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Variational differential inclusions without ellipticity condition

Zhenhai Liu ✉ [1, 2], **Roberto Livrea**[3], **Dumitru Motreanu**[2,4] and **Shengda Zeng**[2,5]

[1]Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, P. R. China
[2]Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China
[3]Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy
[4]Department of Mathematics, Université de Perpignan, 52 Avenue Paul Alduy, 66860, Perpignan, France
[5]Jagiellonian University in Krakow, Faculty of Mathematics and Computer Science, ul. Lojasiewicza 6, 30348 Krakow, Poland

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Abstract. The paper sets forth a new type of variational problem without any ellipticity or monotonicity condition. A prototype is a differential inclusion whose driving operator is the competing weighted (p, q)-Laplacian $-\Delta_p u + \mu\Delta_q u$ with $\mu \in \mathbb{R}$. Local and nonlocal boundary value problems fitting into this nonstandard setting are examined.

Keywords: variational problem, hemivariational inequality, lack of ellipticity, competing (p, q)-Laplacian, local and nonlocal operators.

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1 Introduction

Let X and Y be Banach spaces and let $j : X \to Y$ be a linear compact map. There are given on X a Gâteaux differentiable function $F : X \to \mathbb{R}$ with its Gâteaux differential $DF : X \to X^*$ and on Y a locally Lipschitz function $\Phi : Y \to \mathbb{R}$ whose generalized directional derivative is denoted $\Phi^0 : Y \times Y \to \mathbb{R}$. With these data we formulate the following problem in the form of a hemivariational inequality: find $u \in X$ such that

$$\langle DF(u), w\rangle + \Phi^0(ju; jw) \geq 0, \qquad \forall w \in X. \tag{1.1}$$

Problem (1.1) qualifies as a hemivariational inequality due to the presence of the term $\Phi^0(ju; jw)$. This problem is equivalent to the differential inclusion

$$-DF(u) \in j^*\partial\Phi(ju),$$

✉ Corresponding author. Email: zhhliu@hotmail.com

where the notation $\partial\Phi(u)$ stands for the generalized gradient of Φ at $u \in X$ and j^* denotes the adjoint operator of j. The hemivariational inequalities provide accurate modeling of contact phenomena involving nonconvex and nonsmooth mechanical processes. For an extensive study on applications of hemivariational inequalities we cite [10,13,14],

Problem (1.1) has a variational structure, which is nonsmooth, whose associated energy functional $I : X \to \mathbb{R}$ is

$$I = F + \Phi \circ j. \tag{1.2}$$

There is a huge literature devoted to variational problems, smooth or nonsmooth, mainly employing minimax techniques based on critical point theory (see, e.g., [11], [3], [10, Chapter 3]). Since F is only Gâteaux differentiable, no available result can be applied to problem (1.1) and its corresponding energy functional I in (1.2).

The main novelty of the present work is represented by the fact that we don't assume any ellipticity condition on the leading term $DF(u)$ in (1.1). In order to highlight this essential aspect, let us consider a particular situation in (1.1) related to boundary value problems with discontinuous nonlinearities. Their study was initiated by Chang [3].

For a fixed $\mu \in \mathbb{R}$, we state the quasilinear differential inclusion

$$\begin{cases} -\Delta_p u + \mu \Delta_q u \in [\underline{f}(u), \overline{f}(u)] & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with the boundary $\partial\Omega$. Here Δ_p and Δ_q denote the p-Laplacian and the q-Laplacian, respectively, with $1 < q < p < +\infty$, and for a function $f \in L^\infty_{\text{loc}}(\mathbb{R})$ we set

$$\underline{f}(s) = \lim_{\delta \to 0} \operatorname*{ess\,inf}_{|\tau - s| < \delta} f(\tau), \qquad \forall s \in \mathbb{R} \tag{1.4}$$

and

$$\overline{f}(s) = \lim_{\delta \to 0} \operatorname*{ess\,sup}_{|\tau - s| < \delta} f(\tau), \qquad \forall s \in \mathbb{R}. \tag{1.5}$$

If the function f is continuous, then the interval $[\underline{f}(u(x)), \overline{f}(u(x))]$ reduces to the singleton $f(u(x))$ and (1.3) becomes the quasilinear Dirichlet equation

$$\begin{cases} -\Delta_p u + \mu \Delta_q u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

An important case in problems (1.3) and (1.6) is when $\mu = 0$ with the p-Laplacian Δ_p as driving operator. Another important case is when $\mu = -1$, where the quasilinear equation is governed by the (p,q)-Laplacian $\Delta_p + \Delta_q$. We emphasize that the behavior of $-\Delta_p + \mu\Delta_q$ with $\mu > 0$ is completely different with respect to the one of $-\Delta_p + \mu\Delta_q$ with $\mu \leq 0$, the latter being an elliptic operator. In the case of $-\Delta_p + \mu\Delta_q$ with $\mu > 0$ the ellipticity is lost as can be easily seen: for $u = \lambda u_0$ with a nonzero $u_0 \in W_0^{1,p}(\Omega)$ and a number $\lambda > 0$ the expression

$$\langle -\Delta_p u + \mu \Delta u, u \rangle = \lambda^p \|\nabla u_0\|_p^p - \mu \lambda^q \|\nabla u_0\|_q^q$$

is positive for λ large and negative for λ small. Therefore the leading operator in (1.3) is a competing (p,q)-Laplacian when $\mu > 0$. This makes (1.3), thus (1.1), a nonstandard problem where a sort of hyperbolic feature is incorporated.

We further discuss a nonlocal counterpart of problem (1.3), namely

$$\begin{cases} -\Delta_p u + \mu(-\Delta)^s_q u \in [\underline{f}(u), \overline{f}(u)] & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.7}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, where $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ with (1.4), (1.5) as above, and a parameter $\mu \in \mathbb{R}$. Inclusion (1.7) is driven by the nonlocal operator formed with the ordinary p-Laplacian Δ_p and the (negative) s-fractional q-Laplacian $(-\Delta)^s_q$, taking $0 < s < 1$ and $1 < q < p < +\infty$, with $sq < N$. The differential operator $-\Delta_p + \mu(-\Delta)^s_q$ is the optimal fractional substitute for the (p,q)-Laplacian $-\Delta_p - \mu\Delta_q$ as noticed below in Remark 5.2. Likewise in the case of fractional p-Laplacian (see, e.g., [15]), a motivation for studying it comes from the theory of Markov processes. In this respect, we refer to [8, Example 1.2.1] describing a typical Markovian symmetric form. A brief survey of the nonlocal setting related to (1.7) can be found in Section 2. If the function f is continuous, (1.7) reduces to the equation

$$\begin{cases} -\Delta_p u + \mu(-\Delta)^s_q u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.8}$$

In the nonlocal problems (1.7) and (1.8) the ellipticity is preserved if $\mu \geq 0$, but not if $\mu < 0$ for which the usual methods fail to apply.

The natural notion of solution (in the weak sense) to problem (1.1) is apparent: any $u \in X$ for which inequality (1.1) holds whenever $w \in X$. Since we do not assume any ellipticity/monotonicity condition upon the principal part of (1.1) or any compactness condition of Palais–Smale type on I in (1.2) or that I be sequentially weakly lower semicontinuous (as basically is required in [1]), in order to establish the solvability of equation (1.1) we need to relax the notion of solution to fit the specific character of problem (1.1).

Definition 1.1. A function $u \in X$ is called a *generalized solution* to (1.1) if there exists a sequence $\{u_n\}_{n\geq 1} \subset X$ with the properties:

(S_1) $u_n \rightharpoonup u$ in X as $n \to \infty$;

(S_2) $\limsup_{n\to\infty} F(u_n) \leq F(u)$;

(S_3)
$$\liminf_{n\to\infty} \langle DF(u_n), v - u_n\rangle + \Phi^0(ju; jv - ju) \geq 0, \qquad \forall v \in X.$$

Remark 1.2. The idea of weakening the notion of solution to cover more general frames is frequent (see, e.g., [12, p. 183]). Different situations where the solution is a limit of (approximate) solutions are discussed in [16, 17].

Remark 1.3. Every solution to (1.1) is a generalized solution in the sense of Definition 1.1. It suffices to take the constant sequence $u_n = u$. For the converse assertion, additional assumptions should be imposed, for instance that the differential $DF : X \to X^*$ be completely sequentially continuous (i.e., $u_n \rightharpoonup u$ implies $DFu_n \to DFu$). A key role might have property (S_2) in Definition 1.1 as will be illustrated for problems (1.3), (1.6), (1.7), (1.8).

Our main result stated as Theorem 3.2 in Section 3 provides the existence of a generalized solution to problem (1.1). The approach relies on minimization of the energy functional I in (1.2) on finite dimensional subspaces of X belonging to a Galerkin basis. Denoting by $\{v_n\}_{n\geq 1} \subset X$ the resulting minimizing sequence of I, in a further step we construct through

Ekeland's variational principle (see [6,7]) applied to I and $\{v_n\}_{n\geq 1}$ a second minimizing sequence $\{u_n\}_{n\geq 1} \subset X$ of I, with finer properties, that will be shown to comply with Definition 1.1. The proof is concluded by a passing to the limit process.

The abstract result in Theorem 3.2 for problem (1.1) is applied in two different directions. First, we establish the existence of a generalized solution to the local quasilinear differential inclusion with discontinuities (1.3), in particular (1.6) (see Theorem 4.2). Second, we obtain the existence of a generalized solution to the nonlocal quasilinear inclusion (1.7), in particular (1.8) (see Theorem 5.1). In both cases, a special attention is paid to clarify when the generalized solution is a weak solution.

2 Mathematical background

Our approach on problem (1.1) relies on two fundamental tools: Galerkin basis and Ekeland's variational principle. For easy reference we recall some basic material.

A Galerkin basis of a Banach space X is a sequence $\{X_n\}_{n\geq 1}$ of vector subspaces of X for which

(i) $\dim(X_n) < \infty, \quad \forall n;$

(ii) $X_n \subset X_{n+1}, \quad \forall n;$

(iii)
$$\overline{\bigcup_{n=1}^{\infty} X_n} = X.$$

If X is separable, there exists a Galerkin basis of X. For an extensive use of Galerkin bases to various existence theorems we refer to [12,16,17].

We shall apply Ekeland's Variational Principle (see [6,7]) in the following form.

Theorem 2.1. *Assume that the functional $I : X \to \mathbb{R}$ is lower semicontinuous and bounded from below on a Banach space X. If $\{v_n\}_{n\geq 1}$ is a minimizing sequence of I, then there exists a sequence $\{u_n\}_{n\geq 1}$ in X with the properties:*

(a) *$I(u_n) \leq I(v_n)$ for all n;*

(b) *$\|u_n - v_n\| \to 0$ as $n \to \infty$;*

(c) *for all $n \geq 1$, it holds*
$$I(w) > I(u_n) - \frac{1}{n}\|w - u_n\|, \qquad \forall w \in X,\ w \neq u_n.$$

Next we outline some prerequisites of nonsmooth analysis regarding the subdifferentiability of locally Lipschitz functions (for more details we recommend [4] and also [3,10]). A function $\Phi : Y \to \mathbb{R}$ on a Banach space Y is called locally Lipschitz if for every $u \in Y$ one can find a neighborhood U of u in Y and a constant $L_u > 0$ such that
$$|\Phi(v) - \Phi(w)| \leq L_u\|v - w\|, \qquad \forall v, w \in U.$$

The generalized directional derivative of a locally Lipschitz function Φ at $u \in Y$ in the direction $v \in Y$ is defined as
$$\Phi^0(u; v) := \limsup_{w\to u,\ t\to 0^+} \frac{1}{t}(\Phi(w + tv) - \Phi(w))$$

and the generalized gradient of Φ at $u \in Y$ is the set

$$\partial\Phi(u) := \left\{u^* \in Y^* : \langle u^*, v\rangle \leq \Phi^0(u;v),\ \forall v \in Y\right\}.$$

A continuous and convex function $\Phi : Y \to \mathbb{R}$ is locally Lipschitz and its generalized gradient $\partial\Phi : Y \to 2^{Y^*}$ coincides with the subdifferential of Φ in the sense of convex analysis.

We need these notions in connection with the nonsmooth problems (1.3), (1.6), (1.7), (1.8). Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ for which we set

$$g(s) = \int_0^s f(t)\,\mathrm{d}t \quad \text{for all } s \in \mathbb{R} \tag{2.1}$$

and note that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Then the generalized gradient $\partial g(s)$ of g at $s \in \mathbb{R}$ is the compact interval in $\mathbb{R}$ expressed by

$$\partial g(s) = [\underline{f}(s), \overline{f}(s)], \tag{2.2}$$

where $\underline{f}(s)$ and $\overline{f}(s)$ are defined in (1.4) and (1.5), respectively.

We also address a few things about the operators in the Dirichlet problems (1.3), (1.6), (1.7), (1.8). Given $1 < q < p < +\infty$, we denote $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$ and consider the Sobolev spaces $W^{1,p}_0(\Omega)$ and $W^{1,q}_0(\Omega)$ endowed with the norms $\|\nabla u\|_p$ and $\|\nabla u\|_q$, respectively, where $\|\cdot\|_r$ stands for the usual L^r-norm. The negative p-Laplacian $-\Delta_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ is defined by

$$\langle -\Delta_p u, \varphi\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \quad \text{for all } u, \varphi \in W^{1,p}_0(\Omega).$$

This operator is strictly monotone and continuous, so pseudomonotone. If $p = 2$ we retrieve the ordinary Laplacian operator. Similarly, we have the definition of the negative q-Laplacian $-\Delta_q : W^{1,q}_0(\Omega) \to W^{-1,q'}(\Omega)$. By virtue of the embedding $W^{1,p}_0(\Omega) \hookrightarrow W^{1,q}_0(\Omega)$, the differential operator $-\Delta_p + \mu\Delta_q$ driving inclusion (1.3) and equation (1.6) is well posed in $W^{1,p}_0(\Omega)$. There exists a constant $k > 0$ such that

$$\|\nabla u\|_q \leq k\|\nabla u\|_p, \qquad \forall u \in W^{1,p}_0(\Omega). \tag{2.3}$$

By a weak solution to problem (1.3) with $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ we mean a $u \in W^{1,p}_0(\Omega)$ for which it holds $\underline{f}(u), \overline{f}(u) \in L^{p'}(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\\ \geq \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W^{1,p}_0(\Omega)\end{aligned} \tag{2.4}$$

or equivalently

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x\\ \leq \int_\Omega \max\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W^{1,p}_0(\Omega).\end{aligned} \tag{2.5}$$

The equivalence between (2.4) and (2.5) arises by replacing $\varphi \in W^{1,p}_0(\Omega)$ with $-\varphi$. For the Dirichlet equation (1.6), the ordinary notion of weak solution is recovered. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $u \in W_0^{1,p}(\Omega)$ is a weak solution to equation (1.6) provided $f(u) \in L^{p'}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x - \mu\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x$$
$$= \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{2.6}$$

This follows readily from (2.4) (or (2.5)), (1.4) and (1.5).

Finally, we sketch the framework of nonlocal problems (1.7) and (1.8). The fractional Sobolev space $W^{s,q}(\Omega)$ of differentiability order $s \in (0,1)$ and summability exponent $1 < q < +\infty$ for a bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz continuous boundary $\partial\Omega$ is introduced as

$$W^{s,q}(\Omega) := \left\{ u \in L^q(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy < \infty \right\},$$

which is a separable and reflexive Banach space endowed with the norm

$$\|u\|_{W^{s,q}(\Omega)} := \left(\|u\|_q^q + \frac{C_{N,q,s}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy \right)^{\frac{1}{q}},$$

with a normalization constant $C_{N,q,s} > 0$. If $sq < N$, the embedding $W^{s,q}(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for all $1 \le \nu \le q_s^*$, and compact for all $1 \le \nu < q_s^*$, with $q_s^* = Np/(N-sq)$ called the fractional critical exponent (see [5, Theorem 6.5, Corollary 7.2]). Under the conditions $0 < s < 1$, $1 < q < p < +\infty$ and $sq < N$, the embeddings $W^{1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ are continuous and thus for a constant $C = C(N,s,q) \ge 1$ one has

$$\|u\|_{W^{s,q}(\Omega)} \le C\|u\|_{W^{1,q}(\Omega)}, \qquad \forall u \in W^{1,p}(\Omega). \tag{2.7}$$

(see [5, Proposition 2.2])).

The closed linear subspace

$$W_0^{s,q}(\Omega) := \{u \in W^{s,q}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

can be endowed with the equivalent norm (determined by the Gagliardo seminorm)

$$\|u\|_{W_0^{s,q}(\Omega)} := \left(\frac{C_{N,q,s}}{2}\right)^{\frac{1}{q}} [u]_{D^{s,q}(\mathbb{R}^N)} := \left(\frac{C_{N,q,s}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy \right)^{\frac{1}{q}}$$

becoming a uniformly convex Banach space with the dual $W^{-s,q'}(\Omega)$.

The (negative) s-fractional q-Laplacian is the nonlinear operator $(-\Delta)_q^s : W_0^{s,q}(\Omega) \to W^{-s,q'}(\Omega)$ defined for all $u, v \in W_0^{s,q}(\Omega)$ by

$$\langle(-\Delta)_q^s(u), v\rangle = \frac{C_{N,q,s}}{2}\int_{R^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sq}}\,dx\,dy \tag{2.8}$$

(see [5,15] for more insight).

Along the pattern of the corresponding local problems, $u \in W_0^{1,p}(\Omega)$ is called a weak solution to inclusion (1.7) with $0 < s < 1$, $1 < q < p < +\infty$, $sq < N$ and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ provided $\underline{f}(u), \overline{f}(u) \in L^{p'}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x$$
$$+\mu\frac{C_{N,q,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+qs}}\,dxdy$$
$$\ge \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \tag{2.9}$$

where we set $u = \varphi = 0$ outside Ω. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $u \in W_0^{1,p}(\Omega)$ is a weak solution to the nonlocal equation (1.8) provided $f(u) \in L^{p'}(\Omega)$ and

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}x \\ &\qquad + \mu\frac{C_{N,q,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+qs}}\,dxdy \\ &= \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \end{aligned} \tag{2.10}$$

3 Existence of a generalized solution

In order to simplify the presentation, we denote by the same symbol $\|\cdot\|$ different norms that occur below. The meaning will be clear from the context. Our hypotheses on the data in problem (1.1) are as follows:

$(H1)$ *The Banach space X is separable and reflexive, and $j : X \to Y$ is a linear compact map from X to a Banach space Y.*

$(H2)$ *The function $F : X \to \mathbb{R}$ is Gâteaux differentiable, continuous, and the function $\Phi : Y \to \mathbb{R}$ is locally Lipschitz* such that

$$I = F + \Phi \circ j \text{ is bounded from below on } X \tag{3.1}$$

and I is coercive on every finite dimensional subspace of X, i.e., if X_0 is a finite dimensional subspace of X, then $I(u) \to +\infty$ as $\|u\| \to \infty$ with $u \in X_0$.

$(H3)$ *The set*

$$\{v \in X : \langle DF(v), v\rangle \le \Phi^0(jv; -jv)\}$$

is bounded in X.

The next example shows that the coercivity on every finite dimensional subspace in hypothesis $(H2)$ is a condition weaker than the coercivity on the whole space.

Example 3.1. Let X be a separable Hilbert space. Fix an orthonormal basis $\{e_m\}_{m\ge1}$ of X. Then every vector $u \in X$ can be written uniquely as $u = \sum_{m=1}^{\infty} x_m(u)e_m$, with $x_m(u) \in \mathbb{R}$, and there holds $\|u\|^2 = \sum_{m=1}^{\infty} x_m(u)^2$. The functional $J : X \to \mathbb{R}$ given by

$$J(u) = \sum_{m=1}^{\infty} \frac{1}{m^2}|x_m(u)|$$

is well defined. It is coercive on each finite dimensional subspace X_1 of X since corresponding to X_1 there is an integer m_1 such that

$$J(u) = \sum_{m=1}^{m_1} \frac{1}{m^2}|x_m(u)|, \qquad \forall u \in X_1.$$

For $u_n = ne_n$ we have $\|u_n\| = n$ and $J(u_n) = \frac{1}{n}$, so J is not coercive on X.

Now we state our existence result for problem (1.1).

Theorem 3.2. *Assume that conditions $(H1)$–$(H3)$ hold. Then problem (1.1) admits at least one generalized solution in the sense of Definition 1.1.*

Proof. We construct a special minimizing sequence $\{v_n\}_{n\geq 1} \subset X$ of the functional I in (1.2). The construction is done through a Galerkin basis $\{X_n\}_{n\geq 1}$ of X, which exists because the Banach space X is separable as known from assumption $(H1)$.

It follows from (3.1) that for every n the functional $I|_{X_n}$ obtained restricting I to X_n is bounded from below on X_n. Due to the coercivity of I on X_n as guaranteed by assumption $(H2)$, any minimizing sequence of $I|_{X_n}$ is bounded. Since $I|_{X_n}$ is also continuous and X_n is finite dimensional (note requirement (i) in the definition of Galerkin basis in Section 2), there exists $v_n \in X_n$ satisfying

$$I(v_n) = \min_{v\in X_n} I(v). \tag{3.2}$$

Then (3.2) implies

$$I(v_n + t(v - v_n)) \geq I(v_n), \qquad \forall t > 0,\ \forall v \in X_n,$$

which reads as

$$\frac{1}{t}(F(v_n + t(v - v_n)) - F(v_n)) + \frac{1}{t}(\Phi(jv_n + t(jv - jv_n)) - \Phi(jv_n)) \geq 0.$$

Passing to the limit as $t \to 0$ and then setting $v = 0$ lead to

$$\langle DF(v_n), v_n\rangle \leq \Phi^0(jv_n; -jv_n), \qquad \forall n. \tag{3.3}$$

On account of hypothesis $(H3)$, we can infer from (3.3) that the sequence $\{v_n\}$ is bounded in X. In view of the reflexivity of X (see hypothesis $(H1)$), along a relabeled subsequence we have

$$v_n \rightharpoonup u \quad \text{in } X \tag{3.4}$$

for some $u \in X$. We shall show that u is a generalized solution to (1.1).

From condition (ii) in the definition of Galerkin basis (see Section 2) and (3.2), for every n we can write

$$I(v_n) = \min_{v\in X_n} I(v) \geq \min_{v\in X_{n+1}} I(v) = I(v_{n+1}) \geq \inf_{v\in X} I(v).$$

Therefore the sequence $\{I(v_n)\}$ is nonincreasing and bounded due to (3.1). Set

$$l := \lim_{n\to\infty} I(v_n) = \inf_{n\geq 1} I(v_n).$$

We claim that

$$\lim_{n\to\infty} I(v_n) = \inf_{w\in X} I(w). \tag{3.5}$$

In order to prove (3.5), we argue by contradiction supposing that

$$l > \inf_{v\in X} I(v).$$

So, there exists $\hat{w} \in X$ such that $I(\hat{w}) < l$. By the continuity of I, there exists an open neighborhood U of $\hat{w}$ in X such that

$$I(w) < l, \qquad \forall w \in U. \tag{3.6}$$

Then through condition (iii) in the definition of Galerkin basis (see Section 2) we derive

$$U \cap \left(\bigcup_{n=1}^{\infty} X_n \right) \neq \emptyset.$$

Hence there exists $\tilde{w} \in U \cap X_{\tilde{n}}$ for some $\tilde{n}$. Recalling that $v_{\tilde{n}}$ is a minimizer of $I_{|X_{\tilde{n}}}$ (see (3.2)), from (3.6) we get the contradiction

$$\min_{v \in X_{\tilde{n}}} I(v) \leq I(\tilde{w}) < l \leq \min_{v \in X_{\tilde{n}}} I(v).$$

The obtained contradiction ensures the validity of (3.5).

Now we construct another minimizing sequence $\{u_n\}$ of I in (1.2) that will satisfy conditions (S_1)–(S_3) in Definition 1.1. To this end we notice from (3.1) that we can apply Theorem 2.1 (Ekeland's Variational Principle, see [6, 7]) to the functional I in (1.2). Through this result, using the minimizing sequence $\{v_n\}_{n \geq 1}$, we can find a sequence $\{u_n\}_{n \geq 1}$ in X with the properties (a), (b), (c) in Theorem 2.1. From property (a) and (3.5) it is clear that

$$\lim_{n \to \infty} I(u_n) = \inf_{v \in X} I(v), \tag{3.7}$$

so $\{u_n\}_{n \geq 1}$ is a minimizing sequence of the functional I. Consequently, from (3.7) it turns out

$$\lim_{n \to \infty} I(u_n) \leq I(u), \tag{3.8}$$

with $u \in X$ in (3.4). By property (b) in Theorem 2.1 and (3.4) we infer that

$$u_n \rightharpoonup u \quad \text{in } X, \tag{3.9}$$

thus condition (S_1) in Definition 1.1 is verified.

Using the compactness of the map $j : X \to Y$ and the weak convergence in (3.9), we note that (3.8) amounts to saying that

$$\begin{aligned} \limsup_{n \to \infty} F(u_n) + \Phi(j(u)) &= \limsup_{n \to \infty} (F(u_n) + \Phi(j(u_n))) \\ &\leq F(u) + \Phi(j(u)). \end{aligned}$$

This proves property (S_2) in Definition 1.1.

Insert $w = u_n + t(v - u_n)$ in assertion (c) of Theorem 2.1, with $t > 0$ and an arbitrary $v \in X$, finding that

$$\frac{1}{t}(F(u_n + t(v - u_n)) - F(u_n)) + \frac{1}{t}(\Phi(ju_n + t(jv - ju_n)) - \Phi(ju_n)) \geq -\frac{1}{n}\|v - u_n\|. \tag{3.10}$$

The Gâteaux differentiability of F yields

$$\lim_{t \to 0} \frac{1}{t}(F(u_n + t(v - u_n)) - F(u_n)) = \langle DF(u_n), v - u_n \rangle, \tag{3.11}$$

while the definition of generalized directional derivative Φ^0 of Φ (see Section 2) shows

$$\limsup_{t \to 0} \frac{1}{t}(\Phi(ju_n + t(jv - ju_n)) - \Phi(ju_n)) \leq \Phi^0(ju_n; jv - ju_n). \tag{3.12}$$

Letting $t \to 0$ in (3.10), by making use of (3.11) and (3.12), we arrive at

$$\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n) \geq -\frac{1}{n}\|v - u_n\|. \tag{3.13}$$

Notice that (3.9) and the compactness of $j : X \to Y$ yield

$$ju_n \to ju \quad \text{in } Y. \tag{3.14}$$

Then the upper semicontinuity of the generalized directional derivative Φ^0 and the strong convergence in (3.14) give

$$\limsup_{n\to\infty} \Phi^0(ju_n; jv - ju_n) \leq \Phi^0(ju; jv - ju). \tag{3.15}$$

Letting $n \to \infty$ in (3.13) and taking into account (3.15) as well as the boundedness of the sequence $\{u_n\}_{n\geq 1}$ we find that

$$\begin{aligned}
&\liminf_{n\to\infty} \langle DF(u_n), v - u_n \rangle \\
&\quad = \liminf_{n\to\infty} (\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n) - \Phi^0(ju_n; jv - ju_n)) \\
&\quad \geq \liminf_{n\to\infty} (\langle DF(u_n), v - u_n \rangle + \Phi^0(ju_n; jv - ju_n)) + \liminf_{n\to\infty} (-\Phi^0(ju_n; jv - ju_n)) \\
&\quad \geq -\limsup_{n\to\infty} \Phi^0(ju_n; jv - ju_n) \geq -\Phi^0(ju; jv - ju), \qquad \forall v \in X.
\end{aligned}$$

Thus we are led to

$$\liminf_{n\to\infty} \langle DF(u_n), v - u_n \rangle + \Phi^0(ju; jv - ju) \geq 0, \qquad \forall v \in X,$$

which is just property (S_3) in Definition 1.1. Therefore $u \in X$ is a generalized solution to problem (1.1). The proof is complete. □

We illustrate the applicability of Theorem 3.2 with verifiable growth conditions.

Corollary 3.3. (i) *Assume that the Gâteaux differentiable, continuous function $F : X \to \mathbb{R}$ and the locally Lipschitz function $\Phi : Y \to \mathbb{R}$ satisfy*

$$F(v) \geq a\|v\|^r - a_0 \quad \textit{for all } v \in X, \tag{3.16}$$

with constants $a > 0$, $a_0 > 0$, $r > 0$, and

$$\Phi(w) \geq -b\|w\|^\sigma - b_0 \quad \textit{for all } w \in Y, \tag{3.17}$$

with constants $b > 0$, $b_0 > 0$ and $\sigma \in (0, r)$. Then condition $(H2)$ holds true.

(ii) *Assume that the Gâteaux differentiable, continuous function $F : X \to \mathbb{R}$, the linear compact map $j : X \to Y$ and the locally Lipschitz function $\Phi : Y \to \mathbb{R}$ satisfy*

$$\langle DF(v), v \rangle \geq \tilde{a}\|v\|^{\tilde{r}} - \tilde{a}_0 \quad \textit{for all } v \in X, \tag{3.18}$$

with constants $\tilde{a} > 0$, $\tilde{a}_0 > 0$, $\tilde{r} > 0$, and

$$\langle \xi, jv \rangle \geq -\tilde{b}\|jv\|^{\tilde{\sigma}} - \tilde{b}_0 \quad \textit{for all } v \in X \textit{ and } \xi \in \partial\Phi(jv), \tag{3.19}$$

with constants $\tilde{b} > 0$, $\tilde{b}_0 > 0$ and $\tilde{\sigma} \in (0, \tilde{r})$. Then condition $(H3)$ is fulfilled.

Proof. (i) From (3.16) and (3.17), we estimate the functional I in (1.2) from below

$$I(v) = F(v) + \Phi(jv) \geq a\|v\|^r - a_0 - b\|j\|^\sigma \|v\|^\sigma - b_0$$

for all $v \in X$. Since $r > \sigma$, we infer that (3.1) holds true. Moreover, the preceding estimate entails

$$I \text{ is coercive on } X, \text{ i.e., } I(u) \to +\infty \text{ as } \|u\| \to \infty,$$

which ensures that condition $(H2)$ is verified.

(ii) We are going to show that the set

$$X_0 := \{v \in X : \langle DF(v), v\rangle \leq \Phi^0(jv; -jv)\}$$

is bounded in X. On the basis of (3.18) and (3.19), for every $v \in X_0$ we obtain

$$\begin{aligned}\tilde{a}\|v\|^{\tilde{r}} - \tilde{a}_0 &\leq \langle DF(v), v\rangle \leq \Phi^0(jv; -jv) = \max\{\langle \xi, -jv\rangle : \xi \in \partial\Phi(jv)\} \\ &= -\min\{\langle \xi, jv\rangle : \xi \in \partial\Phi(jv)\} \leq \tilde{b}\|jv\|^{\tilde{\sigma}} + \tilde{b}_0 \leq \tilde{b}\|j\|^{\tilde{\sigma}}\|v\|^{\tilde{\sigma}} + \tilde{b}_0.\end{aligned}$$

Taking into account that $\tilde{\sigma} < \tilde{r}$, the boundedness of the set X_0 in X follows. □

Remark 3.4. Conditions (3.16), (3.17), (3.18) and (3.19) are compatible offering a large range of applicability for Theorem 3.2.

4 Local boundary value problems without ellipticity

In this section we focus on the boundary value inclusion with discontinuities (1.3), which extends the Dirichlet equation (1.6). For $1 < q < p < +\infty$ and $\mu \in \mathbb{R}$, we shall show that problem (1.3), so a fortiori (1.6), is a special case of problem (1.1) treated in Section 3. The principal point is that the leading operator $-\Delta_p + \mu\Delta_q$ exhibits a competing (p, q)-Laplacian if μ is positive, thus the ellipticity fails.

We assume to be fulfilled:

$(H)_f$ the function $f : \mathbb{R} \to \mathbb{R}$ is measurable and there exist constants $c > 0$ and $\sigma \in (1, p)$ such that

$$|f(t)| \leq c(1 + |t|^{\sigma-1}) \quad \text{for a.e. } t \in \mathbb{R}.$$

From assumption $(H)_f$ it follows that $f \in L^\infty_{\text{loc}}(\mathbb{R})$, hence the functions $\underline{f} : \mathbb{R} \to \mathbb{R}$ and $\overline{f} : \mathbb{R} \to \mathbb{R}$ introduced in (1.4) and (1.5), respectively, are well-defined.

The notion of generalized solution to problem (1.1) introduced in Definition 1.1 reads in the case of (1.3) as follows: $u \in W^{1,p}_0(\Omega)$ is a generalized solution to (1.3) if there exists a sequence $\{u_n\}_{n\geq 1} \subset W^{1,p}_0(\Omega)$ such that

(S_1') $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega)$;

(S_2')

$$\limsup_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] \leq \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q; \tag{4.1}$$

(S_3') $\liminf\limits_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \geq \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,dx, \qquad \forall \varphi \in W^{1,p}_0(\Omega).$

Passing from (S_3) in Definition 1.1 to formulation (S_3') is based on the Aubin–Clarke Theorem for an integral functional (see [4, Theorem 2.7.5]).

Remark 4.1. If f is continuous, then the interval $[\underline{f}(u(x)), \overline{f}(u(x))]$ reduces to the singleton $f(u(x))$ and (S_3') becomes

$(\tilde{S}_3')$ $-\Delta_p u_n + \mu\Delta_q u_n \rightharpoonup f(u)$ in $W^{-1,p'}(\Omega)$, i.e.,

$$\lim_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle = \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).$$

Indeed, (S_3') entails

$$\liminf_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \geq \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).$$

Changing φ into $-\varphi$ produces

$$\limsup_{n\to\infty}\langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle \leq \int_\Omega f(u(x))\varphi(x)\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega),$$

whence the result.

If $q = 2 < p < +\infty$, from (S_1') and the linearity of the Laplacian Δ we deduce that $(\tilde{S}_3')$ requires $-\Delta_p u_n \rightharpoonup -\mu\Delta u + f(u)$ in $W^{-1,p'}(\Omega)$.

Now we state our result on problems (1.3) and (1.6).

Theorem 4.2. *Assume that condition $(H)_f$ holds. Then, for every $\mu \in \mathbb{R}$, problem (1.3) admits at least one generalized solution. Every generalized solution is a weak solution provided $\mu \leq 0$. In particular, problem (1.6) with f continuous possesses at least a generalized solution, which is a weak solution when $\mu \leq 0$.*

Proof. Our goal is to apply Theorem 3.2 by means of Corollary 3.3. To this end we choose $X = W_0^{1,p}(\Omega)$, which is a separable and reflexive Banach space. Further, we take $Y = L^p(\Omega)$ and let $j : W_0^{1,p}(\Omega) \to L^p(\Omega)$ be the inclusion map. By Rellich–Kondrachov Theorem j is compact. Therefore assumption $(H1)$ is satisfied.

With a fixed $\mu \in \mathbb{R}$, define the functional $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ as

$$F(v) = \frac{1}{p}\|\nabla v\|_p^p - \frac{\mu}{q}\|\nabla v\|_q^q \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

It is clear that $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ is continuously differentiable, so Gâteaux differentiable and continuous. By (2.3), Young's inequality and $p > q$, we infer that

$$F(v) \geq \frac{1}{p}\|\nabla v\|_p^p - \frac{|\mu|k}{q}\|\nabla v\|_p^q \geq \frac{1}{2p}\|\nabla v\|_p^p - a_0 \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

with a constant $a_0 > 0$. Hence condition (3.16) is verified with $r = p$.

Next we consider the function $g : \mathbb{R} \to \mathbb{R}$ in (2.1) corresponding to $f : \mathbb{R} \to \mathbb{R}$ in the right-hand side of (1.3). Thanks to assumption $(H)_f$, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and in turn the functional $\Phi : L^p(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = -\int_\Omega g(v(x))\,\mathrm{d}x \quad \text{for all } v \in L^p(\Omega) \tag{4.2}$$

is locally Lipschitz. Precisely, Φ is Lipschitz continuous on the bounded subsets of $L^{\sigma}(\Omega)$ and we use the continuous embedding $L^p(\Omega) \hookrightarrow L^{\sigma}(\Omega)$ with $\sigma < p$.

Hypothesis $(H)_f$ implies

$$\begin{aligned}|\Phi(v)| &\leq \int_{\Omega} |g(v(x))|\,\mathrm{d}x \leq c\|v\|_1 + \frac{c}{\sigma}\|v\|_{\sigma}^{\sigma} \leq c|\Omega|^{\frac{1}{\sigma'}}\|v\|_{\sigma} + \frac{c}{\sigma}\|v\|_p^{\sigma} \\ &\leq c_0(1 + \|v\|_{\sigma}^{\sigma}), \qquad \forall v \in L^p(\Omega),\end{aligned}$$

with a constant $c_0 > 0$ and $\sigma' = \sigma/(\sigma - 1)$. We derive (3.17) due to the continuous embedding $L^p(\Omega) \hookrightarrow L^{\sigma}(\Omega)$. By Corollary 3.3 part (i), condition $(H2)$ is fulfilled.

We note that

$$\langle DF(v), v\rangle = \|\nabla v\|_p^p - \mu\|\nabla v\|_q^q \quad \text{for all } v \in X,$$

so condition (3.18) is satisfied with $\tilde{r} = p$ because $p > q$. Pick any $v \in W_0^{1,p}(\Omega)$ and $\xi \in \partial\Phi(jv)$, with Φ in (4.2). The Aubin–Clarke Theorem (see [4, Theorem 2.7.5]) and (2.2) guarantee that $\xi \in L^{p'}(\Omega)$ and

$$-\xi(x) \in \partial g(v(x)) = [\underline{f}(v(x)), \overline{f}(v(x))] \quad \text{for a.e. } x \in \Omega. \tag{4.3}$$

Then by (4.2), $(H)_f$, (4.3) (see also (1.4), (1.5)) and the continuous embedding $L^p(\Omega) \hookrightarrow L^{\sigma}(\Omega)$, we infer that

$$\begin{aligned}\langle \xi, jv\rangle &= \int_{\Omega} \xi(x) jv(x) dx \geq -\int_{\Omega} |\xi(x)||jv(x)| dx \\ &\geq -\int_{\Omega} c(1 + |jv(x)|^{\sigma-1})|jv(x)| dx \\ &\geq -\tilde{b}\|jv\|^{\sigma} - \tilde{b}_0 \quad \text{for all } v \in W_0^{1,p}(\Omega) \text{ and } \xi \in \partial\Phi(jv),\end{aligned}$$

with constants $\tilde{b} > 0$ and $\tilde{b}_0 > 0$. This confirms the validity of (3.19) with $\tilde{\sigma} = \sigma$. From Corollary 3.3 part (ii), assumption $(H3)$ holds true.

We are in a position to apply Theorem 3.2, which ensures the existence of a generalized solution to problem (1.3) in the sense of Definition 1.1. Specifically, we find $u \in W_0^{1,p}(\Omega)$ and a sequence $\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega)$ satisfying (S_1'), (S_2') and

$$\liminf_{n\to\infty} \langle -\Delta_p u_n + \mu\Delta_q u_n, \varphi\rangle + \Phi^0(u;\varphi) \geq 0, \qquad \forall \varphi \in W_0^{1,p}(\Omega), \tag{4.4}$$

with Φ in (4.2). By the Aubin–Clarke Theorem applied to Φ in (4.2), $(H)_f$ and (2.2), we find

$$\begin{aligned}\Phi^0(u;\varphi) &\leq \int_{\Omega} \max[-\partial g(u(x))\varphi(x)]\,\mathrm{d}x \\ &= -\int_{\Omega} \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).\end{aligned} \tag{4.5}$$

At this point it is enough to insert (4.5) in (4.4) to get that (S_3') holds, which proves the first part of Theorem 4.2.

Suppose that $u \in W_0^{1,p}(\Omega)$ is a generalized solution to problem (1.3) with $\mu \leq 0$. We note from property (ii) in Definition 1.1 that

$$\limsup_{n\to\infty} \left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] \leq \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q;.$$

On the other hand, using the weak lower semicontinuity of the norm in conjunction with $\mu \leq 0$ and (i) of Definition 1.1, it turns out

$$\begin{aligned}\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u_n\|_q^q\right] &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p - \frac{\mu}{q}\liminf_{n\to\infty}\|\nabla u_n\|_p^p \\ &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p - \frac{\mu}{q}\|\nabla u\|_q^q.\end{aligned}$$

By a simple comparison we are led to

$$\limsup_{n\to\infty}\|\nabla u_n\|_p \leq \|\nabla u\|_p,$$

which implies the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$ because the space $W_0^{1,p}(\Omega)$ is uniformly convex (see, e.g., [2, Proposition 3.32]). On the basis of the strong convergence $u_n \to u$, we can utilize the continuity of $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ with $q < p$, to pass to the limit in (S_3') obtaining (2.4). This amounts to saying that u is a weak solution of (1.3). Since (2.6) is a particular case of (2.4), the proof is complete. □

5 Nonlocal boundary value problems without ellipticity

This section deals with the nonlocal boundary value problem with discontinuities (1.7) and its particular case (1.8) under the conditions $0 < s < 1$, $1 < q < p < +\infty$, $sq < N$ and $\mu \in \mathbb{R}$, thus allowing that the local operator $-\Delta_p$ and the nonlocal operator $(-\Delta)_q^s$ be competing.

The function $f : \mathbb{R} \to \mathbb{R}$ in the right-hand side of (1.7) and (1.8) is required to satisfy condition $(H)_f$ in Section 4. Subsequently, we use the notation in Section 2, in particular the associated functions $\underline{f} : \mathbb{R} \to \mathbb{R}$ and $\overline{f} : \mathbb{R} \to \mathbb{R}$ have the meaning in (1.4) and (1.5), respectively.

We rely on the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,q}(\Omega)$. As in (2.7), there is a constant $C > 0$ such that

$$\|u\|_{W_0^{s,q}(\Omega)} \leq C\|\nabla u\|_p, \qquad \forall u \in W_0^{1,p}(\Omega) \tag{5.1}$$

making the sum $-\Delta_p u + \mu(-\Delta)_q^s u$ well defined for $u \in W_0^{1,p}(\Omega)$ in problems (1.7) and (1.8).

In accordance with Definition 1.1, by a generalized solution to nonlocal problem (1.7) we mean any $u \in W_0^{1,p}(\Omega)$ for which one can find a sequence $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ satisfying

(S_1'') $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$;

(S_2'')
$$\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u_n\|_{W_0^{s,q}(\Omega)}^q\right] \leq \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{q}\|u\|_{W_0^{s,q}(\Omega)}^q; \tag{5.2}$$

(S_3'')
$$\begin{aligned}&\liminf_{n\to\infty}\langle -\Delta_p(u_n) + \mu(-\Delta)_q^s(u_n), \varphi\rangle \\ &\quad\geq \int_\Omega \min\{\underline{f}(u(x))\varphi(x), \overline{f}(u(x))\varphi(x)\}\,\mathrm{d}x, \qquad \forall \varphi \in W_0^{1,p}(\Omega).\end{aligned}$$

Here (S_3'') is obtained from (S_3) in Definition 1.1 by applying the Aubin–Clarke Theorem (see [4, Theorem 2.7.5]).

Our result on the nonlocal problems (1.7) and (1.8) is as follows.

Theorem 5.1. *Assume that condition $(H)_f$ holds. Then, for every $\mu \in \mathbb{R}$, problem (1.7) admits at least one generalized solution, which is a weak solution provided $\mu \geq 0$. In particular, this is valid for problem (1.8) with f continuous.*

Proof. In order to address Theorem 3.2 and Corollary 3.3, we choose: $X = W_0^{1,p}(\Omega)$, $Y = L^p(\Omega)$ and $j : W_0^{1,p}(\Omega) \to L^p(\Omega)$ be the inclusion map, which is compact. Consequently, assumption $(H1)$ is verified.

For a fixed $\mu \in \mathbb{R}$, we introduce the functional $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$F(v) = \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{q}\|u\|^q_{W_0^{s,q}(\Omega)} \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

This is possible thanks to (5.1). Using (2.8), it is seen that F is continuously differentiable with the differential

$$\langle DF(u), v\rangle = \langle -\Delta_p(u_n) + \mu(-\Delta)_q^s(u_n), v\rangle, \qquad \forall u, v \in W_0^{1,p}(\Omega).$$

By (5.1), Young's inequality and $p > q$, we find the estimate

$$F(v) \geq \frac{1}{p}\|\nabla v\|_p^p - \frac{|\mu|}{q}\|v\|^q_{W_0^{s,q}(\Omega)} \geq \frac{1}{2p}\|\nabla v\|_p^p - a_0 \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

with a constant $a_0 > 0$. Condition (3.16) is thus verified with $r = p$.

Consider the function $\Phi : L^p(\Omega) \to \mathbb{R}$ introduced in (4.2). Taking into account $(H)_f$, condition (3.19) was already checked in the proof of Theorem 4.2. Gathering (3.16) and (3.19), we are able to refer to Corollary 3.3, which yields that Theorem 3.2 can be applied. A reasoning similar to the one in the proof of Theorem 4.2 enables us to conclude that there exists a generalized solution to problem (1.7) and thus (1.8).

The last step in the proof is to show that any generalized solution of problems (1.7) and (1.8) is a weak solution provided $\mu \geq 0$. We argue on the basis of assertion (S_2'') in the definition of generalized solution. Given a generalized solution $u \in W_0^{1,p}(\Omega)$ of problem (1.7) with $\mu \geq 0$, we compare inequality (5.2) in the definition of generalized solution and the following inequality derived from weak lower semicontinuity of the norm (note (S_1''))

$$\begin{aligned}\limsup_{n\to\infty}\left[\frac{1}{p}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u_n\|^q_{W_0^{s,q}(\Omega)}\right] &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p + \frac{\mu}{q}\liminf_{n\to\infty}\|u_n\|^q_{W_0^{s,q}(\Omega)} \\ &\geq \frac{1}{p}\limsup_{n\to\infty}\|\nabla u_n\|_p^p + \frac{\mu}{q}\|u\|^q_{W_0^{s,q}(\Omega)}\end{aligned}$$

to deduce that

$$\limsup_{n\to\infty}\|\nabla u_n\|_p \leq \|\nabla u\|_p.$$

In view of the uniform convexity of the space $W_0^{1,p}(\Omega)$, property (S_1'') entitles the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. From here and (S_3''), through the continuity of $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $(-\Delta)_q^s : W_0^{s,q}(\Omega) \to W^{-s,q'}(\Omega)$, we reach in the limit (2.9). Therefore u is a weak solution to nonlocal problem (1.7). If f is continuous, we get (2.10), which completes the proof. □

Remark 5.2. As established in [9], one always has $W_0^{s,p}(\Omega) \not\subset W_0^{s,q}(\Omega)$. For this reason we cannot replace $-\Delta_p$ by the nonlocal operator $(-\Delta)_p^s$ in problems (1.7) and (1.8).

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References

[1] G. Bonanno, P. Candito, D. Motreanu, A coincidence point theorem for sequentially continuous mappings, *J. Math. Anal. Appl.* **435**(2016), No. 1, 606–615. https://doi.org/10.1016/j.jmaa.2015.10.039; MR3423416; Zbl 1342.47072

[2] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829; Zbl 1220.46002

[3] K. C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80**(1981), No. 1, 102–129. https://doi.org/10.1016/0022-247X(81)90095-0; MR0614246; Zbl 0487.49027

[4] F. H. Clarke, *Optimization and nonsmooth analysis*, John Wiley & Sons, Inc., New York, 1983. https://doi.org/10.1137/1.9781611971309; MR1019086; Zbl 0727.90045

[5] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**(2012), No. 5, 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004; MR2944369; Zbl 1252.46023

[6] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47**(1974), No. 2, 324–353. https://doi.org/10.1016/0022-247X(74)90025-0; MR0346619; Zbl 0286.49015

[7] I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1**(1979), No. 3, 443–474. https://doi.org/10.1090/S0273-0979-1979-14595-6; MR0526967; Zbl 0441.49011

[8] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*, De Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, 2011. https://doi.org/10.1515/9783110889741; MR2778606; Zbl 1227.31001

[9] P. Mironescu, W. Sickel, A Sobolev non embedding, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **26**(2015), No. 3, 291–298. https://doi.org/10.4171/RLM/707; MR3357858; Zbl 1341.46024

[10] D. Motreanu, P. D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Nonconvex Optimization and its Applications, Vol. 29, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/978-1-4615-4064-9; MR1675895; Zbl 1060.49500

[11] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conference Series in Mathematics, Vol. 65, CBMS, American Mathematical Society, Providence, 1986. https://doi.org/10.1090/cbms/065; MR0845785; Zbl 0609.58002

[12] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, Vol. 49, American Mathematical Society, Providence, RI, 1997. https://doi.org/10.1090/surv/049; MR1422252; Zbl 0870.35004

[13] M. Sofonea, W. Han, M. Shillor, *Analysis and approximation of contact problems with adhesion or damage*, Pure and Applied Mathematics, Vol. 276. Chapman & Hall/CRC, Boca Raton, FL, 2006. https://doi.org/10.1201/9781420034837; MR2183435; Zbl 1089.74004

[14] M. Sofonea, S. Migorski, *Variational-hemivariational inequalities with applications*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2018. https://doi.org/10.1201/9781315153261; MR3752610; Zbl 1384.49002

[15] M. Warma, Local Lipschitz continuity of the inverse of the fractional p-Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains, *Nonlinear Anal.* **135**(2016), 129–157. https://doi.org/10.1016/j.na.2016.01.022; MR3473114; Zbl 1333.35332

[16] E. Zeidler, *Nonlinear functional analysis and its applications. II/A. Linear monotone operators*, Springer-Verlag, New York, 1990. https://doi.org/10.1007/978-1-4612-0985-0; MR1033497; Zbl 0684.47028

[17] E. Zeidler, *Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators*, Springer-Verlag, New York, 1990. https://doi.org/10.1007/978-1-4612-0985-0; MR1033498; Zbl 0684.47029