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Strong solutions to the nonhomogeneous Boussinesq equations for magnetohydrodynamics convection without thermal diffusion

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Abstract. We are concerned with the Cauchy problem of nonhomogeneous Boussinesq equations for magnetohydrodynamics convection in $\mathbb{R}^2$. We show that there exists a unique local strong solution provided the initial density, the magnetic field, and the initial temperature decrease at infinity sufficiently quickly. In particular, the initial data can be arbitrarily large and the initial density may contain vacuum states.

Keywords: nonhomogeneous Boussinesq-MHD system, strong solutions, Cauchy problem.

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1 Introduction

Consider the following nonhomogeneous Boussinesq system for magnetohydrodynamic convection (Boussinesq-MHD) in $\mathbb{R}^2$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b} + \rho \theta \mathbf{e}_2, \\ \theta_t + \mathbf{u} \cdot \nabla \theta = 0, \\ \mathbf{b}_t - \nu \Delta \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{b} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is time, $x = (x_1, x_2) \in \mathbb{R}^2$ is the spatial coordinate, and $\rho = \rho(x,t)$, $\mathbf{u} = (u^1, u^2)(x,t)$, $\mathbf{b} = (b^1, b^2)(x,t)$, $\theta = \theta(x,t)$, and $P = P(x,t)$ denote the density, velocity, magnetic field, temperature, and pressure of the fluid, respectively. The coefficients μ and ν are positive constants. $\mathbf{e}_2 = (0,1)^T$, where T is the transpose.

We consider the Cauchy problem for (1.1) with the far field behavior

$$(\rho, \mathbf{u}, \theta, \mathbf{b}) \to (0, \mathbf{0}, 0, \mathbf{0}), \quad \text{as } |x| \to \infty, \tag{1.2}$$

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and the initial condition

$$\rho(x,0)=\rho_0(x),\quad \rho\mathbf{u}(x,0)=\rho_0\mathbf{u}_0(x),\quad \theta(x,0)=\theta_0(x),\quad \mathbf{b}(x,0)=\mathbf{b}_0(x),\quad x\in\mathbb{R}^2, \tag{1.3}$$

for given initial data ρ_0, $\mathbf{u}_0$, θ_0, and $\mathbf{b}_0$.

The system (1.1) is a combination of the nonhomogeneous Boussinesq equations of fluid dynamics and Maxwell's equations of electromagnetism, where the displacement current can be neglected. The Boussinesq-MHD system models the convection of an incompressible flow driven by the buoyant effect of a thermal or density field, and the Lorenz force, generated by the magnetic field of the fluid and the Lorentz force. Specifically, it closely relates to a natural type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with the presence of a magnetic field. For more physics background, one may refer to [7,14,16] and references therein.

When ρ is constant, the system (1.1) reduces to the homogeneous Boussinesq-MHD system. Recently, the well-posedness issue of solutions has attracted much attention. Bian [3] studied the initial boundary value problem of two-dimensional (2D) viscous Boussinesq-MHD system and obtained a unique classical solution for H^3 initial data. Without smallness assumption on the initial data, Bian and Gui [4] proved the global unique solvability of 2D Boussinesq-MHD system with the temperature-dependent viscosity, thermal diffusivity, and electrical conductivity. Later on, the authors [5] established the global existence of weak solutions with H^1 initial data. By imposing a higher regularity assumption on the initial data, they also obtained a unique global strong solution. In [10], Larios and Pei proved the local well-posedness of solutions to the fully dissipative 3D Boussinesq-MHD system, and also the fully inviscid, irresistive, non-diffusive Boussinesq-MHD system. Moreover, they also provided a Prodi–Serrin-type global regularity condition for the 3D Boussinesq-MHD system without thermal diffusion, in terms of only two velocity and two magnetic components. By Fourier localization techniques, Zhai and Chen [20] investigated well-posedness to the Cauchy problem of the Boussinesq-MHD system with the temperature-dependent viscosity in Besov spaces. Very recently, Liu et al. [13] showed the global existence and uniqueness of strong and smooth large solutions to the 3D Boussinesq-MHD system with a damping term. Meanwhile, Bian and Pu [6] proved global axisymmetric smooth solutions for the 3D Boussinesq-MHD equations without magnetic diffusion and heat convection.

If the fluid is not affected by the Lorentz force (i.e., $\mathbf{b}=\mathbf{0}$), then the system (1.1) becomes the nonhomogeneous Boussinesq system. The authors [9,21] studied regularity criteria for 3D nonhomogeneous incompressible Boussinesq equations, while Qiu and Yao [17] showed the local existence and uniqueness of strong solutions of multi-dimensional nonhomogeneous incompressible Boussinesq equations in Besov spaces. A blow-up criterion was also obtained in [17]. We should point out here that the results in [9,17,21] always require the initial density is bounded away from zero. For the initial density allowing vacuum states, Zhong [22] recently showed local existence of strong solutions of the Cauchy problem in $\mathbb{R}^2$ by making use of weighted energy estimate techniques. In this paper, we will investigate the local existence of strong solutions to the problem (1.1)–(1.3) with zero density at infinity. The initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, in particular, the initial density can even have compact support.

Before stating our main result, we first explain the notations and conventions used throughout this paper. For $r>0$, set

$$B_r \triangleq \left\{x\in\mathbb{R}^2 \mid |x|<r\right\}.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by:

$$L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = H^{k,2}(\mathbb{R}^2), \quad D^{k,p} = \{u \in L^1_{\text{loc}} \mid \nabla^k u \in L^p\}.$$

Our main result can be stated as follows:

Theorem 1.1. *Let η_0 be a positive constant and*

$$\bar{x} \triangleq \left(3 + |x|^2\right)^{\frac{1}{2}} \log^{1+\eta_0}\left(3 + |x|^2\right). \tag{1.4}$$

For constants $q > 2$ and $a > 1$, we assume that the initial data $(\rho_0 \ge 0, \mathbf{u}_0, \theta_0 \ge 0, \mathbf{b}_0)$ satisfy

$$\begin{cases} \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \ \theta_0 \in H^1 \cap W^{1,q}, \\ \sqrt{\rho_0}\mathbf{u}_0 \in L^2, \ \nabla \mathbf{u}_0 \in L^2, \ \operatorname{div} \mathbf{u}_0 = 0, \\ \mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2, \ \nabla \mathbf{b}_0 \in L^2, \ \operatorname{div} \mathbf{b}_0 = 0. \end{cases} \tag{1.5}$$

Then there exists a positive time $T_0 > 0$ such that the problem (1.1)–(1.3) has a strong solution $(\rho \ge 0, \mathbf{u}, \theta \ge 0, \mathbf{b})$ on $\mathbb{R}^2 \times (0, T_0]$ satisfying

$$\begin{cases} \rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \\ \rho \bar{x}^a \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho}\mathbf{u}, \ \nabla \mathbf{u}, \ \sqrt{t}\sqrt{\rho}\mathbf{u}_t, \ \sqrt{t}\nabla^2 \mathbf{u} \in L^\infty(0, T_0; L^2), \\ \theta \in C([0, T_0]; H^1 \cap W^{1,q}), \\ \mathbf{b}, \mathbf{b}\bar{x}^{\frac{a}{2}}, \nabla \mathbf{b}, \sqrt{t}\mathbf{b}_t, \ \sqrt{t}\nabla^2 \mathbf{b} \in L^\infty(0, T_0; L^2), \\ \nabla \mathbf{u} \in L^2(0, T_0; H^1) \cap L^{\frac{q+1}{q}}(0, T_0; W^{1,q}), \\ \nabla \mathbf{b} \in L^2(0, T_0; H^1), \ \mathbf{b}_t, \ \nabla \mathbf{b}\bar{x}^{\frac{a}{2}} \in L^2(0, T_0; L^2), \\ \sqrt{t}\nabla \mathbf{u} \in L^2(0, T_0; W^{1,q}), \\ \sqrt{\rho}\mathbf{u}_t, \ \sqrt{t}\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}, \ \sqrt{t}\nabla \mathbf{u}_t, \ \sqrt{t}\nabla \mathbf{b}_t \in L^2(\mathbb{R}^2 \times (0, T_0)), \end{cases} \tag{1.6}$$

and

$$\inf_{0 \le t \le T_0} \int_{B_{N_1}} \rho(x,t)dx \ge \frac{1}{4}\int_{\mathbb{R}^2} \rho_0(x)dx, \tag{1.7}$$

for some positive constant N_1. Moreover, if $\theta_0 \bar{x}^a \in H^1 \cap W^{1,q}$, then the strong solution just established is unique.

Remark 1.2. When there is no electromagnetic field effect, that is $\mathbf{b} = \mathbf{0}$, (1.1) turns to be the nonhomogeneous Boussinesq equations, and Theorem 1.1 is the same as that of in [22]. Hence we generalize the main result of [22] to the nonhomogeneous Boussinesq-MHD system (1.1). However, compared with [22], for the system (1.1) treated here, the strong coupling between the velocity field and the magnetic field, such as $\mathbf{u} \cdot \nabla \mathbf{b}$, as well as strong nonlinearity $\mathbf{b} \cdot \nabla \mathbf{b}$, will bring out some new difficulties. To this end, we require $\mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2$ and $\nabla \mathbf{b}_0 \in L^2$ beyond the typical hypothesis of $\mathbf{b}_0 \in H^1$. This additional hypothesis is needed in order to obtain the estimate (3.10), which plays a crucial role in dealing with coupling between the velocity field and the magnetic field.

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 is devoted to the a priori estimates which are needed to obtain the local existence of strong solutions. The main result Theorem 1.1 is proved in Section 4.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as in [19].

Lemma 2.1. *For $R>0$ and $B_R=\{x\in\mathbb{R}^2 \mid |x|<R\}$, assume that $(\rho_0,\mathbf{u}_0,\theta_0\geq 0,\mathbf{b}_0)$ satisfies*

$$(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0)\in H^2(B_R),\quad \inf_{x\in B_R}\rho_0(x)>0,\quad \operatorname{div}\mathbf{u}_0=\operatorname{div}\mathbf{b}_0=0. \tag{2.1}$$

Then there exists a small time $T_R>0$ and a unique classical solution $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ to the following initial-boundary-value problem

$$\begin{cases}\rho_t+\operatorname{div}(\rho\mathbf{u})=0,\\ (\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}+\nabla P=\mathbf{b}\cdot\nabla\mathbf{b}+\rho\theta\mathbf{e}_2,\\ \theta_t+\mathbf{u}\cdot\nabla\theta=0,\\ \mathbf{b}_t-\nu\Delta\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}=\mathbf{0},\\ \operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{b}=0,\\ (\rho,\mathbf{u},\theta,\mathbf{b})(x,t=0)=(\rho_0,\mathbf{u}_0,\theta_0,\mathbf{b}_0), & x\in B_R,\\ \mathbf{u}(x,t)=\mathbf{b}(x,t)=\mathbf{0}, & x\in\partial B_R,\ t>0,\end{cases} \tag{2.2}$$

on $B_R\times(0,T_R]$ such that

$$\begin{cases}(\rho,\theta)\in C\left([0,T_R];H^2\right),\\ (\mathbf{u},\mathbf{b})\in C\left([0,T_R];H^2\right)\cap L^2\left(0,T_R;H^3\right),\\ P\in C\left([0,T_R];H^1\right)\cap L^2\left(0,T_R;H^2\right),\end{cases} \tag{2.3}$$

where we denote $H^k=H^k(B_R)$ for positive integer k.

Next, for $\Omega\subset\mathbb{R}^2$, the following weighted L^m-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega)\triangleq\{v\in H^1_{\rm loc}(\Omega)|\nabla v\in L^2(\Omega)\}$ can be found in [12, Theorem B.1].

Lemma 2.2. *For $m\in[2,\infty)$ and $s\in(1+\frac{m}{2},\infty)$, there exists a positive constant C such that for either $\Omega=\mathbb{R}^2$ or $\Omega=B_R$ with $R\geq 1$ and for any $v\in\tilde{D}^{1,2}(\Omega)$,*

$$\left(\int_\Omega\frac{|v|^m}{3+|x|^2}(\log(3+|x|^2))^{-s}dx\right)^{\frac{1}{m}}\leq C\|v\|_{L^2(B_1)}+C\|\nabla v\|_{L^2(\Omega)}. \tag{2.4}$$

A useful consequence of Lemma 2.2 is the following crucial weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$, which have been proved in [11, Lemma 2.3].

Lemma 2.3. *Let $\bar{x}$ and η_0 be as in (1.4) and Ω be as in Lemma 2.2. Assume that $\rho\in L^1(\Omega)\cap L^\infty(\Omega)$ is a non-negative function such that*

$$\int_{B_{N_1}}\rho dx\geq M_1,\quad \|\rho\|_{L^1(\Omega)\cap L^\infty(\Omega)}\leq M_2, \tag{2.5}$$

for positive constants M_1, M_2, and $N_1\geq 1$ with $B_{N_1}\subset\Omega$. Then for $\varepsilon>0$ and $\eta>0$, there is a positive constant C depending only on $\varepsilon,\eta,M_1,M_2,N_1$, and η_0 such that every $v\in\tilde{D}^{1,2}(\Omega)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\tilde{\eta}}(\Omega)}\leq C\|\sqrt{\rho}v\|_{L^2(\Omega)}+C\|\nabla v\|_{L^2(\Omega)} \tag{2.6}$$

with $\tilde{\eta}=\min\{1,\eta\}$.

Next, the following L^p-bound for elliptic systems, whose proof is similar to that of [8, Lemma 12], is a direct result of the combination of the well-known elliptic theory [1,2] and a standard scaling procedure.

Lemma 2.4. *For $p > 1$ and $k \geq 0$, there exists a positive constant C depending only on p and k such that*

$$\|\nabla^{k+2} v\|_{L^p(B_R)} \leq C\|\Delta v\|_{W^{k,p}(B_R)}, \tag{2.7}$$

for every $v \in W^{k+2,p}(B_R)$ satisfying

$$v = 0 \quad on\ B_R.$$

3 A priori estimates

Throughout this section, for $r \in [1, \infty]$ and $k \geq 0$, we denote

$$\int \cdot\, dx = \int_{B_R} \cdot\, dx, \quad L^r = L^r(B_R), \quad W^{k,r} = W^{k,r}(B_R), \quad H^k = W^{k,2}.$$

Moreover, for $R > 4N_0 \geq 4$ with N_0 fixed, assume that $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ satisfies, in addition to (2.1), that

$$\frac{1}{2} \leq \int_{B_{N_0}} \rho_0(x)dx \leq \int_{B_R} \rho_0(x)dx \leq 1. \tag{3.1}$$

Thus Lemma 2.1 yields that there exists some $T_R > 0$ such that the initial-boundary-value problem (1.1) and (2.2) has a unique classical solution $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ on $B_R \times [0, T_R]$ satisfying (2.3).

Let $\bar{x}, \eta_0, a,$ and q be as in Theorem 1.1, the main aim of this section is to derive the following key a priori estimate on ψ defined by

$$\psi(t) \triangleq 1 + \|\sqrt{\rho}\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2} + \|\theta\|_{H^1\cap W^{1,q}} + \|\nabla\mathbf{b}\|_{L^2} + \|\bar{x}^{\frac{a}{2}}\mathbf{b}\|_{L^2} + \|\bar{x}^a\rho\|_{L^1\cap H^1\cap W^{1,q}}. \tag{3.2}$$

Proposition 3.1. *Assume that $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ satisfies (2.1) and (3.1). Let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ be the solution to the initial-boundary-value problem (1.1) and (2.2) on $B_R \times (0, T_R]$ obtained by Lemma 2.1. Then there exist positive constants T_0 and M both depending only on $\mu, \nu, \eta_0, q,\ a,\ N_0,$ and E_0 such that*

$$\begin{aligned}
&\sup_{0\leq t\leq T_0}\left[\psi(t) + \sqrt{t}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla^2\mathbf{u}\|_{L^2} + \|\mathbf{b}_t\|_{L^2} + \|\nabla^2\mathbf{b}\|_{L^2} + \|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)\right] \\
&+ \int_0^{T_0}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)dt \\
&+ \int_0^{T_0}\left(\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla^2\mathbf{u}\|_{L^q}^2 + t\|\nabla P\|_{L^q}^2\right)dt \\
&+ \int_0^{T_0}\left(t\|\nabla\mathbf{u}_t\|_{L^2}^2 + t\|\nabla\mathbf{b}_t\|_{L^2}^2 + t\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)dt \leq M,
\end{aligned} \tag{3.3}$$

where

$$E_0 \triangleq \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2} + \|\nabla\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{H^1\cap W^{1,q}} + \|\nabla\mathbf{b}_0\|_{L^2} + \|\bar{x}^{\frac{a}{2}}\mathbf{b}_0\|_{L^2} + \|\bar{x}^a\rho_0\|_{L^1\cap H^1\cap W^{1,q}}.$$

To show Proposition 3.1, whose proof will be postponed to the end of this subsection, we begin with the following standard energy estimate for $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and the estimate on the L^p-norm of the density.

Lemma 3.2. *Under the conditions of Proposition 3.1, let* $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ *be a smooth solution to the initial-boundary-value problem* (1.1) *and* (2.2). *Then for any* $t \in (0, T_1]$,

$$\sup_{0\le s\le t}\left(\|\rho\|_{L^1\cap L^\infty} + \|\theta\|_{L^2\cap L^\infty} + \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^t \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2\right) ds \le C, \tag{3.4}$$

where (and in what follows) C denotes a generic positive constant depending only on $\mu, \nu, q, a, N_0, \eta_0$ *and* E_0. T_1 *is as that of Lemma 3.3.*

Proof. 1. Since $\operatorname{div} \mathbf{u} = 0$, we deduce from $(1.1)_1$ that

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0. \tag{3.5}$$

Define particle path

$$\begin{cases} \frac{d}{dt}\mathbf{X}(x,t) = \mathbf{u}(\mathbf{X}(x,t),t), \\ \mathbf{X}(x,0) = x. \end{cases}$$

Thus, along particle path, we obtain from (3.5) that

$$\frac{d}{dt}\rho(\mathbf{X}(x,t),t) = 0,$$

which implies

$$\rho(\mathbf{X}(x,t),t) = \rho_0. \tag{3.6}$$

Similarly, one derives from $(1.1)_3$ that

$$\theta(\mathbf{X}(x,t),t) = \theta_0. \tag{3.7}$$

2. Multiplying $(1.1)_2$ by $\mathbf{u}$ and then integrating the resulting equation over B_R, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}|^2 dx + \mu \int |\nabla \mathbf{u}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx + \int \rho\theta\mathbf{e}_2\cdot\mathbf{u}dx. \tag{3.8}$$

Multiplying $(1.1)_4$ by $\mathbf{b}$ and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{b}|^2 dx + \nu \int |\nabla \mathbf{b}|^2 dx + \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx = 0,$$

which combined with (3.8) and (3.7) implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \left(\mu\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2\right) &= \int \rho\theta\mathbf{u}\cdot\mathbf{e}_2 dx \\ &\le \|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\theta\|_{L^2} \\ &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + C. \end{aligned} \tag{3.9}$$

Thus, Gronwall's inequality leads to

$$\sup_{0\le s\le t}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^t \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2\right) ds \le C,$$

which together with (3.6) and (3.7) yields (3.4) and completes the proof of Lemma 3.2. □

Next, we will give some spatial weighted estimates on the density and the magnetic.

Lemma 3.3. *Under the conditions of Proposition 3.1, let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ be a smooth solution to the initial-boundary-value problem (1.1) and (2.2). Then there exists a $T_1 = T_1(N_0, E_0) > 0$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t}\left(\|\rho\bar{x}^a\|_{L^1} + \|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right) + \int_0^t \|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C. \tag{3.10}$$

Proof. 1. For $N > 1$, let $\varphi_N \in C_0^\infty(B_N)$ satisfy

$$0 \le \varphi_N \le 1, \quad \varphi_N(x) = 1, \quad \text{if } |x| \le \frac{N}{2}, \ |\nabla\varphi_N| \le CN^{-1}. \tag{3.11}$$

It follows from $(1.1)_1$ and (3.4) that

$$\begin{aligned}\frac{d}{dt}\int \rho\varphi_{2N_0}dx &= \int \rho\mathbf{u}\cdot\nabla\varphi_{2N_0}dx \\ &\ge -CN_0^{-1}\left(\int\rho dx\right)^{\frac{1}{2}}\left(\int\rho|\mathbf{u}|^2dx\right)^{\frac{1}{2}} \ge -\tilde{C}(E_0).\end{aligned} \tag{3.12}$$

Integrating (3.12) and using (3.1) give rise to

$$\inf_{0\le t\le T_1}\int_{B_{2N_0}}\rho dx \ge \inf_{0\le t\le T_1}\int\rho\varphi_{2N_0}dx \ge \int\rho_0\varphi_{2N_0}dx - \tilde{C}T_1 \ge \frac{1}{4}. \tag{3.13}$$

Here, $T_1 \triangleq \min\{1, (4\tilde{C})^{-1}\}$. From now on, we will always assume that $t \le T_1$. The combination of (3.13), (3.4), and (2.6) implies that for $\varepsilon > 0$ and $\eta > 0$, every $v \in \tilde{D}^{1,2}(B_R)$ satisfies

$$\|v\bar{x}^{-\eta}\|^2_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}} \le C(\varepsilon,\eta)\|\sqrt{\rho}v\|_{L^2}^2 + C(\varepsilon,\eta)\|\nabla v\|_{L^2}^2, \tag{3.14}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

2. Noting that

$$|\nabla\bar{x}| \le (3+2\eta_0)\log^{1+\eta_0}(3+|x|^2) \le C(a,\eta_0)\bar{x}^{\frac{4}{8+a}},$$

multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating by parts imply that

$$\begin{aligned}\frac{d}{dt}\|\rho\bar{x}^a\|_{L^1} &= \int\rho(\mathbf{u}\cdot\nabla)\bar{x}a\bar{x}^{a-1}dx \\ &\le C\int\rho|\mathbf{u}|\bar{x}^{a-1+\frac{4}{8+a}}dx \\ &\le C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|\mathbf{u}\bar{x}^{-\frac{4}{8+a}}\|_{L^{8+a}} \\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{8+a}}\|\rho\bar{x}^a\|_{L^1}^{\frac{7+a}{8+a}}(\|\sqrt{\rho}\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}) \\ &\le C\left(1+\|\rho\bar{x}^a\|_{L^1}\right)\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\end{aligned}$$

due to (3.4) and (3.14). This combined with Gronwall's inequality and (3.4) leads to

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{L^1} \le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)ds\right\} \le C. \tag{3.15}$$

3. Multiplying $(1.1)_3$ by $\mathbf{b}\bar{x}^a$ and integrating by parts yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\bar{x}^{a/2}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\bar{x}^{a/2}\|_{L^2}^2 &= \frac{\nu}{2}\int|\mathbf{b}|^2\Delta\bar{x}^a dx + \int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\bar{x}^a dx + \frac{1}{2}\int|\mathbf{b}|^2\mathbf{u}\cdot\nabla\bar{x}^a dx \\ &\triangleq \bar{I}_1 + \bar{I}_2 + \bar{I}_3,\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
|\bar I_1| &\le C\int |\mathbf{b}|^2\bar x^a\bar x^{-2}\log^{2(1-\eta_0)}(3+|x|^2)dx \le C\int |\mathbf{b}|^2\bar x^a dx,\\
|\bar I_2| &\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar x^{\frac a2}\|_{L^4}^2\\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}(\|\nabla\mathbf{b}\bar x^{\frac a2}\|_{L^2}+\|\mathbf{b}\nabla\bar x^{\frac a2}\|_{L^2})\\
&\le C(\|\nabla\mathbf{u}\|_{L^2}^2+1)\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2,\\
|\bar I_3| &\le C\|\mathbf{b}\bar x^{\frac a2}\|_{L^4}\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}\|\mathbf{u}\bar x^{-\frac34}\|_{L^4}\\
&\le C\|\mathbf{b}\bar x^{\frac a2}\|_{L^4}^2+C\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2\left(\|\sqrt\rho\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2\right)\\
&\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

due to Gagliardo–Nirenberg inequality, (3.4), and (3.14). Putting (3.17) into (3.16), we get after using Gronwall's inequality and (3.4) that

$$
\sup_{0\le s\le t}\|\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2+\int_0^t\|\nabla\mathbf{b}\bar x^{\frac a2}\|_{L^2}^2ds \le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)ds\right\}\le C,
\tag{3.18}
$$

which together with (3.15) gives (3.10) and finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha>1$ such that for all $t\in(0,T_1]$,*

$$
\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+\int_0^t\left(\|\sqrt\rho\mathbf{u}_s\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\mathbf{b}_s\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\quad\le C+C\int_0^t\psi^\alpha(s)ds.
\end{aligned}
\tag{3.19}
$$

Proof. 1. It follows from (3.4), (3.10), and (3.14) that for any $\varepsilon>0$ and any $\eta>0$,

$$
\begin{aligned}
\|\rho^\eta v\|_{L^{\frac{2+\varepsilon}{\tilde\eta}}} &\le C\|\rho^\eta\bar x^{\frac{3\tilde\eta a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{3\tilde\eta}}}\|v\bar x^{-\frac{3\tilde\eta a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde\eta}}}\\
&\le C\left(\int\rho^{\frac{4(2+\varepsilon)\eta}{3\tilde\eta}-1}\rho\bar x^a dx\right)^{\frac{3\tilde\eta}{4(2+\varepsilon)}}\|v\bar x^{-\frac{3\tilde\eta a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde\eta}}}\\
&\le C\|\rho\|_{L^\infty}^{\frac{4(2+\varepsilon)\eta-3\tilde\eta}{4(2+\varepsilon)}}\|\rho\bar x^a\|_{L^1}^{\frac{3\tilde\eta}{4(2+\varepsilon)}}\left(\|\sqrt\rho v\|_{L^2}+\|\nabla v\|_{L^2}\right)\\
&\le C\|\sqrt\rho v\|_{L^2}+C\|\nabla v\|_{L^2},
\end{aligned}
\tag{3.20}
$$

where $\tilde\eta=\min\{1,\eta\}$ and $v\in\tilde D^{1,2}(B_R)$. In particular, this together with (3.4) and (3.14) yields

$$
\|\rho^\eta\mathbf{u}\|_{L^{\frac{2+\varepsilon}{\tilde\eta}}}+\|\mathbf{u}\bar x^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde\eta}}}\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}\right),
\tag{3.21}
$$

$$
\|\rho^\eta\theta\|_{L^{\frac{2+\varepsilon}{\tilde\eta}}}+\|\theta\bar x^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde\eta}}}\le C\left(1+\|\nabla\theta\|_{L^2}\right).
\tag{3.22}
$$

2. Multiplying $(1.1)_2$ by $\mathbf{u}_t$ and integrating by parts, one has

$$
\mu\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\le C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx+\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx+\int\rho\theta|\mathbf{u}_t|dx.
\tag{3.23}
$$

We derive from (3.21), Hölder's inequality, and Gagliardo–Nirenberg inequality that

$$\begin{aligned}\int \rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^{\frac{8}{3}}}^2\\ &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\nabla\mathbf{u}\|_{H^1}^{\frac{1}{2}}\\ &\le C\psi^\alpha+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2, \end{aligned}\tag{3.24}$$

where (and in what follows) we use $\alpha>1$ to denote a genetic constant, which may be different from line to line. For the second term on the right-hand side of (3.23), integration by parts together with $(1.1)_5$ and Gagliardo–Nirenberg inequality indicates that for any $\varepsilon>0$,

$$\begin{aligned}\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx &= -\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx\\ &\le -\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le -\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\ &\le -\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha. \end{aligned}\tag{3.25}$$

From Cauchy–Schwarz inequality and (3.4), we have

$$\int\rho\theta|\mathbf{u}_t|dx\le\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{1}{2}\|\rho\|_{L^\infty}\|\theta\|_{L^2}^2\le\frac{1}{2}\int\rho|\mathbf{u}_t|^2dx+C.\tag{3.26}$$

Thus, inserting (3.24)–(3.26) into (3.23) gives

$$\frac{d}{dt}B(t)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\le\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha,\tag{3.27}$$

where

$$B(t)\triangleq\mu\|\nabla\mathbf{u}\|_{L^2}^2+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx$$

satisfies

$$\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2-C_1\|\nabla\mathbf{b}\|_{L^2}^2\le B(t)\le C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2,\tag{3.28}$$

owing to Hölder's inequality, Gagliardo–Nirenberg inequality, and (3.4).

3. It follows from $(1.1)_3$ that

$$\begin{aligned}&\nu\frac{d}{dt}\|\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\nu^2\|\Delta\mathbf{b}\|_{L^2}^2\\ &\le C\||\mathbf{b}||\nabla\mathbf{u}|\|_{L^2}^2+C\||\mathbf{u}||\nabla\mathbf{b}|\|_{L^2}^2\\ &\le C\|\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2+C\|\bar{x}^{-\frac{a}{4}}\mathbf{u}\|_{L^8}^2\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^4}\\ &\le\frac{\nu^2}{2}\|\Delta\mathbf{b}\|_{L^2}^2+C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2 \end{aligned}\tag{3.29}$$

due to (2.7), (3.21), and Gagliardo–Nirenberg inequality. Multiplying (3.29) by $\nu^{-1}(C_1+1)$ and adding the resulting inequality to (3.27) imply

$$\begin{aligned}\frac{d}{dt}\left(B(t)+(C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\frac{\nu}{2}\|\Delta\mathbf{b}\|_{L^2}^2&\\ \le C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2.&\end{aligned}\tag{3.30}$$

Since $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ satisfies the following Stokes system

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla P = -\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \rho\theta\mathbf{e}_2, & x \in B_R, \\ \operatorname{div}\mathbf{u} = 0, & x \in B_R, \\ \mathbf{u}(x) = 0, & x \in \partial B_R, \end{cases} \tag{3.31}$$

applying regularity theory of Stokes system to (3.31) (see [18]) yields that for any $p \in [2, \infty)$,

$$\|\nabla^2\mathbf{u}\|_{L^p} + \|\nabla P\|_{L^p} \le C\|\rho\mathbf{u}_t\|_{L^p} + C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p} + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^p} + C\|\rho\theta\|_{L^p}. \tag{3.32}$$

Hence, we infer from (3.32), (3.4), (3.21), and Gagliardo–Nirenberg inequality that

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \\ &\quad \le C\|\rho\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2 + C\|\rho\theta\|_{L^2}^2 \\ &\quad \le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{b}\|_{L^4}^2 + C\|\rho\|_{L^\infty}^2\|\theta\|_{L^2}^2 \\ &\quad \le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1} + C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{b}\|_{H^1} + C \\ &\quad \le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\left(1 + \|\nabla\mathbf{b}\|_{L^2}^4 + \|\nabla\mathbf{u}\|_{L^2}^6\right) \\ &\quad \le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\psi^\alpha. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.30) and choosing ε suitably small, one gets

$$\frac{d}{dt}\left(B(t) + (C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right) + \frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2 + \frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2 \le C\psi^\alpha + C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2.$$

Integrating the above inequality over $(0, t)$, then we obtain (3.19) from (2.7), (3.28), (3.10), and (3.33). The proof of Lemma 3.4 is finished. □

Lemma 3.5. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t} s\left(\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2 + \|\mathbf{b}_s\|_{L^2}^2\right) + \int_0^t s\left(\|\nabla\mathbf{u}_s\|_{L^2}^2 + \|\nabla\mathbf{b}_s\|_{L^2}^2\right) ds \le C\exp\left\{C\int_0^t \psi^\alpha ds\right\}. \tag{3.34}$$

Proof. 1. Differentiating $(1.1)_2$ with respect to t gives

$$\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t = -\rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} - \nabla P_t + (\mathbf{b}\cdot\nabla\mathbf{b})_t + (\rho\theta\mathbf{e}_2)_t. \tag{3.35}$$

Multiplying (3.35) by $\mathbf{u}_t$ and integrating the resulting equality by parts over B_R, we obtain after using $(1.1)_1$ and $(1.1)_5$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}_t|^2 dx + \mu\int|\nabla\mathbf{u}_t|^2 dx \\ &\quad \le C\int \rho|\mathbf{u}||\mathbf{u}_t|\left(|\nabla\mathbf{u}_t| + |\nabla\mathbf{u}|^2 + |\mathbf{u}||\nabla^2\mathbf{u}|\right) dx + C\int \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t| dx \\ &\qquad + C\int\rho|\mathbf{u}_t|^2|\nabla\mathbf{u}| dx + \int \mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t dx + \int \mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_t dx \\ &\qquad + \int \rho_t\theta\mathbf{e}_2\cdot\mathbf{u}_t dx + \int\rho\theta_t\mathbf{e}_2\cdot\mathbf{u}_t dx \triangleq \sum_{i=1}^7 \hat{I}_i. \end{aligned} \tag{3.36}$$

It follows from (3.20), (3.21), and Gagliardo–Nirenberg inequality that

$$\begin{aligned}\hat{I}_1 &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^4}^2\right)\\&\quad+C\|\rho^{\frac{1}{4}}\mathbf{u}\|_{L^{12}}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}\\&\le C(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}_t\|_{L^2}\right)^{\frac{1}{2}}\\&\quad\times\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}\right)\\&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2.\end{aligned}\tag{3.37}$$

Hölder's inequality combined with (3.20) and (3.21) leads to

$$\begin{aligned}\hat{I}_2+\hat{I}_3 &\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{3}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\\&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2}^2\right).\end{aligned}\tag{3.38}$$

Integration by parts together with $(1.1)_5$, Hölder's and Gagliardo–Nirenberg inequalities indicates that

$$\begin{aligned}\hat{I}_4+\hat{I}_5 &= -\int\mathbf{b}_t\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}dx-\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx\\&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\mathbf{b}_t\|_{L^4}^2\\&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\mu\nu}{4(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2.\end{aligned}\tag{3.39}$$

Integration by parts together with $(1.1)_1$, $(1.1)_5$, Hölder's inequality, Gagliardo–Nirenberg inequality, and (3.7) indicates that

$$\begin{aligned}\hat{I}_6 &= \int\rho\mathbf{u}\cdot\nabla(\theta\mathbf{e}_2\cdot\mathbf{u}_t)dx\\&\le\int\rho|\mathbf{u}||\nabla\theta||\mathbf{u}_t|dx+\int\rho|\mathbf{u}|\theta|\nabla\mathbf{u}_t|dx\\&\le\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\theta\|_{L^q}+\|\nabla\mathbf{u}_t\|_{L^2}\|\rho\mathbf{u}\|_{L^4}\|\theta\|_{L^4}\\&\le\frac{\mu}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha.\end{aligned}\tag{3.40}$$

We get from Hölder's inequality, (3.4), and (3.21) that

$$\begin{aligned}\hat{I}_7 &\le \int\rho|\mathbf{u}||\nabla\theta||\mathbf{u}_t|dx\\&\le\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\theta\|_{L^q}\\&\le C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha.\end{aligned}\tag{3.41}$$

Substituting (3.37)–(3.41) into (3.36), we obtain after using (3.33) that

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2 &\le C\psi^\alpha\left(1+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)\\&\quad+\frac{\mu\nu}{2(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2.\end{aligned}\tag{3.42}$$

2. Differentiating $(1.1)_3$ with respect to t shows

$$\mathbf{b}_{tt}-\mathbf{b}_t\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{u}_t+\mathbf{u}_t\cdot\nabla\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}_t=\nu\Delta\mathbf{b}_t.\tag{3.43}$$

Multiplying (3.43) by $\mathbf{b}_t$ and integrating the resulting equality over B_R yield that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\mathbf{b}_t|^2dx+\nu\int|\nabla\mathbf{b}_t|^2dx\\&\quad=\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx-\int\mathbf{u}_t\cdot\nabla\mathbf{b}\cdot\mathbf{b}_tdx+\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx-\int\mathbf{u}\cdot\nabla\mathbf{b}_t\cdot\mathbf{b}_tdx\\&\quad\triangleq\sum_{i=1}^4S_i.\end{aligned}\tag{3.44}$$

On the one hand, we deduce from (3.14) and (3.18) that

$$\begin{aligned}\sum_{i=1}^2S_i&\le C\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^4}\|\mathbf{b}\|_{L^4}+C\|\nabla\mathbf{b}_t\|_{L^2}\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}\\&\le C\|\mathbf{b}_t\|_{L^4}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{8}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}^2\\&\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}_t\bar{x}^{-\frac{a}{4}}\|_{L^8}^2\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{b}\|_{L^4}\\&\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2,\end{aligned}\tag{3.45}$$

where one has used the following estimate

$$\sup_{0\le s\le t}\||\mathbf{b}|^2\|_{L^2}^2+\int_0^t\||\nabla\mathbf{b}||\mathbf{b}|\|_{L^2}^2ds\le C.\tag{3.46}$$

Indeed, multiplying $(1.1)_3$ by $\mathbf{b}|\mathbf{b}|^2$ and integrating by parts lead to

$$\begin{aligned}&\frac{1}{4}\left(\||\mathbf{b}|^2\|_{L^2}^2\right)_t+\nu\||\nabla\mathbf{b}||\mathbf{b}|\|_{L^2}^2+\frac{\nu}{2}\|\nabla|\mathbf{b}|^2\|_{L^2}^2\\&\quad\le C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^4}^2\le C\|\nabla\mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^2}\|\nabla|\mathbf{b}|^2\|_{L^2}\\&\quad\le\frac{\nu}{4}\|\nabla|\mathbf{b}|^2\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\||\mathbf{b}|^2\|_{L^2}^2,\end{aligned}\tag{3.47}$$

which together with Gronwall's inequality and (3.4) gives (3.46).

On the other hand, integration by parts combined with $(1.1)_5$ and Gagliardo–Nirenberg inequality yields

$$\sum_{i=3}^4S_i=\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx\le C\|\mathbf{b}_t\|_{L^2}\|\nabla\mathbf{b}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\le\frac{\nu}{4}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2.\tag{3.48}$$

Inserting (3.45) and (3.48) into (3.44), one has

$$\frac{d}{dt}\|\mathbf{b}_t\|_{L^2}^2+\nu\|\nabla\mathbf{b}_t\|_{L^2}^2\le C\psi^\alpha\left(\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+C_2\|\nabla\mathbf{u}_t\|_{L^2}^2.\tag{3.49}$$

3. From (3.42) multiplied by $\mu^{-1}(C_2+1)$ and (3.49), we get

$$\begin{aligned}&\frac{d}{dt}\left(\mu^{-1}(C_2+1)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2\\&\quad\le C\psi^\alpha\left(1+\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2.\end{aligned}\tag{3.50}$$

Multiplying (3.50) by t, we obtain (3.34) after using Gronwall's inequality and (3.19). The proof of Lemma 3.5 is finished. □

Lemma 3.6. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t} s\left(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2 + \|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right) + \int_0^t s\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C\exp\left\{C\exp\left\{C\int_0^t \psi^\alpha ds\right\}\right\}. \tag{3.51}$$

Proof. 1. Multiplying $(1.1)_4$ by $\Delta\mathbf{b}\bar{x}^a$ and integrating by parts lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx + \nu\int|\Delta\mathbf{b}|^2\bar{x}^a dx\\ &\le C\int|\nabla\mathbf{b}||\mathbf{b}||\nabla\mathbf{u}||\nabla\bar{x}^a|dx + C\int|\nabla\mathbf{b}|^2|\mathbf{u}||\nabla\bar{x}^a|dx + C\int|\nabla\mathbf{b}||\Delta\mathbf{b}||\nabla\bar{x}^a|dx\\ &\quad + C\int|\mathbf{b}||\nabla\mathbf{u}||\Delta\mathbf{b}|\bar{x}^a dx + C\int|\nabla\mathbf{u}||\nabla\mathbf{b}|^2\bar{x}^a dx \triangleq \sum_{i=1}^5 J_i.\end{aligned} \tag{3.52}$$

Applying (3.10), (3.14), Hölder's inequality, and Gagliardo–Nirenberg inequality, one gets by some direct calculations that

$$\begin{aligned}J_1 &\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\ &\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac12}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2} + \|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{H^1}^{\frac12}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\ &\le C\psi^\alpha + C\|\nabla^2\mathbf{u}\|_{L^2}^2 + C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\ J_2 &\le C\||\nabla\mathbf{b}|^{2-\frac{2}{3a}}\bar{x}^{a-\frac13}\|_{L^{\frac{6a}{6a-2}}}\|\mathbf{u}\bar{x}^{-\frac13}\|_{L^{6a}}\||\nabla\mathbf{b}|^{\frac{2}{3a}}\|_{L^{6a}}\\ &\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{6a-2}{3a}}\|\nabla\mathbf{b}\|_{L^4}^{\frac{2}{3a}} \le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\nabla\mathbf{b}\|_{L^4}^2\\ &\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\ J_3 + J_4 &\le \frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\ &\le \frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\ &\quad + C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2} + \|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\ &\le \varepsilon\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\psi^\alpha + C\|\nabla^2\mathbf{u}\|_{L^2}^2,\\ J_5 &\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 \le C\left(\psi^\alpha + \|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (3.52) and noting the following fact

$$\begin{aligned}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx &= \int|\Delta\mathbf{b}|^2\bar{x}^a dx - \int\partial_i\partial_k\mathbf{b}\cdot\partial_k\mathbf{b}\partial_i\bar{x}^a dx + \int\partial_i\partial_i\mathbf{b}\cdot\partial_k\mathbf{b}\partial_k\bar{x}^a dx\\ &\le \int|\Delta\mathbf{b}|^2\bar{x}^a dx + \frac12\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx + C\int|\nabla\mathbf{b}|^2\bar{x}^a dx,\end{aligned}$$

we derive that

$$\begin{aligned}&\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx + \frac{\nu}{2}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx\\ &\le C\left(\psi^\alpha + \|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\left(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \psi^\alpha\right).\end{aligned} \tag{3.53}$$

2. We now claim that

$$\int_0^t \left(\|\nabla^2 \mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 \mathbf{u}\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2 \right) ds \le C \exp\left\{ C\int_0^t \psi^\alpha(s)ds \right\}, \tag{3.54}$$

whose proof will be given at the end of this proof. Thus, multiplying (3.53) by t, we infer from (3.10), (3.4), (3.54), and Gronwall's inequality that

$$\sup_{0\le s\le t} \left(s\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right) + \int_0^t s\|\nabla^2 \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C\exp\left\{ C\exp\left\{ C\int_0^t \psi^\alpha ds \right\} \right\}. \tag{3.55}$$

3. It deduces from $(1.1)_4$, (2.7), (3.4), (3.21), Hölder's inequality, and Gagliardo–Nirenberg inequality that

$$\begin{aligned}
\|\nabla^2 \mathbf{b}\|_{L^2}^2 &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\||\mathbf{u}||\nabla \mathbf{b}|\|_{L^2}^2 + C\||\mathbf{b}||\nabla \mathbf{u}|\|_{L^2}^2 \\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^2 \|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2} \|\nabla \mathbf{b}\|_{L^4} + C\|\mathbf{b}\|_{L^2}\|\nabla^2 \mathbf{b}\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^2 \\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^4 \|\nabla \mathbf{b}\|_{L^4}^2 + C\|\nabla^2 \mathbf{b}\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^2 \\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{1}{4}\|\nabla^2 \mathbf{b}\|_{L^2}^2 + C\left(1+\|\nabla \mathbf{u}\|_{L^2}^8\right)\left(1+\|\nabla \mathbf{b}\|_{L^2}^2\right),
\end{aligned} \tag{3.56}$$

which together with (3.33) gives that

$$\begin{aligned}
\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \mathbf{b}\|_{L^2}^2 &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla \mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 \right) \\
&\quad + C\left(1+\|\nabla \mathbf{u}\|_{L^2}^8\right)\left(1+\|\nabla \mathbf{b}\|_{L^2}^4\right).
\end{aligned} \tag{3.57}$$

Then, multiplying (3.57) by s, one gets from (3.19), (3.34), and (3.55) that

$$\begin{aligned}
&\sup_{0\le s\le t} \left(s\|\nabla^2 \mathbf{u}\|_{L^2}^2 + s\|\nabla P\|_{L^2}^2 + s\|\nabla^2 \mathbf{b}\|_{L^2}^2 \right) \\
&\quad \le C\exp\left\{ C\exp\left\{ C\int_0^t \psi^\alpha ds \right\} \right\} + C\left(1 + \int_0^t \psi^\alpha(s)ds \right)^{12} \\
&\quad \le C\exp\left\{ C\exp\left\{ C\int_0^t \psi^\alpha ds \right\} \right\}.
\end{aligned} \tag{3.58}$$

4. To finish the proof of Lemma 3.6, it suffices to show (3.54). Indeed, choosing $p = q$ in (3.32), we deduce from (3.19), (3.20), and Gagliardo–Nirenberg inequality that

$$\begin{aligned}
&\|\nabla^2 \mathbf{u}\|_{L^q} + \|\nabla P\|_{L^q} \\
&\quad \le C\left(\|\rho \mathbf{u}_t\|_{L^q} + \|\rho \mathbf{u}\cdot\nabla \mathbf{u}\|_{L^q} + \||\mathbf{b}||\nabla \mathbf{b}|\|_{L^q} + \|\rho\theta\|_{L^q} \right) \\
&\quad \le C\left(\|\rho \mathbf{u}_t\|_{L^q} + \|\rho \mathbf{u}\|_{L^{2q}}\|\nabla \mathbf{u}\|_{L^{2q}} + \|\mathbf{b}\|_{L^{2q}}\|\nabla \mathbf{b}\|_{L^{2q}} + \|\sqrt{\rho}\theta\|_{L^2} + \|\nabla\theta\|_{L^2} \right) \\
&\quad \le C\|\rho \mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\rho \mathbf{u}_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}} + C\psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{1-\frac{1}{q}} \right) \\
&\quad \le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\nabla \mathbf{u}_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2} \right) \\
&\qquad + C\psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{1-\frac{1}{q}} \right),
\end{aligned} \tag{3.59}$$

which together with (3.19) and (3.34) implies that

$$\begin{aligned}
&\int_0^t \left(\|\nabla^2 \mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} \right) ds \\
&\le C \int_0^t s^{-\frac{q+1}{2q}} \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}} \left(s\|\nabla \mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
&\quad + C \int_0^t \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{q+1}{q}} ds + C \int_0^t \psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{q^2-1}{q^2}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{\frac{q^2-1}{q^2}} \right) ds \\
&\le C \sup_{0\le s\le t} \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}} \int_0^t s^{-\frac{q+1}{2q}} \left(s\|\nabla \mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
&\quad + C \int_0^t \left(\psi^\alpha + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{b}\|_{L^2}^2\right) ds \\
&\le C \exp\left\{ C \int_0^t \psi^\alpha ds \right\} \left(1 + \int_0^t \left(s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} + s\|\nabla \mathbf{u}_t\|_{L^2}^2 \right) ds \right) \\
&\le C \exp\left\{ C \int_0^t \psi^\alpha ds \right\}
\end{aligned} \tag{3.60}$$

and

$$\begin{aligned}
&\int_0^t \left(s\|\nabla^2 \mathbf{u}\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2 \right) ds \\
&\le C \int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 ds + C \int_0^t \left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{2(q-1)}{q^2-2}} \left(s\|\nabla \mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-2q}{q^2-2}} ds \\
&\quad + C \int_0^t s\psi^\alpha \left(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \mathbf{b}\|_{L^2}^{1-\frac{1}{q}} \right)^2 ds \\
&\le C \int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 ds + C \int_0^t s\|\nabla \mathbf{u}_t\|_{L^2}^2 ds + C \int_0^t \left(\psi^\alpha + s\|\nabla^2 \mathbf{u}\|_{L^2}^2 + s\|\nabla^2 \mathbf{b}\|_{L^2}^2\right) ds \\
&\le C \exp\left\{ C \int_0^t \psi^\alpha ds \right\}.
\end{aligned} \tag{3.61}$$

One thus obtains (3.54) from (3.60)–(3.61) and finishes the proof of Lemma 3.6. □

Lemma 3.7. *Let T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t} \left(\|\rho\bar{x}^a\|_{H^1\cap W^{1,q}} + \|\nabla\theta\|_{L^2\cap L^q} \right) \le \exp\left\{ C \exp\left\{ C \int_0^t \psi^\alpha ds \right\} \right\}. \tag{3.62}$$

Proof. 1. It follows from Sobolev's inequality and (3.21) that for $0 < \delta < 1$,

$$\begin{aligned}
\|\mathbf{u}\bar{x}^{-\delta}\|_{L^\infty} &\le C(\delta) \left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla(\mathbf{u}\bar{x}^{-\delta})\|_{L^3} \right) \\
&\le C(\delta) \left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla \mathbf{u}\|_{L^3} + \|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} \|\bar{x}^{-1}\nabla\bar{x}\|_{L^{\frac{12}{4-3\delta}}} \right) \\
&\le C(\delta) \left(\psi^\alpha + \|\nabla^2 \mathbf{u}\|_{L^2} \right).
\end{aligned} \tag{3.63}$$

One derives from $(1.1)_1$ and $(1.1)_4$ that $\rho\bar{x}^a$ satisfies

$$(\rho\bar{x}^a)_t + \mathbf{u}\cdot\nabla(\rho\bar{x}^a) - a\rho\bar{x}^a\mathbf{u}\cdot\nabla\log\bar{x} = 0, \tag{3.64}$$

which along with (3.63) gives that for any $r \in [2, q]$,

$$\begin{aligned}\frac{d}{dt}\|\nabla(\rho\bar{x}^a)\|_{L^r} &\le C\left(1+\|\nabla\mathbf{u}\|_{L^\infty}+\|\mathbf{u}\cdot\nabla\log\bar{x}\|_{L^\infty}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r}\\ &\quad + C\|\rho\bar{x}^a\|_{L^\infty}\left(\||\nabla\mathbf{u}||\nabla\log\bar{x}|\|_{L^r}+\||\mathbf{u}||\nabla^2\log\bar{x}|\|_{L^r}\right)\\ &\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r}\\ &\quad + C\|\rho\bar{x}^a\|_{L^\infty}\left(\|\nabla\mathbf{u}\|_{L^r}+\|\mathbf{u}\bar{x}^{-\frac{2}{5}}\|_{L^{4r}}\|\bar{x}^{-\frac{3}{2}}\|_{L^{\frac{4r}{3}}}\right)\\ &\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\left(1+\|\nabla(\rho\bar{x}^a)\|_{L^r}+\|\nabla(\rho\bar{x}^a)\|_{L^q}\right). \end{aligned} \tag{3.65}$$

Then we derive from (3.65), (3.54), and Gronwall's inequality that

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{H^1\cap W^{1,q}} \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.66}$$

2. Operating ∇ to $(1.1)_3$ and then multiplying $|\nabla\theta|^{r-2}\nabla\theta$ for $r \in [2, q]$ gives that

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta\|_{L^r} &\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^r}+C\|\theta\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^r}\\ &\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\|\nabla\theta\|_{L^r}+C\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}^{\frac{q+1}{q}}\\ &\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}^{\frac{q+1}{q}}\right)\left(1+\|\nabla\theta\|_{L^r}\right), \end{aligned} \tag{3.67}$$

which along with Gronwall's inequality leads to

$$\sup_{0\le s\le t}\|\nabla\theta\|_{L^2\cap L^q} \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.68}$$

Hence the desired (3.62) follows from (3.66) and (3.68). □

Now, Proposition 3.1 is a direct consequence of Lemmas 3.2–3.7.

Proof of Proposition 3.1. It follows from (3.4), (3.19), and (3.62) that

$$\psi(t) \le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.$$

Standard arguments yield that for $M \triangleq e^{Ce}$ and $T_0 \triangleq \min\{T_2, (CM^\alpha)^{-1}\}$,

$$\sup_{0\le t\le T_0}\psi(t) \le M,$$

which together with (3.62), (3.19), (3.34), and (3.54) gives (3.3). The proof of Proposition 3.1 is thus completed. □

4 Proof of Theorem 1.1

With the a priori estimates in Section 3 at hand, it is a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ be as in Theorem 1.1. Without loss of generality, we assume that the initial density ρ_0 satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \tag{4.1}$$

We construct $\rho_0^R = \hat{\rho}_0^R + R^{-1} e^{-|x|^2}$, where $0 \leq \hat{\rho}_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\begin{cases} \int_{B_{N_0}} \hat{\rho}_0^R dx \geq 1/2, \\ \bar{x}^a \hat{\rho}_0^R \to \bar{x}^a \rho_0 \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \text{ as } R \to \infty. \end{cases} \tag{4.2}$$

Due to $\mathbf{b}_0 \bar{x}^{\frac{a}{2}} \in L^2(\mathbb{R}^2)$ and $\nabla \mathbf{b}_0 \in L^2(\mathbb{R}^2)$, we choose $\mathbf{b}_0^R \in \{\mathbf{w} \in C_0^\infty(B_R) \mid \operatorname{div} \mathbf{w} = 0\}$ satisfying

$$\mathbf{b}_0^R \bar{x}^{\frac{a}{2}} \to \mathbf{b}_0 \bar{x}^{\frac{a}{2}}, \quad \nabla \mathbf{b}_0^R \to \nabla \mathbf{b}_0 \quad \text{in } L^2(\mathbb{R}^2), \quad \text{as } R \to \infty. \tag{4.3}$$

Noting that $\theta_0 \in H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)$, we choose $\theta_0^R \in C_0^\infty(B_R)$ such that

$$\theta_0^R \to \theta_0 \quad \text{in } H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \quad \text{as } R \to \infty. \tag{4.4}$$

Since $\nabla \mathbf{u}_0 \in L^2(\mathbb{R}^2)$, we select $\mathbf{v}_i^R \in C_0^\infty(B_R)$ $(i = 1, 2)$ such that for $i = 1, 2$,

$$\lim_{R \to \infty} \|\mathbf{v}_i^R - \partial_i \mathbf{u}_0\|_{L^2(\mathbb{R}^2)} = 0. \tag{4.5}$$

We consider the unique smooth solution $\mathbf{u}_0^R$ of the following elliptic problem:

$$\begin{cases} -\Delta \mathbf{u}_0^R + \rho_0^R \mathbf{u}_0^R + \nabla P_0^R = \sqrt{\rho_0^R} \mathbf{h}^R - \partial_i \mathbf{v}_i^R, & \text{in } B_R, \\ \operatorname{div} \mathbf{u}_0^R = 0, & \text{in } B_R, \\ \mathbf{u}_0^R = \mathbf{0}, & \text{on } \partial B_R, \end{cases} \tag{4.6}$$

where $\mathbf{h}^R = (\sqrt{\rho_0} \mathbf{u}_0) * j_{\frac{1}{R}}$ with j_δ being the standard mollifying kernel of width δ.

Extending $\mathbf{u}_0^R$ to $\mathbb{R}^2$ by defining $\mathbf{0}$ outside B_R and denoting it by $\tilde{\mathbf{u}}_0^R$, we claim that

$$\lim_{R \to \infty} \left(\|\nabla (\tilde{\mathbf{u}}_0^R - \mathbf{u}_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R - \sqrt{\rho_0} \mathbf{u}_0\|_{L^2(\mathbb{R}^2)} \right) = 0. \tag{4.7}$$

In fact, it is easy to find that $\tilde{\mathbf{u}}_0^R$ is also a solution of (4.6) in $\mathbb{R}^2$. Multiplying (4.6) by $\tilde{\mathbf{u}}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned} &\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 dx \\ &\leq \|\sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)} \|\mathbf{h}^R\|_{L^2(B_R)} + C \|\mathbf{v}_i^R\|_{L^2(B_R)} \|\partial_i \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)} \\ &\leq \frac{1}{2} \|\nabla \tilde{\mathbf{u}}_0^R\|_{L^2(B_R)}^2 + \frac{1}{2} \int_{B_R} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + C \left(\|\mathbf{h}^R\|_{L^2(B_R)}^2 + \|\mathbf{v}_i^R\|_{L^2(B_R)}^2 \right), \end{aligned}$$

which implies

$$\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 dx \le C \tag{4.8}$$

for some C independent of R. This together with (4.2) yields that there exist a subsequence $R_j \to \infty$ and a function $\tilde{\mathbf{u}}_0 \in \{\tilde{\mathbf{u}}_0 \in H^1_{\text{loc}}(\mathbb{R}^2) | \sqrt{\rho_0}\tilde{\mathbf{u}}_0 \in L^2(\mathbb{R}^2), \nabla \tilde{\mathbf{u}}_0 \in L^2(\mathbb{R}^2)\}$ such that

$$\begin{cases} \sqrt{\rho_0^{R_j}}\tilde{\mathbf{u}}_0^{R_j} \rightharpoonup \sqrt{\rho_0}\tilde{\mathbf{u}}_0 \text{ weakly in } L^2(\mathbb{R}^2), \\ \nabla \tilde{\mathbf{u}}_0^{R_j} \rightharpoonup \nabla \tilde{\mathbf{u}}_0 \text{ weakly in } L^2(\mathbb{R}^2). \end{cases} \tag{4.9}$$

Next, we will show

$$\tilde{\mathbf{u}}_0 = \mathbf{u}_0. \tag{4.10}$$

Indeed, multiplying (4.6) by a test function $\boldsymbol{\pi} \in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{div} \boldsymbol{\pi} = 0$, it holds that

$$\int_{\mathbb{R}^2} (\partial_i \tilde{\mathbf{u}}_0^{R_j} - \mathbf{v}_i^{R_j}) \cdot \partial_i \boldsymbol{\pi} dx + \int_{\mathbb{R}^2} \sqrt{\rho_0^{R_j}} (\sqrt{\rho_0^{R_j}} \tilde{\mathbf{u}}_0^{R_j} - \mathbf{h}^{R_j}) \cdot \boldsymbol{\pi} dx = 0. \tag{4.11}$$

Let $R_j \to \infty$, it follows from (4.2), (4.5), and (4.9) that

$$\int_{\mathbb{R}^2} \partial_i (\tilde{\mathbf{u}}_0 - \mathbf{u}_0) \cdot \partial_i \boldsymbol{\pi} dx + \int_{\mathbb{R}^2} \rho_0 (\tilde{\mathbf{u}}_0 - \mathbf{u}_0) \cdot \boldsymbol{\pi} dx = 0, \tag{4.12}$$

which implies (4.10).

Furthermore, multiplying (4.6) by $\tilde{\mathbf{u}}_0^{R_j}$ and integrating the resulting equation over $\mathbb{R}^2$, by the same arguments as (4.12), we have

$$\lim_{R_j \to \infty} \int_{\mathbb{R}^2} \left(|\nabla \tilde{\mathbf{u}}_0^{R_j}|^2 + \rho_0^{R_j} |\tilde{\mathbf{u}}_0^{R_j}|^2 \right) dx = \int_{\mathbb{R}^2} \left(|\nabla \mathbf{u}_0|^2 + \rho_0 |\mathbf{u}_0|^2 \right) dx,$$

which combined with (4.9) leads to

$$\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0|^2 dx, \ \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0^{R_j} |\tilde{\mathbf{u}}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} \rho_0 |\tilde{\mathbf{u}}_0|^2 dx.$$

This, along with (4.10) and (4.9), gives (4.7).

Hence, by virtue of Lemma 2.1, the initial-boundary-value problem (2.2) with the initial data $(\rho_0^R, \mathbf{u}_0^R, \theta_0^R, \mathbf{b}_0^R)$ has a classical solution $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$ on $B_R \times [0, T_R]$. Moreover, Proposition 3.1 shows that there exists a T_0 independent of R such that (3.3) holds for $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$.

For simplicity, in what follows, we denote

$$L^p = L^p(\mathbb{R}^2), \ W^{k,p} = W^{k,p}(\mathbb{R}^2).$$

Extending $(\rho^R, \mathbf{u}^R, P^R, \theta^R, \mathbf{b}^R)$ by zero on $\mathbb{R}^2 \setminus B_R$ and denoting it by

$$\left(\tilde{\rho}^R \triangleq \varphi_R \rho^R, \tilde{\mathbf{u}}^R, \tilde{P}^R, \tilde{\theta}^R, \tilde{\mathbf{b}}^R \right)$$

with φ_R satisfying (3.11). First, (3.3) leads to

$$\begin{aligned} &\sup_{0 \le t \le T_0} \left(\|\sqrt{\tilde{\rho}^R} \tilde{\mathbf{u}}^R\|_{L^2} + \|\nabla \tilde{\mathbf{u}}^R\|_{L^2} + \|\nabla \tilde{\theta}^R\|_{L^2 \cap L^q} + \|\nabla \tilde{\mathbf{b}}^R\|_{L^2} + \|\tilde{\mathbf{b}}^R \bar{x}^{\frac{a}{2}}\|_{L^2} \right) \\ &\le \sup_{0 \le t \le T_0} \Big(\|\sqrt{\rho^R} \mathbf{u}^R\|_{L^2(B_R)} + \|\nabla \mathbf{u}^R\|_{L^2(B_R)} \\ &\qquad + \|\nabla \theta^R\|_{L^2(B_R) \cap L^q(B_R)} + \|\nabla \mathbf{b}^R\|_{L^2(B_R)} + \|\mathbf{b}^R \bar{x}^{\frac{a}{2}}\|_{L^2(B_R)} \Big) \\ &\le C, \end{aligned} \tag{4.13}$$

and

$$\sup_{0\le t\le T_0}\|\tilde\rho^R\bar x^a\|_{L^1\cap L^\infty}\le C. \tag{4.14}$$

Similarly, it follows from (3.3) that for $q>2$,

$$\begin{aligned}&\sup_{0\le t\le T_0}\sqrt{t}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}+\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}+\|\tilde{\mathbf{b}}_t^R\|_{L^2}\right)\\&\quad+\int_0^{T_0}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}^2+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}^2+\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}^2+\|\nabla\tilde{\mathbf{b}}^R\bar x^{\frac a2}\|_{L^2}^2\right)dt\\&\quad+\int_0^{T_0}\left(\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^2+t\|\nabla\tilde{\mathbf{u}}_t^R\|_{L^2}^2+t\|\nabla\tilde{\mathbf{b}}_t^R\|_{L^2}^2\right)dt\le C.\end{aligned} \tag{4.15}$$

Next, for $p\in[2,q]$, we obtain from (3.3) and (3.62) that

$$\begin{aligned}\sup_{0\le t\le T_0}\|\nabla(\tilde\rho^R\bar x^a)\|_{L^p}&\le C\sup_{0\le t\le T_0}\left(\|\nabla(\rho^R\bar x^a)\|_{L^p(B_R)}+R^{-1}\|\rho^R\bar x^a\|_{L^p(B_R)}\right)\\&\le C\sup_{0\le t\le T_0}\|\rho^R\bar x^a\|_{H^1(B_R)\cap W^{1,p}(B_R)}\le C,\end{aligned} \tag{4.16}$$

which together with (3.63) and (3.3) yields

$$\begin{aligned}\int_0^{T_0}\|\bar x\tilde\rho_t^R\|_{L^p}^2dt&\le C\int_0^{T_0}\|\bar x|\mathbf{u}^R||\nabla\rho^R|\|_{L^p(B_R)}^2)dt\\&\le C\int_0^{T_0}\|\bar x^{1-a}\mathbf{u}^R\|_{L^\infty(B_R)}^2\|\bar x^a\nabla\rho^R\|_{L^p(B_R)}^2dt\\&\le C.\end{aligned} \tag{4.17}$$

With the estimates (4.13)–(4.17) together with $(2.2)_1$ and $(2.2)_3$, we find that the sequence $(\tilde\rho^R,\tilde{\mathbf{u}}^R,\tilde P^R,\tilde\theta^R,\tilde{\mathbf{b}}^R)$ converges, up to the extraction of subsequences, to some limit $(\rho,\mathbf{u},P,\theta,\mathbf{b})$ in the obvious weak sense, that is, as $R\to\infty$, we have

$$\tilde\rho^R\bar x\to\rho\bar x,\ \tilde\theta^R\to\theta,\ \text{in } C(\overline{B_N}\times[0,T_0]),\ \text{for any } N>0, \tag{4.18}$$

$$\tilde\rho^R\bar x^a\rightharpoonup\rho\bar x^a,\ \text{weakly * in } L^\infty(0,T_0;H^1\cap W^{1,q}), \tag{4.19}$$

$$\nabla\tilde\theta^R\rightharpoonup\nabla\theta,\ \text{weakly * in } L^\infty(0,T_0;L^2\cap L^q), \tag{4.20}$$

$$\tilde{\mathbf{b}}^R\bar x^{\frac a2}\rightharpoonup\mathbf{b}\bar x^{\frac a2},\ \text{weakly * in} L^\infty(0,T_0;L^2), \tag{4.21}$$

$$\tilde{\mathbf{b}}_t^R\rightharpoonup\mathbf{b}_t,\ \nabla\tilde{\mathbf{b}}^R\bar x^{\frac a2}\rightharpoonup\nabla\mathbf{b}\bar x^{\frac a2},\ \nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\nabla^2\mathbf{b},\ \text{weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.22}$$

$$\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{\rho}\mathbf{u},\ \nabla\tilde{\mathbf{u}}^R\rightharpoonup\nabla\mathbf{u},\ \nabla\tilde{\mathbf{b}}^R\rightharpoonup\nabla\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.23}$$

$$\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\nabla^2\mathbf{u},\ \text{weakly in } L^{\frac{q+1}{q}}(0,T_0;L^q)\cap L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.24}$$

$$\sqrt{t}\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{u},\ \text{weakly in } L^2(0,T_0;L^q),\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.25}$$

$$\sqrt{t}\tilde{\mathbf{b}}_t^R\rightharpoonup\sqrt{t}\mathbf{b}_t,\ \sqrt{t}\nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.26}$$

$$\sqrt{t}\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\rightharpoonup\sqrt{t}\sqrt{\rho}\mathbf{u}_t,\ \text{weakly * in } L^\infty(0,T_0;L^2), \tag{4.27}$$

$$\sqrt{t}\nabla\tilde{\mathbf{u}}_t^R\rightharpoonup\sqrt{t}\nabla\mathbf{u}_t,\ \sqrt{t}\nabla\tilde{\mathbf{b}}_t^R\rightharpoonup\sqrt{t}\nabla\mathbf{b}_t,\ \text{weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.28}$$

with

$$\rho\bar x^a\in L^\infty(0,T_0;L^1),\quad \inf_{0\le t\le T_0}\int_{B_{2N_0}}\rho(x,t)dx\ge\frac14. \tag{4.29}$$

Next, for any function $\phi \in C_0^\infty(\mathbb{R}^2 \times [0,T_0))$, we take $\phi\varphi_R$ as test function in the initial-boundary-value problem (2.2) with the initial data $(\rho_0^R, \mathbf{u}_0^R, \theta_0^R, \mathbf{b}_0^R)$. Then, letting $R \to \infty$, standard arguments together with (4.18)–(4.29) show that $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ is a strong solution of (1.1)–(1.3) on $\mathbb{R}^2 \times (0, T_0]$ satisfying (1.6) and (1.7). Indeed, the existence of a pressure P follows immediately from the $(1.1)_2$ and $(1.1)_4$ by a classical consideration. The proof of the existence part of Theorem 1.1 is finished.

It remains only to prove the uniqueness of the strong solutions provided that $\theta_0\bar{x}^a \in H^1 \cap W^{1,q}$. Let $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar\rho, \bar{\mathbf{u}}, \bar P, \bar\theta, \bar{\mathbf{b}})$ be two strong solutions satisfying (1.6) and (1.7) with the same initial data, and denote

$$\Theta \triangleq \rho - \bar\rho,\ \mathbf{U} \triangleq \mathbf{u} - \bar{\mathbf{u}},\ \Psi \triangleq \theta - \bar\theta,\ \mathbf{\Phi} \triangleq \mathbf{b} - \bar{\mathbf{b}}.$$

First, subtracting the mass equation satisfied by $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar\rho, \bar{\mathbf{u}}, \bar P, \bar\theta, \bar{\mathbf{b}})$ gives

$$\Theta_t + \bar{\mathbf{u}} \cdot \nabla\Theta + \mathbf{U} \cdot \nabla\rho = 0. \tag{4.30}$$

Multiplying (4.30) by $2\Theta\bar{x}^{2r}$ for $r \in (1, \tilde a)$ with $\tilde a = \min\{2, a\}$ and integrating by parts yield

$$\begin{aligned}\frac{d}{dt}\int |\Theta\bar{x}^r|^2 dx &\le C\|\bar{\mathbf{u}}\bar{x}^{-\frac12}\|_{L^\infty}\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-(\tilde a - r)}\|_{L^{\frac{2q}{(q-2)(\tilde a - r)}}}\|\bar{x}^{\tilde a}\nabla\rho\|_{L^{\frac{2q}{q-(q-2)(\tilde a - r)}}}\\ &\le C\left(1 + \|\nabla\bar{\mathbf{u}}\|_{W^{1,q}}\right)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}\left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right)\end{aligned}$$

due to Sobolev's inequality, (1.7), (3.14), and (3.63). This combined with Gronwall's inequality shows that for all $0 \le t \le T_0$,

$$\|\Theta\bar{x}^r\|_{L^2} \le C\int_0^t \left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right) ds. \tag{4.31}$$

Similarly to (4.31), one has

$$\|\Psi\bar{x}^r\|_{L^2} \le C\int_0^t \left(\|\nabla\mathbf{U}\|_{L^2} + \|\sqrt{\rho}\mathbf{U}\|_{L^2}\right) ds. \tag{4.32}$$

Next, subtracting $(1.1)_2$ and $(1.1)_4$ satisfied by $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ and $(\bar\rho, \bar{\mathbf{u}}, \bar P, \bar\theta, \bar{\mathbf{b}})$ leads to

$$\begin{aligned}\rho\mathbf{U}_t + \rho\mathbf{U}\cdot\nabla\mathbf{U} - \mu\Delta\mathbf{U} = &-\rho\mathbf{U}\cdot\nabla\bar{\mathbf{u}} - \Theta(\bar{\mathbf{u}}_t + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}) - \nabla(P - \bar P)\\ &+ \mathbf{b}\cdot\nabla\mathbf{\Phi} + \mathbf{\Phi}\cdot\nabla\bar{\mathbf{b}} + \Theta\theta\mathbf{e}_2 + \bar\rho\Psi\mathbf{e}_2\end{aligned} \tag{4.33}$$

and

$$\mathbf{\Phi}_t - \nu\Delta\mathbf{\Phi} = \mathbf{b}\cdot\nabla\mathbf{U} + \mathbf{\Phi}\cdot\nabla\bar{\mathbf{u}} - \mathbf{u}\cdot\nabla\mathbf{\Phi} - \mathbf{U}\cdot\nabla\bar{\mathbf{b}}, \tag{4.34}$$

Multiplying (4.33) by $\mathbf{U}$ and (4.34) by $\mathbf{\Phi}$ respectively, and adding the resulting equations together, we obtain after integration by parts that

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right)dx + \int\left(\mu|\nabla\mathbf{U}|^2 + \nu|\nabla\mathbf{\Phi}|^2\right)dx\\ &\le C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int\left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right)dx + C\int|\Theta||\mathbf{U}|\left(|\bar{\mathbf{u}}_t| + |\bar{\mathbf{u}}||\nabla\bar{\mathbf{u}}|\right)dx\\ &\quad + C\int|\mathbf{U}|\left(|\Theta|\theta + \bar\rho|\Psi|\right)dx - \int\mathbf{\Phi}\cdot\nabla\mathbf{U}\cdot\bar{\mathbf{b}}dx - \int\mathbf{U}\cdot\nabla\bar{\mathbf{b}}\cdot\mathbf{\Phi}dx\\ &\triangleq C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int\left(\rho|\mathbf{U}|^2 + |\mathbf{\Phi}|^2\right)dx + \sum_{i=1}^4 K_i.\end{aligned} \tag{4.35}$$

We first estimate K_1. Hölder's inequality combined with (1.7), (2.6), (3.3), (4.31), and Young's inequality yields that for $r \in (1, \tilde{a})$,

$$\begin{aligned}
K_1 &\le C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\left(\|\bar{\mathbf{u}}_t\bar{x}^{-\frac{r}{2}}\|_{L^4} + \|\nabla\bar{\mathbf{u}}\|_{L^\infty}\|\bar{\mathbf{u}}\bar{x}^{-\frac{r}{2}}\|_{L^4}\right) \\
&\le C(\varepsilon)\left(\|\sqrt{\bar{\rho}}\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}\|_{L^\infty}^2\right)\|\Theta\bar{x}^r\|_{L^2}^2 \\
&\quad + \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) \\
&\le C(\varepsilon)\left(1 + t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + t\|\nabla^2\bar{\mathbf{u}}\|_{L^q}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds \\
&\quad + \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right).
\end{aligned} \tag{4.36}$$

For the term K_2, we derive from Hölder's inequality, (3.3), and (4.32) that

$$\begin{aligned}
K_2 &\le C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\|\theta\|_{L^4}\|\bar{x}^{-\frac{r}{2}}\|_{L^\infty} + C\|\sqrt{\bar{\rho}}\|_{L^\infty}\|\sqrt{\bar{\rho}}\mathbf{U}\|_{L^2}\|\Psi\|_{L^2} \\
&\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Psi\|_{L^2}^2 \\
&\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Psi\bar{x}^r\|_{L^2}^2\|\bar{x}^{-r}\|_{L^\infty}^2 \\
&\le \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds.
\end{aligned} \tag{4.37}$$

We derive from Gagliardo–Nirenberg inequality and (3.46) that

$$K_3 \le C\|\bar{\mathbf{b}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4}\|\nabla\mathbf{U}\|_{L^2} \le \varepsilon\|\nabla\mathbf{U}\|_{L^2}^2 + \varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2 + C(\varepsilon)\|\mathbf{\Phi}\|_{L^2}^2. \tag{4.38}$$

Owing to (1.7), (2.6), and (3.3), K_4 can be estimated as follows

$$\begin{aligned}
K_4 \le& C\|\mathbf{U}\bar{x}^{-a}\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\bar{x}^a\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4} \\
\le& C\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2} + \|\nabla\mathbf{U}\|_{L^2}\right)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{1}{2}}\|\mathbf{\Phi}\|_{L^4} \\
\le& \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{\Phi}\|_{L^4}^2 \\
\le& \varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{U}\|_{L^2}^2\right) + \varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2 + C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\|\mathbf{\Phi}\|_{L^2}^2.
\end{aligned} \tag{4.39}$$

Denoting

$$G(t) \triangleq \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2 + \|\mathbf{\Phi}\|_{L^2}^2 + \int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2 + \|\nabla\mathbf{\Phi}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds,$$

then substituting (4.36)–(4.39) into (4.35) and choosing ε suitably small lead to

$$G'(t) \le C\left(1 + \|\nabla\bar{\mathbf{u}}\|_{L^\infty} + \|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2 + \|\nabla\bar{\mathbf{u}}\|_{L^2}^2 + t\|\nabla^2\mathbf{u}\|_{L^q}^2\right)G(t),$$

which together with Gronwall's inequality and (1.6) implies $G(t) = 0$. Hence, $(\mathbf{U}, \mathbf{\Phi})(x,t) = (\mathbf{0}, \mathbf{0})$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$. Finally, one can deduce from (4.31)–(4.32) that $\Theta(x,t) = 0$ and $\Psi(x,t) = 0$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$.

The proof of Theorem 1.1 is completed. □

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