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A remark on Philos-type oscillation criteria for differential equations

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Abstract. The purpose of this short note is to call attention to expressions of Philos-type criteria for the oscillation of solutions of a simple differential equation. A perfect square expression is used to obtain an evaluation that plays an essential role in the proof of Philos-type oscillation theorems. The required condition is pointed out when using the perfect square expression. To simplify the discussion, here we deal with two second-order linear differential equations, but its content is also applied to a variety of equations.

Keywords: oscillation of solutions, perfect square expression, Riccati transformation, integral averaging technique.

2020 Mathematics Subject Classification: Primary 34C10, 34C29, 34K11; Secondary 35B05, 39A21.

1 Introduction

Oscillation problems represent one of the main themes of qualitative theory of differential equations. Over many years, a large number of sufficient (or necessary) conditions have been reported by numerous researchers for the oscillation of solutions of the second-order linear differential equation

$$x'' + c(t)x = 0, \qquad t \geq t_0, \tag{1.1}$$

and equations that generalize it to various directions. Here, the coefficient c is a continuous real-valued function on $[t_0, \infty)$. Since equation (1.1) is linear, all solutions are guaranteed to exist until an infinite amount of time. For this reason, all nontrivial solutions of (1.1) can be classified into two groups. A nontrivial solution x of (1.1) is said to be *oscillatory* if there exists a divergence sequence $\{t_n\}$ such that $x(t_n) = 0$, and otherwise, it said to be *nonoscillatory*. Sturm's separation theorem ensures that if there is an oscillatory solution of (1.1), then all nontrivial solutions of (1.1) are oscillatory. Equation (1.1) is often called oscillatory if all nontrivial solutions of (1.1) are oscillatory.

As an example of the many superior conditions to ensure that equation (1.1) is oscillatory, we can cite Philos's criterion as follows.

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Theorem A. *Let* $H\colon D \stackrel{\text{def}}{=} \{(t,s)\colon t \geq s \geq t_0\} \to \mathbb{R}$ *be a continuous function, which is such that*

$$H(t,t) = 0 \quad \text{for } t \geq t_0 \quad \text{and} \quad H(t,s) > 0 \quad \text{for } t > s \geq t_0$$

and has a continuous and nonpositive partial derivative on D *with respect to the second variable. Moreover, let* $h\colon D \to [0,\infty)$ *be a continuous function with*

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D. \tag{1.2}$$

Then equation (1.1) *is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s) \right) ds = \infty. \tag{1.3}$$

The feature of Theorem A is to use an auxiliary function H that is not directly related to equation (1.1) in order to examine the oscillation of solutions of (1.1). One thing to note here is that the domain of h is the same D as that of H. When we choose $(t-s)^\alpha$ as the auxiliary function H, from condition (1.2), the function h becomes $\alpha(t-s)^{(\alpha-2)/2}$. Hence, in order for the function h to be continuous on D, the exponent α must be greater than or equal to 2. As long as $\alpha > 1$, we can easily confirm that $\int_{t_0}^{t} h^2(t,s)ds/H(t,t_0)$ tends to 0 as $t \to \infty$ and $t^\alpha/H(t,t_0)$ converges to 1 as $t \to \infty$. From Theorem A and these facts, we see that equation (1.1) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha c(s)ds = \infty, \tag{1.4}$$

where $\alpha \geq 2$. In other words, Theorem A can be said to be a partial extension of the criterion was given by Kamenev [3]:

Theorem B. *Equation* (1.1) *is oscillatory if condition* (1.4) *holds for some* $\alpha > 1$.

2 Additional condition for generalization

In order for Theorem A to completely cover Theorem B, it needs to change the domain D of h to $D_0 \stackrel{\text{def}}{=} \{(t,s)\colon t > s \geq t_0\}$ and rewrite condition (1.2) to

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D_0. \tag{2.1}$$

The reason is that the function $\alpha(t-s)^{(\alpha-2)/2}$ becomes nonnegative and continuous on D_0 provided that $\alpha \geq 0$. Of course, even when α is 1, the function h satisfying condition (2.1) is nonnegative and continuous on D_0. From such a consideration, if the domain of h is changed as described previously, Theorem A may seem to hold in the form as follows:

Proposition C. *Let* H *be the same function as in Theorem A. Suppose that there exists a continuous function* $h\colon D_0 \to [0,\infty)$ *satisfying condition* (2.1)*. Then equation* (1.1) *is oscillatory if condition* (1.3) *is satisfied.*

However, there is an issue to be discussed here. Philos [6] used the method of proof by contradiction together with the Riccati transformation and integral averaging techniques to obtain Theorem A. Let x be a nonoscillatory solution of (1.1) and let

$$w(t) = \frac{x'(t)}{x(t)} \quad \text{for } t \geq T,$$

where T is a sufficiently large number. Then equation (1.1) becomes

$$c(t) = -w'(t) - w^2(t) \quad \text{for every } \ t \geq T.$$

Using this Riccati equation and condition (1.2), we obtain

$$\int_T^t H(t,s)c(s)ds = H(t,T)w(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds - \int_T^t H(t,s)w^2(s)ds$$

for $t \geq T$. The right-hand side of this evaluation is rewritten as follows:

$$\begin{aligned}
&H(t,T)w(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds - \int_T^t H(t,s)w^2(s)ds \\
&\quad = H(t,T)w(T) + \frac{1}{4}\int_T^t h^2(t,s)ds - \frac{1}{4}\int_T^t h^2(t,s)ds - \int_T^t h(t,s)\sqrt{H(t,s)}w(s)ds \\
&\qquad - \int_T^t H(t,s)w^2(s)ds \\
&\quad = H(t,T)w(T) + \frac{1}{4}\int_T^t h^2(t,s)ds - \int_T^t \left(\frac{1}{2}h(t,s) + \sqrt{H(t,s)}w(s)\right)^2 ds.
\end{aligned}$$

In Theorem A, the function h was assumed to be continuous on D; thus, the integral value

$$\int_{t_0}^t h^2(t,s)ds \left(= \int_T^t h^2(t,s)ds + \int_{t_0}^T h^2(t,s)ds\right)$$

exists for each fixed $t \geq t_0$. Hence, the previous perfect square expression is correct. Even if the function h is continuous only on D_0 included in D, there is a possibility that the integral value of h^2 exists for each fixed value $t \geq t_0$. For example, consider $h(t,s) = \alpha(t-s)^{(\alpha-2)/2}$ with $\alpha > 1$. Then we have

$$\int_{t_0}^t h^2(t,s)ds = \frac{\alpha^2}{\alpha-1}(t-t_0)^{\alpha-1} < \infty \quad \text{for each fixed value } \ t \geq t_0.$$

However, in the case where $0 \leq \alpha \leq 1$, this integral value does not exist for each fixed value $t \geq t_0$. Hence, in this case, it is not possible to use a perfect square expression to obtain the evaluation as described previously.

We can therefore conclude that Theorems A and B are correct, but Proposition C cannot be proved simply by changing condition (1.2) to condition (2.1). Proposition C lacks an important condition that is unnoticeable and it needs to be modified as follows:

Theorem 2.1. *Let H be the same function as in Theorem A. Suppose that there exists a continuous function $h\colon D_0 \to [0,\infty)$ satisfying condition (2.1) and*

$$\int_{t_0}^t h^2(t,s)ds < \infty \quad \textit{for each fixed value } \ t \geq t_0. \tag{2.2}$$

Then equation (1.1) is oscillatory if condition (1.3) is satisfied.

Proof. As mentioned above, we use the contradiction method. Suppose that equation (1.1) has a nonoscillatory solution x. We may assume without loss of generality that there exists a $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$. Using the Riccati transformation

$$w(t) = \frac{x'(t)}{x(t)} \quad \text{for } \ t \geq T,$$

we can rewrite equation (1.1) to

$$c(t) = -w'(t) - w^2(t) \quad \text{for} \;\; t \geq T.$$

Note that $H(t,t) = 0$. Applying integration by parts together with condition (2.1), we obtain

$$\begin{aligned}
\int_T^t H(t,s)c(s)ds &= -\int_T^t H(t,s)w'(s)ds - \int_T^t H(t,s)w^2(s)ds \\
&= H(t,T)w(T) + \int_T^t \left(\frac{\partial}{\partial s}H(t,s)\right)w(s)ds - \int_T^t H(t,s)w^2(s)ds \\
&= H(t,T)w(T) - \int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s)\right)ds \qquad (2.3)
\end{aligned}$$

for $t \geq T$. Using a perfect square expression, we have

$$\begin{aligned}
h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s) &= -\frac{1}{4}h^2(t,s) + \left(\frac{1}{2}h(t,s) + \sqrt{H(t,s)}w(s)\right)^2 \\
&\geq -\frac{1}{4}h^2(t,s)
\end{aligned}$$

for all $(t,s) \in D_0$. The function h is continuous on D_0, but not necessarily continuous on D. For this reason, the integral of h on D may be an improper integral. However, condition (2.2) guarantees that this integral always converges to a finite value. Hence, we can obtain the inequality

$$-\int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)w^2(s)\right)ds \leq \frac{1}{4}\int_T^t h^2(t,s)ds < \infty$$

for each fixed $t \geq T$. Combining (2.3) and this inequality, we get

$$\int_T^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \leq H(t,T)w(T).$$

Hence, we have

$$\begin{aligned}
\int_{t_0}^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds &= \int_{t_0}^T \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \\
&\quad + \int_T^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds \\
&\leq \int_{t_0}^T H(t,s)c(s)ds + H(t,T)w(T).
\end{aligned}$$

From the assumption of $\partial H(t,s)/\partial s$, we see that

$$H(t,t_0) \geq H(t,s) > 0 \quad \text{for} \;\; t > s \geq t_0.$$

Hence, we have

$$\begin{aligned}
\frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)c(s) - \frac{1}{4}h^2(t,s)\right)ds &\leq \int_{t_0}^T \frac{H(t,s)}{H(t,t_0)}c(s)ds + \frac{H(t,T)}{H(t,t_0)}w(T) \\
&\leq \int_{t_0}^T c(s)ds + w(T) < \infty
\end{aligned}$$

for $t > t_0$. This contradicts condition (1.3). Thus, the proof of Theorem 2.1 is complete. $\square$

3 Discussion

As the auxiliary function H, we choose $t-s$; namely, the case where the power index α used in Sections 1 and 2 corresponds to 1. In this case, as already mentioned, all assumptions of Proposition C are satisfied. Also, condition (1.3) becomes

$$\limsup_{t\to\infty} \frac{1}{t}\left(\int_{t_0}^{t}(t-s)c(s)ds - \frac{1}{4}\int_{t_0}^{t}\frac{1}{t-s}ds\right) = \infty \tag{3.1}$$

because $H(t,s) = t-s$ and $h^2(t,s) = 1/(t-s)$, and

$$\frac{t}{H(t,t_0)} = \frac{t}{t-t_0} \to 1 \quad \text{as } t \to \infty.$$

There are two integrals in evaluation (3.1). The former is a proper (or normal) integral but the latter is an improper integral. The latter improper integral diverges to infinity for each fixed value $t > t_0$ because

$$\int_{t_0}^{t}\frac{1}{t-s}ds = \lim_{\varepsilon\to 0^+}\int_{t_0}^{t-\varepsilon}\frac{1}{t-s}ds = \lim_{\varepsilon\to 0^+}\big(\ln(t-t_0) - \ln\varepsilon\big) = \infty$$

for each fixed value $t > t_0$. Hence, this improper integral cannot be defined for any $t > t_0$. For this reason, the evaluation (3.1) has no meaning.

If the above expression is meaningful and has a finite value, condition (3.1) is identical with the assumption

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)c(s)ds = \infty.$$

This assumption is equivalent to

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds = \infty. \tag{3.2}$$

Wintner [10] proved that equation (1.1) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds = \infty.$$

Three years later, Hartman [2] reported that equation (1.1) is oscillatory if

$$-\infty < \liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds < \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}c(\tau)d\tau ds \le \infty.$$

Even after that, many researchers have continued to improve sufficient conditions for equation (1.1) to be oscillatory. However, it is not yet settled whether equation (1.1) is oscillatory or not in the case where condition (3.2) alone is satisfied.

As described in Section 2, Proposition C cannot be proved using a perfect square expression. The proof becomes incomplete. The above discussion shows that if Proposition C holds, condition (3.2) is a sufficient condition for equation (1.1) to be oscillatory. But, the author thinks that it is not possible to judge whether equation (1.1) is oscillatory or not without adding another condition to condition (3.2), as Hartman [2] showed.

4 Attention

Li [4] considered a second-order linear differential equation of the self-adjoint form

$$\big(r(t)x'\big)' + c(t)x = 0, \qquad t \geq t_0. \tag{4.1}$$

Here, the coefficients r and c are continuous real-valued functions on $[t_0, \infty)$ and it is assumed that $r(t) > 0$ for all $t \geq t_0$. He gave the following Philos-type oscillation criterion.

Theorem D. *Let H and h be the same functions as in Proposition C. Suppose that there exists a continuous differentiable function $f\colon [t_0, \infty) \to \mathbb{R}$ such that*

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty \quad \textit{for all} \ \ t \geq t_0 \tag{4.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\psi(s) - \frac{1}{4}a(s)r(s)h^2(t,s) \right) ds = \infty,$$

where $a(s) = \exp\{-2\int^s f(\xi)d\xi\}$ and $\psi(s) = a(s)\{c(s)r(s)f^2(s) - \big(r(s)f(s)\big)'\}$. Then, equation (4.1) is oscillatory.

Note that if the coefficient r is smooth enough (continuously differentiable more than twice), then equations (1.1) and (4.1) can be transformed into the form of each other. In fact, by letting $x = \sqrt{r(t)}u$ for any positive and smooth enough function r, equation (1.1) becomes

$$\big(r(t)u'\big)' + \left(c(t)r(t) - \frac{\big(r'(t)\big)^2}{4r(t)} + \frac{1}{2}r''(t) \right) u = 0.$$

Conversely, by changing $u = \sqrt{r(t)}x$, equation (4.1) becomes

$$u'' + \left(\frac{c(t)}{r(t)} + \frac{\big(r'(t)\big)^2}{4r^2(t)} - \frac{r''(t)}{2r(t)} \right) u = 0.$$

Rogovchenko [7] observed that condition (4.2) appears to be superfluous. Certainly, the expression of condition (4.2) is incorrect (there is the same mistake in [5]). However, Theorem D does not hold only by deleting condition (4.2), because equation (4.1) contains equation (1.1). It is necessary to assume the condition

$$\int_{t_0}^{t} a(s)r(s)h^2(t,s)ds < \infty \quad \text{for each fixed value} \ \ t \geq t_0, \tag{4.3}$$

which plays the same role as condition (2.2), instead of condition (4.2). By using condition (4.3), a perfect square expression will have the correct meaning in the proof of Theorem D. We omit the proof of the result of changing condition (4.2) to condition (4.3).

Philos's criterion has been improved, extending its applicability to a variety of equations. For example, those results can be found in studies on nonlinear differential equations including the Emden–Fowler equation, half-linear differential equations with and without the self-adjoint form, damped differential equations with and without time-delay, higher-order differential equations, matrix differential systems, elliptic partial differential equations, Hamiltonian systems, difference equations, dynamic equations and others. When the function h is continuous on D, there seems to be no problem with the result. However, one needs to pay close attention to an additional condition when defining the domain of h to D_0 and assuming a condition such as (2.1). Unfortunately, there are mistakes arising from this carelessness in some previous research papers. For example, Proposition C which is wrong is included in [1, Corollary 3.3], [8, Theorem 2.1], [9, Theorem 2.1], [11, Theorem 3.4] and [12, Theorem 2.8].

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