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Positive solutions for $(p, 2)$-equations with superlinear reaction and a concave boundary term

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Abstract. We consider a nonlinear boundary value problem driven by the $(p, 2)$-Laplacian, with a $(p-1)$-superlinear reaction and a parametric concave boundary term (a "concave-convex" problem). Using variational tools (critical point theory) together with truncation and comparison techniques, we prove a bifurcation type theorem describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. We also show that for every admissible parameter $\lambda > 0$, the problem has a minimal positive solution $\overline{u}_\lambda$ and determine the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

Keywords: concave boundary term, superlinear reaction, $(p, 2)$-Laplacian, nonlinear regularity, nonlinear maximum principle, positive solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric $(p, 2)$-equation

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z) u(z)^{p-1} = f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega\,, \\ u > 0,\ \lambda > 0,\ 1 < \tau < 2 < p < N. \end{cases} \tag{P_λ}$$

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < N.$$

The potential function $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a.

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$z \in \Omega$, $x \mapsto f(z,x)$ is continuous). We assume that $f(z,\cdot)$ is $(p-1)$-superlinear satisfying the Ambrosetti–Rabinowitz condition (the *AR-condition* for short). In the boundary condition, $\frac{\partial u}{\partial n_{p2}}$ denotes the conormal derivative of u corresponding to the $(p,2)$-Laplace differential operator. This directional derivative of u, is interpreted via the nonlinear Green's identity (see Papageorgiou–Rădulescu–Repovš [21], pp. 34, 35). If $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{p2}} = \left[|Du|^{p-2}+1\right]\frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Also $\lambda > 0$ is a parameter and $\tau \in (1,2)$. So, in problem (P_λ) we have the competing effects of two nonlinearities of different nature. One is the reaction term which is superlinear ("convex" nonlinearity) and the other is the parametric boundary term, which is sublinear ("concave" nonlinearity). Therefore, problem (P_λ) is a variant of the classical "concave-convex" problem, with the concave term coming from the boundary condition.

The study of "concave-convex" problems was initiated with the seminal paper of Ambrosetti–Brezis–Cerami [2] (semilinear Dirichlet equations). Their work was extended to nonlinear Dirichlet problems driven by the p-Laplacian by García Azorero–Manfredi–Peral Alonso [7] and Guo-Zhang [9]. In these works the reaction has the special form

$$x \mapsto \lambda x^{\tau-1} + x^{r-1} \quad \text{for all } x \geq 0,$$

with $\lambda > 0$ (the parameter) and $1 < \tau < p < r < p^*$,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Recently more general reactions and different boundary conditions were considered by Papageorgiou–Rădulescu–Repovš [18] (semilinear Robin problems), by Leonardi–Papageorgiou [12], Marano–Marino–Papageorgiou [14] (nonlinear Dirichlet problems) and by Papageorgiou–Scapellato [23] (nonlinear Robin problems). In these works the competition phenomena occur in the reaction of the equation, where we have the presence of concave and convex nonlinearities. Problems with parametric concave boundary term were considered by Hu–Papageorgiou [11] (semilinear equations), Papageorgiou–Rădulescu [16], Papageorgiou–Rădulescu–Repovš [20], Sabina de Lis–Segura de Leon [25] (nonlinear problems driven by the p-Laplacian). Finally we mention the recent work of Papageorgiou–Scapellato [22] where in the reaction we have the combined effects of linear and superlinear terms.

Our work here extends those of Hu–Papageorgiou [11] and of Sabina de Lis–Segura de Leon [25].

Using variational tools based on the critical point theory, together with truncation and comparison techniques, we prove a bifurcation-type result describing in a precise way the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies. Also we show that for every admissible $\lambda > 0$, problem (P_λ) has a smallest positive solution.

We mention that boundary value problems driven by a combination of differential operators of different nature (such as $(p,2)$-equations), arise in many mathematical models of physical processes. Among the first such examples we mention the Cahn–Hilliard equation (see [4]) describing the process of separation of binary alloys. More recently, we mention the works of Benci–D'Avenia–Fortunato–Pisani [3] (quantum physics) and Cherfils–Il'yasov [5] (reaction-diffusion systems).

2 Mathematical background – hypotheses

In the study of problem (P_λ), we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \le s < \infty)$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$ given by

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \ \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using $\sigma(\cdot)$, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \le s \le \infty)$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the *trace map*, such that

$$\gamma_0(u) = u\Big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. This map is compact into $L^s(\partial\Omega)$ for all $1 \le s < \frac{(N-1)p}{N-p}$ when $p < N$ and into $L^s(\Omega)$ for all $1 \le s < \infty$ when $N \le p$. Moreover, we have

$$\begin{aligned} \operatorname{im} \gamma_0 &= W^{\frac{1}{p'},p}(\partial\Omega) \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right), \\ \ker \gamma_0 &= W_0^{1,p}(\Omega). \end{aligned}$$

In what follows for the sake of notational simplicity we drop the use of the trace map. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

If $u, v \in W^{1,p}(\Omega)$ with $u(z) \le v(z)$ for a.a. $z \in \Omega$, then we define

$$\begin{aligned} [u,v] &= \{h \in W^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \quad \text{for a.a. } z \in \Omega\}, \\ [u) &= \{h \in W^{1,p}(\Omega) : u(z) \le h(z) \quad \text{for a.a. } z \in \Omega\}. \end{aligned}$$

Given $g_1, g_2 \in L^\infty(\Omega)$, we write $g_1 \prec g_2$ if for every $K \subseteq \Omega$ compact we can find $c_K > 0$ such that

$$c_K \le g_2(z) - g_1(z) \quad \text{for a.a. } z \in K.$$

Note that if $g_1, g_2 \in C(\Omega)$ and $g_1(z) < g_2(z)$ for all $z \in \Omega$, then $g_1 \prec g_2$.

We say that a set $S \subseteq W^{1,p}(\Omega)$ is *downward directed*, if given $u_1, u_2 \in S$, we can find $u \in S$ such that $u \le u_1$, $u \le u_2$.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W^{1,p}(\Omega), W^{1,p}(\Omega)^*)$ and let $A_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

Proposition 2.1. *The operator $A_p(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \le 0 \quad \Rightarrow \quad u_n \to u \text{ in } W^{1,p}(\Omega).$$

If $p = 2$, then $A_2 = A \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$.

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, given $u \in W^{1,p}(\Omega)$, we define

$$u^\pm(z) = u(z)^\pm \quad \text{for all } z \in \Omega.$$

We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Finally, if X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of $\varphi(\cdot)$, that is,

$$K_\varphi = \{u \in W^{1,p}(\Omega) : \varphi'(u) = 0\}.$$

Now we introduce our hypotheses on the data of problem (P_λ).

H(ξ): $\xi \in L^\infty(\Omega)$, $\xi(z) \ge 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$.

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \le f(z,x) \le \eta x^{r-1}$ for a.a. $z \in \Omega$, all $x \ge 0$, with $0 < \eta$, $p < r < p^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then there exist $\vartheta_0 \in (p,r)$ and $M > 0$ such that

$$0 < \vartheta_0 F(z,x) \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M,$$
$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M).$$

Remark 2.2. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0. \tag{2.1}$$

Hypothesis H(f)(i) implies that

$$\lim_{x\to 0^+} \frac{f(z,x)}{x^{\tau-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega. \tag{2.2}$$

Hypothesis H(f)(ii) is the well known AR-condition (unilateral version due to (2.1)). The AR-condition implies that

$$\begin{aligned} & c_0 x^{\vartheta_0} \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \text{ some } c_0 > 0 \\ \Rightarrow\ & c_0 x^{\vartheta_0 - 1} \le f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M \\ \Rightarrow\ & f(z,\cdot) \text{ is } (p-1)\text{-superlinear (since } \vartheta_0 > p). \end{aligned}$$

It is an interesting open problem whether we can replace the AR-condition by a less restrictive one as in Papageorgiou–Rădulescu [17].

The following functions satisfy hypotheses H(f). For the sake of simplicity we drop the z-dependence:

$$f_1(x) = \begin{cases} (x^+)^{r-1} + \ln(1 + (x^+)^{q-1}) & \text{if } x \le 1 \\ x^{s-1} & \text{if } 1 < x \end{cases} \quad \text{with } p < r \le q < \infty,\ p < s < p^*,$$

$$f_2(x) = (x^+)^{r-1} \quad \text{with } p < r < p^*.$$

In the sequel, by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma_p(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

On account of hypothesis H(ξ) and Lemma 4.11 of Mugnai–Papageorgiou [15], we have

$$c_1\|u\|^p \leq \gamma_p(u) \quad \text{for all } u \in W^{1,p}(\Omega), \text{ some } c_1 > 0. \tag{2.3}$$

3 Positive solutions

We introduce the following sets

$$\begin{aligned} \mathscr{L} &= \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ admits a positive solution}\}, \\ S_\lambda &= \text{ set of positive solutions of } (\mathrm{P}_\lambda). \end{aligned}$$

Proposition 3.1. *If hypotheses* H(ξ), H(f) *hold, then* $\mathscr{L} \neq \varnothing$ *and* $S_\lambda \subseteq \operatorname{int} C_+$ *for all* $\lambda > 0$.

Proof. For every $\lambda > 0$, let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u^+)\, dz - \frac{\lambda}{\tau}\int_{\partial\Omega} (u^+)^\tau \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

On account of (2.2) and hypothesis H(f)(i), we see that given $\epsilon > 0$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z,x) \leq \epsilon x^\tau + c_2|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then we have

$$\varphi_\lambda(u) \geq \frac{c_1}{p}\|u\|^p - c_3\left[\epsilon\|u\|^\tau + \|u\|^r + \lambda\|u\|^\tau\right] \quad \text{for some } c_3 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{3.1}$$

Here we used (2.3) and the fact that via the trace map the Sobolev space $W^{1,p}(\Omega)$ is embedded continuously (in fact compactly) into $L^\tau(\partial\Omega)$.

For $\rho > 0$, we let $\epsilon = \frac{1}{2}\frac{c_1}{p}\frac{\rho^{p-\tau}}{c_3}$. Then we have

$$\left[\frac{c_1}{p}\rho^{p-\tau} - \epsilon c_3\right]\rho^\tau = \frac{1}{2}\frac{c_1}{p}\rho^p. \tag{3.2}$$

Using (3.2) in (3.1) we obtain

$$\varphi_\lambda(u) \geq \frac{1}{2}\frac{c_1}{p}\rho^p - c_3[\rho^r + \lambda\rho^\tau] \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho.$$

Since $p < r$, we can choose $\rho \in (0,1)$ small such that

$$\frac{1}{2}\frac{c_1}{p}\rho^p - c_3\rho^r \geq \overline{\eta} > 0.$$

Then we choose $\lambda_0 > 0$ small so that

$$\begin{aligned} & \overline{\eta} - \lambda_0 c_3 \rho^\tau \geq \frac{1}{2}\overline{\eta} > 0 \\ \Rightarrow \quad & \overline{\eta} - \lambda c_3 \rho^\tau \geq \frac{1}{2}\overline{\eta} > 0 \quad \text{for all } \lambda \in (0,\lambda_0] \\ \Rightarrow \quad & \varphi_\lambda(u) \geq \frac{1}{2}\overline{\eta} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho, \text{ all } 0 < \lambda \leq \lambda_0. \end{aligned} \tag{3.3}$$

We introduce the open ball

$$B_\rho = \{u \in W^{1,p}(\Omega) : \|u\| < \rho\}.$$

By the Alaoglu and Eberlein-Šmulian theorems, we have that $\overline{B_\rho}$ is sequentially weakly compact. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Invoking the Weierstrass–Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \min\left[\varphi_\lambda(u) : u \in \overline{B_\rho}\right]. \tag{3.4}$$

Since $\tau < 2 < p$, we see that

$$\begin{aligned} &\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0) < \frac{1}{2}\overline{\eta} \\ \Rightarrow\quad & u_0 \in B_\rho \setminus \{0\} \quad \text{(see (3.3))}. \end{aligned} \tag{3.5}$$

Then from (3.4) and (3.5) it follows that

$$\begin{aligned} &\varphi'_\lambda(u_0) = 0, \\ \Rightarrow\quad & \langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h\,dz \\ &\qquad = \int_\Omega f(x, u_0^+)h\,dz + \lambda\int_{\partial\Omega}(u_0^+)^{\tau-1}h\,d\sigma \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.6}$$

In (3.6) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\gamma_p(u_0^-) + \|Du_0^-\|_2^2 = 0 \\ \Rightarrow\quad & c_1\|u_0^-\|^p \le 0 \quad \text{(see (2.3))} \\ \Rightarrow\quad & u_0 \ge 0,\ u_0 \ne 0. \end{aligned}$$

From (3.6) we see that $u_0 \in W^{1,p}(\Omega)$ is a positive solution of (P_λ) and we have

$$\begin{cases} -\Delta_p u_0(z) - \Delta u_0(z) + \xi(z)u_0(z)^{p-1} = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_{p2}} = \lambda u_0^{\tau-1} & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Proposition 2.10 of Papageorgiou–Rădulescu [17] implies that $u_0 \in L^\infty(\Omega)$ and then from Theorem 2 of Lieberman [13], we have that $u_0 \in C_+ \setminus \{0\}$. From (3.7) we see that

$$\begin{aligned} &\Delta_p u_0(z) + \Delta u_0(z) \le \|\xi\|_\infty u_0(z)^{p-1} \quad \text{for a.a. } x \in \Omega \\ \Rightarrow\quad & u_0 \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [24], pp. 111, 120).} \end{aligned}$$

So, we have proved that

$$\begin{aligned} &(0, \lambda_0] \subseteq \mathscr{L}, \text{ that is, } \mathscr{L} \ne \varnothing, \\ &S_\lambda \subseteq \operatorname{int} C_+ \text{ for all } \lambda > 0. \end{aligned}$$ □

Next we show that $\mathscr{L}$ is an interval.

Proposition 3.2. *If hypotheses* H(ξ), H(f) *hold,* $\lambda \in \mathscr{L}$ *and* $\mu \in (0, \lambda)$, *then* $\mu \in \mathscr{L}$.

Proof. Since $\lambda \in \mathscr{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 3.1). We introduce the following truncations of the data of problem (P_μ):

$$\widehat{f}(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq u_\lambda(z) \\ f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.8}$$

$$e_\mu(z,x) = \begin{cases} \mu(x^+)^{\tau-1} & \text{if } x \leq u_\lambda(z) \\ \mu u_\lambda(z)^{\tau-1} & \text{if } u_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.9}$$

Both are Carathéodory functions. We set

$$\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds, \qquad E_\mu(z,x) = \int_0^x e_\mu(z,s)\,ds$$

and consider the C^1-functional $\psi_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}(z,u)\,dz - \int_{\partial\Omega} E_\mu(z,u)\,d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (2.3), (3.8) and (3.9), we see that $\psi_\mu(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. Therefore we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf\left[\psi_\mu(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.10}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small (at least so that $tu \leq u_\lambda$, recall that $u_\lambda \in \operatorname{int} C_+$). Then since $\tau < 2 < p$, we will have

$$\begin{aligned} & \psi_\mu(tu) < 0 \\ \Rightarrow\quad & \psi_\mu(u_\mu) < 0 = \psi_\mu(0) \quad \text{(see (3.10))} \\ \Rightarrow\quad & u_\mu \neq 0. \end{aligned}$$

From (3.10) we have

$$\begin{aligned} & \psi'_\mu(u_\mu) = 0 \\ \Rightarrow\quad & \langle A_p(u_\mu),h\rangle + \langle A(u_\mu),h\rangle + \int_\Omega \xi(z)|u_\mu|^{p-2}u_\mu h\,dz \\ & \quad = \int_\Omega \widehat{f}(z,u_\mu)h\,dz + \int_{\partial\Omega} e(z,u_\mu)h\,d\sigma \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.11}$$

In (3.11) first we choose $h = -u_\mu^- \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned} & \gamma_p(u_\mu^-) + \|Du_\mu^-\|_2^2 = 0 \\ \Rightarrow\quad & c_1\|u_\mu^-\|^p \leq 0 \quad \text{(see (2.3))} \\ \Rightarrow\quad & u_\mu \geq 0,\ u_\mu \neq 0. \end{aligned}$$

Next in (3.11) we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} & \langle A_p(u_\mu),(u_\mu-u_\lambda)^+\rangle + \langle A(u_\mu),(u_\mu-u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\mu^{p-1}(u_\mu-u_\lambda)^+\,dz = \\ & = \int_\Omega f(x,u_\lambda)(u_\mu-u_\lambda)^+\,dz + \int_{\partial\Omega} \mu u_\lambda^{\tau-1}(u_\mu-u_\lambda)^+\,d\sigma \quad \text{(see (3.8), (3.9))} \\ & \leq \int_\Omega f(z,u_\lambda)(u_\mu-u_\lambda)^+\,dz + \int_{\partial\Omega} \lambda u_\lambda^{\tau-1}(u_\mu-u_\lambda)^+\,dz \quad \text{(since } \mu < \lambda) \\ & = \langle A_p(u_\lambda),(u_\mu-u_\lambda)^+\rangle + \langle A(u_\lambda),(u_\mu-u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\mu-u_\lambda)^+\,dz \\ & \text{(since } u_\lambda \in S_\lambda) \end{aligned}$$

$$\Rightarrow \quad u_\mu \leq u_\lambda \quad \text{(see Proposition 2.1).}$$

So we have proved that

$$u_\mu \in [0, u_\lambda] \setminus \{0\}. \tag{3.12}$$

From (3.11), (3.12), (3.8), (3.9) it follows that

$$\begin{aligned} & u_\mu \in S_\mu \subseteq \operatorname{int} C_+, \\ \Rightarrow \quad & \mu \in \mathscr{L}. \end{aligned}$$

□

An interesting byproduct of the above proof is the following corollary.

Corollary 3.3. *If hypotheses* H(ξ), H(f) *hold,* $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\mu \in (0, \lambda)$, *then* $\mu \in \mathscr{L}$ *and there exists* $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ *such that* $u_\mu \leq u_\lambda$.

We can improve this corollary, by imposing an additional mild condition on $f(z, \cdot)$. So, the new hypotheses on the reaction $f(z, x)$ are the following:

H(f)′: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H(f)′(i), (ii) are the same as the corresponding hypotheses H(f)(i), (ii) and

(i) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \mapsto f(z, x) + \widehat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 3.4. The extra condition is a one-sided local Lipschitz condition (recall that $p > 2$). If $f(z, \cdot)$ is differentiable for a.a. $z \in \Omega$ and for every $\rho > 0$, there exists $c_\rho > 0$ such that

$$f'_x(z, x) \geq -c_\rho x^{p-2} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \rho],$$

then hypothesis H(f)′(i) is satisfied.

Proposition 3.5. *If hypotheses* H(ξ), H(f)′ *hold,* $\lambda \in \mathscr{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\mu \in (0, \lambda)$, *then* $\mu \in \mathscr{L}$ *and we can find* $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ *such that*

$$u_\lambda - u_\mu \in D_+.$$

Proof. From Corollary 3.3 we already know that $\mu \in \mathscr{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\mu \leq u_\lambda. \tag{3.13}$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a(y) = |y|^{p-2} y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Evidently $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and

$$\begin{aligned} & \nabla a(y) = |y|^{p-2} \left[I + (p-2) \frac{y \otimes y}{|y|^2} \right] + I \\ \Rightarrow \quad & (\nabla a(y) \vartheta, \vartheta)_{\mathbb{R}^N} \geq |\vartheta|^2 \quad \text{for all } y, \vartheta \in \mathbb{R}^N. \end{aligned} \tag{3.14}$$

Observe that

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W^{1,p}(\Omega). \tag{3.15}$$

From (3.13), (3.14), (3.15) and the tangency principle of Pucci-Serrin [24], p. 35, we have

$$u_\mu(z) < u_\lambda(z) \quad \text{for all } z \in \Omega. \tag{3.16}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H$(f)'$(i). Let $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$. We have

$$\begin{aligned}
& -\Delta_p u_\mu - \Delta u_\mu + \left[\xi(z) + \widetilde{\xi}_\rho\right] u_\mu^{p-1} \\
&\quad = f(z, u_\mu) + \widehat{\xi}_\rho u_\mu^{p-1} + \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\mu^{p-1} \\
&\quad \leq f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} + \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\lambda^{p-1} \quad \text{(see (3.13) and hypothesis H}(f)'\text{(i))} \\
&\quad = -\Delta_p u_\lambda - \Delta u_\lambda + \left[\xi(z) + \widetilde{\xi}_\rho\right] u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega.
\end{aligned} \tag{3.17}$$

On account of (3.16), we see that

$$\left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\mu^{p-1} \prec \left[\widetilde{\xi}_\rho - \widehat{\xi}_\rho\right] u_\lambda^{p-1}.$$

Then from (3.17) and Proposition 3.2 of Gasiński–Papageorgiou [8] we have

$$u_\lambda - u_\mu \in D_+.$$

□

From Papageorgiou–Rădulescu–Repovš [19] (see the proof of Proposition 7), we know that S_λ is downward directed. We will use this fact to show that for every $\lambda \in \mathscr{L}$ problem (P_λ) has a smallest positive solution $\overline{u}_\lambda \in S_\lambda$, that is, $\overline{u}_\lambda \leq u$ for all $u \in S_\lambda$.

Proposition 3.6. *If hypotheses* H(ξ), H(f) *hold and* $\lambda \in \mathscr{L}$, *then problem* (P_λ) *admits a smallest positive solution*

$$\overline{u}_\lambda \in \operatorname{int} C_+.$$

Proof. Since S_λ is downward directed, using Lemma 3.10, p. 178, of Hu–Papageorgiou [10], we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda$ decreasing such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda.$$

We have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h \, dz = \int_\Omega f(z, u_n) h \, dz + \lambda \int_{\partial\Omega} u_n^{\tau-1} h \, d\sigma \tag{3.18}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (3.18) we choose $h = u_n \in W^{1,p}(\Omega)$. Since $0 \leq u_n \leq u_1$ for all $n \in \mathbb{N}$, using (2.3) and hypothesis H(f)(i), we see that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

From Lieberman [13] (Theorem 2), we see that there exist $\alpha \in (0, 1)$ and $c_4 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_4 \quad \text{for all } n \in \mathbb{N}.$$

Recall that $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly. This fact and the monotonicity of the sequence $\{u_n\}_{n\geq 1}$ imply that there exists $\overline{u}_\lambda \in C^1(\overline{\Omega})$ such that

$$u_n \to \overline{u}_\lambda \text{ in } C^1(\Omega) \text{ as } n \to \infty. \tag{3.19}$$

We need to show that $\overline{u}_\lambda \neq 0$. To this end we consider the following auxiliary boundary value problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z) u(z)^{p-1} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega\,. \\ u > 0, \lambda > 0,\ \tau < 2 < p \end{cases} \tag{Q_λ}$$

<u>Claim 1.</u> *For every $\lambda > 0$ problem (Q_λ) admits a unique solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$.*

First we show the existence of a positive solution for problem (Q_λ). For this purpose we introduce the C^1-functional $\beta_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\beta_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{\tau}\int_{\partial\Omega} (u^+)^\tau \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (2.3) and since $\tau < 2 < p$, we see that

$$\beta_\lambda(\cdot) \text{ is coercive.}$$

Also the Sobolev embedding theorem and the compactness of the trace map, imply that

$$\beta_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\beta_\lambda(\widetilde{u}_\lambda) = \min\left[\beta_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.20}$$

Since $\tau < 2 < p$, we infer that

$$\begin{aligned} & \beta_\lambda(\widetilde{u}_\lambda) < 0 = \beta_\lambda(0) \\ \Rightarrow\ & \widetilde{u}_\lambda \neq 0. \end{aligned}$$

From (3.20) we have

$$\begin{aligned} & \beta'_\lambda(\widetilde{u}_\lambda) = 0 \\ \Rightarrow\ & \langle A_p(\widetilde{u}_\lambda), h\rangle + \langle A(\widetilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\widetilde{u}_\lambda|^{p-2}\widetilde{u}_\lambda h \, dz = \lambda \int_{\partial\Omega} (\widetilde{u}_\lambda^+)^{\tau-1} h \, d\sigma \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

Choosing $h = -\widetilde{u}_\lambda^- \in W^{1,p}(\Omega)$ and using (2.3), we infer that

$$\widetilde{u}_\lambda \geq 0, \quad \widetilde{u}_\lambda \neq 0.$$

Moreover, as before (see the proof of Proposition 3.1), using the nonlinear regularity theory of Lieberman [13] (Theorem 2) and the nonlinear maximum principle of Pucci–Serrin [24] (p. 120), we conclude that

$$\widetilde{u}_\lambda \in \operatorname{int} C_+. \tag{3.21}$$

Now we show the uniqueness of this positive solution of problem (Q_λ). To this end, we consider the integral functional $j_\lambda : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j_\lambda(u) = \begin{cases} \frac{1}{p}\|Du^{\frac{1}{2}}\|_p^p + \frac{1}{2}\|Du^{\frac{1}{2}}\|_2^2 + \frac{1}{p}\int_\Omega \xi(z) u^{\frac{p}{2}}\,\mathrm{d}z, & \text{if } u \geq 0,\, u^{\frac{1}{2}} \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

From Diaz–Saá [6] (Lemma 1), we know that $j_\lambda(\cdot)$ is convex.

Let $\operatorname{dom} j_\lambda = \{u \in L^1(\Omega) : j_\lambda(u) < \infty\}$ (the effective domain of $j_\lambda(\cdot)$). Let $\widetilde{v}_\lambda$ be another positive solution of (Q_λ). Reasoning as we did for $\widetilde{u}_\lambda$, we show that

$$\widetilde{v}_\lambda \in \operatorname{int} C_+. \tag{3.22}$$

Then from (3.21), (3.22) and Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [21], we have $\frac{\widetilde{u}_\lambda}{\widetilde{v}_\lambda}, \frac{\widetilde{v}_\lambda}{\widetilde{u}_\lambda} \in L^\infty(\Omega)$. Let $h = \widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2$. For $t \in [0,1]$ we have

$$\widetilde{u}_\lambda^2 - th \in \operatorname{dom} j_\lambda \quad \text{and} \quad \widetilde{v}_\lambda^2 + th \in \operatorname{dom} j_\lambda.$$

Then $j_\lambda(\cdot)$ is Gâteaux differentiable at $\widetilde{u}_\lambda^2$ and at $\widetilde{v}_\lambda^2$ in the direction h. Moreover, using the nonlinear Green's identity, we have

$$\begin{aligned} j_\lambda'(\widetilde{u}_\lambda^2)(h) &= \frac{\lambda}{2}\int_{\partial\Omega} \widetilde{u}_\lambda^{\tau-2}(\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,\mathrm{d}\sigma, \\ j_\lambda'(\widetilde{v}_\lambda^2)(h) &= \frac{\lambda}{2}\int_{\partial\Omega} \widetilde{v}_\lambda^{\tau-2}(\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,\mathrm{d}\sigma. \end{aligned}$$

Since $j_\lambda(\cdot)$ is convex, we have that $j_\lambda'(\cdot)$ is monotone. Since $\tau < 2$ we have

$$\begin{aligned} & 0 \leq \frac{\lambda}{2}\int_{\partial\Omega}\left[\frac{1}{\widetilde{u}_\lambda^{2-\tau}} - \frac{1}{\widetilde{v}_\lambda^{2-\tau}}\right](\widetilde{u}_\lambda^2 - \widetilde{v}_\lambda^2)\,\mathrm{d}\sigma \leq 0 \\ \Rightarrow\quad & \widetilde{u}_\lambda = \widetilde{v}_\lambda. \end{aligned}$$

Therefore the positive solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$ is unique. This proves Claim 1.

This solution provides a lower bound for the elements of S_λ.

<u>Claim 2.</u> *$\widetilde{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Let $u \in S_\lambda \subseteq \operatorname{int} C_+$. We introduce the following Carathéodory function

$$b_\lambda(z,x) = \begin{cases} \lambda (x^+)^{\tau-1} & \text{if } x \leq u(z) \\ \lambda u(z)^{\tau-1} & \text{if } u(z) < x \end{cases} \quad \text{for all } (x,z) \in \partial\Omega \times \mathbb{R}. \tag{3.23}$$

We set $B_\lambda(z,x) = \int_0^x b_\lambda(z,s)\,\mathrm{d}s$ and consider the C^1-functional $\vartheta_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\vartheta_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_{\partial\Omega} B_\lambda(z,u)\,\mathrm{d}\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.23) and (2.3) it is clear that $\vartheta_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\vartheta_\lambda(\widehat{u}_\lambda) = \inf\left[\vartheta_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.24}$$

As before (see Claim 1), since $\tau < 2 < p$, we see that

$$\begin{aligned} & \vartheta_\lambda(\widehat{u}_\lambda) < 0 = \vartheta_\lambda(0) \\ \Rightarrow \quad & \widehat{u}_\lambda \neq 0. \end{aligned}$$

From (3.24) we have

$$\begin{aligned} & \vartheta'_\lambda(\widehat{u}_\lambda) = 0 \\ \Rightarrow \quad & \langle A_p(\widehat{u}_\lambda), h\rangle + \langle A(\widehat{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\widehat{u}_\lambda|^{p-2}\widehat{u}_\lambda h \, dz = \int_{\partial\Omega} b_\lambda(z, \widehat{u}_\lambda) h \, d\sigma \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.25}$$

As before (see the proof of Proposition 3.2), if in (3.25) we choose first $h = -\widetilde{u}_\lambda^- \in W^{1,p}(\Omega)$ and then $h = (\widehat{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$ and using (3.23), we show that

$$\widehat{u}_\lambda \in [0, u] \setminus \{0\}. \tag{3.26}$$

From (3.26), (3.23), (3.25) and Claim 1, it follows that

$$\begin{aligned} & \widehat{u}_\lambda = \widetilde{u}_\lambda \\ \Rightarrow \quad & \widetilde{u}_\lambda \leq u \quad \text{for all } u \in S_\lambda \text{ (see (3.26)).} \end{aligned}$$

This proves Claim 2.

From (3.19) and Claim 2, we have

$$\begin{aligned} & \widetilde{u}_\lambda \leq \overline{u}_\lambda \\ \Rightarrow \quad & \overline{u}_\lambda \neq 0 \text{ and so } \overline{u}_\lambda \in S_\lambda \subseteq \operatorname{int} C_+, \ \overline{u}_\lambda = \inf S_\lambda. \end{aligned}$$

□

Proposition 3.7. *If hypotheses* H(ξ), H(f) *hold and* $0 < \mu < \lambda \in \mathscr{L}$, *then*

(a) $\overline{u}_\mu \leq \overline{u}_\lambda$;

(b) $\widetilde{u}_\mu \leq \widetilde{u}_\lambda$.

Proof.

(a) Let $\overline{u}_\lambda \in \operatorname{int} C_+$ be the minimal positive solution of problem (P_λ) (see Proposition 3.6). On account of Corollary 3.3, we can find $u_\mu \in S_\mu \in \operatorname{int} C_+$ such that

$$\begin{aligned} & u_\mu \leq \overline{u}_\lambda \\ \Rightarrow \quad & \overline{u}_\mu \leq \overline{u}_\lambda \quad \text{recall that } \overline{u}_\mu \leq u \text{ for all } u \in S_\mu. \end{aligned}$$

(b) Let $\widetilde{e}_\mu(z, x)$ be the Carathéodory function defined by

$$\widetilde{e}_\mu(z,x) = \begin{cases} \mu(x^+)^{\tau-1} & \text{if } x \leq \widetilde{u}_\lambda(z) \\ \mu\widetilde{u}_\lambda(z)^{\tau-1} & \text{if } \widetilde{u}_\lambda(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.27}$$

We set $\widetilde{E}_\mu(z, x) = \int_0^x \widetilde{e}_\mu(z, s) \, ds$ and consider the C^1-functional $\widetilde{\varphi}_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_{\partial\Omega} \widetilde{E}_\mu(z, u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $\widetilde{\varphi}_\mu(\cdot)$ is coercive (see (3.27) and (2.3)) and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\mu \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widetilde{\varphi}_\mu(\widehat{u}_\mu) = \inf\left[\widetilde{\varphi}_\mu(u) : u \in W^{1,p}(\Omega)\right] < 0 = \widetilde{\varphi}_\mu(0) \quad (\text{since } \tau < 2 < p) \\ \Rightarrow\quad &\widehat{u}_\mu \neq 0. \end{aligned}$$

We have

$$\langle \widetilde{\varphi}'_\mu(\widehat{u}_\mu), h\rangle = 0 \quad \text{for all } h \in W^{1,p}(\Omega).$$

Choosing $h = -\widehat{u}_\mu^- \in W^{1,p}(\Omega)$ and $h = (\widehat{u}_\mu - \widetilde{u}_\lambda)^+ \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} &\widehat{u}_\mu \in [0, \widetilde{u}_\lambda],\ \widehat{u}_\mu \neq 0 \\ \Rightarrow\quad &\widehat{u}_\mu = \widetilde{u}_\mu \in \operatorname{int} C_+ \quad (\text{see (3.27) and Claim 1 in the proof of Proposition 3.6}) \\ \Rightarrow\quad &\widetilde{u}_\mu \leq \widetilde{u}_\lambda. \end{aligned}$$

□

Let $0 < \mu < \lambda$ and $\eta_0 = \frac{\eta}{\mu}$. Then $\eta \leq \lambda\eta_0$. Motivated by hypothesis H(f)(i), we consider the following auxiliary boundary value problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi(z)u(z)^{p-1} = \lambda\eta_0 u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{p2}} = \lambda u^{\tau-1} & \text{on } \partial\Omega, \\ u > 0, \lambda > 0, \tau < 2 < p < r. \end{cases} \tag{R$_\lambda$}$$

Reasoning as in the proofs of Propositions 3.1 and 3.6, we obtain the following result.

Proposition 3.8. *If hypothesis* H(ξ) *holds and* $\lambda \in \mathscr{L}$, *then problem* (R$_\lambda$) *admits a smallest positive solution* $u_\lambda^* \in \operatorname{int} C_+$ *and there exists* $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *such that*

$$\widetilde{u}_\lambda \leq u_\lambda \leq u_\lambda^*.$$

Let $\lambda^* = \sup \mathscr{L}$.

Proposition 3.9. *If hypotheses* H(ξ), H(f) *hold, then* $\lambda^* < \infty$.

Proof. Let $\mu \in (0, \lambda)$ and set $0 < \widetilde{m}_\mu = \min\limits_{\overline{\Omega}} \widetilde{u}_\mu$ (recall that $\widetilde{u}_\mu \in \operatorname{int} C_+$). From Propositions 3.8 and 3.7(b), we have

$$0 < \widetilde{m}_\mu \leq \widetilde{u}_\lambda \leq u_\lambda^*.$$

We have

$$\begin{cases} -\Delta_p u_\lambda^* - \Delta u_\lambda^* + \xi(z)(u_\lambda^*)^{p-1} = \lambda\eta_0 (u_\lambda^*)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u_\lambda^*}{\partial n_{p2}} = \lambda (u_\lambda^*)^{\tau-1} & \text{on } \partial\Omega, \\ \lambda > 0,\ \tau < 2 < p < r. \end{cases} \tag{3.28}$$

Let $a(z) = \eta_0(u_\lambda^*(z))^{r-2}$ and $d(z) = u_\lambda^*(z)^{\tau-2}$. Then $a \in L^\infty(\Omega)$ and $d \in C(\overline{\Omega})$. We rewrite (3.28) using $a(\cdot)$ and $d(\cdot)$. So, we have

$$\begin{cases} -\Delta_p u_\lambda^* - \Delta u_\lambda^* + \xi(z)(u_\lambda^*)^{p-1} = \lambda a(z) u_\lambda^* & \text{in } \Omega, \\ \dfrac{\partial u_\lambda^*}{\partial n_{p2}} = \lambda d(z) u_\lambda^* & \text{on } \partial\Omega, \\ \lambda > 0. \end{cases} \tag{3.29}$$

Let

$$\widehat{W}_p = \left\{ w \in W^{1,p}(\Omega) : k(w) = \int_\Omega a(z) w \, dz + \int_{\partial\Omega} d(z) w \, d\sigma = 0 \right\}.$$

We have $W^{1,p}(\Omega) = \mathbb{R} \oplus \widehat{W}_p$ (see Abreu-Madeira [1], Lemma 2.2). Then from (3.29) and Theorem 1.1 of [1], we have

$$0 < \lambda \leq \widehat{c} \inf \left[\frac{\frac{1}{p}\gamma_p(w) + \frac{1}{2}\|Dw\|_2^2}{k(w)} : w \in \widehat{W}_p,\ w \neq 0 \right] < \infty \quad \text{for some } \widehat{c} > 0.$$

This fact combined with Proposition 3.8 implies that we have $\lambda^* < \infty$. $\square$

Proposition 3.10. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(f)'$ *hold and* $\lambda \in (0, \lambda^*)$, *then problem* (P_λ) *admits at least two positive solutions:*

$$u_0, \widehat{u} \in \operatorname{int} C_+,\ u_0 \leq \widehat{u},\ u_0 \neq \widehat{u}.$$

Proof. Let $\vartheta \in (\lambda, \lambda^*)$. Using Proposition 3.5 we can find $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ and $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta - u_0 \in D_+. \tag{3.30}$$

We introduce the following truncations of the data of (P_λ)

$$\widehat{\mu}(z,x) = \begin{cases} f(z, u_0(z)) & \text{if } x \leq u_0(z) \\ f(z,x) & \text{if } u_0(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.31}$$

$$\widehat{w}_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{\tau-1} & \text{if } x \leq u_0(z) \\ \lambda x^{\tau-1} & \text{if } u_0(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.32}$$

These are Carathéodory functions. We set

$$\widehat{M}(z,x) = \int_0^x \widehat{\mu}(z,s) \, ds \quad \text{and} \quad \widehat{W}_\lambda(z,x) = \int_0^x \widehat{w}_\lambda(z,s) \, ds$$

and consider the C^1-functional $\widehat{d}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{d}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{M}(z,u) \, dz - \int_{\partial\Omega} \widehat{W}_\lambda(z,u) \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

In addition, we introduce the following truncations of $\widehat{\mu}(z,\cdot)$ and of $\widehat{w}_\lambda(z,\cdot)$

$$\widehat{\mu}_0(z,x) = \begin{cases} \widehat{\mu}(z,x) & \text{if } x \leq u_\vartheta(z) \\ \widehat{\mu}(z, u_\vartheta(z)) & \text{if } u_\vartheta(z) < x \end{cases} \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}, \tag{3.33}$$

$$\widehat{w}_\lambda^0(z,x) = \begin{cases} \widehat{w}_\lambda(z,x) & \text{if } x \leq u_\vartheta(z) \\ \widehat{w}_\lambda(z, u_\vartheta(z)) & \text{if } u_\vartheta(z) < x \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{3.34}$$

These are Carathéodory functions. We set

$$\widehat{M}_0(z,x) = \int_0^x \widehat{\mu}_0(z,s) \, ds \quad \text{and} \quad \widehat{W}_\lambda^0(z,x) = \int_0^x \widehat{w}_\lambda^0(z,s) \, ds$$

and consider the C^1-functional $\widehat{d}_\lambda^0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{d}_\lambda^0(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{M}_0(z,u) \, dz - \int_{\partial\Omega} \widehat{W}_\lambda^0(z,u) \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.31), (3.32), (3.33) and (3.34) it is clear that

$$\widehat{d}_\lambda\Big|_{[0,u_\vartheta]} = \widehat{d}^{\,0}_\lambda\Big|_{[0,u_\vartheta]} \quad \text{and} \quad \widehat{d}'_\lambda\Big|_{[0,u_\vartheta]} = \left(\widehat{d}^{\,0}_\lambda\right)'\Big|_{[0,u_\vartheta]}. \tag{3.35}$$

Moreover, we have

$$K_{\widehat{d}_\lambda} \subseteq [u_0) \cap \operatorname{int} C_+ \quad \text{(see (3.31), (3.32))} \tag{3.36}$$

$$K_{\widehat{d}^{\,0}_\lambda} \subseteq [u_0, u_\vartheta] \cap \operatorname{int} C_+ \quad \text{(see (3.33), (3.34))}. \tag{3.37}$$

From (3.35) and (3.36) we see that without any loss of generality we may assume that

$$K_{\widehat{d}_\lambda} \cap [0, u_\vartheta] = \{u_0\}. \tag{3.38}$$

Otherwise we already have a second positive smooth solution of (P_λ) bigger than u_0 (see (3.36)) and so we are done.

From (3.33), (3.34) and (2.3) it is clear that $\widehat{d}^{\,0}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \widehat{d}^{\,0}_\lambda(\widetilde{u}_0) = \min\left[\widehat{d}^{\,0}_\lambda(u) : u \in W^{1,p}(\Omega)\right] \\
\Rightarrow\quad & \widetilde{u}_0 \in [u_0, u_\vartheta] \cap \operatorname{int} C_+ \quad \text{(see (3.37))} \\
\Rightarrow\quad & \widetilde{u}_0 \in K_{\widehat{d}_\lambda} \quad \text{(see (3.35))} \\
\Rightarrow\quad & \widetilde{u}_0 = u_0 \quad \text{(see (3.38))}.
\end{aligned}$$

From (3.30) and (3.35) it follows that

$$\begin{aligned}
& u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } d_\lambda \\
\Rightarrow\quad & u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } d_\lambda \\
& \text{(see Papageorgiou–Rădulescu [17], Proposition 2.12).}
\end{aligned}$$

We assume that $K_{\widehat{d}_\lambda}$ is finite or otherwise on account of (3.36) we already have an infinity of positive smooth solutions bigger than u_0 and so we are done. Invoking Theorem 5.7.6, p. 449, of Papageorgiou–Rădulescu–Repovš [21], we can find $\rho \in (0, 1)$ small such that

$$\widehat{d}_\lambda(u_0) < \inf\left[d_\lambda(u) : \|u - u_0\| = \rho\right] = \widehat{m}_\lambda. \tag{3.39}$$

Moreover, on account of hypothesis $\mathrm{H}(f)'$(ii)=$\mathrm{H}(f)$(ii), we have that

$$\widehat{d}_\lambda(\cdot) \text{ satisfies the Palais–Smale condition} \tag{3.40}$$

and if $u \in \operatorname{int} C_+$, then

$$\widehat{d}_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{3.41}$$

Then (3.39), (3.40) and (3.41) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \widehat{u} \in K_{\widehat{d}_\lambda} \quad \text{and} \quad \widehat{m}_\lambda \le d_\lambda(\widehat{u}) \\
\Rightarrow\quad & u_0 \le \widehat{u} \in \operatorname{int} C_+ \text{ (see (3.36))},\ u_0 \ne \widehat{u} \text{ (see (3.39))},\ \widehat{u} \in S_\lambda \text{ (see (3.31), (3.32))}.
\end{aligned}$$

□

Proposition 3.11. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(f)$ *hold, then* $\lambda^* \in \mathscr{L}$.

Proof. Let $\lambda_n \uparrow \lambda^*$ as $n \to \infty$. We can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$, such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N} \text{ (see the proof of Proposition 3.2)}, \tag{3.42}$$

$$\varphi'_{\lambda_n}(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.43}$$

From (3.42), (3.43) and hypothesis H(f)(ii) (the AR-condition) we deduce that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_{\lambda^*} \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_{\lambda^*} \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.44}$$

From (3.43) we have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h \,\mathrm{d}z = \int_\Omega f(z, u_n) h \,\mathrm{d}z + \lambda_n \int_{\partial\Omega} u_n^{\tau-1} h \,\mathrm{d}\sigma \tag{3.45}$$

for all $h \in W^{1,p}(\Omega)$.

We choose $h = u_n - u_{\lambda^*} \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.44). Then

$$\begin{aligned}
& \lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u_{\lambda^*}\rangle + \langle A(u_n), u_n - u_{\lambda^*}\rangle\right] = 0 \\
\Rightarrow\quad & \limsup_{n\to+\infty} \left[\langle A_p(u_n), u_n - u_{\lambda^*}\rangle + \langle A(u_{\lambda^*}), u_n - u_{\lambda^*}\rangle\right] \leq 0 \quad \text{(since } A(\cdot) \text{ is monotone)} \\
\Rightarrow\quad & \limsup_{n\to\infty} \langle A_p(u_n), u_n - u_{\lambda^*}\rangle \leq 0 \quad \text{(see (3.44))} \\
\Rightarrow\quad & u_n \to u_{\lambda^*} \quad \text{in } W^{1,p}(\Omega) \text{ (see Proposition 2.1).}
\end{aligned} \tag{3.46}$$

Passing to the limit as $n \to \infty$ in (3.45) and using (3.46), we obtain

$$\langle A_p(u_{\lambda^*}), h\rangle + \langle A(u_{\lambda^*}), h\rangle + \int_\Omega \xi(z) u_{\lambda^*}^{p-1} h \,\mathrm{d}z = \int_\Omega f(z, u_{\lambda^*}) h \,\mathrm{d}z + \lambda^* \int_{\partial\Omega} u_{\lambda^*}^{\tau-1} h \,\mathrm{d}\sigma \tag{3.47}$$

for all $h \in W^{1,p}(\Omega)$,

$$\widetilde{u}_{\lambda_1} \leq u_\lambda \quad \text{(see Claim 2 in the proof of Proposition 3.6 and Proposition 3.7(b))}. \tag{3.48}$$

From (3.47) and (3.48) we infer that

$$u_{\lambda^*} \in S_{\lambda^*}, \text{ that is, } \lambda^* \in \mathscr{L}. \qquad \square$$

Therefore we have

$$\mathscr{L} = (0, \lambda^*].$$

Next we examine the properties of the minimal solution map $\lambda \mapsto \overline{u}_\lambda$ from $\mathscr{L}$ into $C^1(\overline{\Omega})$.

Proposition 3.12. *If hypotheses* H(ξ), H(f)′ *hold, then the minimal solution map* $\lambda \mapsto \overline{u}_\lambda$ *from* $\mathscr{L}$ *into* $C^1(\overline{\Omega})$ *is*

(a) *strictly increasing in the sense that*

$$0 < \mu < \lambda \leq \lambda^* \quad \Rightarrow \quad \overline{u}_\lambda - \overline{u}_\mu \in D_+;$$

(b) *left continuous.*

Proof.

(a) Let $0 < \mu < \lambda \le \lambda^*$. According to Proposition 3.5, we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} & \overline{u}_\lambda - u_\mu \in D_+ \\ \Rightarrow \quad & \overline{u}_\lambda - \overline{u}_\mu \in D_+ \quad (\text{since } \overline{u}_\mu \le u \text{ for all } u \in S_\mu). \end{aligned}$$

(b) Let $\lambda_n \uparrow \lambda \in \mathscr{L}$. We have $\overline{u}_n = \overline{u}_{\lambda_n} \le \overline{u}_{\lambda^*} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. So, from Theorem 2 of Lieberman [13], we know that there exist $\alpha \in (0,1)$ and $c_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le c_5 \quad \text{for all } n \in \mathbb{N}. \tag{3.49}$$

Exploiting the fact that $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly and the monotonicity of $\{\overline{u}_n\}_{n\ge 1}$ (see part (a)), from (3.49) we have

$$\overline{u}_n \to \widehat{u}_\lambda \quad \text{in } C^1(\overline{\Omega}). \tag{3.50}$$

If $\widehat{u}_\lambda \neq \overline{u}_\lambda$, then we can find $z_0 \in \overline{\Omega}$ such that $\overline{u}_\lambda(z_0) < \widehat{u}_\lambda(z_0)$. On account of (3.50) we have

$$\overline{u}_\lambda(z_0) < \overline{u}_n(z_0) \quad \text{for all } n \ge n_0,$$

which contradicts part (a). So, we conclude that $\lambda \mapsto \overline{u}_\lambda$ is left continuous. $\square$

The following bifurcation-type theorem describes the dependence on the parameter $\lambda > 0$ of the set of positive solutions of (P_λ).

Theorem 3.13. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(f)'$ *hold, then there exists* $\lambda^* > 0$ *such that*

(a) *for all* $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *admits at least two positive solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+,\ u_0 \le \widehat{u},\ u_0 \neq \widehat{u};$$

(b) *for* $\lambda = \lambda^*$ *problem* (P_λ) *has at least one positive solution* $u_{\lambda^*} \in \operatorname{int} C_+$;

(c) *for all* $\lambda > \lambda^*$ *there are no positive solutions;*

(d) *for all* $\lambda \in \mathscr{L} = (0, \lambda^*]$ *problem* (P_λ) *has a smallest positive solution*

$$\overline{u}_\lambda \in \operatorname{int} C_+$$

and the map $\lambda \mapsto \overline{u}_\lambda$ *from* $\mathscr{L}$ *into* $C^1(\overline{\Omega})$ *is*

- *strictly increasing, that is,* $0 < \mu < \lambda \le \lambda^* \Rightarrow \overline{u}_\lambda - \overline{u}_\mu \in D_+$;
- *left continuous.*

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