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Infinitely many solutions for a quasilinear Schrödinger equation with Hardy potentials

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Abstract. In this article, we study the following quasilinear Schrödinger equation

$$-\Delta u - \mu \frac{u}{|x|^2} + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N,$$

where $V(x)$ is a given positive potential and the nonlinearity $f(x,u)$ is allowed to be sign-changing. Under some suitable assumptions, we obtain the existence of infinitely many nontrivial solutions by a change of variable and Symmetric Mountain Pass Theorem.

Keywords: quasilinear Schrödinger equation, Hardy potential, Mountain Pass Theorem.

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1 Introduction and main results

In this paper, we consider the following equation

$$-\Delta u - \mu \frac{u}{|x|^2} + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $0 \leq \mu < \bar{\mu} := \frac{(N-2)^2}{4}$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is a given potential and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

For problem (1.1), if $\mu = 0$, $f(x,u) = f(u)$, then (1.1) becomes

$$-\Delta u + V(x)u - (\Delta(u^2))u = f(u), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Recently, the existence of solutions for (1.2) has drawn much attention, see for example [5,7, 19,21,22,25]. Particularly, it was established the existence of both one-sign and nodal ground states of soliton type solutions in [21] by Nehari method. Furthermore, using a constrained minimization argument, the existence of a positive ground state solution has been proved in [25]. Later, by using a change of variables, [19] and [7] studied the existence of solutions in

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different working spaces with different classes of nonlinearities. For more results we can refer to [18,20,23,33,34].

Moreover, if we take $\mu \equiv 0$ in (1.1), we have

$$-\Delta u + V(x)u - (\Delta(u^2))u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.3}$$

In [39], Zhang and Tang proved there are infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential by Mountain Pass Theorem. When $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s}$, where $0 \le s < 2$ and $2^*(s) = \frac{2(N-s)}{N-2}$ is the critical Sobolev–Hardy exponents, the problem (1.3) was studied in [10,12]. If $f(x,u) = \lambda|u|^{q-2}u + \frac{|u|^{p-2}u}{|x|^s}$, the authors in [40] have proved the existence of solutions by using a change of variable.

Recently, great attention has been attracted to the study of the following problem

$$-\Delta u - \mu\frac{u}{|x|^2} + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

This class of quasilinear equations are often referred as modified nonlinear Schrödinger equations, whose solutions are related to the existence of standing wave solutions. For example, by use of variational method, Kang and Deng in [13] proved the existence of solutions for $V(x) = 0$ and $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s} + K(x)|u|^{r-2}u$. Using the similar method, Li in [14] proved the existence of nontrivial solutions for $V(x) = 0$ and $f(x,u) = \frac{|u|^{2^*(s)-2}u}{|x|^s} + K(x)|u|^{r-2}u + \lambda u$. In [4], Cao and Zhou studied the problem (1.4) with $V(x) \equiv 1$ and general subcritical nonlinearity $f(x,u)$, they obtained the existence and multiplicity of positive solutions in some different conditions, their method relies upon the proof of Tarantello in [30]. Under certain conditions, using Ekeland's variational principle, Chen and Peng in [6] obtained the existence of a positive solution with $V(x) \equiv 1$ and nonlinearity $\lambda(f(x,u) + h(x))$. For more results about (1.4), we can refer to [9,11,29] and the references therein.

As regards other relevant papers, we mention here [8,15–17,27,28,31,35,38]. Motivated by facts mentioned above, in this paper, we study the existence of infinitely many solutions for problem (1.1) by Mountain Pass Theorem. Before giving the main result of this paper, we give the assumptions of the potential $V(x)$ and the nonlinear term $f(x,u)$ as follows, firstly

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;

(V_2) for any $L > 0$, there exists a constant $\vartheta > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\{x \in \mathbb{R}^N : |x-y| \le \vartheta, V(x) \le L\} = 0;$$

(F_0) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_1, c_2 > 0$ and $4 < p < 22^*$ such that

$$|f(x,u)| \le c_1|u| + c_2|u|^{p-1}, \qquad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(F_1) $\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = \infty$ uniformly in x, and there exists $a_0 \ge 0$ such that $F(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ and $|u| \ge a_0$, where $F(x,u) = \int_0^u f(x,s)ds$;

(F_2) $\tilde{F}(x,u) := \frac{1}{4}f(x,u)u - F(x,u) \ge 0$ and there exist $c_0 > 0$ and $\sigma \in \left(\max\{1, \frac{2N}{N+2}\}, 2\right)$ such that

$$|F(x,u)|^\sigma \le c_0|u|^{2\sigma}\tilde{F}(x,u)$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ with u large enough;

(F_3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that* (V_1)–(V_2), (F_0)–(F_3) *are satisfied, then problem* (1.1) *has infinitely many nontrivial solutions* $\{u_n\}$ *such that* $\|u_n\| \to \infty$ *and* $I(u_n) \to \infty$*(I will be defined later).*

Remark 1.2 (see [30]). It follows from (F_1) and (F_2) that

$$\tilde{F}(x,u) \geq \frac{1}{c_0}\left(\frac{|F(x,u)|}{|u|^2}\right)^{\sigma} \to \infty, \tag{1.5}$$

uniformly in x as $|u| \to \infty$.

This paper is organized as follows. In Section 2, we will introduce the variational setting for the problem and some preliminary results. In Section 3, we give the proof of main result.

Notations. In what follows we will adopt the following notations

- C, C_i, $i = 1,2,3,\ldots$ denote possibly different positive constants which may change from line to line;
- For $1 \leq p < \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue spaces with norms

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}, \qquad 1 \leq p < \infty;$$

- $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces modeled in $L^2(\mathbb{R}^N)$ with norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx\right)^{1/2}.$$

- B_R denotes the open ball centered at the origin and radius $R > 0$.

2 Variational setting and preliminary results

Before establishing the variational setting for problem (1.1), we give our working space firstly. Under the assumption (V_1) we define

$$E := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\right\},$$

then E is a Hilbert space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^N}\left(\nabla u \nabla v - \mu\frac{uv}{|x|^2} + V(x)uv\right)dx, \qquad \|u\| = (u,u)^{1/2}.$$

In view of (V_1) and for $u \in E$, the following norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx\right)^{\frac{1}{2}}$$

is equivalent to the classic one in $H^1(\mathbb{R}^N)$.

Now, let us recall the Hardy inequality, which is the main tool and allows us to deal with Hardy-type potentials.

Lemma 2.1 (see [1]). *Assume that $1 < p < N$ and $u \in W^{1,p}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \le \left(\frac{p}{(N-p)}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Thus, by Lemma 2.1, $\|u\|$ is well defined. In fact

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx \\
&\quad \ge \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{4}{(N-2)^2} |\nabla u|^2\right) dx \\
&\quad = \left(1 - \mu \frac{4}{(N-2)^2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&\quad > \left(1 - \frac{(N-2)^2}{4} \frac{4}{(N-2)^2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&\quad = 0.
\end{aligned} \tag{2.1}$$

Lemma 2.2. *Assume that $0 \le \mu < \bar{\mu} = \frac{(N-2)^2}{4}$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \|u\|_E^2 \le \|u\|^2 \le C_2 \|u\|_E^2,$$

for any $u \in H^1(\mathbb{R}^N)$.

Proof. For $\mu \ge 0$, we have

$$\begin{aligned}
\|u\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + V(x) u^2\right) dx \\
&\le \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx \\
&= \|u\|_E^2.
\end{aligned} \tag{2.2}$$

On the other hand, since $0 \le \mu < \bar{\mu} = \frac{(N-2)^2}{4}$, we can get

$$1 \ge 1 - \frac{4\mu}{(N-2)^2} > 0.$$

Then, we have

$$\begin{aligned}
\|u\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} + V(x) u^2\right) dx \\
&\ge \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{4\mu}{(N-2)^2} |\nabla u|^2 + V(x) u^2\right) dx \\
&\ge \left(1 - \frac{4\mu}{(N-2)^2}\right) \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx \\
&= \left(1 - \frac{4\mu}{(N-2)^2}\right) \|u\|_E^2.
\end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$C_1 \|u\|_E^2 \le \|u\|^2 \le C_2 \|u\|_E^2. \qquad \square$$

As we all known, under the assumption (V_1), the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2^*]$ and $E \hookrightarrow L^r_{loc}(\mathbb{R}^N)$ is compact for $[2, 2^*)$ i.e. there is a constant $d_r > 0$ such that

$$\|u\|_s \leq d_r \|u\|_E, \qquad \forall u \in E,\ r \in [2, 2^*].$$

From this, by Lemma 2.2, there is $C_3 > 0$ such that

$$\|u\|_r \leq d_r \|u\|_E \leq C_3 \|u\|, \qquad \forall u \in E,\ r \in [2, 2^*].$$

Furthermore, under the assumptions (V_1) and (V_2), we have the following compactness lemma due to [3] (see also [2, 41]).

Lemma 2.3. *Assume that (V_1) and (V_2) hold, the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $2 \leq r < 2^*$.*

In order to solve problem (1.1), we define the energy functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 + 2|u|^2)|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

It is well known that I is not well defined in E. To overcome this difficulty, we make the change of variables by $v = h^{-1}(u)$, where h is defined by

$$h'(t) = \frac{1}{\sqrt{1 + 2|h(t)|^2}} \quad \text{on } [0, \infty),$$

and

$$h(-t) = -h(t) \quad \text{on } (-\infty, 0].$$

Therefore, after the change of variables, from $I(u)$ we obtain the following functional

$$\begin{aligned} J(v) &:= I(h(v)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mu}{|x|^2} h^2(v) dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) h^2(v) dx - \int_{\mathbb{R}^N} F(x, h(v)) dx. \end{aligned} \tag{2.4}$$

It is easy to check that J is well defined on E. Under our hypotheses, $J \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'(v), \phi \rangle = &\int_{\mathbb{R}^N} \nabla v \nabla \phi dx - \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} h(v) h'(v) \phi dx \\ &+ \int_{\mathbb{R}^N} V(x) h(v) h'(v) \phi dx - \int_{\mathbb{R}^N} f(x, h(v)) h'(v) \phi dx. \end{aligned} \tag{2.5}$$

for all $\phi \in E$.

Moreover, the critical points of J are the weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2|h(v)|^2}} \left(f(x, h(v)) - V(x) h(v) + \frac{\mu}{|x|^2} h(v) \right) \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

We also observe that if v is a critical point of the functional J, then $u = h(v)$ is a critical point of the functional I, i.e. $u = h(v)$ is a solution of (1.1).

Now, let us recall some properties of the change of variables $h : \mathbb{R} \to \mathbb{R}$.

Lemma 2.4. (see [24]) *The function $h(t)$ and its derivative satisfy the following properties*

(h_1) *h is uniquely defined, C^∞ and invertible;*

(h_2) $|h'(t)| \le 1$ *for all* $t \in \mathbb{R}$;

(h_3) $|h(t)| \le |t|$ *for all* $t \in \mathbb{R}$;

(h_4) $\frac{h(t)}{t} \to 1$ *as* $t \to 0$;

(h_5) $\frac{h(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ *as* $t \to \infty$;

(h_6) $\frac{h(t)}{2} \le th'(t) \le h(t)$ *for all* $t > 0$;

(h_7) $\frac{h^2(t)}{2} \le th(t)h'(t) \le h^2(t)$ *for all* $t \in \mathbb{R}$;

(h_8) $|h(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$;

(h_9) *there exists a positive constant* C *such that*

$$|h(t)| \ge \begin{cases} C|t|, & |t| \le 1 \\ C|t|^{\frac{1}{2}}, & |t| \ge 1; \end{cases}$$

(h_{10}) *for each* $\alpha > 0$, *there exists a positive constant* $C(\alpha)$ *such that*

$$|h(\alpha t)|^2 \le C(\alpha)|h(t)|^2;$$

(h_{11}) $|h(t)h'(t)| \le \frac{1}{\sqrt{2}}$.

For convenience of our proof, we give the following basic and important definition.

Definition 2.5 (see [36])**.** Assume that $J \in C^1(E, \mathbb{R})$, sequence $\{u_n\} \subset E$ is called $(C)_c$ sequence if

$$J(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)J'(v_n) \to 0.$$

If any $(C)_c$ sequence has a convergent subsequence, we say J satisfies Cerami condition at level c.

Lemma 2.6. *Assume that* (V_1), (V_2), (F_0)–(F_2) *hold, then any* $(C)_c$*-sequence of* J *is bounded in* E *for each* $c \in \mathbb{R}$.

Proof. Let $\{v_n\} \subset E$ be a $(C)_c$-sequence of J, we have

$$J(v_n) \to c, \qquad (1 + \|v_n\|)J'(v_n) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Then, there is a constant $C_4 > 0$ such that

$$J(v_n) - \frac{1}{4}\langle J'(v_n), v_n \rangle \le C_4. \tag{2.8}$$

Let

$$\|v_n\|_h^2 := \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x)h^2(v_n) \right) dx.$$

First, we prove that there exists $C_5 > 0$ such that

$$\|v_n\|_h^2 \le C_5. \tag{2.9}$$

On the contrary, we suppose that

$$\|v_n\|_h^2 \to \infty.$$

Taking $\tilde{h}(v_n) = \frac{h(v_n)}{\|v_n\|_h}$, then $\|\tilde{h}(v_n)\| \leq 1$. Passing to a subsequence, we assume that

$$\tilde{h}(v_n) \rightharpoonup v \quad \text{in } E,$$

$$\tilde{h}(v_n) \to v \quad \text{in } L^r(\mathbb{R}^N),\ 2 \leq r < 2^*,$$

and

$$\tilde{h}(v_n) \to v \quad \text{a.e. on } \mathbb{R}^N.$$

From (2.4) and (2.7), we have

$$\lim_{|n|\to\infty} \int_{\mathbb{R}^N} \frac{|F(x,h(v_n))|}{\|v_n\|_h^2} dx = \frac{1}{2}. \tag{2.10}$$

On the other hand, set $\xi_n = \frac{h(v_n)}{h'(v_n)}$, then there exists $C_6 > 0$ such that $\|\xi_n\| \leq C_6\|v_n\|$. Since $\{v_n\}$ is a $(C)_c$ sequence of J, by (2.8) we have

$$\begin{aligned}
C_6 &\geq J(v_n) - \frac{1}{4}\langle J'(v_n), \xi_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^N} \left(|\nabla h(v_n)|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x)h^2(v_n) \right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{4} f(x,h(v_n))h(v_n) - F(x,h(v_n)) \right) dx \\
&= \frac{1}{4}\|h(v_n)\|^2 + \int_{\mathbb{R}^N} \tilde{F}(x,h(v_n))dx \\
&\geq \int_{\mathbb{R}^N} \tilde{F}(x,h(v_n))dx.
\end{aligned} \tag{2.11}$$

Take $l(a) = \inf\{\tilde{F}(x,h(v_n)) \mid x \in \mathbb{R}^N, |h(v_n)| \geq a\}$, for $a \geq 0$. By (1.5), we have $l(a) \to \infty$ as $a \to \infty$. For $0 \leq b_1 < b_2$, let

$$B_n(b_1,b_2) = \{x \in \mathbb{R}^N : b_1 \leq |h(v_n(x))| < b_2\}.$$

Combining with (2.11) that

$$\begin{aligned}
C_6 &\geq \int_{B_n(0,a)} \tilde{F}(x,h(v_n))dx + \int_{B_n(a,\infty)} \tilde{F}(x,h(v_n))dx \\
&\geq \int_{B_n(0,a)} \tilde{F}(x,h(v_n))dx + l(a)\text{meas}\{B_n(a,\infty)\},
\end{aligned}$$

from this we get $\text{meas}\{B_n(a,\infty)\} \to 0$ as $a \to \infty$ uniformly in n. Hence, for $r \in [2, 2^*)$ and

(h_{11}), there exist $C, C_7 > 0$ such that

$$\begin{aligned}
\int_{B_n(a,\infty)} \tilde{h}^r(v_n)dx &\le \left(\int_{B_n(a,\infty)} \tilde{h}^{22^*}(v_n)dx \right)^{\frac{r}{22^*}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&= \frac{1}{\|v_n\|_h^r} \left(\int_{B_n(a,\infty)} h^{22^*}(v_n)dx \right)^{\frac{r}{22^*}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\le \frac{C}{\|v_n\|_h^r} \left(\int_{B_n(a,\infty)} |\nabla h^2(v_n)|^2 dx \right)^{\frac{r}{4}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\le \frac{C_6}{\|v_n\|_h^r} \left(\int_{B_n(a,\infty)} |\nabla v_n|^2 dx \right)^{\frac{r}{4}} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \\
&\le C_7 \|v_n\|_h^{-r/2} \left(\text{meas}\{B_n(a,\infty)\}\right)^{\frac{22^*-r}{22^*}} \to 0,
\end{aligned} \tag{2.12}$$

as $a \to \infty$ uniformly in n.

If $\nu = 0$, then $\tilde{h}(v_n) \to 0$ in $L^r(\mathbb{R}^N)$, $2 \le r < 2^*$. For any $0 < \varepsilon < \frac{1}{8}$, there exist a_1, L large enough, such that

$$\begin{aligned}
\int_{B_n(0,a_1)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx &\le \int_{B_n(0,a_1)} \frac{c_1|h(v_n)|^2 + c_2|h(v_n)|^p}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \\
&\le (c_1 + c_2 a_1^{p-2}) \int_{B_n(0,r_1)} |\tilde{h}(v_n)|^2 dx \\
&\le (c_1 + c_2 a_1^{p-2}) \int_{\mathbb{R}^N} |\tilde{h}(v_n)|^2 dx \\
&< \varepsilon,
\end{aligned} \tag{2.13}$$

for $n > L$. Set $\tau' = \frac{\sigma}{\sigma-1}$, since $\sigma \in \left(\max\{1, \frac{2N}{N+2}\}, 2\right)$, then $2\tau' \in (2, 22^*)$. Thus, by (F_2) and (2.12) we have

$$\begin{aligned}
&\int_{B_n(a_1,\infty)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \\
&\le \left(\int_{B_n(a_1,\infty)} \left(\frac{|F(x,h(v_n))|}{|h(v_n)|^2} \right)^{\sigma} dx \right)^{\frac{1}{\sigma}} \left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'} dx \right)^{\frac{1}{\tau'}} \\
&\le c_0^{\frac{1}{\sigma}} \left(\int_{B_n(a_1,\infty)} \tilde{F}(x,h(v_n)) dx \right)^{\frac{1}{\sigma}} \left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'} dx \right)^{\frac{1}{\tau'}} \\
&\le C_8 \left(\int_{B_n(a_1,\infty)} |\tilde{h}(v_n)|^{2\tau'} dx \right)^{\frac{1}{\tau'}} \\
&< \varepsilon.
\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we can get

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|F(x,h(v_n))|}{\|v_n\|_h^2} dx &= \int_{B_n(0,a_1)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx + \int_{B_n(a_1,\infty)} \frac{|F(x,h(v_n))|}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \\
&< 2\varepsilon < \frac{1}{4},
\end{aligned}$$

for $n > L$, which contradicts (2.10).

If $\nu \neq 0$, then meas$\{B\} > 0$, where $B = \{x \in \mathbb{R}^N : \nu \neq 0\}$. For $x \in B$, we have $|h(v_n)| \to \infty$ as $n \to \infty$. Hence $B \subset B_n(a_0, \infty)$ for $n \in \mathbb{N}$ large enough, where a_0 is given in (F_1). By (F_1), we have

$$\frac{F(x, h(v_n))}{|h(v_n)|^4} \to \infty \quad \text{as } n \to \infty.$$

Using Fatou's Lemma, then

$$\int_B \frac{F(x, h(v_n))}{|h(v_n)|^4} dx \to \infty \quad \text{as } n \to \infty. \tag{2.15}$$

We see from (2.7) and (2.15)

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|v_n\|_h^2} = \lim_{n\to\infty} \frac{J(v_n)}{\|v_n\|_h^2} \\
&= \lim_{n\to\infty} \frac{1}{\|v_n\|_h^2} \left(\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x) h^2(v_n)\right) dx - \int_{\mathbb{R}^N} F(x, h(v_n)) dx \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} - \int_{B_n(0,a_0)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx - \int_{B_n(a_0,\infty)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \right) \\
&\leq \frac{1}{2} + \limsup_{n\to\infty} \left((c_1 + c_2 a_0^{p-2}) \int_{\mathbb{R}^N} |\tilde{h}(v_n)|^2 dx - \int_{B_n(a_0,\infty)} \frac{F(x, h(v_n))}{|h(v_n)|^2} |\tilde{h}(v_n)|^2 dx \right) \\
&\leq C_9 - \liminf_{n\to\infty} \int_B \frac{F(x, h(v_n))}{|h(v_n)|^4} |h(v_n)\tilde{h}(v_n)|^2 dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Hence, (2.9) holds.

In order to prove that $\{v_n\}$ is bounded, we only need to show that there is $C_{10} > 0$ such that

$$\|v_n\|_h^2 \geq C_{10} \|v_n\|^2. \tag{2.16}$$

Arguing indirectly, for a subsequence, we assume $\frac{\|v_n\|_h^2}{\|v_n\|^2} \to 0$, where $v_n \neq 0$ (if not, the result is obvious). Take $\xi_{n,1} = \frac{v_n}{\|v_n\|}$, $\eta_{n,1} = \frac{h^2(v_n)}{\|v_n\|^2}$, then

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \eta_{n,1}(x) + V(x) \eta_{n,1}(x) \right) dx \to 0. \tag{2.17}$$

It follows from (h_3) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \eta_{n,1}(x) + V(x) \eta_{n,1}(x) \right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \frac{h^2(v_n)}{\|v_n\|^2} + V(x) \eta_{n,1}(x) \right) dx \\
&\quad \geq \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \frac{v_n^2}{\|v_n\|^2} + V(x) \eta_{n,1}(x) \right) dx \\
&\quad = \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,1}|^2 - \frac{\mu}{|x|^2} \xi_{n,1}^2 + V(x) \eta_{n,1}(x) \right) dx \\
&\quad \geq 0.
\end{aligned} \tag{2.18}$$

Combining (2.1), (2.17) and (2.18), we

$$\int_{\mathbb{R}^N}\left(|\nabla\xi_{n,1}|^2-\frac{\mu}{|x|^2}\xi_{n,1}^2+V(x)\eta_{n,1}(x)\right)dx\to 0.$$

Hence

$$\int_{\mathbb{R}^N}\left(|\nabla\xi_{n,1}|^2-\frac{\mu}{|x|^2}\xi_{n,1}^2\right)dx\to 0,\qquad \int_{\mathbb{R}^N}V(x)\eta_{n,1}(x)dx\to 0\quad\text{and}\quad\int_{\mathbb{R}^N}V(x)\xi_{n,1}^2dx\to 1.$$

Similar to the idea of [37], let $B_n=\{x\in\mathbb{R}^N:|v_n(x)|\geq C_{11}\}$, where $C_{11}>0$ is independent of n. We suppose that for $\varepsilon>0$, $\text{meas}\{B_n\}<\varepsilon$. If not, there exists $\varepsilon'>0$ and $\{v_{n_i}\}\subset\{v_n\}$ such that

$$\text{meas}\{x\in\mathbb{R}^N:|v_{n_i}(x)|\geq i\}\geq\varepsilon',$$

where $i>0$ is a integer. Set $B_{n_i}=\{x\in\mathbb{R}^N:|v_{n_i}(x)|\geq i\}$. From (2.1), (h_3) and (h_9) we have

$$\begin{aligned}\|v_{n_i}\|_h^2&=\int_{\mathbb{R}^N}\left(|\nabla v_{n_i}|^2-\frac{\mu}{|x|^2}h^2(v_{n_i})+V(x)h^2(v_{n_i})\right)dx\\&\geq\int_{\mathbb{R}^N}\left(|\nabla v_{n_i}|^2-\frac{\mu}{|x|^2}v_{n_i}^2+V(x)h^2(v_{n_i})\right)dx\\&>\int_{\mathbb{R}^N}V(x)h^2(v_{n_i})dx\\&>Ci\varepsilon'\to\infty.\end{aligned}$$

as $i\to\infty$, which is a contradiction. For constants $C_{12},C_{13}>0$, it follows $|v_n(x)|\leq C_{12}$, (h_9) and (h_{10}) that

$$\frac{C}{C_{12}^2}v_n^2\leq h^2\left(\frac{1}{C_{12}}v_n\right)\leq C_{13}h^2(v_n).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_n}V(x)\xi_{n,1}^2dx&\leq C_{14}\int_{\mathbb{R}^N\setminus B_n}V(x)\frac{h^2(v_n)}{\|v_n\|}dx\\&\leq C_{14}\int_{\mathbb{R}^N}V(x)\eta_{n,1}(x)dx\to 0,\end{aligned}\tag{2.19}$$

where $C_{14}>0$ is a constant. For another, by absolute continuity of integral, there exists $\varepsilon>0$ such that

$$\int_{B'}V(x)\xi_{n,1}^2dx\leq\frac{1}{2},\tag{2.20}$$

where $B'\subset\mathbb{R}^N$ and $\text{meas}\{B'\}<\varepsilon$. By (2.19) and (2.20), we have

$$\int_{\mathbb{R}^N}V(x)\xi_{n,1}^2dx=\int_{\mathbb{R}^N\setminus B_n}V(x)\xi_{n,1}^2dx+\int_{B_n}V(x)\xi_{n,1}^2dx\leq\frac{1}{2}+o(1).$$

We can get a contradiction. Hence (2.16) holds. Combining (2.9) with (2.16), we complete the proof of this lemma. □

Lemma 2.7. *Assume that (V_1), (V_2), (F_0)–(F_2) hold, then J satisfies $(C)_c$-condition.*

Proof. Lemma 2.6 implies that $\{v_n\}$ is bounded in E. For a subsequence, we can assume that $v_n \rightharpoonup v$ in E. From Lemma 2.3, $v_n \to v$ in $L^r(\mathbb{R}^N)$ for all $2 \le r < 2^*$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. First, we claim that there exists $C_{15} > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla(v_n - v)|^2 + \left(V(x) - \frac{\mu}{|x|^2} \right)(h(v_n)h'(v_n) - h(v)h'(v)) \right)(v_n - v)dx \ge C_{15}\|v_n - v\|_E^2 \tag{2.21}$$

Indeed, we may assume $v_n \neq v$ (otherwise the conclusion is trivial). Set

$$\xi_{n,2} = \frac{v_n - v}{\|v_n - v\|} \quad \text{and} \quad \eta_{n,2} = \frac{h(v_n)h'(v_n) - h(v)h'(v)}{v_n - v},$$

we argue by contradiction and assume that

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2}\eta_{n,2}(x)\xi_{n,2}^2 + V(x)\eta_{n,2}(x)\xi_{n,2}^2 \right) dx \to 0. \tag{2.22}$$

Since

$$\frac{d}{dt}(h(t)h'(t)) = h(t)h''(t) + (h'(t))^2 = \frac{1}{(1 + 2h^2(t))^2} > 0,$$

$h(t)h'(t)$ is strictly increasing and for each $C_{16} > 0$, there is $\delta_1 > 0$ such that

$$\frac{d}{dt}(h(t)h'(t)) \ge \delta_1,$$

at $|t| \le C_{16}$. From this, we see that $\eta_{n,2}(x)$ is positive. On the other hand, for $v_n > v$, there exists $\theta \in (v, v_n)$ such that

$$\eta_{n,2} = \frac{h(v_n)h'(v_n) - h(v)h'(v)}{v_n - v} = \frac{d}{dt}(h(\theta)h'(\theta)) = \frac{1}{(1 + 2h^2(\theta))^2} \le 1.$$

Similarly, we can prove the case $v_n < v$.

Hence,

$$\eta_{n,2}(x) \le 1 \quad \text{for all } v_n \neq v. \tag{2.23}$$

It follows from (2.1), (2.22) and (2.23) that

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2}\xi_{n,2}^2 + V(x)\eta_{n,2}(x)\xi_{n,2}^2 \right) dx \\ &\le \int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2}\eta_{n,2}(x)\xi_{n,2}^2 + V(x)\eta_{n,2}(x)\xi_{n,2}^2 \right) dx \\ &\to 0. \end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^N} \left(|\nabla \xi_{n,2}|^2 - \frac{\mu}{|x|^2}\xi_{n,2}^2 \right) dx \to 0, \qquad \int_{\mathbb{R}^N} V(x)\eta_{n,2}(x)\xi_{n,2}^2 dx \to 0,$$

and

$$\int_{\mathbb{R}^N} V(x)\xi_{n,2}^2 dx \to 1.$$

By a similar fashion as (2.19) and (2.20), we can conclude a contradiction.

On the other hand, by (h_2), (h_3), (h_8), (h_{11}), (F_0) and (1.5), there is $C_{17} > 0$ such that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (f(x,h(v_n))h'(v_n) - f(x,h(v))h'(v))(v_n - v)dx \right| \\
&\quad \le \int_{\mathbb{R}^N} C_{17}(|v_n| + |v_n|^{\frac{p}{2}-1} + |v| + |v|^{\frac{p}{2}-1})|v_n - v|dx \\
&\quad \le C_{17}(\|v_n\|_2 + \|v\|_2)\|v_n - v\|_2 + \left(\|v_n\|_{\frac{p}{2}}^{\frac{p-2}{2}} + \|v\|_{\frac{p}{2}}^{\frac{p-2}{2}}\right)\|v_n - v\|_{\frac{p}{2}} \\
&\quad = o(1).
\end{aligned} \tag{2.24}$$

Therefore, by (2.21) and (2.24), we have

$$\begin{aligned}
o(1) &= \langle J'(v_n) - J'(v), v_n - v\rangle \\
&= \int_{\mathbb{R}^N} \left(|\nabla(v_n - v)|^2 + (V(x) - \frac{\mu}{|x|^2})(h(v_n)h'(v_n) - h(v)h'(v))(v_n - v) \right) dx \\
&\quad - \int_{\mathbb{R}^N} (f(x,h(v_n))h'(v_n) - f(x,h(v))h'(v))(v_n - v)dx \\
&\ge C_{15}\|v_n - v\| + o(1).
\end{aligned}$$

This implies that $\|v_n - v\| \to 0$ as $n \to \infty$. Thus, the proof is complete. □

To prove our main result in this paper, we need the following lemma.

Lemma 2.8 (Symmetric Mountain Pass Theorem [26])**.** *Let X be an infinite dimensional Banach space, $X = Y \bigoplus Z$, where Y is finite dimensional. If $\Psi \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- (I_1) *$\Psi(0) = 0$, $\Psi(-u) = u$ for all $u \in X$;*
- (I_2) *there exist constants ρ, $\alpha > 0$ such that $\Psi |_{\partial B_\rho \cap Z} \ge \alpha$;*
- (I_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $\Psi(u) \le 0$ on $\tilde{X} \setminus B_R$;*

then Ψ possesses an unbounded sequence of critical values.

3 Proof of Theorem 1.1

Let $\{e_i\}$ is a total orthonormal basis of E and define $X_i = \mathbb{R}e_i$, then $E = \bigoplus_{i=1}^{\infty} X_i$. Let

$$Y_j = \bigoplus_{i=1}^{i} X_i, \qquad Z_j = \overline{\bigoplus_{j+1}^{\infty} X_i}, \qquad j \in \mathbb{Z},$$

then $E = Y_j \bigoplus Z_j$ and Y_j is finite-dimensional. Similar to Lemma 3.8 in [36], we have the following lemma.

Lemma 3.1 ([36])**.** *Under assumptions (V_1) and (V_2), for $2 \le r < 2^*$,*

$$\beta_j(r) := \sup_{u \in Z_j, \|v\|=1} \|v\|_r \to 0, \qquad j \to \infty.$$

Before going further, we need to show that there exists $C_{18} > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{\mu}{|x|^2} h^2(v) + V(x) h^2(v) \right) dx \geq C_{18} \|v\|^2, \qquad \forall v \in S_\rho, \tag{3.1}$$

where $S_\rho = \{v \in E : \|v\| = \rho\}$. Indeed, by a similar argument as (2.16), we can get this conclusion. Moreover, by Lemma 3.1, we can choose an integer $\kappa \geq 1$ such that

$$\|v\|_2^2 \leq \frac{C_{18}}{4c_1} \|v\|, \qquad \|v\|_{\frac{p}{2}}^{\frac{p}{2}} \leq \frac{C_{18}}{4c_2} \|v\|^{\frac{p}{2}}, \qquad \forall v \in Z_\kappa. \tag{3.2}$$

Lemma 3.2. *Assume that (V_1), (V_2) and (F_0) hold, then there exist constants ρ, $\alpha > 0$ such that $J|_{S_\rho \cap Z_\kappa} \geq \alpha$.*

Proof. For any $v \in Z_\kappa$ with $\|v\| = \rho < 1$, by (h_3), (h_8), (3.1) and (3.2), we have

$$\begin{aligned} J(v) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{\mu}{|x|^2} h^2(v) + V(x) h^2(v) \right) dx - \int_{\mathbb{R}^N} F(x, h(v)) dx \\ &\geq \frac{C_{18}}{2} \|v\|^2 - \int_{\mathbb{R}^N} (c_1 |h(v)|^2 + c_2 |h(v)|^p) dx \\ &\geq \frac{C_{18}}{2} \|v\|^2 - \int_{\mathbb{R}^N} (c_1 |v|^2 + c_2 |v|^{\frac{p}{2}}) dx \\ &\geq \frac{C_{18}}{2} \|v\|^2 - \frac{C_{18}}{4} \|v\|^2 - \frac{C_{18}}{4} \|v\|^{\frac{p}{2}} \\ &= \frac{C_{18}}{4} \|v\|^2 \left(1 - \|v\|^{\frac{p-4}{2}} \right) \\ &> 0. \end{aligned}$$

since $p \in (4, 22^*)$. This completes the proof. □

Lemma 3.3. *Assume that (V_1), (V_2), (F_0) and (F_1) hold, for any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that*

$$J(v) \leq 0, \qquad \forall v \in \tilde{E} \setminus B_R.$$

Proof. For any finite dimensional subspace $\tilde{E} \subset E$, there is a positive integral number κ such that $\tilde{E} \subset Y_\kappa$. Suppose to the contrary that there is a sequence $\{v_n\} \subset \tilde{E}$ such that $\|v_n\| \to \infty$ and $J(v_n) > 0$. Hence

$$\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 - \frac{\mu}{|x|^2} h^2(v_n) + V(x) h^2(v_n) \right) dx > \int_{\mathbb{R}^N} F(x, h(v_n)) dx. \tag{3.3}$$

Jointly with (h_3), we have

$$\frac{\int_{\mathbb{R}^N} F(x, h(v_n)) dx}{\|v_n\|^2} < \frac{1}{2}. \tag{3.4}$$

Set $\eta_n = \frac{v_n}{\|v_n\|}$. Then up to a subsequence, we can assume that

$$\begin{aligned} &\eta_n \rightharpoonup \eta \quad \text{in } E, \\ &\eta_n \to \eta \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for } 2 \leq r < 2^* \end{aligned}$$

and

$$\eta_n \to \eta \quad \text{a.e. on } \mathbb{R}^N.$$

Set $A_1 = \{x \in \mathbb{R}^N : \eta(x) \neq 0\}$ and $A_2 = \{x \in \mathbb{R}^N : \eta(x) = 0\}$. If meas$\{A_1\} > 0$, then by (F_1), (h_5) and Fatou's Lemma, we have

$$\int_{A_1} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = \int_{A_1} \frac{F(x, h(v_n))}{h^4(v_n)} \frac{h^4(v_n)}{v_n^2} \eta_n^2 dx \to \infty.$$

By (F_0) and (F_1), there exists $C_{19} > 0$ such that

$$F(x, t) \geq -C_{19} t^2, \qquad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{A_2} \frac{F(x, h(v_n))}{\|v_n\|^2} dx \geq -C_{19} \int_{A_2} \frac{h^2(v_n)}{\|v_n\|^2} dx \geq -C_{19} \int_{A_2} \eta_n^2 dx.$$

Since $\eta_n \to \eta$ in $L^2(\mathbb{R}^N)$, it is clear that

$$\liminf_{n \to \infty} \int_{A_2} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = 0.$$

Consequently,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, h(v_n))}{\|v_n\|^2} dx = \infty.$$

By (3.4) we obtain $\frac{1}{2} > \infty$, a contradiction. This shows meas$\{A_1\} = 0$ i.e. $\eta(x) = 0$ a.e. on $\mathbb{R}^N$. By the equivalency of all norms in $\tilde{E}$, there exists $C > 0$ such that

$$\|v\|_2^2 \geq C\|v\|^2, \qquad \forall v \in \tilde{E}.$$

Hence

$$0 = \lim_{n \to \infty} \|\eta_n\|_2^2 \geq C \lim_{n \to +\infty} \|\eta_n\|^2 = C,$$

a contradiction. This completes the proof. □

Now, we prove our main result.

Proof of Theorem 1.1. Let $\Psi = J$, $X = E$, $Y = Y_\kappa$ and $Z = Z_\kappa$. Obviously, $J(0) = 0$ and (F_3) implies that J is even. By Lemma 2.7, 3.2 and Lemma 3.3, all conditions of Lemma 2.5 are satisfied. Thus, problem (2.6) has infinitely many nontrivial solutions sequence $\{v_n\}$ such that $J(v_n) \to \infty$ as $n \to \infty$. Namely, problem (1.1) also has infinitely many solutions sequence $\{u_n\}$ such that $I(u_n) \to \infty$ as $n \to \infty$. □

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References

[1] M. Badiale, G. Tarantello, A Sobolev–Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics, *Arch. Rat. Mechanics Anal.* **163**(2002), 259–293. https://doi.org/10.1007/s002050200201; MR1918928; Zbl 1010.35041

[2] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations* **20**(1995), 1725–1741. https://doi.org/10.1080/03605309508821149; MR1349229; Zbl 0837.35043

[3] T. Bartsch, Z. Q. Wang, M. Willem, The Dirichlet problem for superlinear elliptic equations, in: *Stationary partial differential equations. Vol. II*, Handbook of Differential Equations, Elsevier/North Holland, Amsterdam, 2005, pp. 1–55. https://doi.org/10.1016/S1874-5733(05)80009-9; MR2181481; Zbl 1207.35001

[4] D. M. Cao, H. S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh. Sect. A.* **126**(1996), 443–463. https://doi.org/10.1017/S0308210500022836; MR1386873; Zbl 0846.35042

[5] J. Chen, B. Guo, Multiple nodal bound states for a quasilinear Schrödinger equation, *J. Math. Phys*. **46**(2005), 1–11. https://doi.org/10.1063/1.2138045; MR2194030; Zbl 1111.34017

[6] K. J. Chen, C. C. Peng, Multipliticy and bifurcation of positive solutions for nonhomogeneous semilinear elliptic problems, *J. Differential Equations*. **240**(2007), 58–91. https://doi.org/10.1016/j.jde.2007.05.023; MR2349165; Zbl 1174.35006

[7] M. Colin, L. Janjean, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal*. **56**(2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008; MR2029068; Zbl 1035.35038

[8] Y. B. Deng, Y. Li, W. Shuai, Existence of solutions for a class of p-Laplacian type equation wiyh critical growth and potential vanishing at infinitely, *Discrete. Contin. Dyn. Syst*. **36**(2016), 683–699. https://doi.org/10.3934/dcds.2016.36.683; MR3392899; Zbl 1323.35038

[9] Y. B. Deng, L. Y. Liu, S. J. Peng, Solutions of Schrödinger equations with inverse square potential and critical nonlinearity, *J. Differential Equations* **253**(2012), 1376–1398. https://doi.org/10.1016/j.jde.2012.05.009; MR2927385; Zbl 1248.35058

[10] N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents, *Trans. Amer. Math. Soc.* **352**(2000), 5703-5743. https://doi.org/10.1090/S0002-9947-00-02560-5; MR1695021; Zbl 0956.35056

[11] Q. Q. Guo, J. Mederski, Ground states of nonlinear Schrödinger equations with sum of periodic and inverse square potentials, *J. Differential Equations* **260**(2016), 4180–4202. https://doi.org/10.1016/j.jde.2015.11.006; MR3437584; Zbl 1335.35232

[12] D. S. Kang, Solutions for semilinear elliptic problems with critical Sobolev–Hardy exponents in $\mathbb{R}^N$, *Nonlinear Anal.* **66**(2007), 241–252. https://doi.org/10.1016/j.na.2005.11.028; MR2271652; Zbl 1387.35242

[13] D. S. Kang, Y. B. Deng, Existence of solutions for a singular critical elliptic equation, *J. Math. Anal. Appl.* **284**(2003), 724–732. https://doi.org/10.1016/S0022-247X(03)00394-9; MR1998664; Zbl 1094.35050

[14] J. Li, Existence of solution for a singular elliptic equation with critical Sobolev–Hardy exponents, *Int. J. Math. Math. Sci.* **2005**(2005), 3213–3223. https://doi.org/10.1155/IJMMS.2005.3213; MR2206527; Zbl 1119.35019

[15] H. L. Liu, H. B. Chen, Multiple solutions for a nonlinear Schrödinger–Poisson system with sign-changing potential, *Comput. Math. Appl.* **71**(2016), 1405–1416. https://doi.org/10.1016/j.camwa.2016.02.010; MR3477712

[16] J. Liu, J. F. Liao, C. L. Tang, A positive ground state solutions for a class of asyptotically periodic Schrödinger equations, *Comput. Math. Appl.* **71**(2016), 965–976. https://doi.org/10.1016/j.camwa.2016.01.004; MR3461272;

[17] J. Liu, J. F. Liao, C. L. Tang, A positive ground state solutions for a class of asyptotically periodic Schrödinger equations with critical exponent, *Comput. Math. Appl.* **72**(2016), 1851–1864. https://doi.org/10.1016/j.camwa.2016.08.010; MR3547688; Zbl 1362.35114

[18] J. Q. Liu, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations. I, *Proc. Amer. Math. Soc.* **131**(2003), 441–448. https://doi.org/10.1090/S0002-9939-02-06783-7; MR1933335; Zbl 1229.35269

[19] J. Liu, Y. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations. II, *J. Differential Equations*. **187**(2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5; MR1949452; Zbl 1229.35268

[20] J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Comm. Paritial Differential Equations* **29**(2004), 879–901. https://doi.org/10.1081/PDE-120037335; MR2059151; Zbl 1140.35399

[21] V. G. Makhankov, V. K. Fedyanin, Nonlinear effects in quasi-one-dimensional models of condensed matter theory, *Phys. Rep.* **104**(1984), No. 1, 1–86. https://doi.org/10.1016/0370-1573(84)90106-6; MR0740342

[22] J. M. B. do Ó, O. H. Miyagaki, S. H. M. Soares, Soliton solutions for quasilinear Schrödinger equations: the critical exponential case, *Nonlinear Anal.* **67**(2007), 3357–3372. https://doi.org/10.1016/j.na.2006.10.018; MR2350892; Zbl 1151.35016

[23] J. M. B. do Ó, U. Severo, Quasilinear Schrödinger equations involving concave and convex nonlinearities, *Commum. Pure Appl. Anal.* **8**(2009), 621–644. https://doi.org/10.3934/cpaa.2009.8.621; MR2461565; Zbl 1171.35118

[24] J. M. B. do Ó, U. Severo, Solitary waves for a class of quasilinear Schrödinger equations in dimension two, *Calc. Var.* **38**(2010), 275–315. https://doi.org/10.1007/s00526-009-0286-6; MR2647122; Zbl 1194.35118

[25] M. Poppenberg, K. Schmitt, Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differential Equtions* **14**(2002), 329–344. https://doi.org/10.1007/s005260100105; MR1899450; Zbl 1052.35060

[26] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math. Providence, RI: American Mathematical Society **65**(1986). MR0845785; Zbl 0609.58002

[27] H. X. Shi, H. B. Chen, Positive solutions for generalized quasilinear Schrödinger equations with potential vanishing at infinity, *Appl. Math. Lett.* **61**(2016), 137–142. https://doi.org/10.1016/j.aml.2016.06.004; MR3518460; Zbl 1347.35116

[28] H. X. Shi, H. B. Chen, Existence and multiplicity of solutions for a class of generalized quasilinear Schrödinger equations, *J. Math. Anal. Appl.* **452**(2017), 578–594. https://doi.org/10.1016/j.jmaa.2017.03.020; MR3628036; Zbl 1376.35017

[29] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, *Trans. Amer. Math. Soc.* **357**(2005), 2909–2938. https://doi.org/10.1090/S0002-9947-04-03769-9; MR2139932; Zbl 1134.35348

[30] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9**(1992), 281–304. https://doi.org/10.1016/S0294-1449(16)30238-4; MR1168304; Zbl 0785.35046

[31] S. Y. Tian, The eigenvalue problem and infinitely many sign-changing solutions for an elliptic equations with critical Hardy constant, *J. Math. Anal. Appl.* **447**(2017), 354–370. https://doi.org/10.1016/j.jmaa.2016.10.022; MR3566476; Zbl 1364.35216

[32] C. Wang, Y. Y. Shang, Existence and multiplicity of solutions for Schrödinger equations with inverse square potential and Hardy–Sobolev critical exponent, *Nonlinear Anal. Real World Appl.* **46**(2019), 525–544. https://doi.org/10.1016/j.nonrwa.2018.10.002; MR3887142; Zbl 1414.35063

[33] Y. J. Wang, J. Yang, Y. M. Zhang, Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in $\mathbb{R}^N$, *Nonlinear. Anal.* **71**(2009), 6157–6169. https://doi.org/10.1016/j.na.2009.06.006; MR2566522; Zbl 1180.35262

[34] Y. J. Wang, Y. M. Zhang, Y. T. Shen, Multiple solutions for quasilinear Schrödinger equations involving critical exponent, *Appl. Math. Comp.* **216**(2010), 849–856. https://doi.org/10.1016/j.amc.2010.01.091; MR2606992; Zbl 1189.35317

[35] Z. P. Wang, H. S. Zhou, Solutions for a nonhomogeneous elliptic problem involving critical Sobolev–Hardy exponent in $\mathbb{R}^N$, *Acta. Math. Sci.* **26**(2006), 525–536. https://doi.org/10.1016/S0252-9602(06)60078-7; MR2244690; Zbl 1284.35180

[36] M. Willem, *Minimax theorems*, Birkhäuser. Berlin 1996. MR1400007; Zbl 0856.49001

[37] X. Wu, Multiple solutions for quasilinear Schrödinger equations with a parameter, *J. Differential Equations* **256**(2014), 2619–2632. https://doi.org/10.1016/j.jde.2014.01.026; MR3160456; Zbl 1286.35089

[38] J. Zhang, X. H. Tang, W. Zhang, Existence of infinitely many solutions for a quasilinear elliptic equation, *Appl. Math. Lett.* **37**(2014), 131–135. https://doi.org/10.1016/j.aml.2014.06.010; MR3231740; Zbl 1318.35043

[39] J. Zhang, X. H. Tang, W. Zhang, Infinitely many solutions of quasilinear Schrödinger equation with sign-shanging potential, *J. Math. Anal. Appl.* **420**(2014), 1762–1775. https://doi.org/10.1016/j.jmaa.2014.06.055; MR3240105; Zbl 1298.35080

[40] Y. M. Zhag, Y. J. Wang, Y. T. Shen, Solutions for quasilinear Schrödinger equations with critical Sobolev–Hardy exponents, *Commun. Pure Appl. Anal.* **10**(2011), 1037–1054. https://doi.org/10.3934/cpaa.2011.10.1037; MR2787433; Zbl 1234.35084

[41] W. M. Zou, M. Schechter, *Critical point theory and its applications*, Springer, New York, 2006. https://doi.org/10.1007/0-387-32968-4; MR2232879; Zbl 1125.58004