde{\varphi}_t - \bar{\varphi}|_C = \max_{-r\le\zeta\le 0} |\widetilde{\varphi}(t+\zeta) - \widetilde{\varphi}(\zeta)| \le |\bar{\varphi}|_{C^1}\alpha_1^* \le \kappa_1^*, \qquad t \in [0,\alpha_1^*].$$

Hence $\Phi_t \in \Omega_1$ and $\widetilde{\varphi}_t \in \Omega_1$ for $t \in [0,\alpha_1^*]$, and

$$|\mathcal{P}(\dot{\Phi}_t) - \mathcal{P}(\dot{\bar{\varphi}})|_{C_{r_0}} = \sup_{-r\le\zeta\le -r_0} |\dot{\bar{\varphi}}(t+\zeta) - \dot{\bar{\varphi}}(\zeta)| \le |\ddot{\bar{\varphi}}|_{L^\infty}\alpha_1^* \le \varepsilon_1^*, \qquad t \in [0,\alpha_1^*].$$

Therefore, it follows from (3.14) that

$$D_1\rho(t,\Phi_t,\lambda)+D_2\rho(t,\Phi_t,\lambda)\dot{\Phi}_t<1-\varepsilon_0^*,\qquad t\in[0,\alpha_1^*],\ \lambda\in\overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_1^*), \tag{3.17}$$

so

$$\frac{d}{dt}\Big(t-\rho(t,\widetilde{\bar{\varphi}}_t,\lambda)\Big)=\frac{d}{dt}(t-\rho(t,\Phi_t,\lambda))=1-D_1\rho(t,\Phi_t,\lambda)-D_2\rho(t,\Phi_t,\lambda)\dot{\Phi}_t>\varepsilon_0^*,$$

for $t\in[0,\alpha_1^*]$ and $\lambda\in\overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_1^*)$. Next we show that a similar lower estimate can be obtained in a small neighbourhood of $\bar{\varphi}$.

The set $\{\widetilde{\bar{\varphi}}_t\colon t\in[0,\alpha_1^*]\}\subset\Omega_1$ is a compact subset of C, since the map $[0,\alpha_1^*]\ni t\mapsto\widetilde{\bar{\varphi}}_t\in C$ is continuous. Then there exists $0<\delta_2^*<\delta_1^*$ such that its closed neighbourhood with radius δ_2^* belongs to Ω_1. Define the set $M_1^*:=\{\widetilde{\varphi}_t\colon t\in[0,\alpha_1^*],|\varphi-\bar{\varphi}|_{W^{1,\infty}}\le\delta_2^*\}\cup\{\Phi_t\colon t\in[0,\alpha_1^*]\}$. Then $M_1^*\subset\Omega_1$, since $|\widetilde{\varphi}_t-\widetilde{\bar{\varphi}}_t|_C\le|\varphi-\bar{\varphi}|_C\le\delta_2^*$ for $t\in[0,\alpha_1^*]$ and $\varphi\in\overline{\mathcal{B}}_{W^{1,\infty}}(\bar{\varphi};\delta_2^*)$. It is easy to check that M_1^* is closed in C and it is bounded in $W^{1,\infty}$, so it is a compact subset of C. Define the closed ball $M_7^*:=\overline{\mathcal{B}}_\Lambda(\bar{\lambda};\delta_2^*)$. Since $\delta_2^*<\delta_1^*$, it follows $M_7^*\subset\Omega_7$.

We introduce the notation

$$\omega_\rho^*(\hat{t},\hat{\psi},t,\psi,\lambda):=\rho(t,\psi,\lambda)-\rho(\hat{t},\hat{\psi},\lambda)-D_1\rho(\hat{t},\hat{\psi},\lambda)(t-\hat{t})-D_2\rho(\hat{t},\hat{\psi},\lambda)(\psi-\hat{\psi})$$

for $t,\hat{t}\in[0,\alpha_1^*]$, $\psi,\hat{\psi}\in M_1^*$ and $\lambda\in M_7^*$. Since ρ is continuously Fréchet differentiable, we have

$$\begin{aligned}\omega_\rho^*(\hat{t},\hat{\psi},t,\psi,\lambda):=\int_0^1\Big\{&\Big[D_1\rho\Big(\hat{t}+\nu(t-\hat{t}),\hat{\psi}+\nu(\psi-\hat{\psi}),\lambda)\Big)-D_1\rho(\hat{t},\hat{\psi},\lambda)\Big](t-\hat{t})\\&+\Big[D_2\rho\Big(\hat{t}+\nu(t-\hat{t}),\hat{\psi}+\nu(\psi-\hat{\psi}),\lambda)\Big)-D_2\rho(\hat{t},\hat{\psi},\lambda)\Big](\psi-\hat{\psi})\Big\}\,d\nu.\end{aligned} \tag{3.18}$$

Define the function

$$\begin{aligned}\Omega_\rho(\delta):=\sup\Big\{&\max\Big(|D_1\rho(t,\psi,\lambda)-D_1\rho(\hat{t},\hat{\psi},\lambda)|,|D_2\rho(t,\psi,\lambda)-D_2\rho(\hat{t},\hat{\psi},\lambda)|_{\mathcal{L}(C,\mathbb{R})}\Big)\colon\\&t,\hat{t}\in[0,\alpha_1^*],\ \psi,\hat{\psi}\in M_1^*,\ \lambda\in M_7^*,\ |t-\hat{t}|+|\psi-\hat{\psi}|_C\le\delta\Big\}\end{aligned}$$

for $\delta>0$. Since M_1^* is closed in C and it is bounded in $W^{1,\infty}$, and M_7^* is a closed and bounded subset of Λ, assumption (A4) (iii) yields that $D_1\rho$ and $D_2\rho$ are uniformly continuous on $[0,\alpha_1^*]\times M_1^*\times M_7^*$, therefore

$$\Omega_\rho(\delta)\to 0,\quad\text{as }\delta\to 0+. \tag{3.19}$$

Relation (3.18) implies

$$|\omega_\rho^*(\hat{t},\hat{\psi},t,\psi,\lambda)|\le\Omega_\rho\Big(|t-\hat{t}|+|\psi-\hat{\psi}|_C\Big)\Big(|t-\hat{t}|+|\psi-\hat{\psi}|_C\Big) \tag{3.20}$$

for $t,\hat{t}\in[0,\alpha_1^*]$, $\psi,\hat{\psi}\in M_1^*$, $\lambda\in M_7^*$.

Fix $0<\varepsilon'<\varepsilon_0^*$. Relation

$$\left|\frac{\Phi_{t+h}-\Phi_t}{h}-\dot{\Phi}_t\right|_C\to 0,\quad\text{as }h\to 0$$

uniformly on $[0,\alpha_1^*]$ implies that there exists $\nu_1>0$ such that

$$m^*\left|\frac{\Phi_{t+h}-\Phi_t}{h}-\dot{\Phi}_t\right|_C\le\varepsilon_0^*-\varepsilon',\qquad 0<|h|<\nu_1,\quad t,t+h\in[0,\alpha_1^*].$$

Then (3.13) and (3.17) yield

$$\begin{aligned} &D_1\rho(t,\Phi_t,\lambda) + D_2\rho(t,\Phi_t,\lambda)\frac{\Phi_{t+h}-\Phi_t}{h} \\ &\qquad \le D_1\rho(t,\Phi_t,\lambda) + D_2\rho(t,\Phi_t,\lambda)\dot\Phi_t + m^*\left|\frac{\Phi_{t+h}-\Phi_t}{h} - \dot\Phi_t\right|_C \\ &\qquad \le 1-\varepsilon', \qquad 0<|h|<\nu_1, \quad t,t+h\in[0,\alpha_1^*], \quad \lambda\in M_7^*. \end{aligned} \tag{3.21}$$

It is easy to check that $t\mapsto t-\rho(t,\widetilde\varphi_t,\lambda)$ is Lipschitz continuous on $[0,\alpha_1^*]$ for $\varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_2^*)$ and $\lambda\in M_7^*$, hence the map is a.e. differentiable on $[0,\alpha_1^*]$.

Fix $0<\varepsilon^*<\varepsilon'$. To show (3.9), it is enough to find $0<\delta^*\le\delta_2^*$ and $\nu>0$ such that

$$\frac{1}{h}\Big(\rho(t+h,\widetilde\varphi_{t+h},\lambda)-\rho(t,\widetilde\varphi_t,\lambda)\Big)\le 1-\varepsilon^*, \qquad t,t+h\in[0,\alpha_1^*],\ 0<|h|<\nu, \tag{3.22}$$

for all $\varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta^*)$, $\lambda\in M_7^*$.

Simple manipulations, (3.3) and (3.16) yield

$$\begin{aligned} &\rho(t+h,\widetilde\varphi_{t+h},\lambda)-\rho(t,\widetilde\varphi_t,\lambda) \\ &\quad = D_1\rho(t,\widetilde\varphi_t,\lambda)h + D_2\rho(t,\widetilde\varphi_t,\lambda)(\widetilde\varphi_{t+h}-\widetilde\varphi_t) + \omega_\rho^*(t,\widetilde\varphi_t,t+h,\widetilde\varphi_{t+h},\lambda) \\ &\quad = D_1\rho(t,\Phi_t,\lambda)h + D_2\rho(t,\Phi_t,\lambda)(\Phi_{t+h}-\Phi_t) + \Big(D_1\rho(t,\widetilde\varphi_t,\lambda)-D_1\rho(t,\Phi_t,\lambda)\Big)h \\ &\qquad + \Big(D_2\rho(t,\widetilde\varphi_t,\lambda)-D_2\rho(t,\Phi_t,\lambda)\Big)\Big(\widetilde\varphi_{t+h}-\widetilde\varphi_t\Big) \\ &\qquad + D_2\rho(t,\Phi_t,\lambda)\Big(\widetilde\varphi_{t+h}-\widetilde\varphi_t-\Phi_{t+h}+\Phi_t\Big) + \omega_\rho^*(t,\widetilde\varphi_t,t+h,\widetilde\varphi_{t+h},\lambda) \\ &\quad = D_1\rho(t,\Phi_t,\lambda)h + D_2\rho(t,\Phi_t,\lambda)(\Phi_{t+h}-\Phi_t) + \Big(D_1\rho(t,\widetilde\varphi_t,\lambda)-D_1\rho(t,\widetilde{\bar\varphi}_t,\lambda)\Big)h \\ &\qquad + \Big(D_2\rho(t,\widetilde\varphi_t,\lambda)-D_2\rho(t,\widetilde{\bar\varphi}_t,\lambda)\Big)\Big(\widetilde\varphi_{t+h}-\widetilde\varphi_t\Big) \\ &\qquad + D_2\rho(t,\Phi_t,\lambda)\Big(\widetilde\varphi_{t+h}-\widetilde\varphi_t-\Phi_{t+h}+\Phi_t\Big) + \omega_\rho^*(t,\widetilde\varphi_t,t+h,\widetilde\varphi_{t+h},\lambda). \end{aligned} \tag{3.23}$$

The Mean Value Theorem yields

$$|\widetilde\varphi_{t+h}-\widetilde\varphi_t|_C\le|\dot\varphi|_{L^\infty}|h|\le(|\dot{\bar\varphi}|_C+\delta_2^*)|h|, \qquad t,t+h\in[0,\alpha_1^*],\ \varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_2^*). \tag{3.24}$$

Then, using (3.20) and (3.24), we get

$$\frac{|\omega_\rho^*(t,\widetilde\varphi_t,t+h,\widetilde\varphi_{t+h},\lambda)|}{|h|}\le N^*\Omega_\rho\Big(N^*|h|\Big), \qquad t,t+h\in[0,\alpha_1^*],\ \varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_2^*),\ \lambda\in M_7^* \tag{3.25}$$

with $N^*:=1+|\dot{\bar\varphi}|_C+\delta_2^*$. Let ν_1 be the corresponding constant from (3.21). Then it follows from (3.4), (3.13), (3.21), (3.23), (3.24), (3.25) and the definition of Ω_ρ for $t,t+h\in[0,\alpha_1^*]$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_2^*)$, $\lambda\in M_7^*$ and $0<|h|<\nu_1$

$$\begin{aligned} &\frac{\rho(t+h,\widetilde\varphi_{t+h},\lambda)-\rho(t,\widetilde\varphi_t,\lambda)}{h} \\ &\quad\le 1-\varepsilon'+\Omega_\rho\Big(|\widetilde\varphi_t-\widetilde{\bar\varphi}_t|_C\Big)+\Omega_\rho\Big(|\widetilde\varphi_t-\widetilde{\bar\varphi}_t|_C\Big)(|\dot{\bar\varphi}|_C+\delta_2^*) \\ &\qquad + \frac{m^*}{|h|}|\mathcal{P}(\widetilde\varphi_{t+h}-\Phi_{t+h}-(\widetilde\varphi_t-\Phi_t))|_{C_{r_0}} + N^*\Omega_\rho\Big(N^*|h|\Big) \\ &\quad\le 1-\varepsilon'+N^*\Omega_\rho\Big(|\varphi-\bar\varphi|_C\Big)+m^*|\varphi-\bar\varphi|_{W^{1,\infty}}+N^*\Omega_\rho\Big(N^*|h|\Big). \end{aligned} \tag{3.26}$$

Let $0 < \delta^* < \delta_2^*$ be such that $N^*\Omega_\rho(\delta^*) + m^*\delta^* < (\varepsilon' - \varepsilon^*)/2$, and $0 < \nu < \nu_1$ be such that $N^*\Omega_\rho\big(N^*\nu\big) < (\varepsilon' - \varepsilon^*)/2$. Then (3.26) implies (3.22). This completes the proof of the lemma. □

Next we show that, under the assumptions listed at the beginning of this section, the IVP (3.1)–(3.2) has a unique solution. The solution of the IVP (3.1)–(3.2) corresponding to a parameter γ and its segment function at t are denoted by $x(t,\gamma)$ and $x_t(\cdot,\gamma)$, respectively.

Note that on a small time interval $[0,\alpha_1]$ with $0 < \alpha_1 \le \min\{r_0, T\}$ the existence of a unique solution can be easily obtained using Equation (3.7) with fixed parameters, and apply known existence and uniqueness results for SD-DDEs with Carathéodory type of conditions (see, e.g., [25]). Since for the proof of differentiability wrt parameters we need the Lipschitz continuity of the solutions wrt parameters in a restricted sense (when one of the parameters belong to $\mathcal{M}$), and other special estimates of the solutions (see parts (iii)–(v) in Theorem 3.3 below) uniformly in a neighbourhood P of a fixed special parameter, we will present the detailed proof. Also, we show that the solutions can be extended to larger interval $[0,\alpha]$, with possibly $\alpha > r_0$, independent of the selection of the parameters from P so that the uniform estimates presented in (iii)–(v) are preserved.

Theorem 3.3. *Assume f, τ, μ and ρ satisfy (A1) (i), (A2) (i), (A3) and (A4) (i)–(iii), respectively, and let $\bar\gamma \in \mathcal{M}$. Then there exist a radius $\delta > 0$ and a finite time $0 < \alpha \le T$ such that*

(i) $P := \mathcal{B}_\Gamma(\bar\gamma;\delta) \subset \Pi$, and there exist $\varepsilon_1^* > 0$ and $0 < \alpha_1 \le \min\{r_0,\alpha\}$ such that $\widetilde\varphi_t \in \Omega_1$ for $t \in [0,\alpha_1]$, and

$$\frac{d}{dt}\Big(t - \rho(t,\widetilde\varphi_t,\lambda)\Big) \ge \varepsilon_1^*, \quad \dot\varphi(t-\mu(t)) \in \Omega_3 \quad \textit{and} \quad \dot\varphi(t-\rho(t,\widetilde\varphi_t,\lambda)) \in \Omega_4 \tag{3.27}$$

for a.e. $t \in [0,\alpha_1]$ and $\gamma \in P$;

(ii) the IVP (3.1)–(3.2) has a unique solution $x(t,\gamma)$ on $[-r,\alpha]$ for all $\gamma \in P$;

(iii) there exist a closed subset $M_1 \subset C$ which is also a bounded and convex subset of $W^{1,\infty}$, and $M_i \subset \Omega_i$ ($i = 2,3,4$) compact and convex subsets of $\mathbb{R}^n$, and $\varepsilon^* > 0$ such that $x(t) := x(t,\gamma)$ satisfies

$$x_t \in M_1, \quad x(t-\tau(t,x_t,\xi)) \in M_2, \qquad t \in [0,\alpha], \tag{3.28}$$

$$\frac{d}{dt}\Big(t - \rho(t,x_t,\lambda)\Big) \ge \varepsilon^*, \quad \text{a.e. } t \in [0,\alpha], \tag{3.29}$$

and

$$\dot x(t-\mu(t)) \in M_3, \quad \dot x(t-\rho(t,x_t,\lambda)) \in M_4, \quad \text{a.e. } t \in [0,\alpha] \tag{3.30}$$

for $\gamma = (\varphi,\xi,\lambda,\theta) \in P$;

(iv) $x_t(\cdot,\gamma) \in W^{1,\infty}$ for $t \in [0,\alpha]$, $\gamma \in P$. Moreover, there exist nonnegative constants $N = N(\alpha,\delta)$ and $L = L(\alpha,\delta)$ such that

$$|x_t(\cdot,\gamma)|_{W^{1,\infty}} \le N, \qquad t \in [0,\alpha], \quad \gamma \in P, \tag{3.31}$$

and

$$|x_t(\cdot,\gamma) - x_t(\cdot,\hat\gamma)|_{W^{1,\infty}} \le L|\gamma - \hat\gamma|_\Gamma, \qquad t \in [0,\alpha], \quad \gamma \in P, \quad \hat\gamma \in \mathcal{M} \cap P. \tag{3.32}$$

(v) *For every $\gamma \in \mathcal{M} \cap P$ the function $x(\cdot,\gamma) : [-r,\alpha] \to \mathbb{R}^n$ is continuously differentiable. Moreover, for every $\gamma \in \mathcal{M} \cap P$ there exists $L^* = L^*(\gamma)$ such that*

$$|\dot{x}(t,\gamma) - \dot{x}(\bar{t},\gamma)| \le L^*|t-\bar{t}|, \qquad t,\bar{t} \in [0,\alpha], \tag{3.33}$$

and, in particular, $x_t(\cdot,\gamma) \in W^{2,\infty}$ for $t \in [0,\alpha]$ and $\gamma \in \mathcal{M} \cap P$.

Proof. (1) First we prove part (i) of the statement of the theorem. Moreover, we show some relations which will be used later in the proof.

Let $\bar{\gamma} := (\bar{\varphi},\bar{\xi},\bar{\lambda},\bar{\theta}) \in \mathcal{M}$ be fixed. Let α_1^*, δ_1^* and ε_1^* be the corresponding constants from Lemma 3.2 for which (3.9) holds. Note that Lemma 3.2 implies $\mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1^*) \subset \Omega_1$. We note that the definition of $\mathcal{M}$ yields $\bar{\varphi} \in C^1$. We introduce the vectors $\bar{u}_1 := \bar{\varphi}(-\tau(0,\bar{\varphi},\bar{\xi}))$, $\bar{v}_1 := \dot{\bar{\varphi}}(-\mu(0))$ and $\bar{w}_1 := \dot{\bar{\varphi}}(-\rho(0,\bar{\varphi},\bar{\lambda}))$. Note that the definitions of Π and $\mathcal{M}$ yield $\bar{\varphi} \in \Omega_1$, $\bar{u}_1 \in \Omega_2$, $\bar{v}_1 \in \Omega_3$, $\bar{v}_2 \in \Omega_4$, $\bar{\theta} \in \Omega_5$, $\bar{\xi} \in \Omega_6$ and $\bar{\lambda} \in \Omega_7$. Since $\Omega_1,\ldots,\Omega_7$ are open subsets of their respective spaces, there exist positive constants κ_1', ε_1' and δ_1' such that $\overline{\mathcal{B}}_C(\bar{\varphi};\kappa_1') \subset \Omega_1$, $\overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{u}_1;\varepsilon_1') \subset \Omega_2$, $\overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{v}_1;\varepsilon_1') \subset \Omega_3$, $\overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{w}_1;\varepsilon_1') \subset \Omega_4$, $\overline{\mathcal{B}}_{\Theta}(\bar{\theta};\delta_1') \subset \Omega_5$, $\overline{\mathcal{B}}_{\Xi}(\bar{\xi};\delta_1') \subset \Omega_6$ and $\overline{\mathcal{B}}_{\Lambda}(\bar{\lambda};\delta_1') \subset \Omega_7$, respectively.

Fix $m_1 > 0$. The assumed continuity of f yields that there exist finite constants $0 < T_1' \le T$, $0 < \kappa_1'' \le \kappa_1'$, $0 < \varepsilon_1 \le \varepsilon_1'$ and $0 < \delta_1'' \le \delta_1'$ such that

$$|f(t,\psi,u,v,w,\theta) - f(0,\bar{\varphi},\bar{u}_1,\bar{v}_1,\bar{w}_1,\bar{\theta})| \le m_1 \tag{3.34}$$

for $t \in [0,T_1']$, $\psi \in \overline{\mathcal{B}}_C(\bar{\varphi};\kappa_1'')$, $u \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{u}_1;\varepsilon_1)$, $v \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{v}_1;\varepsilon_1)$, $w \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar{w}_1;\varepsilon_1)$, $\theta \in \overline{\mathcal{B}}_{\Theta}(\bar{\theta};\delta_1'')$.

For $\varphi \in \Omega_1$, $\xi \in \Omega_6$ and $t \in [0,T]$ it follows

$$\begin{aligned}|\varphi(-\tau(t,\varphi,\xi)) - \bar{u}_1| &\le |\varphi(-\tau(t,\varphi,\xi)) - \bar{\varphi}(-\tau(t,\varphi,\xi))| + |\bar{\varphi}(-\tau(t,\varphi,\xi)) - \bar{\varphi}(-\tau(0,\bar{\varphi},\bar{\xi}))| \\ &\le |\varphi - \bar{\varphi}|_C + |\bar{\varphi}(-\tau(t,\varphi,\xi)) - \bar{\varphi}(-\tau(0,\bar{\varphi},\bar{\xi}))|.\end{aligned}$$

Let $\varphi \in \Omega_1 \cap W^{1,\infty}$. Then Lemma 2.5 and the monotonicity of $t - \mu(t)$ implied by assumption (A3) yield $\dot{\varphi}(t-\mu(t))$ exists for a.e. $t \in [0,r_0]$, and

$$\begin{aligned}|\dot{\varphi}(t-\mu(t)) - \bar{v}_1| &\le |\dot{\varphi}(t-\mu(t)) - \dot{\bar{\varphi}}(t-\mu(t))| + |\dot{\bar{\varphi}}(t-\mu(t)) - \dot{\bar{\varphi}}(-\mu(0))| \\ &\le |\varphi - \bar{\varphi}|_{W^{1,\infty}} + |\dot{\bar{\varphi}}(t-\mu(t)) - \dot{\bar{\varphi}}(-\mu(0))|, \quad \text{a.e. } t \in [0,r_0].\end{aligned}$$

Suppose $\varphi \in \mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1^*) \subset \Omega_1$ and $\lambda \in \mathcal{B}_{\Lambda}(\bar{\lambda};\delta^*) \subset \Omega_7$. Then Lemma 2.5 and Lemma 3.2 imply that $\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda))$ is defined for a.e. $t \in [0,\alpha_1^*]$, and

$$\begin{aligned}|\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \bar{w}_1| &\le |\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \dot{\bar{\varphi}}(t-\rho(t,\widetilde{\varphi}_t,\lambda))| \\ &\quad + |\dot{\bar{\varphi}}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \dot{\bar{\varphi}}(-\rho(0,\bar{\varphi},\bar{\lambda}))| \\ &\le |\varphi - \bar{\varphi}|_{W^{1,\infty}} + |\dot{\bar{\varphi}}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \dot{\bar{\varphi}}(-\rho(0,\bar{\varphi},\bar{\lambda}))|, \qquad t \in [0,\alpha_1^*].\end{aligned}$$

The continuity of $\bar{\varphi}$, $\dot{\bar{\varphi}}$, τ, μ and ρ and the above inequalities yield that there exist $0 < \kappa_1 \le \kappa_1''$, $0 < T_1 \le \min\{T_1',\alpha_1^*\}$ and $0 < \delta_1''' \le \min\{\delta_1'',\delta^*\}$ such that

$$|\varphi(-\tau(t,\varphi,\xi)) - \bar{u}_1| < \varepsilon_1, \qquad t \in [0,T_1],\ \varphi \in \mathcal{B}_C(\bar{\varphi};\kappa_1),\ \xi \in \mathcal{B}_{\Xi}(\bar{\xi};\delta_1'''), \tag{3.35}$$

$$|\dot{\varphi}(t-\mu(t)) - \bar{v}_1| < \varepsilon_1, \qquad \text{a.e. } t \in [0,T_1],\ \varphi \in \mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1'''), \tag{3.36}$$

and

$$|\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \bar{w}_1| < \varepsilon_1, \quad \text{a.e. } t \in [0,T_1],\ \varphi \in \mathcal{B}_C(\bar{\varphi};\kappa_1) \cap \mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1'''),\ \lambda \in \mathcal{B}_{\Lambda}(\bar{\lambda};\delta_1'''). \tag{3.37}$$

Let

$$\delta_1 := \min\left\{\frac{\kappa_1}{3}, \frac{\varepsilon_1}{2}, \delta_1'''\right\}.$$

Then we get that $\mathcal{B}_\Gamma(\bar\gamma;\delta) \subset \Pi$ and (3.27) hold with any $0 < \alpha_1 \le T_1$ and $0 < \delta \le \delta_1$.

(2) Next we show the existence of a unique solution of IVP (3.1)-(3.2) on a small time interval $[-r, \alpha_1]$ for some $0 < \alpha_1 \le r_0$, and we also prove part (iii) and the first relation of part (iv) of the statement on $[0, \alpha_1]$.

With the help of the notations introduced in part (1) of the proof, we define the following constants and sets

$$\begin{aligned}
\beta_1 &:= \frac{\kappa_1}{3},\\
a_0 &:= |\bar\varphi|_{C^1} + \delta_1,\\
a_1 &:= \max\{a_0, |f(0,\bar\varphi,\bar u_1,\bar v_1,\bar w_1,\bar\theta)| + m_1\},\\
M_{1,1} &:= \{\psi \in W^{1,\infty}\colon\ |\psi-\bar\varphi|_C \le \kappa_1,\ |\dot\psi|_{L^\infty} \le a_1\},\\
M_2 &:= \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar u_1;\varepsilon_1),\\
M_3 &:= \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar v_1;\varepsilon_1),\\
M_4 &:= \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar w_1;\varepsilon_1),\\
M_5 &:= \overline{\mathcal{B}}_{\Theta}(\bar\theta;\delta_1),\\
M_6 &:= \overline{\mathcal{B}}_{\Xi}(\bar\xi;\delta_1) \quad \text{and}\\
M_7 &:= \overline{\mathcal{B}}_{\Lambda}(\bar\lambda;\delta_1).
\end{aligned}$$

Then M_j is a closed and bounded subsets of Ω_j for $j = 2,\dots,7$ since $0 < \delta_1 \le \delta_1'$ and $0 < \varepsilon_1 \le \varepsilon_1'$. Moreover, the set $M_{1,1}$ is closed in C and it is bounded in $W^{1,\infty}$. We further define

$$\begin{aligned}
\alpha_1 &:= \min\Big\{\frac{\beta_1}{a_1}, \frac{\kappa_1}{3a_0}, T_1, r_0\Big\},\\
E_1 &:= \Big\{y \in C([-r,\alpha_1],\mathbb{R}^n)\colon\ y(s) = 0,\ s\in[-r,0] \text{ and } |y(s)| \le \beta_1,\ s \in [0,\alpha_1]\Big\}.
\end{aligned}$$

We show that the IVP (3.1)–(3.2) corresponding to $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\delta_1)$ has a solution x on the interval $[0,\alpha_1]$. Since $\alpha_1 \le r_0$, we have that (3.6) holds, so Lemma 2.5 and Lemma 3.2 yield that $\dot\varphi(t-\mu(t))$ and $\dot\varphi(t-\rho(t,\widetilde\varphi_t,\lambda))$ are defined for a.e. $t \in [0,\alpha_1]$. Hence, using the initial condition (3.2), on the interval $[0,\alpha_1]$ and for $\gamma \in P$, Equation (3.1) can be replaced with (3.7), or with the integral equation

$$x(t) = \varphi(0) + \int_0^t f(s,x_s,x(s-\tau(s,x_s,\xi)),\dot\varphi(s-\mu(s)),\dot\varphi(s-\rho(s,\widetilde\varphi_s,\lambda)),\theta)\,ds, \quad t\in[0,\alpha_1]. \tag{3.38}$$

We have $|\dot\varphi|_{L^\infty} \le |\varphi|_{W^{1,\infty}} \le |\bar\varphi|_{C^1} + |\varphi-\bar\varphi|_{W^{1,\infty}} \le a_0$ for $\varphi \in \mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_1)$, and therefore $\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_1) \subset M_{1,1}$. Then for $y \in E_1$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_1)$, $t\in[0,\alpha_1]$ and $\zeta \in [-r,0]$ we get

$$\begin{aligned}
|y(t+\zeta) + \widetilde\varphi(t+\zeta) - \bar\varphi(\zeta)| &\le |y(t+\zeta)| + |\widetilde\varphi(t+\zeta) - \varphi(\zeta)| + |\varphi(\zeta) - \bar\varphi(\zeta)|\\
&< \beta_1 + \alpha_1 a_0 + \delta_1\\
&\le \kappa_1,
\end{aligned}$$

and hence

$$|y_t + \widetilde\varphi_t - \bar\varphi|_C < \kappa_1, \qquad y \in E_1,\ \varphi\in\mathcal{B}_{W^{1,\infty}}(\bar\varphi;\delta_1),\ t\in[0,\alpha_1], \tag{3.39}$$

i.e., $y_t+\widetilde{\varphi}_t \in \mathcal{B}_C(\bar{\varphi};\kappa_1)$ for $y\in E_1$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1)$ and $t\in[0,\alpha_1]$. Note that the above inequalities also give

$$|\widetilde{\varphi}_t-\bar{\varphi}|_C<\kappa_1,\qquad \varphi\in\mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1),\quad t\in[0,\alpha_1]. \tag{3.40}$$

Relations $0<\alpha_1\le r_0$, $0<\delta_1\le\delta_1'''$, (3.35), (3.37), (3.39) and (3.40) yield for $y\in E_1$ and $\gamma=(\varphi,\xi,\lambda,\theta)\in\mathcal{B}_\Gamma(\bar{\gamma};\delta_1)$ that

$$|(y_t+\widetilde{\varphi}_t)(-\tau(t,y_t+\widetilde{\varphi}_t,\xi))-\bar{u}_1|<\varepsilon_1,\qquad t\in[0,\alpha_1] \tag{3.41}$$

and

$$|\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda))-\bar{w}_1|<\varepsilon_1,\quad \text{a.e. } t\in[0,\alpha_1], \tag{3.42}$$

and so

$$(y_t+\widetilde{\varphi}_t)(-\tau(t,y_t+\widetilde{\varphi}_t,\xi))\in M_2,\quad t\in[0,\alpha_1],\qquad \dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda))\in M_4,\quad \text{a.e. } t\in[0,\alpha_1].$$

Note that (3.36) implies $\dot{\varphi}(t-\mu(t))\in M_3$ for a.e. $t\in[0,\alpha_1]$.

The above observations and relations $0<\kappa_1\le\kappa_1''$, $0<\delta_1\le\delta_1''$ and (3.34) imply that

$$\begin{aligned}
&\Big|f\Big(t,y_t+\widetilde{\varphi}_t,(y_t+\widetilde{\varphi}_t)(-\tau(t,y_t+\widetilde{\varphi}_t,\xi)),\dot{\varphi}(t-\mu(t)),\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)),\theta\Big)\Big|\\
&\quad\le\Big|f(0,\bar{\varphi},\bar{u}_1,\bar{v}_1,\bar{w}_1,\bar{\theta})\Big|\\
&\qquad+\Big|f\Big(t,y_t+\widetilde{\varphi}_t,(y_t+\widetilde{\varphi}_t)(-\tau(t,y_t+\widetilde{\varphi}_t,\xi)),\dot{\varphi}(t-\mu(t)),\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)),\theta\Big)\\
&\qquad\quad-f(0,\bar{\varphi},\bar{u}_1,\bar{v}_1,\bar{w}_1,\bar{\theta})\Big|\\
&\quad\le\Big|f(0,\bar{\varphi},\bar{u}_1,\bar{v}_1,\bar{w}_1,\bar{\theta})\Big|+m_1\\
&\quad\le a_1,\quad \text{a.e. } t\in[0,\alpha_1]
\end{aligned}$$

for $y\in E_1$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\bar{\varphi};\delta_1)$, $\lambda\in\mathcal{B}_\Lambda(\bar{\lambda};\delta_1)$ and $\xi\in\mathcal{B}_\Xi(\bar{\xi};\delta_1)$.

We introduce the new variable $y(t):=x(t)-\widetilde{\varphi}(t)$, and we define the operator

$$\begin{aligned}
&\mathcal{T}^1(y,\gamma)(t)\\
&:=\begin{cases}\displaystyle\int_0^t f\Big(s,y_s+\widetilde{\varphi}_s,(y_s+\widetilde{\varphi}_s)(-\tau(s,y_s+\widetilde{\varphi}_s,\xi)),\dot{\varphi}(s-\mu(s)),\dot{\varphi}(s-\rho(s,\widetilde{\varphi}_s,\lambda)),\theta\Big)ds, & t\in[0,\alpha_1],\\ 0, & t\in[-r,0].\end{cases}
\end{aligned}$$

Then, on the interval $[-r,\alpha_1]$, the IVP (3.1)–(3.2) is equivalent to the fixed point problem

$$y=\mathcal{T}^1(y,\gamma).$$

We have $|\mathcal{T}^1(y,\gamma)(t)|\le a_1\alpha_1\le\beta_1$ for $t\in[0,\alpha_1]$, and $\left|\frac{d}{dt}\mathcal{T}^1(y,\gamma)(t)\right|\le a_1$ for a.e. $t\in[0,\alpha_1]$, hence $\mathcal{T}^1(\cdot,\gamma)$ maps the closed, bounded and convex subset E_1 of C into E_1 for all $\gamma\in\mathcal{B}_\Gamma(\bar{\gamma};\delta_1)$, and $\mathcal{T}^1(E_1,\gamma)$ is relatively compact. Therefore, it follows from the Schauder's Fixed Point Theorem that the operator $\mathcal{T}^1(\cdot,\gamma)$ has a fixed point $y=y(\cdot,\gamma)$, and therefore, the IVP (3.1)-(3.2) has a solution $x=x(\cdot,\gamma)=y(\cdot,\gamma)+\widetilde{\varphi}$ on the interval $[-r,\alpha_1]$. It satisfies $|\dot{x}(t)|=|\dot{y}(t)|\le a_1$ for a.e. $t\in[0,\alpha_1]$, and $|\dot{x}(t)|=|\dot{\varphi}(t)|\le a_0\le a_1$ for a.e. $t\in[-r,0]$. It follows from (3.39) that $|x(t)|\le|\bar{\varphi}|_C+\kappa_1$, $t\in[-r,\alpha_1]$, hence $x_t\in W^{1,\infty}$ and $|x_t|_{W^{1,\infty}}\le$

$\max\{|\bar{\varphi}|_C + \kappa_1, a_1\}$, $t \in [0, \alpha_1]$. Moreover, $x_t \in M_{1,1}$, $x(t - \tau(t, x_t, \xi)) \in M_2$, hold for $t \in [0, \alpha_1]$, and $\dot{x}(t - \mu(t)) \in M_3$ and $\dot{x}(t - \rho(t, x_t, \lambda)) \in M_4$ for a.e. $t \in [0, \alpha_1]$, and (3.6) and (3.9) yield (3.29) with $\alpha = \alpha_1$ and $\varepsilon^* = \varepsilon_1^*$.

Since on the interval $[0, \alpha_1]$ the initial condition (3.2) yields that Equation (3.1) reduces to (3.7), the assumed Lipschitz continuity of f, τ and φ and standard results of SD-DDEs (see, e.g., [25]) give the uniqueness of the solution.

(3) Next we prove that any solution $x(\cdot, \gamma)$ obtained in part (2) of the proof satisfies part (v) of the theorem on the interval $[0, \alpha_1]$.

Let $\gamma \in \mathcal{M} \cap P$, and let x denote any solution of the IVP (3.1)-(3.2) on $[0, \alpha_1]$ corresponding to γ obtained in part (1) of the proof. The continuity of f, τ, μ, ρ and $\dot{\varphi}$ yields that the function $t \mapsto f(t, x_t, x(t - \tau(t, x_t, \xi)), \dot{\varphi}(t - \mu(t)), \dot{\varphi}(t - \rho(t, \widetilde{\varphi}_t, \lambda)), \theta)$ is continuous on $[0, \alpha_1]$. Then $\varphi \in C^1$, (3.7) and the compatibility condition in the definition of $\mathcal{M}$ imply that x is continuously differentiable on the interval $[-r, \alpha_1]$.

Next we show that $\dot{x}$ is Lipschitz continuous on $[-r, \alpha_1]$. Since $\varphi \in W^{2,\infty}$, it is enough to show that $\dot{x}$ is Lipschitz continuous on $[0, \alpha_1]$. Define the Lipschitz constants from (A1) (i), (A2) (i), (A3) and (A4) (i) corresponding to the time $0 < \alpha_1 \le T$ and the sets introduced above:

$$
\begin{aligned}
L_{1,1} &:= L_1(\alpha_1, M_{1,1}, M_2, M_3, M_4, M_5)\\
L_{2,1} &:= L_2(\alpha_1, M_{1,1}, M_6)\\
L_{3,1} &:= L_3(\alpha_1)\\
L_{4,1} &:= L_4(\alpha_1, M_{1,1}, M_7).
\end{aligned}
$$

Let $t, \bar{t} \in [0, \alpha_1]$. We have

$$
\begin{aligned}
|\dot{x}(t) - \dot{x}(\bar{t})| &= \Big| f(t, x_t, x(t - \tau(t, x_t, \xi)), \dot{\varphi}(t - \mu(t)), \dot{\varphi}(t - \rho(t, \widetilde{\varphi}_t, \lambda)), \theta)\\
&\qquad - f(\bar{t}, x_{\bar{t}}, x(\bar{t} - \tau(\bar{t}, x_{\bar{t}}, \xi)), \dot{\varphi}(\bar{t} - \mu(\bar{t})), \dot{\varphi}(\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \lambda)), \theta) \Big|\\
&\le L_{1,1}\Big(|t - \bar{t}| + |x_t - x_{\bar{t}}|_C + |x(t - \tau(t, x_t, \xi)) - x(\bar{t} - \tau(\bar{t}, x_{\bar{t}}, \xi))|\\
&\qquad + |\dot{\varphi}(t - \mu(t)) - \dot{\varphi}(\bar{t} - \mu(\bar{t}))|\\
&\qquad + |\dot{\varphi}(t - \rho(t, \widetilde{\varphi}_t, \lambda)) - \dot{\varphi}(\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \lambda))| \Big)\\
&\le L_{1,1}\Big((1 + a_1)|t - \bar{t}| + a_1\Big(|t - \bar{t}| + |\tau(t, x_t, \xi) - \tau(\bar{t}, x_{\bar{t}}, \xi)| \Big)\\
&\qquad + |\ddot{\varphi}|_{L^\infty}(1 + L_{3,1})|t - \bar{t}| + |\ddot{\varphi}|_{L^\infty}\Big(|t - \bar{t}| + |\rho(t, \widetilde{\varphi}_t, \lambda) - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \lambda)| \Big) \Big)\\
&\le L_{1,1}\Big((1 + 2a_1)|t - \bar{t}| + a_1 L_{2,1}\Big(|t - \bar{t}| + |x_t - x_{\bar{t}}|_C \Big)\\
&\qquad + |\ddot{\varphi}|_{L^\infty}(2 + L_{3,1})|t - \bar{t}| + |\ddot{\varphi}|_{L^\infty} L_{4,1}\Big(|t - \bar{t}| + |\widetilde{\varphi}_t - \widetilde{\varphi}_{\bar{t}}|_C \Big) \Big)\\
&\le L_1^*|t - \bar{t}|,
\end{aligned}
\tag{3.43}
$$

where $L_1^* := \max\{L_{1,1}(1 + a_1(2 + L_{2,1}(1 + a_1)) + |\ddot{\varphi}|_{L^\infty}(2 + L_{3,1} + L_{4,1}(1 + a_1))), |\ddot{\varphi}|_{L^\infty}\}$. We get that $x_t \in W^{2,\infty}$ and $|\ddot{x}_t|_{L^\infty} \le L_1^*$ for $t \in [0, \alpha_1]$ and $\gamma \in \mathcal{M} \cap P$.

(4) Next we show the special Lipschitz continuity property (3.32) of the solutions on the interval $[0, \alpha_1]$.

Let $\gamma = (\varphi, \xi, \lambda, \theta) \in P$ and $\hat{\gamma} = (\hat{\varphi}, \hat{\xi}, \hat{\lambda}, \hat{\theta}) \in \mathcal{M} \cap P$, let $x = x(\cdot, \gamma)$ and $\hat{x} = x(\cdot, \hat{\gamma})$ be the solution of the IVP (3.1)–(3.2) on the interval $[-r, \alpha_1]$ corresponding to γ and $\hat{\gamma}$, respectively.

Then part (2) of the proof yields

$$x_t, \hat{x}_t \in M_{1,1}, \qquad x(t-\tau(t,x_t,\xi)), \hat{x}(t-\tau(t,\hat{x}_t,\hat{\xi})) \in M_2, \quad \text{for } t \in [0,\alpha_1]$$

and

$$\dot{x}(t-\mu(t)), \dot{\hat{x}}(t-\mu(t)) \in M_3, \qquad \dot{x}(t-\rho(t,x_t,\lambda)), \dot{\hat{x}}(t-\rho(t,\hat{x}_t,\hat{\lambda})) \in M_4 \quad \text{for a.e. } t \in [0,\alpha_1].$$

Integrating (3.7) from 0 to t and using the Lipschitz continuity of f, we get for $t \in [0,\alpha_1]$

$$\begin{aligned}
|x(t)-\hat{x}(t)| &\le |\varphi(0)-\hat{\varphi}(0)| + \int_0^t \Big| f(s,x_s,x(s-\tau(s,x_s,\xi)),\dot{\varphi}(s-\mu(s)),\dot{\varphi}(s-\rho(s,\widetilde{\varphi}_s,\lambda)),\theta) \\
&\qquad - f(s,\hat{x}_s,\hat{x}(s-\tau(s,\hat{x}_s,\hat{\xi})),\dot{\hat{\varphi}}(s-\mu(s)),\dot{\hat{\varphi}}(s-\rho(s,\widetilde{\hat{\varphi}}_s,\hat{\lambda})),\hat{\theta})\Big|\, ds \\
&\le |\varphi-\hat{\varphi}|_C + L_{1,1}\int_0^t \Big(|x_s-\hat{x}_s|_C + |x(s-\tau(s,x_s,\xi)) - \hat{x}(s-\tau(s,\hat{x}_s,\hat{\xi}))| \\
&\qquad + |\dot{\varphi}-\dot{\hat{\varphi}}|_{L^\infty} + |\dot{\varphi}(s-\rho(s,\widetilde{\varphi}_s,\lambda)) - \dot{\hat{\varphi}}(s-\rho(s,\widetilde{\hat{\varphi}}_s,\hat{\lambda}))| + |\theta-\hat{\theta}|_\Theta \Big)\, ds.
\end{aligned}$$

Using estimate $|\dot{\hat{x}}_t|_{L^\infty} \le a_1$ for $t \in [0,\alpha_1]$ and (A2) (ii) we get

$$\begin{aligned}
&|x(s-\tau(s,x_s,\xi)) - \hat{x}(s-\tau(s,\hat{x}_s,\hat{\xi}))| \\
&\quad \le |x(s-\tau(s,x_s,\xi)) - \hat{x}(s-\tau(s,x_s,\xi))| + |\hat{x}(s-\tau(s,x_s,\xi)) - \hat{x}(s-\tau(s,\hat{x}_s,\hat{\xi}))| \\
&\quad \le |x_s-\hat{x}_s|_C + a_1|\tau(s,x_s,\xi)-\tau(s,\hat{x}_s,\hat{\xi})| \\
&\quad \le |x_s-\hat{x}_s|_C + a_1 L_{2,1}(|x_s-\hat{x}_s|_C + |\xi-\hat{\xi}|_\Xi), \qquad s \in [0,\alpha_1].
\end{aligned} \tag{3.44}$$

Similarly, (A4) (i) yields

$$\begin{aligned}
&|\dot{\varphi}(s-\rho(s,\widetilde{\varphi}_s,\lambda)) - \dot{\hat{\varphi}}(s-\rho(s,\widetilde{\hat{\varphi}}_s,\hat{\lambda}))| \\
&\quad \le |\dot{\varphi}(s-\rho(s,\widetilde{\varphi}_s,\lambda)) - \dot{\hat{\varphi}}(s-\rho(s,\widetilde{\varphi}_s,\lambda))| + |\dot{\hat{\varphi}}(s-\rho(s,\widetilde{\varphi}_s,\lambda)) - \dot{\hat{\varphi}}(s-\rho(s,\widetilde{\hat{\varphi}}_s,\hat{\lambda}))| \\
&\quad \le |\varphi-\hat{\varphi}|_{W^{1,\infty}} + |\hat{\varphi}|_{W^{2,\infty}}|\rho(s,\widetilde{\varphi}_s,\lambda) - \rho(s,\widetilde{\hat{\varphi}}_s,\hat{\lambda})| \\
&\quad \le |\varphi-\hat{\varphi}|_{W^{1,\infty}} + |\hat{\varphi}|_{W^{2,\infty}} L_{4,1}(|\widetilde{\varphi}_s - \widetilde{\hat{\varphi}}_s|_C + |\lambda-\hat{\lambda}|_\Lambda) \\
&\quad \le |\varphi-\hat{\varphi}|_{W^{1,\infty}} + |\hat{\varphi}|_{W^{2,\infty}} L_{4,1}(|\varphi-\hat{\varphi}|_{W^{1,\infty}} + |\lambda-\hat{\lambda}|_\Lambda), \quad \text{a.e. } s \in [0,\alpha_1].
\end{aligned} \tag{3.45}$$

Then, combining the above inequalities, we get

$$\begin{aligned}
|x(t)-\hat{x}(t)| &\le (1+2\alpha_1 L_{1,1})|\varphi-\hat{\varphi}|_{W^{1,\infty}} + \alpha_1 L_{1,1} a_1 L_{2,1}|\xi-\hat{\xi}|_\Xi \\
&\qquad + \alpha_1 L_{1,1} L_{4,1}|\hat{\varphi}|_{W^{2,\infty}}(|\varphi-\hat{\varphi}|_{W^{1,\infty}} + |\lambda-\hat{\lambda}|_\Lambda) + L_{1,1}\int_0^t (2+a_1L_{2,1})|x_s-\hat{x}_s|_C\, ds \\
&\le K_{1,1}|\gamma-\hat{\gamma}|_\Gamma + K_{2,1}\int_0^t |x_s-\hat{x}_s|_C\, ds, \qquad t \in [0,\alpha_1],
\end{aligned}$$

where $K_{1,1} := 1 + 2\alpha_1 L_{1,1} + \alpha_1 L_{1,1} a_1 L_{2,1} + \alpha_1 L_{1,1} L_{4,1}|\hat{\varphi}|_{W^{2,\infty}}$ and $K_{2,1} := L_{1,1}(2+a_1L_{2,1})$. Since $K_{1,1} \ge 1$, it follows from Gronwall's Lemma that

$$|x(t)-\hat{x}(t)| \le |x_t-\hat{x}_t|_C \le K_{3,1}|\gamma-\hat{\gamma}|_\Gamma, \qquad t \in [0,\alpha_1], \tag{3.46}$$

where $K_{3,1} := K_{1,1}e^{K_{2,1}\alpha_1}$. Assumption (A1) (i), (3.44), (3.45) and (3.46) yield

$$
\begin{aligned}
|\dot{x}(t) - \dot{\hat{x}}(t)| &\leq \Big| f(t, x_t, x(t-\tau(t,x_t,\xi)), \dot{\varphi}(t-\mu(t)), \dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)), \theta) \\
&\qquad - f(t, \hat{x}_t, \hat{x}(t-\tau(t,\hat{x}_t,\hat{\xi})), \dot{\hat{\varphi}}(t-\mu(t)), \dot{\hat{\varphi}}(t-\rho(t,\widetilde{\hat{\varphi}}_t,\hat{\lambda})), \hat{\theta}) \Big| \\
&\leq L_{1,1}\Big(|x_t - \hat{x}_t|_C + |x(t-\tau(t,x_t,\xi)) - \hat{x}(t-\tau(t,\hat{x}_t,\hat{\xi}))| \\
&\qquad + |\dot{\varphi} - \dot{\hat{\varphi}}|_{L^\infty} + |\dot{\varphi}(t-\rho(t,\widetilde{\varphi}_t,\lambda)) - \dot{\hat{\varphi}}(t-\rho(t,\widetilde{\hat{\varphi}}_t,\hat{\lambda}))| + |\theta-\hat{\theta}|_\Theta \Big) \\
&\leq L_{1,1}\Big(2|x_t - \hat{x}_t|_C + a_1 L_{2,1}(|x_t - \hat{x}_t|_C + |\xi - \hat{\xi}|_\Xi) \\
&\qquad + 2|\varphi - \hat{\varphi}|_{W^{1,\infty}} + |\hat{\varphi}|_{W^{2,\infty}} L_{4,1}(|\widetilde{\varphi}_t - \widetilde{\hat{\varphi}}_t|_C + |\lambda - \hat{\lambda}|_\Lambda) + |\theta - \hat{\theta}|_\Theta \Big) \\
&\leq K_{4,1}|\gamma - \hat{\gamma}|_\Gamma, \quad \text{a.e. } t \in [0,\alpha]
\end{aligned}
$$

with $K_{4,1} := L_{1,1}\Big(2K_{3,1} + a_1 L_{2,1}(K_{3,1}+1) + 2 + |\hat{\varphi}|_{W^{2,\infty}} L_{4,1} + 1\Big)$.

Therefore we get

$$
|x_t - \hat{x}_t|_{W^{1,\infty}} \leq L^{(1)}|\gamma - \hat{\gamma}|_\Gamma, \qquad t \in [0,\alpha_1], \quad \gamma \in \mathcal{B}_\Gamma(\hat{\gamma};\delta_1), \tag{3.47}
$$

where $L^{(1)} := \max\{K_{3,1}, K_{4,1}\} \geq 1$. This concludes the proof of (3.32) on $[0,\alpha_1]$.

(5) Next we show that there exists $0 < \delta_2 \leq \delta_1$ and $0 < \Delta\alpha_2 \leq r_0$ such that for every parameter value $\gamma \in \mathcal{B}_\Gamma(\bar{\gamma};\delta_2)$ the corresponding solution of the IVP (3.1)–(3.2) can be extended to the interval $[\alpha_1, \alpha_1 + \Delta\alpha_2]$, and the solution satisfies claims (ii)–(v) of the theorem with $\alpha = \alpha_2$.

Let $x(\cdot,\gamma)$ be the solution of (3.1)-(3.2) on $[-r,\alpha_1]$, $\sigma := x_{\alpha_1}(\cdot,\gamma)$ and $\bar{\sigma} := x_{\alpha_1}(\cdot,\bar{\gamma})$. Note that $\bar{\sigma} \in C^1$ since $\bar{\gamma} \in \mathcal{M} \cap P$. Consider the equation

$$
\dot{x}(t) = f\Big(t+\alpha_1, x_t, x(t-\tau(t+\alpha_1,x_t,\xi)), \dot{\sigma}(t-\mu(t+\alpha_1)), \dot{\sigma}(t-\rho(t+\alpha_1,\widetilde{\sigma}_t,\lambda)), \theta\Big)
$$

for a.e. $t \in [0,\Delta\alpha_2]$, where $0 < \Delta\alpha_2 \leq r_0$ will be specified later, and the associated initial condition

$$
x(t) = \sigma(t), \qquad t \in [-r,0].
$$

Relation (3.39) yields $|\bar{\sigma} - \bar{\varphi}|_C < \kappa_1$. Therefore, there exists $0 < \kappa_2' < \kappa_1$ such that $\overline{\mathcal{B}}_C(\bar{\sigma};\kappa_2') \subset \mathcal{B}_C(\bar{\varphi};\kappa_1)$. Define the vectors $\bar{u}_2 := \bar{\sigma}(-\tau(\alpha_1,\bar{\sigma},\bar{\xi}))$, $\bar{v}_2 := \dot{\bar{\sigma}}(-\mu(\alpha_1))$ and $\bar{w}_2 := \dot{\bar{\sigma}}(-\rho(\alpha_1,\bar{\sigma},\bar{\lambda}))$. It follows from (3.41), (3.36) and (3.42), respectively, that $|\bar{u}_2 - \bar{u}_1| < \varepsilon_1$, $|\bar{v}_2 - \bar{v}_1| < \varepsilon_1$ and $|\bar{w}_2 - \bar{w}_1| < \varepsilon_1$, hence there exists $0 < \varepsilon_2' < \varepsilon_1$ such that $\mathcal{B}_{\mathbb{R}^n}(\bar{u}_2;\varepsilon_2') \subset \mathcal{B}_{\mathbb{R}^n}(\bar{u}_1;\varepsilon_1)$, $\mathcal{B}_{\mathbb{R}^n}(\bar{v}_2;\varepsilon_2') \subset \mathcal{B}_{\mathbb{R}^n}(\bar{v}_1;\varepsilon_1)$ and $\mathcal{B}_{\mathbb{R}^n}(\bar{w}_2;\varepsilon_2') \subset \mathcal{B}_{\mathbb{R}^n}(\bar{w}_1;\varepsilon_1)$. It follows from part (2) of the proof that (3.29) holds for $t = \alpha_1$ and $\gamma = \bar{\gamma}$, as well, hence

$$
D_1\rho(\alpha_1,\bar{\sigma},\bar{\lambda}) + D_2\rho(\alpha_1,\bar{\sigma},\bar{\lambda})\dot{\bar{\sigma}} < 1.
$$

We apply Lemma 3.2 with $\rho^{(2)}(t,\psi,\lambda) = \rho(t+\alpha_1,\psi,\lambda)$, $\bar{\varphi} = \bar{\sigma}$, and let α_2^*, ε_2^*, δ_2^* be the corresponding constants. Note that we may assume that $\varepsilon_2^* \leq \varepsilon_1^*$. Fix a constant $m_2 > 0$. Then there exist constants $0 < \kappa_2'' \leq \kappa_2'$, $0 < \delta_2' \leq \min\{\delta_1, \delta_2^*\}$, $0 < \varepsilon_2 \leq \varepsilon_2'$, $0 < T_2' \leq \min\{r_0, T - \alpha_1, \alpha_2^*\}$, such that

$$
|f(t+\alpha_1,\sigma,u,v,w,\theta) - f(\alpha_1,\bar{\sigma},\bar{u}_2,\bar{v}_2,\bar{w}_2,\bar{\theta})| \leq m_2,
$$

for $t \in [0, T_2']$, $\sigma \in \overline{\mathcal{B}}_C(\bar\sigma; \kappa_2'')$, $u \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar u_2; \varepsilon_2)$, $v \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar v_2; \varepsilon_2)$, $w \in \overline{\mathcal{B}}_{\mathbb{R}^n}(\bar w_2; \varepsilon_2)$, $\theta \in \overline{\mathcal{B}}_\Theta(\bar\theta; \delta_2')$ and $\chi \in C^1$ satisfying $|\mathcal{P}(\dot\chi) - \mathcal{P}(\dot{\bar\sigma})|_{C_{r_0}} \le \varepsilon_2$. Similarly to (3.35) and (3.37) we have that there exist constants $0 < \kappa_2 \le \kappa_2''$, $0 < \delta_2'' \le \delta_2'$, $0 < T_2 \le T_2'$, such that

$$|\chi(-\tau(t+\alpha_1, \chi, \xi)) - \bar u_2| \le |\chi - \bar\sigma|_C + |\bar\sigma(-\tau(t+\alpha_1, \chi, \xi)) - \bar\sigma(-\tau(\alpha_1, \bar\sigma, \bar\xi))| < \varepsilon_2$$

for $t \in [0, T_2]$, $\chi \in \mathcal{B}_C(\bar\sigma; \kappa_2)$, $\xi \in \mathcal{B}_\Xi(\bar\xi; \delta_2'')$, and

$$|\dot\chi(t - \rho(t+\alpha_1, \widetilde\chi_t, \lambda)) - \bar v_2| \le |\chi - \bar\sigma|_{W^{1,\infty}} + |\dot{\bar\sigma}(t - \rho(t+\alpha_1, \widetilde\chi_t, \lambda)) - \dot{\bar\sigma}(-\rho(\alpha_1, \bar\sigma, \bar\lambda))| < \varepsilon_2$$

for a.e. $t \in [0, T_2]$, $\chi \in \mathcal{B}_C(\bar\sigma; \kappa_2) \cap \mathcal{B}_{W^{1,\infty}}(\bar\sigma; \delta_2'')$, $\lambda \in \mathcal{B}_\Lambda(\bar\lambda; \delta_2'')$.

We define the constants and sets

$$\begin{aligned}
\delta_2 &:= \min\left\{\frac{\kappa_2}{3}, \frac{\varepsilon_2}{2}, \frac{\delta_2''}{L^{(1)}}, \frac{\kappa_2}{L^{(1)}}\right\},\\
\beta_2 &:= \frac{\kappa_2}{3},\\
a_2 &:= \max\{a_1, |f(\alpha_1, \bar\sigma, \bar u_2, \bar v_2, \bar w_2, \bar\theta)| + m_2\},\\
M_{1,2} &:= \{\psi \in W^{1,\infty} \colon |\psi - \bar\varphi|_C \le \kappa_1,\ |\dot\psi|_{L^\infty} \le a_2\},\\
\Delta\alpha_2 &:= \min\left\{\frac{\beta_2}{a_2}, \frac{\kappa_2}{3a_1}, T_2, r_0\right\},\\
\alpha_2 &:= \alpha_1 + \Delta\alpha_2,\\
E_2 &:= \Big\{y \in C([-r, \Delta\alpha_2], \mathbb{R}^n) \colon y(s) = 0,\ s \in [-r, 0] \text{ and } |y(s)| \le \beta_2,\ s \in [0, \Delta\alpha_2]\Big\}.
\end{aligned}$$

Relation (3.47) and the definition of δ_2 yield

$$|\sigma - \bar\sigma|_{W^{1,\infty}} = |x_{\alpha_1}(\cdot, \gamma) - x_{\alpha_1}(\cdot, \bar\gamma)|_{W^{1,\infty}} \le L^{(1)}|\gamma - \bar\gamma|_\Gamma < L^{(1)}\delta_2 \le \delta_2'',$$

and

$$|\sigma - \bar\sigma|_C = |x_{\alpha_1}(\cdot, \gamma) - x_{\alpha_1}(\cdot, \bar\gamma)|_C \le L^{(1)}|\gamma - \bar\gamma|_\Gamma < L^{(1)}\delta_2 \le \kappa_2$$

for $\gamma \in \mathcal{B}_\Gamma(\bar\gamma; \delta_2)$.

Then, it is easy to check that for each $\gamma \in \mathcal{B}_\Gamma(\bar\gamma; \delta_2)$ the operator $\mathcal{T}^2(\cdot, \gamma)$ defined by

$$\mathcal{T}^2(y, \gamma)(t) := \begin{cases} \displaystyle\int_0^t f\Big(s+\alpha_1, y_s + \widetilde\sigma_s, (y_s + \widetilde\sigma_s)(-\tau(s+\alpha_1, y_s + \widetilde\sigma_s, \xi)), \\ \qquad\qquad \dot\sigma(s - \mu(s+\alpha_1)), \dot\sigma(s - \rho(s+\alpha_1, \widetilde\sigma_s, \lambda)), \theta\Big)\, ds, & t \in [0, \Delta\alpha_2], \\ 0, & t \in [-r, 0]. \end{cases}$$

maps E_2 into E_2, therefore Schauder's fixed point theorem yield the existence of a solution of the equation

$$y = \mathcal{T}^2(y, \gamma).$$

But then the function $x(t+\alpha_1) := y(t) + \widetilde\sigma(t)$ is an extension of the solution $x(\cdot, \gamma)$ to the interval $[\alpha_1, \alpha_2]$. Note that $a_1 \le a_2$, therefore $M_{1,1} \subset M_{1,2}$, and $x_t \in M_{1,2}$, $x(t - \tau(t, x_t, \xi)) \in M_2$ and $|x_t|_{W^{1,\infty}} \le \max\{|\bar\varphi|_C + \delta_1, a_2\}$ for $t \in [0, \alpha_2]$, and $\dot x(t - \mu(t)) \in M_3$ and $\dot x(t - \rho(t, x_t, \lambda)) \in M_4$ for a.e. $t \in [0, \alpha_2]$. Moreover, similarly to the proof given in part (5) of the proof, it can be shown that there exists a constant $L^{(2)} \ge L^{(1)}$ such that $|x(\cdot, \gamma)_t - x(\cdot, \bar\gamma)_t|_{W^{1,\infty}} \le L^{(2)}|\gamma - \bar\gamma|_\Gamma$ for $t \in [0, \alpha_2]$. We can show that (3.29) is satisfied with $\alpha = \alpha_2$ and $\varepsilon^* = \varepsilon_2^*$.

Suppose $\gamma \in \mathcal{M} \cap P$. As in part (3) of the proof, we can easily check that the corresponding solution $x = x(\cdot, \gamma)$ is continuously differentiable on $[-r, \alpha_2]$, and $x_t \in \mathcal{M}$ for all $t \in [0, \alpha_2]$. Introduce the Lipschitz constants $L_{1,2} := L_1(\alpha_2, M_{1,2}, M_3, M_4, M_5)$, $L_{2,2} := L_2(\alpha_2, M_{1,2}, M_6)$, $L_{3,2} := L_3(\alpha_2)$, and $L_{4,2} := L_4(\alpha_2, M_{1,2}, M_7)$ defined by (A1) (i), (A2) (i), (A3) and (A4) (i), respectively. Similarly to estimate (3.43) it is easy to show that the constant $L_2^* := \max\{L_{1,2}(1 + a_2(2 + L_{2,2}(1 + a_2)) + L_1^*(2 + L_{3,2} + L_{4,2}(1 + a_2))), L_1^*\}$ satisfies $|\dot{x}(t) - \dot{x}(\bar{t})| \leq L_2^*|t - \bar{t}|$ for $t, \bar{t} \in [t_1, t_2]$. Therefore, we get that $x_t \in W^{2,\infty}$ and $|\ddot{x}_t|_{L^\infty} \leq L_2^*$ for $t \in [0, \alpha_2]$ and for $\gamma \in \mathcal{M} \cap P$.

(6) Finally, we show that the extension procedure described in part (5) of the proof can be repeated arbitrary many times, so the solutions can be extended to a "large" finite interval $[-r, \alpha]$ so that parts (ii)–(v) of the theorem hold.

If the solution is defined on the interval $[-r, \alpha_i]$ and $\alpha_i < T$, then we can again repeat the method described in part (5) of the proof, and get an extension of the solution to a larger interval $[-r, \alpha_{i+1}]$. Suppose we repeated the extension procedure described above k times. Then we defined a sequence of radii $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_k > 0$, a sequence of time values $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$, a sequence of upper estimates of the L^∞-norm of the derivative of the solutions $a_1 \leq a_2 \leq \cdots \leq a_k$, a nested sequence of sets $M_{1,1} \subset M_{1,2} \subset \cdots \subset M_{1,k}$, a sequence of Lipschitz constants $L^{(1)} \leq L^{(2)} \leq \cdots \leq L^{(k)}$, and a sequence $\varepsilon_1^* \geq \varepsilon_2^* \geq \cdots \geq \varepsilon_k^* > 0$. Then statement (i) of the theorem holds with $\delta := \delta_k$, the IVP (3.1)–(3.2) has a unique solution on the interval $[-r, \alpha]$ with $\alpha := \alpha_k$, statement (iii) holds with $M_1 := M_{1,k}$, $\varepsilon^* = \varepsilon_k^*$, statement (iv) holds with $N := \max\{|\bar{\varphi}|_C + \delta_1, a_k\}$ and $L := L^{(k)}$. Also, for $\gamma \in \overline{\mathcal{B}}_\Gamma(\bar{\gamma}; \delta) \cap \mathcal{M}$ we can find a sequence of Lipschitz constants $L_1^* \leq \cdots \leq L_k^*$ of $\dot{x}$, and statement (v) holds with $L^* := L_k^*$. $\square$

We note that if $\alpha \leq r_0$, then the compatibility condition is not used to prove (3.33).

Remark 3.4. It follows from the proof of Theorem 3.3 that the set M_1 defined by the theorem has the form $M_1 = \{\psi \in W^{1,\infty} : |\psi - \bar{\varphi}|_C \leq \kappa_1, |\dot{\psi}|_{L^\infty} \leq a\}$ for some positive constants κ_1 and a. The proof implies a slightly stronger versions of (3.28) and (3.30) too: for $\gamma = (\varphi, \xi, \lambda, \theta) \in P$ the solution $x(t) = x(t, \gamma)$ satisfies $x_t \in \{\psi \in W^{1,\infty} \colon |\psi - \bar{\varphi}|_C < \kappa_1, |\dot{\psi}|_{L^\infty} \leq a\}$, $x(t - \tau(t, x_t, \xi)) \in \operatorname{int}(M_2)$ for $t \in [0, \alpha]$, and $\dot{x}(t - \mu(t)) \in \operatorname{int}(M_3)$, $\dot{x}(t - \rho(t, x_t, \lambda)) \in \operatorname{int}(M_4)$ for a.e. $t \in [0, \alpha]$; moreover, $\xi \in \operatorname{int}(M_5)$, $\lambda \in \operatorname{int}(M_6)$ and $\theta \in \operatorname{int}(M_7)$, where $\operatorname{int}(M_j)$ is the interior of the set M_j, $j = 2, \ldots, 7$.

4 Differentiability wrt parameters

In this section we study differentiability of solutions of the IVP (3.1)–(3.2) wrt the initial function, φ, and the parameters ξ, λ and θ of the functions τ, ρ and f, respectively.

Fix $\bar{\gamma} := (\bar{\varphi}, \bar{\xi}, \bar{\lambda}, \bar{\theta}) \in \mathcal{M}$. Let the positive constants α and δ, the parameter set $P := \overline{\mathcal{B}}_\Gamma(\bar{\gamma}; \delta)$, and the compact and convex sets M_1, M_2, M_3 and M_4 be defined by Theorem 3.3, and let

$$M_5 := \overline{\mathcal{B}}_\Theta(\bar{\theta}; \delta), \quad M_6 := \overline{\mathcal{B}}_\Xi(\bar{\xi}; \delta), \quad \text{and} \quad M_7 := \overline{\mathcal{B}}_\Lambda(\bar{\lambda}; \delta), \tag{4.1}$$

as in the proof of Theorem 3.3.

First we define a few notations will be used throughout this section. We introduce the space

$$\mathbf{X} := \mathbb{R} \times C \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta,$$

and for a vector $\mathbf{x} := (t, \psi, u, v, w, \theta) \in \mathbf{X}$ its norm is defined by

$$|\mathbf{x}|_\mathbf{X} := |t| + |\psi|_C + |u| + |v| + |w| + |\theta|_\Theta.$$

Fix a vector $\bar{\mathbf{x}} := (\bar{t}, \bar{\psi}, \bar{u}, \bar{v}, \bar{w}, \bar{\theta}) \in [0, \alpha] \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \subset \mathbf{X}$. We define the notation

$$\begin{aligned}\omega_f(\bar{\mathbf{x}}, \mathbf{x}) := f(\mathbf{x}) - f(\bar{\mathbf{x}}) - D_2 f(\bar{\mathbf{x}})(\psi - \bar{\psi}) - D_3 f(\bar{\mathbf{x}})(u - \bar{u}) - D_4 f(\bar{\mathbf{x}})(v - \bar{v}) \\ - D_5 f(\bar{\mathbf{x}})(w - \bar{w}) - D_6 f(\bar{\mathbf{x}})(\theta - \bar{\theta})\end{aligned}$$

for $\mathbf{x} := (\bar{t}, \psi, u, v, w, \theta) \in [0, \alpha] \times \Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4 \times \Omega_5 \subset \mathbf{X}$. Note that the first components of $\bar{\mathbf{x}}$ and $\mathbf{x}$ are identical. Assumption (A1) (ii) yields the function $(\psi, u, v, w, \theta) \mapsto f(\bar{t}, \psi, u, v, w, \theta)$ is continuously differentiable on $\Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4 \times \Omega_5$, and in particular, it is differentiable at $(\bar{\psi}, \bar{u}, \bar{v}, \bar{w}, \bar{\theta})$. Hence

$$\frac{|\omega_f(\bar{\mathbf{x}}, \mathbf{x})|}{|\mathbf{x} - \bar{\mathbf{x}}|_X} \to 0, \qquad \text{as } |\mathbf{x} - \bar{\mathbf{x}}|_X \to 0. \tag{4.2}$$

Let $L_1 = L_1(\alpha, M_1, M_2, M_3, M_4, M_5)$ be defined by (A1) (i). Then it follows from (A1) (i) that

$$\begin{aligned}|\omega_f(\bar{\mathbf{x}}, \mathbf{x})| &\leq |f(\mathbf{x}) - f(\bar{\mathbf{x}})| + |D_2 f(\bar{\mathbf{x}})(\psi - \bar{\psi})| + |D_3 f(\bar{\mathbf{x}})(u - \bar{u})| + |D_4 f(\bar{\mathbf{x}})(v - \bar{v})| \\ &\quad + |D_5 f(\bar{\mathbf{x}})(w - \bar{w})| + |D_6 f(\bar{\mathbf{x}})(\theta - \bar{\theta})| \\ &\leq \left(L_1 + \max_{j=2,\ldots,5} |D_j f(\bar{\mathbf{x}})|_{X_j} \right) |\mathbf{x} - \bar{\mathbf{x}}|_{\mathbf{X}} \end{aligned}\tag{4.3}$$

for $\mathbf{x} := (\bar{t}, \psi, u, v, w, \theta) \in [0, \alpha] \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \subset \mathbf{X}$. Note here $X_2, \ldots, X_5$ are defined in assumption (A1) (ii).

Similarly, we define the normed linear space

$$\mathbf{A} := \mathbb{R} \times C \times \Xi, \qquad |\mathbf{a}|_{\mathbf{A}} := |t| + |\psi|_C + |\xi|_\Xi \quad \text{for } \mathbf{a} := (t, \psi, \xi) \in \mathbf{A}.$$

We fix $\bar{\mathbf{a}} := (\bar{t}, \bar{\psi}, \bar{\xi}) \in [0, \alpha] \times M_1 \times M_6 \subset \mathbf{A}$. Introduce the notation

$$\omega_\tau(\bar{\mathbf{a}}, \mathbf{a}) := \tau(\mathbf{a}) - \tau(\bar{\mathbf{a}}) - D_2 \tau(\bar{\mathbf{a}})(\psi - \bar{\psi}) - D_3 \tau(\bar{\mathbf{a}})(\xi - \bar{\xi})$$

for $\mathbf{a} := (\bar{t}, \psi, \xi) \in [0, \alpha] \times \Omega_1 \times \Omega_6 \subset \mathbf{A}$. Let $L_2 = L_2(\alpha, M_1, M_6)$ be defined by (A2) (i). Then, similarly to (4.2) and (4.3), we get

$$\frac{|\omega_\tau(\bar{\mathbf{a}}, \mathbf{a})|}{|\mathbf{a} - \bar{\mathbf{a}}|_{\mathbf{A}}} \to 0, \quad \text{as } |\mathbf{a} - \bar{\mathbf{a}}|_{\mathbf{A}} \to 0, \tag{4.4}$$

and

$$\frac{|\omega_\tau(\bar{\mathbf{a}}, \mathbf{a})|}{|\mathbf{a} - \bar{\mathbf{a}}|_{\mathbf{A}}} \leq L_2 + \max_{j=2,3} |D_j \tau(\bar{\mathbf{a}})|_{Y_j}, \qquad \mathbf{a} \neq \bar{\mathbf{a}}, \quad \mathbf{a} := (\bar{t}, \psi, \xi) \in [0, \alpha] \times M_1 \times M_6 \subset \mathbf{A}, \tag{4.5}$$

where Y_2 and Y_3 are defined in (A2) (ii).

We also define

$$\mathbf{B} := \mathbb{R} \times C \times \Lambda, \qquad |\mathbf{b}|_{\mathbf{B}} := |t| + |\psi|_C + |\lambda|_\Lambda \quad \text{for } \mathbf{b} := (t, \psi, \lambda) \in \mathbf{B}.$$

Fix a vector $\bar{\mathbf{b}} := (\bar{t}, \bar{\psi}, \bar{\lambda}) \in [0, \alpha] \times M_1 \times M_7 \subset \mathbf{B}$. We define

$$\omega_\rho(\bar{\mathbf{b}}, \mathbf{b}) := \rho(\mathbf{b}) - \rho(\bar{\mathbf{b}}) - D_2 \rho(\bar{\mathbf{b}})(\psi - \bar{\psi}) - D_3 \rho(\bar{\mathbf{b}})(\lambda - \bar{\lambda})$$

for $\mathbf{b} := (\bar{t}, \psi, \lambda) \in [0, \alpha] \times \Omega_1 \times \Omega_7 \subset \mathbf{B}$. Let $L_4 = L_4(\alpha, M_1, M_7)$ be defined by (A4) (i). Then we can obtain

$$\frac{|\omega_\rho(\bar{\mathbf{b}}, \mathbf{b})|}{|\mathbf{b} - \bar{\mathbf{b}}|_{\mathbf{B}}} \to 0, \quad \text{as } |\mathbf{b} - \bar{\mathbf{b}}|_{\mathbf{B}} \to 0, \tag{4.6}$$

and

$$\frac{|\omega_\rho(\bar{\mathbf{b}},\mathbf{b})|}{|\mathbf{b}-\bar{\mathbf{b}}|_{\mathbf{B}}} \le L_4 + \max_{j=2,3} |D_j\rho(\bar{\mathbf{b}})|_{Z_j}, \qquad \mathbf{b}\neq\bar{\mathbf{b}}, \quad \mathbf{b} := (\bar{t},\psi,\lambda) \in [0,\alpha]\times M_1\times M_7 \subset \mathbf{B}, \tag{4.7}$$

where Z_2 and Z_3 are defined in (A4) (ii).

Let $x\colon [-r,\alpha] \to \mathbb{R}^n$ be continuously differentiable. We have

$$\begin{aligned} |x(t)-x(\bar{t})-\dot{x}(\bar{t})(t-\bar{t})| &= \left|\int_{\bar{t}}^{t} (\dot{x}(u)-\dot{x}(\bar{t}))\,du\right| \\ &\le \Omega_{\dot{x}}(|t-\bar{t}|)|t-\bar{t}|, \qquad t,\bar{t}\in[-r,\alpha], \end{aligned} \tag{4.8}$$

where

$$\Omega_{\dot{x}}(\varepsilon) := \sup\Big\{|\dot{x}(t)-\dot{x}(\bar{t})|\colon\ |t-\bar{t}|\le\varepsilon,\ t,\bar{t}\in[-r,\alpha]\Big\}.$$

Note that the uniform continuity of $\dot{x}$ on $[-r,\alpha]$ implies $\Omega_{\dot{x}}(\varepsilon)\to 0$ as $\varepsilon\to 0$.

It is easy to show (see [14] or [23]) that the partial derivatives of a function of the form $F\colon [0,\alpha]\times W^{1,\infty}\times\Xi\to\mathbb{R}^n,\ F(t,\psi,\xi) := \psi(-\tau(t,\psi,\xi))$ are given by

$$\begin{aligned} &D_2F(t,\psi,\xi)u = -\dot{\psi}(-\tau(t,\psi,\xi))D_2\tau(t,\psi,\xi)u + u(-\tau(t,\psi,\xi)), \qquad u\in W^{1,\infty}, \\ &D_3F(t,\psi,\xi)v = -\dot{\psi}(-\tau(t,\psi,\xi))D_3\tau(t,\psi,\xi)v, \qquad v\in\Xi \end{aligned}$$

for $t\in[0,\alpha]$, $\psi\in C^1$ and $\xi\in\Xi$. Using these relations, we can formulate the linear variational equation corresponding to Equation (3.1) in the following way. (See also [14, 15, 18] for the form of the variational equations associated to other classes of SD-DDEs.)

Let $\gamma = (\varphi,\xi,\lambda,\theta)\in\mathcal{M}\cap P$ be fixed, and $x(t) := x(t,\gamma)$ be the corresponding solution of the IVP (3.1)–(3.2) on $[-r,\alpha]$. Note that Theorem 3.3 yields that x is continuously differentiable on $[-r,\alpha]$. We will use the short vector notation

$$\mathbf{x}(t) := \Big(t, x_t, x(t-\tau(t,x_t,\xi)), \dot{x}(t-\mu(t)), \dot{x}(t-\rho(t,x_t,\lambda)),\theta\Big)$$

for the argument of f in Equation (3.1), and the vectors

$$\mathbf{a}(t) := \Big(t,x_t,\xi\Big) \quad\text{and}\quad \mathbf{b}(t) := \Big(t,x_t,\lambda\Big)$$

for the arguments of τ and ρ, respectively.

Fix $h = (h^\varphi,h^\xi,h^\lambda,h^\theta)\in\Gamma$, and consider the variational equation

$$\begin{aligned} \dot{z}(t) = {} & D_2f(\mathbf{x}(t))z_t \\ & + D_3f(\mathbf{x}(t))\Big[-\dot{x}(t-\tau(\mathbf{a}(t)))\Big\{D_2\tau(\mathbf{a}(t))z_t + D_3\tau(\mathbf{a}(t))h^\xi\Big\} + z(t-\tau(\mathbf{a}(t)))\Big] \\ & + D_4f(\mathbf{x}(t))\dot{z}(t-\mu(t)) \\ & + D_5f(\mathbf{x}(t))\Big[-\ddot{x}(t-\rho(\mathbf{b}(t)))\Big\{D_2\rho(\mathbf{b}(t))z_t + D_3\rho(\mathbf{b}(t))h^\lambda\Big\} + \dot{z}(t-\rho(\mathbf{b}(t)))\Big] \\ & + D_6f(\mathbf{x}(t))h^\theta, \qquad \text{a.e. } t\in[0,\alpha], \end{aligned} \tag{4.9}$$

$$z(t) = h^\varphi(t), \qquad t\in[-r,0]. \tag{4.10}$$

This is an inhomogeneous linear time-dependent but state-independent NFDE for z. Note that for $\gamma\in\mathcal{M}\cap P$ the term $\ddot{x}(t-\rho(\mathbf{b}(t)))$ is defined only for a.e. $t\in[0,\alpha]$. Since the neutral terms on the right-hand side of (4.9) do not depend on values of $\dot{z}$ on the interval $(t-r_0,t]$, it is easy to prove using the method of steps with the intervals $[ir_0,(i+1)r_0]$, that the IVP (4.9)–(4.10) has a unique solution, $z(t)=z(t,\gamma,h)$, which depends linearly on h. The boundedness of the map $\Gamma\to\mathbb{R}^n$, $h\mapsto z(t,\gamma,h)$ for each $t\in[0,\alpha]$ follows from Lemma 4.1 below.

We introduce the following notation

$$
\begin{aligned}
&L(t,\gamma)(\psi,h^\xi,h^\lambda,h^\theta)\\
&\quad:= D_2f(\mathbf{x}(t))\psi\\
&\qquad + D_3f(\mathbf{x}(t))\Big[-\dot{x}(t-\tau(\mathbf{a}(t)))\Big\{D_2\tau(\mathbf{a}(t))\psi + D_3\tau(\mathbf{a}(t))h^\xi\Big\} + \psi(-\tau(\mathbf{a}(t)))\Big]\\
&\qquad + D_4f(\mathbf{x}(t))\dot{\psi}(-\mu(t))\\
&\qquad + D_5f(\mathbf{x}(t))\Big[-\ddot{x}(t-\rho(\mathbf{b}(t)))\Big\{D_2\rho(\mathbf{b}(t))\psi + D_3\rho(\mathbf{b}(t))h^\lambda\Big\} + \dot{\psi}(-\rho(\mathbf{b}(t)))\Big]\\
&\qquad + D_6f(\mathbf{x}(t))h^\theta
\end{aligned}
\tag{4.11}
$$

for a.e. $t\in[0,\alpha]$, $\gamma\in\mathcal{M}\cap P$ and $\psi\in W^{1,\infty}$, $h^\xi\in\Xi$, $h^\lambda\in\Lambda$, $h^\theta\in\Theta$. With this notation (4.9) can be rewritten as

$$
\dot{z}(t)=L(t,\gamma)(z_t,h^\xi,h^\lambda,h^\theta),\qquad \text{a.e. } t\in[0,\alpha]. \tag{4.12}
$$

We introduce the Lipschitz constants $L_1=L_1(\alpha,M_1,M_2,M_3,M_4,M_5)$, $L_2=L_2(\alpha,M_1,M_6)$, $L_3:=L_3(\alpha)$ and $L_4=L_4(\alpha,M_1,M_7)$ from (A1) (i), (A2) (i), (A3) and (A4) (i), respectively. Let N and $L^*=L^*(\gamma)$ be defined by (3.31) and (3.33), respectively. Then (A1), (A2), (A4) and Remarks 3.1 and 3.4 yield

$$
|D_jf(\mathbf{x}(t))|_{X_j}\le L_1,\quad |D_3\tau(\mathbf{a}(t))|_{Y_3}\le L_2,\quad |D_3\rho(\mathbf{b}(t))|_{Z_3}\le L_3,\qquad t\in[0,\alpha],\quad j=3,4,5,6, \tag{4.13}
$$

where X_3,X_4,X_5,X_6,Y_3 and Z_3 are defined in (A1) (ii), (A2) (ii) and (A4) (ii), respectively. We claim

$$
|D_2f(\mathbf{x}(t))\psi|\le L_1|\psi|_C,\;\; |D_2\tau(\mathbf{a}(t))\psi|\le L_2|\psi|_C,\;\; |D_2\rho(\mathbf{b}(t))\psi|\le L_3|\psi|_C,\;\; \psi\in W^{1,\infty},\; t\in[0,\alpha] \tag{4.14}
$$

for $t\in[0,\alpha]$. We show only the first estimate, the proofs of the second and third relations are similar. Let $\psi\in W^{1,\infty}$, $|\psi|_C\ne 0$ and $t\in[0,\alpha]$ be fixed. Let $\Delta:=(0,\psi,0,0,0,0)\in\mathbf{X}$ (here the 0-s are the zero vectors of the respective spaces). It follows from Remark 3.4 that $x_t\in M_1$, and for large enough $k\in\mathbb{N}$ we have $x_t+\frac{1}{k}\varphi\in M_1$. Then assumption (A1) (1) implies for such k

$$
\begin{aligned}
\frac{|D_2f(\mathbf{x}(t))\psi|}{|\psi|_C} &= \frac{|D_2f(\mathbf{x}(t))\frac{1}{k}\psi|}{|\frac{1}{k}\psi|_C}\\
&\le \frac{|f(\mathbf{x}(t)+\frac{1}{k}\Delta)-f(\mathbf{x}(t))|}{|\frac{1}{k}\psi|_C} + \frac{|f(\mathbf{x}(t)+\frac{1}{k}\Delta)-f(\mathbf{x}(t))-D_2f(\mathbf{x}(t))\frac{1}{k}\psi|}{|\frac{1}{k}\psi|_C}\\
&\le L_1 + \frac{|f(\mathbf{x}(t)+\frac{1}{k}\Delta)-f(\mathbf{x}(t))-D_2f(\mathbf{x}(t))\frac{1}{k}\psi|}{|\frac{1}{k}\psi|_C},
\end{aligned}
$$

which yields the first estimate of (4.14) as $k\to\infty$.

Then, combining (3.31), (3.33), (4.13) and (4.14) with (4.11), we get

$$
\begin{aligned}
|L(t,\gamma)(\psi,h^\xi,h^\lambda,h^\theta)| &\le L_1|\psi|_C + L_1\Big(NL_2(|\psi|_C+|h^\xi|_\Xi)+|\psi|_C\Big) + L_1|\mathcal{P}(\dot{\psi})|_{L^\infty_{r_0}}\\
&\quad + L_1\Big(L^*L_4(|\psi|_C+|h^\lambda|_\Lambda)+|\mathcal{P}(\dot{\psi})|_{L^\infty_{r_0}}\Big) + L_1|h^\theta|_\Theta\\
&\le N_0\Big(|\psi|_C+|\mathcal{P}(\dot{\psi})|_{L^\infty_{r_0}}+|h^\xi|_\Xi+|h^\lambda|_\Lambda+|h^\theta|_\Theta\Big)
\end{aligned}
\tag{4.15}
$$

for a.e. $t \in [0,\alpha]$, $\psi \in W^{1,\infty}$, $h^\xi \in \Xi$, $h^\lambda \in \Lambda$, $h^\theta \in \Theta$, where $N_0 := L_1(NL_2 + L^* L_4 + 2)$.

The next lemma shows that the linear maps $\Gamma \ni h \mapsto z(t,\gamma,h) \in \mathbb{R}^n$ and $\Gamma \ni h \mapsto \dot{z}(t,\gamma,h) \in \mathbb{R}^n$ are bounded for $t \in [-r,\alpha]$ and for a.e. $t \in [-r,\alpha]$, respectively, and for $\gamma \in \mathcal{M} \cap P$. We also prove that $\dot{z}$ satisfies a certain special inequality, which will be important in the proof of Lemma 4.2 below.

Lemma 4.1. *Assume (A1)–(A4), let $\alpha > 0$ and $P \subset \Pi$ be defined by Theorem 3.3. Then for every $\gamma \in \mathcal{M} \cap P$ there exist constants $N_1 \geq 0$ and $N_2 \geq 0$ such that the solution of the IVP (4.9)–(4.10) satisfies*

$$|z(t,\gamma,h)| \leq N_1 |h|_\Gamma, \qquad t \in [-r,\alpha], \quad h \in \Gamma, \tag{4.16}$$
$$|\dot{z}(t,\gamma,h)| \leq N_2 |h|_\Gamma, \qquad \text{a.e. } t \in [-r,\alpha], \quad h \in \Gamma. \tag{4.17}$$

Moreover, for every $\gamma \in \mathcal{M} \cap P$ there exist constants $N_3 \geq 0$, $N_4 \geq 0$, $N_5 \geq 0$, $N_6 \geq 0$ and a monotone increasing function $\Omega\colon [0,\alpha] \to [0,\infty)$ with the property $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0+$ such that

$$\begin{aligned} |\dot{z}(t,\gamma,h) - \dot{z}(\bar{t},\gamma,h)| \leq{}& N_3 |t-\bar{t}| |h|_\Gamma + N_4 \Omega(|t-\bar{t}|) |h|_\Gamma \\ &+ N_5 |\dot{x}(t-\rho(\mathbf{b}(t))) - \dot{x}(\bar{t}-\rho(\mathbf{b}(\bar{t})))| |h|_\Gamma \\ &+ N_6 \Big(|\dot{z}(t-\mu(t),\gamma,h) - \dot{z}(\bar{t}-\mu(\bar{t}),\gamma,h)| \\ &\qquad + |\dot{z}(t-\rho(\mathbf{b}(t)),\gamma,h) - \dot{z}(\bar{t}-\rho(\mathbf{b}(\bar{t})),\gamma,h)| \Big) \end{aligned} \tag{4.18}$$

for a.e. $t,\bar{t} \in [0,\alpha]$, $h \in \Gamma$.

Proof. Let $\gamma \in \mathcal{M} \cap P$ be fixed. For simplicity we use the notations $h = (h^\varphi, h^\xi, h^\theta, h^\lambda) \in \Gamma$, $x(t) := x(t,\gamma)$ and $z(t) := z(t,\gamma,h)$. Let δ, M_1, M_2, M_3 and M_4 be defined by Theorem 3.3, M_5, M_6 and M_7 be defined by (4.1), $L_1,\ldots,L_4$ be the corresponding Lipschitz constants from (A1)–(A4), and let $L^* = L^*(\gamma)$ and $N_0 = N_0(\gamma)$ be defined by (3.33) and (4.15), respectively.

Let $m := [\alpha/r_0]$ (here $[\cdot]$ denotes the greatest integer part), $t_j := jr_0$ for $j = 0,1,\ldots,m$, $t_{m+1} := \alpha$, and let $t \in [t_0,t_1]$. Integrating (4.12) from 0 to t, and using estimate (4.15) we get

$$\begin{aligned} |z(t)| &\leq |h^\varphi(0)| + \int_0^t |L(s,\gamma)(z_s, h^\xi, h^\lambda, h^\theta)|\, ds \\ &\leq |h^\varphi|_C + N_0 r_0 (|h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta) + N_0 \int_0^t (|z_s|_C + |\mathcal{P}(\dot{z}_s)|_{L^\infty_{r_0}}) ds \\ &= |h^\varphi|_C + N_0 r_0 (|h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta) + N_0 \int_0^t (|z_s|_C + \operatorname*{ess\,sup}_{-r \leq \zeta \leq -r_0} |\dot{z}(s+\zeta)|) ds \\ &= |h^\varphi|_C + N_0 r_0 (|h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta) + N_0 \int_0^t (|z_s|_C + \operatorname*{ess\,sup}_{-r \leq \zeta \leq -r_0} |\dot{h}^\varphi(s+\zeta)|) ds \\ &\leq (1 + N_0 r_0) |h|_\Gamma + N_0 \int_0^t |z_s|_C\, ds, \qquad t \in [t_0,t_1]. \end{aligned}$$

Note that $|z_0|_C \leq (1 + N_0 r_0)|h|_\Gamma$, hence Gronwall's inequality yields

$$|z(t)| \leq c_0 |h|_\Gamma, \qquad t \in [t_0,t_1],$$

where $c_0 := (1 + N_0 r_0) e^{N_0 r_0}$. Since $c_0 \geq 1$, it follows that

$$|z(t)| \leq c_0 |h|_\Gamma, \qquad t \in [-r,t_1].$$

Using Equation (4.12) and relation (4.15) we get

$$\begin{aligned}|\dot z(t)| &\le N_0\Big(|z_t|_C + |\mathcal{P}(\dot z_t)|_{L^\infty_{r_0}} + |h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta\Big)\\ &\le N_0\Big(c_0|h|_\Gamma + |h^\varphi|_{W^{1,\infty}} + |h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta\Big)\\ &\le d_0|h|_\Gamma, \qquad \text{a.e. } t\in[t_0,t_1],\end{aligned}$$

where $d_0 := \max\{N_0(c_0+1),1\}$. Then $d_0\ge 1$ yields

$$|\dot z(t)| \le d_0|h|_\Gamma, \qquad \text{a.e. } t\in[-r,t_1].$$

Suppose we have already defined constants $c_0\le\cdots\le c_{j-1}$ and $d_0\le\cdots\le d_{j-1}$ such that $|z(t)|\le c_{j-1}|h|_\Gamma$ for $t\in[-r,t_j]$ and $|\dot z(t)|\le d_{j-1}|h|_\Gamma$ for a.e. $t\in[-r,t_j]$. Then for $t\in[t_j,t_{j+1}]$ it follows

$$\begin{aligned}|z(t)| &\le |z(t_j)| + \int_{t_j}^t |L(s,\gamma)(z_s,h^\xi,h^\lambda,h^\theta)|\,ds\\ &= |z(t_j)| + N_0r_0|h|_\Gamma + N_0\int_{t_j}^t\Big(|z_s|_C + \operatorname*{ess\,sup}_{-r\le\zeta\le -r_0}|\dot z(s+\zeta)|\Big)ds\\ &\le (c_{j-1}+N_0r_0+N_0r_0d_{j-1})|h|_\Gamma + N_0\int_{t_j}^t |z_s|_C\,ds, \qquad t\in[t_j,t_{j+1}].\end{aligned}$$

Therefore Gronwall's inequality implies

$$|z(t)|\le c_j|h|_\Gamma, \qquad t\in[t_j,t_{j+1}],$$

where $c_j := (c_{j-1}+N_0r_0+N_0r_0d_{j-1})e^{N_0r_0}$. Note that $c_{j-1}\le c_j$ holds. We have

$$\begin{aligned}|\dot z(t)| &\le N_0\Big(|z_t|_C + |\mathcal{P}(\dot z_t)|_{L^\infty_{r_0}} + |h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta\Big)\\ &\le N_0\Big(c_j|h|_\Gamma + d_{j-1}|h|_\Gamma + |h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta\Big)\\ &\le d_j|h|_\Gamma, \quad \text{a.e. } t\in[t_j,t_{j+1}],\end{aligned}$$

where $d_j := \max\{N_0(c_j+d_{j-1}+1),d_{j-1}\}$. Repeating these estimates for $j=0,1,\dots,m$, we get $N_1 := c_m$ and $N_2 := d_m$ satisfy (4.16) and (4.17), respectively.

For $t,\bar t\in[0,\alpha]$ such that $\dot z(t)$ and $\dot z(\bar t)$ exist consider

$$\begin{aligned}&|\dot z(t)-\dot z(\bar t)|\\ &\quad= |L(t,\gamma)(z_t,h^\xi,h^\lambda,h^\theta) - L(\bar t,\gamma)(z_{\bar t},h^\xi,h^\lambda,h^\theta)|\\ &\quad\le |D_2f(\mathbf{x}(t))z_t - D_2f(\mathbf{x}(\bar t))z_{\bar t}|\\ &\qquad + \Big|D_3f(\mathbf{x}(t))\Big[-\dot x(t-\tau(\mathbf{a}(t)))\Big\{D_2\tau(\mathbf{a}(t))z_t + D_3\tau(\mathbf{a}(t))h^\xi\Big\} + z(t-\tau(\mathbf{a}(t)))\Big]\\ &\qquad\quad - D_3f(\mathbf{x}(\bar t))\Big[-\dot x(\bar t-\tau(\mathbf{a}(\bar t)))\Big\{D_2\tau(\mathbf{a}(\bar t))z_{\bar t} + D_3\tau(\mathbf{a}(\bar t))h^\xi\Big\} + z(\bar t-\tau(\mathbf{a}(\bar t)))\Big]\Big|\\ &\qquad + \Big|D_4f(\mathbf{x}(t))\dot z(t-\mu(t)) - D_4f(\mathbf{x}(\bar t))\dot z(\bar t-\mu(\bar t))\Big|\\ &\qquad + \Big|D_5f(\mathbf{x}(t))\Big[-\ddot x(t-\rho(\mathbf{b}(t)))\Big\{D_2\rho(\mathbf{b}(t))z_t + D_3\rho(\mathbf{b}(t))h^\lambda\Big\} + \dot z(t-\rho(\mathbf{b}(t)))\Big]\\ &\qquad\quad - D_5f(\mathbf{x}(\bar t))\Big[-\ddot x(\bar t-\rho(\mathbf{b}(\bar t)))\Big\{D_2\rho(\mathbf{b}(\bar t))z_{\bar t} + D_3\rho(\mathbf{b}(\bar t))h^\lambda\Big\} + \dot z(\bar t-\rho(\mathbf{b}(\bar t)))\Big]\Big|\\ &\qquad + |D_6f(\mathbf{x}(t))h^\theta - D_6f(\mathbf{x}(\bar t))h^\theta|\end{aligned}$$

$$
\begin{aligned}
&\le |(D_2 f(\mathbf{x}(t)) - D_2 f(\mathbf{x}(\bar t)))z_t| + |D_2 f(\mathbf{x}(\bar t))(z_t - z_{\bar t})| \\
&\quad + \Big|(D_3 f(\mathbf{x}(t)) - D_3 f(\mathbf{x}(\bar t)))\Big[-\dot x(t-\tau(\mathbf{a}(t)))\Big\{D_2\tau(\mathbf{a}(t))z_t + D_3\tau(\mathbf{a}(t))h^{\xi}\Big\} \\
&\qquad + z(t-\tau(\mathbf{a}(t)))\Big]\Big| \\
&\quad + \Big|D_3 f(\mathbf{x}(\bar t))\Big[-\Big(\dot x(t-\tau(\mathbf{a}(t))) - \dot x(\bar t-\tau(\mathbf{a}(\bar t)))\Big)\Big\{D_2\tau(\mathbf{a}(t))z_t + D_3\tau(\mathbf{a}(t))h^{\xi}\Big\}\Big]\Big| \\
&\quad + \Big|D_3 f(\mathbf{x}(\bar t))\Big[-\dot x(\bar t-\tau(\mathbf{a}(\bar t)))\Big\{\Big(D_2\tau(\mathbf{a}(t)) - D_2\tau(\mathbf{a}(\bar t))\Big)z_t + D_2\tau(\mathbf{a}(\bar t))(z_t - z_{\bar t}) \\
&\qquad + \Big(D_3\tau(\mathbf{a}(t)) - D_3\tau(\mathbf{a}(\bar t))\Big)h^{\xi}\Big\}\Big]\Big| \\
&\quad + \Big|D_3 f(\mathbf{x}(\bar t))\Big[z(t-\tau(\mathbf{a}(t))) - z(\bar t-\tau(\mathbf{a}(\bar t)))\Big]\Big| \\
&\quad + \Big|\Big(D_4 f(\mathbf{x}(t)) - D_4 f(\mathbf{x}(\bar t))\Big)\dot z(t-\mu(t))\Big| + \Big|D_4 f(\mathbf{x}(\bar t))\Big(\dot z(t-\mu(t)) - \dot z(\bar t-\mu(\bar t))\Big)\Big| \\
&\quad + \Big|\Big(D_5 f(\mathbf{x}(t)) - D_5 f(\mathbf{x}(\bar t))\Big)\Big[-\ddot x(t-\rho(\mathbf{b}(t)))\Big\{D_2\rho(\mathbf{b}(t))z_t + D_3\rho(\mathbf{b}(t))h^{\lambda}\Big\} \\
&\qquad + \dot z(t-\rho(\mathbf{b}(t)))\Big]\Big| \\
&\quad + \Big|D_5 f(\mathbf{x}(\bar t))\Big[-\Big(\ddot x(t-\rho(\mathbf{b}(t))) - \ddot x(\bar t-\rho(\mathbf{b}(\bar t)))\Big)\Big\{D_2\rho(\mathbf{b}(t))z_t + D_3\rho(\mathbf{b}(t))h^{\lambda}\Big\}\Big]\Big| \\
&\quad + \Big|D_5 f(\mathbf{x}(\bar t))\Big[-\ddot x(\bar t-\rho(\mathbf{b}(\bar t)))\Big\{\Big(D_2\rho(\mathbf{b}(t)) - D_2\rho(\mathbf{b}(\bar t))\Big)z_t + D_2\rho(\mathbf{b}(\bar t))(z_t - z_{\bar t}) \\
&\qquad + \Big(D_3\rho(\mathbf{b}(t)) - D_3\rho(\mathbf{b}(\bar t))\Big)h^{\lambda}\Big\}\Big]\Big| \\
&\quad + \Big|D_5 f(\mathbf{x}(\bar t))\Big[-\ddot x(\bar t-\rho(\mathbf{b}(\bar t)))\Big(\dot z(t-\rho(\mathbf{b}(t))) - \dot z(\bar t-\rho(\mathbf{b}(\bar t)))\Big)\Big| \\
&\quad + \Big|\Big(D_6 f(\mathbf{x}(t)) - D_6 f(\mathbf{x}(\bar t))\Big)h^{\theta}\Big|.
\end{aligned} \tag{4.19}
$$

We define the monotone increasing functions

$$
\begin{aligned}
\Omega_{Df}(\varepsilon) &:= \max_{j=2,\dots,6}\Big\{|D_j f(\mathbf{x}(s)) - D_j f(\mathbf{x}(\bar s))|_{X_j}\colon\ s,\bar s\in[0,\alpha],\ |s-\bar s|\le\varepsilon\Big\}, \\
\Omega_{D\tau}(\varepsilon) &:= \max_{j=2,3}\Big\{|D_j\tau(\mathbf{a}(s)) - D_j\tau(\mathbf{a}(\bar s))|_{Y_j}\colon\ s,\bar s\in[0,\alpha],\ |s-\bar s|\le\varepsilon\Big\}, \\
\Omega_{D\rho}(\varepsilon) &:= \max_{j=2,3}\Big\{|D_j\rho(\mathbf{b}(s)) - D_j\rho(\mathbf{b}(\bar s))|_{Z_j}\colon\ s,\bar s\in[0,\alpha],\ |s-\bar s|\le\varepsilon\Big\}
\end{aligned}
$$

for $\varepsilon\in[0,\alpha]$, where $X_2,\dots,X_6$, Y_2,Y_3 and Z_2,Z_3 are defined in (A1) (ii), (A2) (ii) and (A4) (ii), respectively.

Using (A2) (i), (A4) (i), (3.31) and (3.33) we have

$$
\begin{aligned}
|\mathbf{x}(t) - \mathbf{x}(\bar t)|_{\mathbf{X}} &= |t-\bar t| + |x_t - x_{\bar t}|_C + |x(t-\tau(\mathbf{a}(t))) - x(\bar t-\tau(\mathbf{a}(\bar t)))| \\
&\quad + |\dot x(t-\mu(t)) - \dot x(\bar t-\mu(\bar t))| + |\dot x(t-\rho(\mathbf{b}(t))) - \dot x(\bar t-\rho(\mathbf{b}(\bar t)))| \\
&\le |t-\bar t| + N|t-\bar t| + N\Big(|t-\bar t| + |\tau(\mathbf{a}(t)) - \tau(\mathbf{a}(\bar t))|\Big) \\
&\quad + L^*\Big(|t-\bar t| + |\mu(t)-\mu(\bar t)|\Big) + L^*\Big(|t-\bar t| + |\rho(\mathbf{b}(t)) - \rho(\mathbf{b}(\bar t))|\Big) \\
&\le |t-\bar t| + N|t-\bar t| + N\Big(|t-\bar t| + L_2(|t-\bar t| + |x_t - x_{\bar t}|_C)\Big) + L^*(1+L_3)|t-\bar t| \\
&\quad + L^*\Big(|t-\bar t| + L_4(|t-\bar t| + |x_t - x_{\bar t}|_C)\Big) \\
&\le K_5|t-\bar t|, \qquad t,\bar t\in[0,\alpha]
\end{aligned} \tag{4.20}
$$

with and appropriate constant K_5. Similarly,

$$|\mathbf{a}(t)-\mathbf{a}(\bar t)|_{\mathbf{A}} = |\mathbf{b}(t)-\mathbf{b}(\bar t)|_{\mathbf{B}} \le |t-\bar t| + |x_t - x_{\bar t}|_C \le K_6|t-\bar t|, \qquad t,\bar t\in[0,\alpha], \tag{4.21}$$

where $K_6 := 1+N$. Then (4.13), (4.14), (4.16), (4.17), (4.19), (4.20), (4.21) and the definitions of Ω_{Df}, $\Omega_{D\tau}$ and $\Omega_{D\rho}$ yield

$$\begin{aligned}
|\dot z(t)-\dot z(\bar t)| \le{}& \Omega_{Df}(|t-\bar t|)N_1|h|_\Gamma + L_1N_2|t-\bar t||h|_\Gamma + \Omega_{Df}(|t-\bar t|)\Big(NL_2(N_1+1)+N_1\Big)|h|_\Gamma\\
&+ L_1L^*(1+L_2K_6)|t-\bar t|L_2(N_1+1)|h|_\Gamma\\
&+ L_1N\Big(\Omega_{D\tau}(|t-\bar t|)N_1|h|_\Gamma + L_2N_2|t-\bar t||h|_\Gamma + \Omega_{D\tau}(|t-\bar t|)|h|_\Gamma\Big)\\
&+ L_1N_2(1+L_2K_6)|t-\bar t||h|_\Gamma\\
&+ \Omega_{Df}(|t-\bar t|)N_2|h|_\Gamma + L_1|\dot z(t-\mu(t)) - \dot z(\bar t-\mu(\bar t))|\\
&+ \Omega_{Df}(|t-\bar t|)\Big(L^*L_4(N_1+1)|h|_\Gamma + N_2|h|_\Gamma\Big)\\
&+ L_1|\ddot x(t-\rho(\mathbf{b}(t))) - \ddot x(\bar t-\rho(\mathbf{b}(\bar t)))|L_4(N_1+1)|h|_\Gamma\\
&+ L_1L^*\Big(\Omega_{D\rho}(|t-\bar t|)N_1|h|_\Gamma + L_4N_2|t-\bar t||h|_\Gamma + \Omega_{D\rho}(|t-\bar t|)|h|_\Gamma\Big)\\
&+ L_1L^*\Big|\dot z(t-\rho(\mathbf{b}(t))) - \dot z(\bar t-\rho(\mathbf{b}(\bar t)))\Big|\\
&+ \Omega_{Df}(|t-\bar t|)|h|_\Gamma, \quad \text{a.e. } t,\bar t\in[0,\alpha].
\end{aligned} \tag{4.22}$$

We define $\Omega(\varepsilon) := \max\{\Omega_{Df}(\varepsilon), \Omega_{D\tau}(\varepsilon), \Omega_{D\rho}(\varepsilon)\}$. Clearly, Ω is monotone increasing on $[0,\alpha]$. The functions $D_jf(\mathbf{x}(s))$ for $j=2,\dots,6$ are continuous, and therefore uniformly continuous on $[0,\alpha]$. Hence (A1) (ii) and the definition of $\Omega_{Df}(\varepsilon)$ yield $\Omega_{Df}(\varepsilon)\to 0$ as $\varepsilon\to 0+$. Similarly, $\Omega_{D\tau}(\varepsilon)\to 0$ and $\Omega_{D\rho}(\varepsilon)\to 0$, hence $\Omega(\varepsilon)\to 0$ as $\varepsilon\to 0+$. Therefore, it follows from (4.22) that there exist nonnegative constants N_3, N_4, N_5 and N_6 such that (4.18) holds for a.e. $t,\bar t\in[0,\alpha]$. □

The next estimate is the key step of the proof of our main result, Theorem 4.3 below. In its proof we need a weak version of Lipschitz continuity of $\dot z(t,\gamma,h)$, wrt t. In order to obtain such a result we apply estimate (4.18), but we also need more smoothness for the first component of $h=(h^\varphi,h^\xi,h^\lambda,h^\theta)$. We introduce the parameter space $\Gamma_2 := W^{2,\infty}\times\Xi\times\Lambda\times\Theta$, and a norm by $|h|_{\Gamma_2} := |h^\varphi|_{W^{2,\infty}} + |h^\xi|_\Xi + |h^\lambda|_\Lambda + |h^\theta|_\Theta$. We note that $\mathcal{M}\subset\Gamma_2$, and the Γ_2-norm is stronger than the Γ-norm on Γ_2.

Lemma 4.2. *Suppose (A1)–(A4), let $\alpha>0$ and $P\subset\Pi$ be defined by Theorem 3.3, and let $\gamma = (\varphi,\xi,\lambda,\theta)\in\mathcal{M}\cap P$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\lambda, h_k^\theta)\in\Gamma_2$ be such that $\gamma+h_k\in P$ for $k\in\mathbb{N}$ and $|h_k|_{\Gamma_2}\to 0$ as $k\to\infty$. Let $x(t):=x(t,\gamma)$, $x^k(t):=x(t,\gamma+h_k)$, $z^k(t):=z(t,\gamma,h_k)$ and*

$$\begin{aligned}
\mathbf{x}(t) &:= \Big(t, x_t, x(t-\tau(t,x_t,\xi)), \dot x(t-\mu(t)), \dot x(t-\rho(t,x_t,\lambda)), \theta\Big),\\
\mathbf{x}^k(t) &:= \Big(t, x_t^k, x^k(t-\tau(t,x_t^k,\xi+h_k^\xi)), \dot{x^k}(t-\mu(t)), \dot{x^k}(t-\rho(t,x_t^k,\lambda+h_k^\lambda)), \theta+h_k^\theta\Big).
\end{aligned}$$

Then there exist nonnegative constants N_7, N_8, and a sequence of nonnegative functions $g_k\colon [0,\alpha]\to[0,\infty)$ satisfying

$$\lim_{k\to\infty}\frac{1}{|h_k|_{\Gamma_2}}\int_0^\alpha g_k(s)\,ds = 0 \tag{4.23}$$

such that

$$\begin{aligned}&\Big|f(\mathbf{x}^k(s))-f(\mathbf{x}(s))-L(s,\gamma)(z_s^k,h_k^\xi,h_k^\lambda,h_k^\theta)\Big|\\&\qquad\le g_k(s)+N_7\Big|x_s^k-x_s-z_s^k\Big|_C+N_8\Big|\mathcal{P}(\dot{x}_s^k-\dot{x}_s-\dot{z}_s^k)\Big|_{L^\infty_{r_0}},\quad \text{a.e. } s\in[0,\alpha],\ k\in\mathbb{N}.\end{aligned}\tag{4.24}$$

Proof. Let α, M_1, M_2, M_3 and M_4 be defined by Theorem 3.3, M_5, M_6 and M_7 be defined by (4.1), and $L_1,\dots,L_4$ be the corresponding Lipschitz constants from (A1)–(A4), and let $L^*=L^*(\gamma)$ and $N_0=N_0(\gamma)$ be defined by (3.33) and (4.15), respectively. We use the short vector notations

$$\mathbf{a}^k(s):=\Big(s,x_s^k,\xi+h_k^\xi\Big)\quad\text{and}\quad\mathbf{b}^k(s):=\Big(s,x_s^k,\lambda+h_k^\lambda\Big),\qquad k\in\mathbb{N}.$$

The definitions of $L(s,\gamma)$, ω_f, ω_τ and ω_ρ yield for a.e. $s\in[0,\alpha]$

$$\begin{aligned}&f(\mathbf{x}^k(s))-f(\mathbf{x}(s))-L(s,\gamma)(z_s^k,h_k^\xi,h_k^\lambda,h_k^\theta)\\&\quad=\omega_f(\mathbf{x}(s),\mathbf{x}^k(s))+D_2f(\mathbf{x}(s))\Big[x_s^k-x_s-z_s^k\Big]\\&\qquad+D_3f(\mathbf{x}(s))\Big\{x^k(s-\tau(\mathbf{a}^k(s)))-x(s-\tau(\mathbf{a}^k(s)))-z^k(s-\tau(\mathbf{a}^k(s)))\\&\qquad\quad+x(s-\tau(\mathbf{a}^k(s)))-x(s-\tau(\mathbf{a}(s)))+\dot{x}(s-\tau(\mathbf{a}(s)))\Big(\tau(\mathbf{a}^k(s))-\tau(\mathbf{a}(s))\Big)\\&\qquad\quad-\dot{x}(s-\tau(\mathbf{a}(s)))\omega_\tau(\mathbf{a}(s),\mathbf{a}^k(s))-\dot{x}(s-\tau(\mathbf{a}(s)))D_2\tau(s,x_s,\xi)\Big[x_s^k-x_s-z_s^k\Big]\\&\qquad\quad+z^k(s-\tau(\mathbf{a}^k(s)))-z^k(s-\tau(\mathbf{a}(s)))\Big\}\\&\qquad+D_4f(\mathbf{x}(s))\Big\{\dot{x}^k(s-\mu(s))-\dot{x}(s-\mu(s))-\dot{z}^k(s-\mu(s))\Big\}\\&\qquad+D_5f(\mathbf{x}(s))\Big\{\dot{x}^k(s-\rho(\mathbf{b}^k(s)))-\dot{x}(s-\rho(\mathbf{b}^k(s)))-\dot{z}^k(s-\rho(\mathbf{b}^k(s)))\\&\qquad\quad+\dot{x}(s-\rho(\mathbf{b}^k(s)))-\dot{x}(s-\rho(\mathbf{b}(s)))+\ddot{x}(s-\rho(\mathbf{b}(s)))\Big(\rho(\mathbf{b}^k(s))-\rho(\mathbf{b}(s))\Big)\\&\qquad\quad-\ddot{x}(s-\rho(\mathbf{b}(s)))\omega_\rho(\mathbf{b}(s),\mathbf{b}^k(s))-\ddot{x}(s-\rho(\mathbf{b}(s)))D_2\rho(s,x_s,\lambda)\Big[x_s^k-x_s-z_s^k\Big]\\&\qquad\quad+\dot{z}^k(s-\rho(\mathbf{b}^k(s)))-\dot{z}^k(s-\rho(\mathbf{b}(s)))\Big\}.\end{aligned}\tag{4.25}$$

Using (3.32) and estimates similar to (3.44) and (3.45), we have that

$$\begin{aligned}|\mathbf{x}^k(s)-\mathbf{x}(s)|_{\mathbf{X}}&=|x_s^k-x_s|_C+|x^k(s-\tau(\mathbf{a}^k(s)))-x(s-\tau(\mathbf{a}(s)))|\\&\quad+|\dot{x}^k(s-\mu(s))-\dot{x}(s-\mu(s))|+|\dot{x}^k(s-\rho(\mathbf{b}^k(s)))-\dot{x}(s-\rho(\mathbf{b}(s)))|+|h_k^\theta|_\Theta\\&\le 2|x_s^k-x_s|_C+L_2N(|x_s^k-x_s|_C+|h_k^\xi|_\Xi)\\&\quad+2|x_s^k-x_s|_{W^{1,\infty}}+L_4L^*(|x_s^k-x_s|_C+|h_k^\lambda|_\Lambda)+|h_k^\theta|_\Theta\\&\le K_7|h_k|_\Gamma,\quad \text{a.e. } s\in[0,\alpha],\quad k\in\mathbb{N},\end{aligned}\tag{4.26}$$

with some constant K_7. Similarly, (3.32) yields

$$|\mathbf{a}^k(s)-\mathbf{a}(s)|_{\mathbf{A}}=|x_s^k-x_s|_C+|h_k^\xi|_\Xi\le(L+1)|h_k|_\Gamma,\qquad s\in[0,\alpha],\quad k\in\mathbb{N},\tag{4.27}$$

and

$$|\mathbf{b}^k(s)-\mathbf{b}(s)|_{\mathbf{B}}=|x_s^k-x_s|_C+|h_k^\lambda|_\Lambda\le(L+1)|h_k|_\Gamma,\qquad s\in[0,\alpha],\quad k\in\mathbb{N}.\tag{4.28}$$

Using (A2) (ii) and (3.32), we get

$$|\tau(\mathbf{a}^k(s)) - \tau(\mathbf{a}(s))| \le L_2\Big(|x_s^k - x_s|_C + |h_k^\xi|_\Xi\Big) \le L_2(L+1)|h_k|_\Gamma, \quad s \in [0,\alpha],\ k \in \mathbb{N}, \tag{4.29}$$

and therefore (4.8) and (4.17) yield

$$\begin{aligned}
&|x(s-\tau(\mathbf{a}^k(s))) - x(s-\tau(\mathbf{a}(s))) + \dot{x}(s-\tau(\mathbf{a}(s)))(\tau(\mathbf{a}^k(s)) - \tau(\mathbf{a}(s)))| \\
&\qquad \le \Omega_{\dot{x}}\Big(L_2(L+1)|h_k|_\Gamma\Big) L_2(L+1)|h_k|_\Gamma,
\end{aligned} \tag{4.30}$$

and

$$|z^k(s-\tau(\mathbf{a}^k(s))) - z^k(s-\tau(\mathbf{a}(s)))| \le N_2|\tau(\mathbf{a}^k(s)) - \tau(\mathbf{a}(s))||h_k|_\Gamma \le N_2L_2(L+1)|h_k|_\Gamma^2 \tag{4.31}$$

for $s \in [0,\alpha]$ and $k \in \mathbb{N}$. Therefore, combining the above estimates and (3.31), (3.33), (4.3), (4.4), (4.6), (4.13), (4.14), (4.26)–(4.31), we get from (4.25)

$$\begin{aligned}
&\Big|f(\mathbf{x}^k(s)) - f(\mathbf{x}(s)) - L(s,\gamma)(z_s^k, h_k^\xi, h_k^\lambda, h_k^\theta)\Big| \\
&\quad \le |\omega_f(\mathbf{x}(s), \mathbf{x}^k(s))| + L_1\Big\{|x_s^k - x_s - z_s^k|_C \\
&\qquad + \Big|x^k(s-\tau(\mathbf{a}^k(s))) - x(s-\tau(\mathbf{a}^k(s))) - z^k(s-\tau(\mathbf{a}^k(s)))\Big| \\
&\qquad + \Omega_{\dot{x}}\Big(L_2(L+1)|h_k|_\Gamma\Big)L_2(L+1)|h_k|_\Gamma + N|\omega_\tau(\mathbf{a}(s), \mathbf{a}^k(s))| \\
&\qquad + NL_2|x_s^k - x_s - z_s^k|_C + N_2L_2(L+1)|h_k|_\Gamma^2 \\
&\qquad + \Big|\dot{x}^k(s-\mu(s)) - \dot{x}(s-\mu(s)) - \dot{z}^k(s-\mu(s))\Big| \\
&\qquad + \Big|\dot{x}^k(s-\rho(\mathbf{b}^k(s))) - \dot{x}(s-\rho(\mathbf{b}^k(s))) - \dot{z}^k(s-\rho(\mathbf{b}^k(s)))\Big| \\
&\qquad + \Big|\dot{x}(s-\rho(\mathbf{b}^k(s))) - \dot{x}(s-\rho(\mathbf{b}(s))) + \ddot{x}(s-\rho(\mathbf{b}(s)))(\rho(\mathbf{b}^k(s)) - \rho(\mathbf{b}(s)))\Big| \\
&\qquad + L^*|\omega_\rho(\mathbf{b}(s), \mathbf{b}^k(s))| + L^*L_4|x_s^k - x_s - z_s^k|_C \\
&\qquad + \Big|\dot{z}^k(s-\rho(\mathbf{b}^k(s))) - \dot{z}^k(s-\rho(\mathbf{b}(s)))\Big|\Big\} \\
&\quad \le C_k|h_k|_\Gamma + g_{k,1}(s) + g_{k,2}(s) + g_{k,3}(s) \\
&\qquad + N_7\Big|x_s^k - x_s - z_s^k\Big|_C + N_8\Big|\mathcal{P}(\dot{x}_s^k - \dot{x}_s - \dot{z}_s^k)\Big|_{L^\infty_{r_0}}
\end{aligned} \tag{4.32}$$

for a.e. $s \in [0,\alpha]$, where $C_k := L_1N_2L_2(L+1)|h_k|_\Gamma + L_1L_2(L+1)\Omega_{\dot{x}}\Big(L_2(L+1)|h_k|_\Gamma\Big)$, $N_7 := L_1(2 + NL_2 + L^*L_4)$, $N_8 := 2L_1$,

$$\begin{aligned}
g_{k,1}(s) &:= |\omega_f(\mathbf{x}(s), \mathbf{x}^k(s))| + L_1N|\omega_\tau(\mathbf{a}(s), \mathbf{a}^k(s))| + L_1L^*|\omega_\rho(\mathbf{b}(s), \mathbf{b}^k(s))|, \\
g_{k,2}(s) &:= \Big|\dot{x}(s-\rho(\mathbf{b}^k(s))) - \dot{x}(s-\rho(\mathbf{b}(s))) + \ddot{x}(s-\rho(\mathbf{b}(s)))(\rho(\mathbf{b}^k(s)) - \rho(\mathbf{b}(s)))\Big|,
\end{aligned}$$

and

$$g_{k,3}(s) := L_1\Big|\dot{z}^k(s-\rho(\mathbf{b}^k(s))) - \dot{z}^k(s-\rho(\mathbf{b}(s)))\Big|.$$

For $s \in [0,\alpha]$ such that $\mathbf{x}(s) \ne \mathbf{x}^k(s)$, $\mathbf{a}(s) \ne \mathbf{a}^k(s)$ and $\mathbf{b}(s) \ne \mathbf{b}^k(s)$ using relations

$|h_k|_\Gamma \le |h_k|_{\Gamma_2}$, (4.26), (4.27) and (4.28), we have

$$\begin{aligned}
\frac{|g_{k,1}(s)|}{|h_k|_{\Gamma_2}} &\le \frac{|\omega_f(\mathbf{x}(s),\mathbf{x}^k(s))|}{|\mathbf{x}^k(s)-\mathbf{x}(s)|_{\mathbf{X}}}\frac{|\mathbf{x}^k(s)-\mathbf{x}(s)|_{\mathbf{X}}}{|h_k|_{\Gamma_2}} + L_1N\frac{|\omega_\tau(\mathbf{a}(s),\mathbf{a}^k(s))|}{|\mathbf{a}^k(s)-\mathbf{a}(s)|_{\mathbf{A}}}\frac{|\mathbf{a}^k(s)-\mathbf{a}(s)|_{\mathbf{A}}}{|h_k|_{\Gamma_2}}\\
&\quad + L_1L^*\frac{|\omega_\rho(\mathbf{b}(s),\mathbf{b}^k(s))|}{|\mathbf{b}^k(s)-\mathbf{b}(s)|_{\mathbf{B}}}\frac{|\mathbf{b}^k(s)-\mathbf{b}(s)|_{\mathbf{B}}}{|h_k|_{\Gamma_2}}\\
&\le K_7\frac{|\omega_f(\mathbf{x}(s),\mathbf{x}^k(s))|}{|\mathbf{x}^k(s)-\mathbf{x}(s)|_{\mathbf{X}}} + L_1N(L+1)\frac{|\omega_\tau(\mathbf{a}(s),\mathbf{a}^k(s))|}{|\mathbf{a}^k(s)-\mathbf{a}(s)|_{\mathbf{A}}} + L_1L^*(L+1)\frac{|\omega_\rho(\mathbf{b}(s),\mathbf{b}^k(s))|}{|\mathbf{b}^k(s)-\mathbf{b}(s)|_{\mathbf{B}}}.
\end{aligned}$$

Then (4.2), (4.4), (4.6), (4.26), (4.27) and (4.28) imply $\frac{|g_{k,1}(s)|}{|h_k|_{\Gamma_2}} \to 0$ as $k\to\infty$ for $s\in[0,\alpha]$. The continuity of the partial derivatives yield that the functions $s\mapsto |D_jf(\mathbf{x}(s))|_{X_j}$ $(j=2,\dots,5)$, $s\mapsto |D_j\tau(\mathbf{a}(s))|_{Y_j}$ $(j=2,3)$ and $s\mapsto |D_j\rho(\mathbf{b}(s))|_{Z_j}$ $(j=2,3)$ are bounded on $[0,\alpha]$. Therefore, (4.3), (4.5) and (4.7) show that $\frac{|g_{k,1}(s)|}{|h_k|_{\Gamma_2}}$ is bounded on $[0,\alpha]$ with a constant independent of k. Therefore, the Lebesgue's Dominated Convergence theorem yields

$$\lim_{k\to\infty}\frac{1}{|h_k|_{\Gamma_2}}\int_0^\alpha g_{k,1}(s)\,ds = 0. \tag{4.33}$$

Assumption (A4) (ii), (4.28) and $|h_k|_\Gamma \le |h_k|_{\Gamma_2}$ imply

$$|\rho(\mathbf{b}^k(s))-\rho(\mathbf{b}(s))| \le K_8|h_k|_{\Gamma_2}, \qquad s\in[0,\alpha],\ k\in\mathbb{N}. \tag{4.34}$$

with $K_8 := L_4(L+1)$. Then Lemma 2.4 with $y=\dot{x}$, $p(s)=s-\rho(\mathbf{b}(s))$, $p^k(s)=s-\rho(\mathbf{b}^k(s))$ and $\omega_k = K_8|h_k|_{\Gamma_2}$ gives

$$\lim_{k\to\infty}\frac{1}{|h_k|_{\Gamma_2}}\int_0^\alpha g_{k,2}(s)\,ds = 0. \tag{4.35}$$

In the remaining part of the proof we show

$$\lim_{k\to\infty}\frac{1}{|h_k|_{\Gamma_2}}\int_0^\alpha g_{k,3}(s)\,ds = 0. \tag{4.36}$$

We introduce the functions

$$u(t) := \begin{cases} t-\mu(t), & t\in[0,\alpha]\\ -\mu(0), & t<0,\end{cases} \qquad v(t) := \begin{cases} t-\rho(\mathbf{b}(t)), & t\in[0,\alpha],\\ -\rho(\mathbf{b}(0)), & t<0,\end{cases}$$

and for $k\in\mathbb{N}$

$$v_k(t) := \begin{cases} t-\rho(\mathbf{b}^k(t)), & t\in[0,\alpha],\\ -\rho(\mathbf{b}^k(0)), & t<0.\end{cases}$$

Note that u, v and v_k are strictly monotone increasing functions on $[0,\alpha]$. Assumptions (A3) and (A4) (i) imply

$$-r\le u(t)\le t-r_0, \qquad -r\le v(t)\le t-r_0 \quad\text{and}\quad -r\le v_k(t)\le t-r_0, \qquad t\in[0,\alpha],\ k\in\mathbb{N}. \tag{4.37}$$

Define the constant $K_9 := \max\{1+L_3, 1+L_4K_6\}$. Then it is easy to check that

$$|u(t)-u(\bar t)|\le K_9|t-\bar t| \quad\text{and}\quad |v(t)-v(\bar t)|\le K_9|t-\bar t|, \qquad t,\bar t\in[-r,\alpha]. \tag{4.38}$$

Also, (A3) and (3.29) yield

$$\operatorname*{ess\,inf}_{t\in[0,\alpha]} \dot u(t) > 0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_{t\in[0,\alpha]} \dot v(t) > 0,$$

hence the constant

$$m^* := \min\Big\{\operatorname*{ess\,inf}_{t\in[0,\alpha]} \dot u(t), \operatorname*{ess\,inf}_{t\in[0,\alpha]} \dot v(t)\Big\} \tag{4.39}$$

is positive. Then the Mean Value Theorem implies

$$|u(t) - u(\bar t)| \ge m^*|t - \bar t| \quad \text{and} \quad |v(t) - v(\bar t)| \ge m^*|t - \bar t|, \qquad t, \bar t \in [0, \alpha]. \tag{4.40}$$

For $k \in \mathbb{N}$ we define $\Delta_k := K_8|h_k|_{\Gamma_2}$. It follows from (4.37) that $v(0) < 0$. We consider the cases $v(\alpha) \le 0$ and $v(\alpha) > 0$ separately.

(1) First suppose $v(\alpha) \le 0$.

We have $-r_0 \le v(0) < v(\alpha)$. So we can suppose that k is large enough that $v(\alpha) - \Delta_k > v(0)$. Then $\eta^*_{k,0} := v^{-1}(v(\alpha) - \Delta_k) \in (0, \alpha)$ is well-defined. For $s \in [0, \alpha]$ the monotonicity of v yields $v(s) \le v(\alpha) \le 0$. Moreover, since $|v_k(s) - v(s)| \le \Delta_k$ by (4.34), we have $v_k(s) = v(s) + v_k(s) - v(s) \le v(\eta^*_{k,0}) + \Delta_k = v(\alpha) - \Delta_k + \Delta_k \le 0$ for $s \in [0, \eta^*_{k,0}]$. Since $v(\alpha) - v(\eta^*_{k,0}) = \Delta_k = K_8|h_k|_{\Gamma_2}$, (4.40) yields

$$\alpha - \eta^*_{k,0} \le \frac{K_8}{m^*}|h_k|_{\Gamma_2}. \tag{4.41}$$

Then the Mean Value Theorem, (4.17), (4.34) and (4.41) imply

$$\begin{aligned}
\int_0^\alpha g_{k,3}(s)\,ds &= \int_0^{\eta^*_{k,0}} g_{k,3}(s)\,ds + \int_{\eta^*_{k,0}}^\alpha g_{k,3}(s)\,ds \\
&\le L_1 \int_0^{\eta^*_{k,0}} |\dot h^\varphi_k(v_k(s)) - \dot h^\varphi_k(v(s))|\,ds + L_1\int_{\eta^*_{k,0}}^\alpha \Big(|\dot z^k(v_k(s))| + |\dot z^k(v(s))|\Big)\,ds \\
&\le L_1 \int_0^{\eta^*_{k,0}} |v_k(s) - v(s)||h_k|_{\Gamma_2}\,ds + \frac{2L_1N_2K_8}{m^*}|h_k|^2_{\Gamma_2} \\
&\le \Big(L_1K_8\alpha + \frac{2L_1N_2K_8}{m^*}\Big)|h_k|^2_{\Gamma_2}.
\end{aligned}$$

This proves the limit relation (4.36) in case (1).

(2) Next suppose $v(\alpha) > 0$.

Since $v(0) < 0 < v(\alpha)$, it follows that $v^{-1}(0)$ is well-defined, and $0 < v^{-1}(0) < \alpha$. We suppose that k is large enough that $v(0) < -\Delta_k < 0 < \Delta_k < v(\alpha)$. Then the constants $\eta'_{k,0} := v^{-1}(-\Delta_k)$ and $\eta''_{k,0} := v^{-1}(\Delta_k)$ are well-defined, and $0 < \eta'_{k,0} < v^{-1}(0) < \eta''_{k,0} < \alpha$. Then it is easy to check that $v(s), v_k(s) \le 0$ for $s \in [0, \eta'_{k,0}]$. Since $v(\eta''_{k,0}) - v(\eta'_{k,0}) = 2\Delta_k = 2K_8|h_k|_{\Gamma_2}$, it follows

$$\eta''_{k,0} - \eta'_{k,0} \le \frac{2K_8}{m^*}|h_k|_{\Gamma_2}. \tag{4.42}$$

Therefore, using (4.34) and (4.42), we get

$$\begin{aligned}
\int_0^\alpha g_{k,3}(s)\,ds &= \int_0^{\eta'_{k,0}} g_{k,3}(s)\,ds + \int_{\eta'_{k,0}}^{\eta''_{k,0}} g_{k,3}(s)\,ds + \int_{\eta''_{k,0}}^\alpha g_{k,3}(s)\,ds \\
&\le \Big(L_1K_8\alpha + \frac{4L_1N_2K_8}{m^*}\Big)|h_k|^2_{\Gamma_2} + L_1\int_{\eta''_{k,0}}^\alpha |\dot z^k(v_k(s)) - \dot z^k(v(s))|\,ds.
\end{aligned} \tag{4.43}$$

On the interval $[\eta''_{k,0}, \alpha]$ both v and v_k take positive values. Therefore, we can use (4.18) to estimate the last integral of (4.43).

$$\begin{aligned}
\int_0^\alpha g_{k,3}(s)\,ds \le{}& \Big(L_1K_8\alpha + \frac{4L_1N_2K_8}{m^*}\Big)|h_k|^2_{\Gamma_2} \\
&+ L_1\int_{\eta''_{k,0}}^{\alpha}\Big(N_3|v_k(s)-v(s)||h_k|_\Gamma + N_4\Omega\Big(|v_k(s)-v(s)|\Big)|h_k|_\Gamma\Big)\,ds \\
&+ L_1\int_{\eta''_{k,0}}^{\alpha} N_5|\ddot{x}(v(v_k(s))) - \ddot{x}(v(v(s)))||h_k|_\Gamma\Big)\,ds \\
&+ L_1\int_{\eta''_{k,0}}^{\alpha} N_6\Big(|\dot{z}^k(u(v_k(s))) - \dot{z}^k(u(v(s)))| + |\dot{z}^k(v(v_k(s))) - \dot{z}^k(v(v(s)))|\Big)\,ds \\
\le{}& \Big(L_1K_8\alpha + \frac{4L_1N_2K_8}{m^*} + L_1N_3K_8\alpha\Big)|h_k|^2_{\Gamma_2} + L_1N_4\alpha\Omega\Big(K_8|h_k|_{\Gamma_2}\Big)|h_k|_{\Gamma_2} \\
&+ L_1N_5|h_k|_{\Gamma_2}\int_0^\alpha |\ddot{x}(v(v_k(s))) - \ddot{x}(v(v(s)))|\,ds \\
&+ L_1N_6\int_{\eta''_{k,0}}^{\alpha} |\dot{z}^k(u(v_k(s))) - \dot{z}^k(u(v(s)))|\,ds \\
&+ L_1N_6\int_{\eta''_{k,0}}^{\alpha} |\dot{z}^k(v(v_k(s))) - \dot{z}^k(v(v(s)))|\,ds.
\end{aligned}\tag{4.44}$$

Clearly, the first term of the right-hand side of last inequality is of order $o(|h_k|_{\Gamma_2})$. Since $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0+$, the second term of the right-hand side of last inequality is also of order $o(|h_k|_{\Gamma_2})$. We have $|v(v_k(s)) - v(v(s))| \le K_9|v_k(s) - v(s)| \le K_9K_8|h_k|_{\Gamma_2}$, $s \in [0, \alpha]$. Therefore Lemma 2.4 with $y = \ddot{x}$, $p(s) = v(v(s))$, $p^k(s) = v(v_k(s))$ and $\omega_k = K_9K_8|h_k|_{\Gamma_2}$ yields that the third term is also of order $o(|h_k|_{\Gamma_2})$. We need to show that the last two integrals of (4.44) are also of order $o(|h_k|_{\Gamma_2})$, i.e.,

$$\lim_{k\to\infty}\frac{1}{|h_k|_{\Gamma_2}}\int_{\eta''_{k,0}}^{\alpha} |\dot{z}^k(w(v_k(s))) - \dot{z}^k(w(v(s)))|\,ds = 0, \qquad \text{for } w \in \{u, v\}. \tag{4.45}$$

Next the function w can be substituted with the functions u or v. Note that both u and v are strictly monotone increasing on $[0, \alpha]$, both are Lipschitz continuous with the Lipschitz constant K_9, and the essential infimum of $\dot{u}$ and $\dot{v}$ are bounded below by the positive constant m^*. So the estimates we present below work for both functions. Next we show relation (4.45) for $w = u$ and $w = v$. We consider two subcases (similarly to cases (1) and (2)): either $w(v(\alpha)) \le 0$ or $w(v(\alpha)) > 0$.

(2.1) Suppose $v(\alpha) > 0$ and $w(v(\alpha)) \le 0$.

We can suppose that k is large enough that $\Delta_k < v(\alpha) - \Delta_k$. We recall that $0 < \eta''_{k,0} < \alpha$, and $v(s) > 0$ for $s \in [\eta''_{k,0}, \alpha]$. Then $\eta^*_{k,1} := v^{-1}(v(\alpha) - \Delta_k) \in (\eta''_{k,0}, \alpha)$ is well-defined. The monotonicity of w and v for nonnegative arguments yields $w(v(s)) \le w(v(\alpha)) \le 0$ for $s \in [\eta''_{k,0}, \alpha]$. Moreover, since $|v_k(s) - v(s)| \le \Delta_k$ by (4.34), we have $w(v_k(s)) = w(v(s) + v_k(s) - v(s)) \le w(v(\eta^*_{k,1}) + \Delta_k) = w(v(\alpha) - \Delta_k + \Delta_k) \le 0$ for $s \in [\eta''_{k,0}, \eta^*_{k,1}]$. Since

$$(m^*)^2(\alpha - \eta^*_{k,1}) \le m^*(v(\alpha) - v(\eta^*_{k,1})) \le w(v(\alpha)) - w(v(\eta^*_{k,1})) \le K_9(v(\alpha) - v(\eta^*_{k,1})) = K_9\Delta_k,$$

it follows

$$\alpha - \eta^*_{k,1} \le \frac{K_9K_8}{(m^*)^2}|h_k|_{\Gamma_2}. \tag{4.46}$$

Then the Mean Value Theorem, (4.17) and (4.46) yield

$$
\begin{aligned}
&\int_{\eta''_{k,0}}^{\alpha} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds \\
&\quad = \int_{\eta''_{k,0}}^{\eta^*_{k,1}} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds + \int_{\eta^*_{k,1}}^{\alpha} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds \\
&\quad \le \int_{\eta''_{k,0}}^{\eta^*_{k,1}} |\dot{h^\varphi_k}(w(v_k(s))) - \dot{h^\varphi_k}(w(v(s)))|\,ds + \int_{\eta^*_{k,1}}^{\alpha} \Big(|\dot{z^k}(w(v_k(s)))| + |\dot{z^k}(w(v(s)))|\Big)\,ds \\
&\quad \le \left(K_9K_8\alpha + \frac{2N_2K_9K_8}{(m^*)^2}\right)|h_k|^2_{\Gamma_2}. \qquad (4.47)
\end{aligned}
$$

This implies (4.45) in this subcase.

(2.2) Suppose $v(\alpha) > 0$ and $w(v(\alpha)) > 0$.

Recall from case (2) that k is such that $v(0) < -\Delta_k < \Delta_k < v(\alpha)$ and $\eta''_{k,0} := v^{-1}(\Delta_k)$. Since in this case $w(0) < 0 < w(v(\alpha))$, and so $0 < w^{-1}(0) < v(\alpha)$, we can further assume that k is large enough that $0 < \Delta_k < w^{-1}(0) - \Delta_k < w^{-1}(0) + \Delta_k < v(\alpha)$. Let $\eta'_{k,1} := v^{-1}(w^{-1}(0) - \Delta_k)$ and $\eta''_{k,1} := v^{-1}(w^{-1}(0) + \Delta_k)$. Note that $\eta'_{k,1}$ and $\eta''_{k,1}$ depend on the selection of the function w, but this dependence is omitted in the notation. Then it is easy to check that $\eta''_{k,0} < \eta'_{k,1} < v^{-1}(w^{-1}(0)) < \eta''_{k,1} < \alpha$, and $w(v(s)), w(v_k(s)) \le 0$ for $s \in [\eta''_{k,0}, \eta'_{k,1}]$ and $w(v(s)), w(v_k(s)) \ge 0$ for $s \in [\eta''_{k,1}, \alpha]$. Since $w(v(\eta''_{k,1})) - w(v(\eta'_{k,1})) \le 2K_9\Delta_k = 2K_9K_8|h_k|_{\Gamma_2}$, it follows

$$
\eta''_{k,1} - \eta'_{k,1} \le \frac{2K_9K_8}{(m^*)^2}|h_k|_{\Gamma_2}. \qquad (4.48)
$$

Therefore, using (4.18), (4.34) and (4.48), we get

$$
\begin{aligned}
&\int_{\eta''_{k,0}}^{\alpha} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds \\
&\quad = \int_{\eta''_{k,0}}^{\eta'_{k,1}} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds + \int_{\eta'_{k,1}}^{\eta''_{k,1}} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds \\
&\qquad + \int_{\eta''_{k,1}}^{\alpha} |\dot{z^k}(w(v_k(s))) - \dot{z^k}(w(v(s)))|\,ds \\
&\quad \le \left(K_9K_8\alpha + \frac{4N_2K_9K_8}{(m^*)^2}\right)|h_k|^2_{\Gamma_2} + N_3\int_{\eta''_{k,1}}^{\alpha} |w(v_k(s)) - w(v(s))||h_k|_{\Gamma_2}\,ds \\
&\qquad + N_4\int_{\eta''_{k,1}}^{\alpha} \Omega\Big(|w(v_k(s)) - w(v(s))|\Big)|h_k|_\Gamma\,ds \\
&\qquad + N_5\int_{\eta''_{k,1}}^{\alpha} |\ddot{x}(v(w(v_k(s)))) - \ddot{x}(v(w(v(s))))||h_k|_\Gamma\,ds \\
&\qquad + N_6\int_{\eta''_{k,1}}^{\alpha} |\dot{z^k}(u(w(v_k(s)))) - \dot{z^k}(u(w(v(s))))|\,ds \\
&\qquad + N_6\int_{\eta''_{k,1}}^{\alpha} |\dot{z^k}(v(w(v_k(s)))) - \dot{z^k}(v(w(v(s))))|\,ds \\
&\quad \le \left(K_9K_8\alpha + \frac{4N_2K_9K_8}{(m^*)^2} + N_3K_9K_8\alpha\right)|h_k|^2_{\Gamma_2} + N_4\alpha\Omega\Big(K_9K_8|h_k|_{\Gamma_2}\Big)|h_k|_{\Gamma_2} \\
&\qquad + N_5|h_k|_{\Gamma_2}\int_0^{\alpha} |\ddot{x}(v(w(v_k(s)))) - \ddot{x}(v(w(v(s))))|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ N_6 \int_{\eta''_{k,1}}^{\alpha} |\dot{z}^k(u(w(v_k(s)))) - \dot{z}^k(u(w(v(s))))|\, ds \\
&+ N_6 \int_{\eta''_{k,1}}^{\alpha} |\dot{z}^k(v(w(v_k(s)))) - \dot{z}^k(v(w(v(s))))|\, ds. \qquad (4.49)
\end{aligned}
$$

We have $|v(w(v_k(s))) - v(w(v(s)))| \leq K_9^2 K_8 |h_k|_{\Gamma_2}$, $s \in [0, \alpha]$. Hence Lemma 2.4 with $y = \ddot{x}$, $p(s) = v(w(v(s)))$, $p^k(s) = v(w(v_k(s)))$ and $\omega_k = K_9^2 K_8 |h_k|_{\Gamma_2}$ yields that the second term is of order $o(|h_k|_{\Gamma_2})$. Therefore, we need to show that

$$
\int_{\eta''_{k,1}}^{\alpha} |\dot{z}^k(w_2(w_1(v_k(s)))) - \dot{z}^k(w_2(w_1(v(s))))|\, ds = o(|h_k|_{\Gamma_2}), \quad \text{for } w_1, w_2 \in \{u, v\}. \qquad (4.50)
$$

As before, we consider two cases.

(2.2.1) $v(\alpha) > 0$, $w_1(v(\alpha)) > 0$ and $w_2(w_1(v(\alpha))) \leq 0$

In this case we have $w_2(w_1(v(s))) \leq 0$ for $s \in [\eta''_{k,1}, \alpha]$, and, similarly to the estimates used in cases (1) and (2.1), it is easy to see that (4.50) holds.

(2.2.2) $v(\alpha) > 0$, $w_1(v(\alpha)) > 0$ and $w_2(w_1(v(\alpha))) > 0$.

Recall that $\eta''_{k,0} = v^{-1}(\Delta_k)$ and $\eta''_{k,1} = v^{-1}(w_1^{-1}(0) + \Delta_k)$, and $\eta''_{k,0} < \eta''_{k,1} < \alpha$. From the assumption it follows $w_1^{-1}(0) < w_1^{-1}(w_2^{-1}(0)) < v(\alpha)$. We assume that k is large enough that $v(0) < -\Delta_k < \Delta_k < w_1^{-1}(0) - \Delta_k < w_1^{-1}(0) + \Delta_k < w_1^{-1}(w_2^{-1}(0)) - \Delta_k < w_1^{-1}(w_2^{-1}(0)) + \Delta_k < v(\alpha)$. Then we define $\eta'_{k,2} := v^{-1}(w_1^{-1}(w_2^{-1}(0)) - \Delta_k)$ and $\eta''_{k,2} := v^{-1}(w_1^{-1}(w_2^{-1}(0)) + \Delta_k)$. The above inequalities yield $\eta''_{k,0} < \eta''_{k,1} < \eta'_{k,2} < \eta''_{k,2} < \alpha$. Then, similarly to (4.49), we can obtain an estimate of (4.50) where on the righ-hand side we have integrals of the form

$$
\int_{\eta''_{k,2}}^{\alpha} |\ddot{x}(v(w_2(w_1(v_k(s))))) - \ddot{x}(v(w_2(w_1(v(s)))))||h_k|_{\Gamma_2}\, ds \qquad (4.51)
$$

and

$$
\int_{\eta''_{k,2}}^{\alpha} |\dot{z}^k(w_3(w_2(w_1(v_k(s))))) - \dot{z}^k(w_3(w_2(w_1(v(s)))))|\, ds \qquad (4.52)
$$

where $w_3, w_2, w_1 \in \{u, v\}$. Lemma 2.4 yields that the integral in (4.51) is of order $o(|h_k|_{\Gamma_2})$. If $w_3(w_2(w_1(v(\alpha)))) \leq 0$, we get an explicit estimate of the integral in (4.52), and it is of order $o(|h_k|_{\Gamma_2})$. But if $w_3(w_2(w_1(v(\alpha)))) > 0$, we can continue the recursive estimating described above. Clearly, this recursion will end, since

$$
w_m(\cdots(w_3(w_2(w_1(v(\alpha)))))\cdots) \leq \alpha - (m+1)r_0 < 0, \qquad w_1, w_2, \ldots, w_m \in \{u, v\}.
$$

Hence, after no more than m number of iterations, the above described iterative procedure ends. Note, that in each step of the second case, we may double the number of integrals similar to (4.52) used in the estimate. But, since we may have only finitely many terms, and each terms have order $o(|h_k|_{\Gamma_2})$, this completes the proof of (4.36). Hence $C_k \to 0$ as $k \to \infty$, and relations (4.32), (4.33), (4.35) and (4.36) prove the statement of the lemma with $g_k(s) := C_k |h_k|_\Gamma + g_{k,1}(s) + g_{k,2}(s) + g_{k,3}(s)$. □

Next we prove differentiability of the functions $x(t, \gamma)$ and $x_t(\cdot, \gamma)$ wrt γ using the Γ_2-norm on the parameter set. We denote this differentiation by $D_2 x$. We note that the differentiability is proved at a parameter value γ which belongs to $\mathcal{M} \cap P$, i.e., where the compatibility condition is satisfied. But in computing the partial derivative we use solutions corresponding to parameter values $\gamma + h$, $h \in \Gamma_2$ with small norm, hence these solutions $x(\cdot, \gamma + h)$, in general, do not satisfy the compatibility condition.

Theorem 4.3. *Assume (A1)–(A4). Let $\bar{\gamma} \in \mathcal{M}$, and let $\delta > 0$, $P := \mathcal{B}_\Gamma(\bar{\gamma};\delta)$, and $\alpha > 0$ be defined by Theorem 3.3, and $x(t;\gamma)$ be the solution of the IVP (3.1)–(3.2) on $[-r,\alpha]$ for $\gamma \in P$. Then the map*

$$\Gamma_2 \supset P \cap \Gamma_2 \to \mathbb{R}^n, \qquad \gamma \mapsto x(t,\gamma)$$

is differentiable at every $\gamma \in \mathcal{M} \cap P$ and $t \in [0,\alpha]$, and

$$D_2 x(t,\gamma)h = z(t,\gamma,h), \qquad h \in \Gamma_2, \quad t \in [0,\alpha], \quad \gamma \in \mathcal{M} \cap P,$$

where z is the solution of the IVP (4.9)–(4.10).

Moreover, the map

$$\Gamma_2 \supset P \cap \Gamma_2 \to C, \qquad \gamma \mapsto x(\cdot,\gamma)_t$$

is also differentiable at $\gamma \in \mathcal{M} \cap P$ and $t \in [0,\alpha]$, and its derivative is given by

$$D_2 x_t(\cdot,\gamma)h = z_t(\cdot,\gamma,h), \qquad h \in \Gamma_2, \quad t \in [0,\alpha], \quad \gamma \in \mathcal{M} \cap P.$$

Proof. Let $\gamma \in \mathcal{M} \cap P$ be fixed, and let $h_k = (h_k^\varphi, h_k^\xi, h_k^\lambda, h_k^\theta) \in \Gamma_2$ be a sequence such that $|h_k|_{\Gamma_2} \to 0$ as $k \to \infty$ and $\gamma + h_k \in P$ for $k \in \mathbb{N}$. For brevity, we use the notations $x(t) := x(t,\gamma)$, $x^k(t) := x(t,\gamma + h_k)$, $z^k(t) := z(t,\gamma,h_k)$, and

$$\begin{aligned}
\mathbf{x}(t) &= \Big(t, x_t, x(t - \tau(t,x_t,\xi)), \dot{x}(t-\mu(t)), \dot{x}(t - \rho(t,x_t,\lambda)), \theta\Big) \\
\mathbf{x}^k(t) &= \Big(t, x_t^k, x^k(t - \tau(t,x_t^k,\xi + h_k^\xi)), \dot{x}^k(t-\mu(t)), \dot{x}^k(t - \rho(t,x_t^k,\lambda + h_k^\lambda)), \theta + h_k^\theta\Big).
\end{aligned}$$

Define the function $w^k(t) := x^k(t) - x(t) - z^k(t)$. Equations (3.1) and (4.12) imply

$$\dot{w}^k(t) = f(\mathbf{x}^k(t)) - f(\mathbf{x}(t)) - L(t,\gamma)(z_t^k, h_k^\xi, h_k^\lambda, h_k^\theta), \qquad \text{a.e. } t \in [0,\alpha]. \tag{4.53}$$

Then Lemma 4.2 yields

$$|\dot{w}^k(t)| \le g_k(t) + N_7 |w_t^k|_C + N_8 |\mathcal{P}(\dot{w}_t^k)|_{L^\infty_{r_0}}, \qquad \text{a.e. } t \in [0,\alpha], \tag{4.54}$$

where $N_7, N_8 \ge 0$, and the nonnegative function g_k satisfies (4.23). Let $m := [\alpha/r_0]$ (here $[\,\cdot\,]$ denotes the greatest integer part), $t_i := ir_0$ for $i = 0,1,..,m$, $t_{m+1} := \alpha$. Integrating (4.53) from t_i to t and using inequality (4.54) we have

$$|w^k(t)| \le |w^k(t_i)| + \int_{t_i}^{t_{i+1}} g_k(s)\,ds + \int_{t_i}^{t} (N_7 |w_s^k|_C + N_8 |\mathcal{P}(\dot{w}_s^k)|_{L^\infty_{r_0}})\,ds, \qquad t \in [t_i, t_{i+1}]. \tag{4.55}$$

Then, using that $w^k(s) = 0$ for $s \in [-r,0]$, we get

$$|w^k(t)| \le \int_{t_0}^{t_1} g_k(s)\,ds + \int_{t_0}^{t} N_7 |w_s^k|_C\,ds, \qquad t \in [t_0, t_1],$$

hence Gronwall's lemma implies

$$|w^k(t)| \le a_0^k, \qquad t \in [t_0,t_1], \quad k \in \mathbb{N},$$

where $a_0^k := \Big(\int_{t_0}^{t_1} g_k(s)\,ds\Big) e^{N_7 r_0}$. Note that (4.23) yields

$$\lim_{k\to\infty} \frac{a_i^k}{|h_k|_{\Gamma_2}} = 0 \tag{4.56}$$

for $i = 0$. Since $w^k(t) = 0$ for $t \in [-r, 0]$, we get from (4.54) that

$$|\dot{w}^k(t)| \le g_k(t) + b_0^k, \quad \text{a.e. } t \in [t_0, t_1],$$

with $b_0^k := N_7 a_0^k$. We note that (4.56) with $i = 0$ yields

$$\lim_{k\to\infty} \frac{b_i^k}{|h_k|_{\Gamma_2}} = 0 \tag{4.57}$$

for $i = 0$. Suppose

$$|w^k(t)| \le a_i^k, \quad t \in [t_i, t_{i+1}], \quad \text{and} \quad |\dot{w}^k(t)| \le g_k(t) + b_i^k, \quad \text{a.e. } t \in [t_i, t_{i+1}]$$

for $i = 0, 1, \ldots, j$ with some $j < m$, moreover, $a_0^k \le a_1^k \le \cdots \le a_j^k$, $b_0^k \le b_1^k \le \cdots \le b_j^k$, and a_i^k and b_i^k satisfy (4.56) and (4.57), respectively, for $i = 0, 1, \ldots, j$. Then (4.55) implies

$$\begin{aligned} |w^k(t)| &\le |w^k(t_{j+1})| + \int_{t_{j+1}}^{t_{j+2}} g_k(s)\,ds + \int_{t_{j+1}}^{t} (N_7|w_s^k|_C + N_8|\mathcal{P}(\dot{w}_s^k)|_{L^\infty_{r_0}})\,ds \\ &\le a_j^k + \int_{t_{j+1}}^{t_{j+2}} g_k(s)\,ds + N_8 b_j^k r_0 + \int_{t_{j+1}}^{t} N_7|w_s^k|_C\,ds, \qquad t \in [t_{j+1}, t_{j+2}], \end{aligned}$$

hence an application of Gronwall's lemma gives

$$|w^k(t)| \le a_{j+1}^k, \qquad t \in [t_{j+1}, t_{j+2}],$$

where $a_{j+1}^k := \left(a_j^k + \int_{t_{j+1}}^{t_{j+2}} g_k(s)\,ds + N_8 b_j^k r_0\right) e^{N_7 r_0}$. We observe that $a_{j+1}^k \ge a_j^k$, and a_{j+1}^k also satisfies (4.56). Then (4.54) implies

$$|\dot{w}^k(t)| \le g_k(t) + b_{j+1}^k, \quad \text{a.e. } t \in [t_{j+1}, t_{j+2}]$$

with $b_{j+1}^k := N_7 a_{j+1}^k + N_8 b_j^k$. Therefore, the constants a_i^k and b_i^k can be defined for $i = 0, 1, \ldots, m$, so that $a_0^k \le \cdots \le a_m^k$ hold, and (4.56) is satisfied for $i = m$ too. Then we get

$$|x^k(t) - x(t) - z^k(t)| \le |x_t^k - x_t - z_t^k|_C \le a_m^k, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}, \tag{4.58}$$

and both claims of the theorem follow from (4.56) with $i = m$. □

We note that the results of this manuscript can be extended to the case of more explicit state-dependent delays in the equation.

Acknowledgements

This research was supported by the Hungarian National Foundation for Scientific Research Grant No. K120186 and the TKP2020-IKA-07 project financed under the 2020-4.1.1-TKP2020 Thematic Excellence Programme by the National Research, Development and Innovation Fund of Hungary.

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