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On discreteness of spectrum of a second order differential operator

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Abstract. A new form of a necessary and sufficient conditions for the discreteness of the spectrum of singular operator $-\frac{1}{\rho(x)}(p(x)u')'$, $-\infty \le a \le x \le b \le +\infty$ is obtained. A simpler proof of the necessity is obtained.

Keywords: discreteness of spectrum, second order differential operator.

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1 Introduction

Let $I = (a, b)$ where $-\infty \le a < b \le +\infty$. The differential operator

$$\mathcal{L}u = \frac{1}{\rho}\left(-(pu')' + qu\right), \qquad x \in I = (a, b), \tag{1.1}$$

was the first to be studied from the point of view of the properties of its spectrum, in particular, the discreteness of the spectrum. Recall that the spectrum of an operator A acting in a Hilbert space H is discrete if it consists only of eigenvalues of finite multiplicity [2]. Operator (1.1) is studied in the space $L_2(I, \rho)$ of functions that are square integrable on I with positive weight ρ.

In the case $(a, b) = (-\infty, \infty)$, and $\rho = 1$ the operator $\mathcal{L}u = -u'' + qu$ has discrete spectrum, if [3] $\lim_{x\to\infty} q(x) = +\infty$. It is a sufficient condition. A. M. Molchanov obtained [11] the following necessary and sufficient condition: for any $\delta > 0$

$$\lim_{x\to\infty} \int_x^{x+\delta} q(x)dx = +\infty. \tag{1.2}$$

Note that Molchanov studied an operator in the n-dimensional space R^n. Here we consider only the case when $q = 0$. In this case for the operator*

$$\mathcal{L}u(x) := -\frac{1}{\rho(x)}(p(x)u')', \qquad x \in I = (a, b), \tag{1.3}$$

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*sign := means *equal by definition*

a necessary and sufficient condition is obtained by I. Kac and M. G. Krein [6]. However, the result in [6] is formulated in such a way that equivalence with the form proposed below (Theorem 2.4) is not obvious (see section 7). Note also that the method in [6] pursued other goals, and is more complicated. We use some method (see Lemma 5.2) close to the Glazman splitting method [5]. The essential point here is a simpler proof of necessity (Lemma 4.1). As test functions, sections $G(x,s)$ of the Green function were chosen, where $s \to a$ or $s \to b$. This simplifies the proof of necessity (see below two-sided estimates (4.5) and (4.6)).

In this regard, we have to note the result of M. Sh. Birman [1, p. 148], [5, p. 93] for an even-order equation on semiaxis $[0,\infty)$. For the operator $\mathcal{L}_0 u = -(1/\rho)u''$ this condition has the following form

$$\lim_{s\to\infty} s \int_s^\infty \rho(x)\,dx = 0. \tag{1.4}$$

It is assumed that $\int_0^\infty \rho(x)\,dx < \infty$. If $\int_0^1 \rho(x)\,dx = \infty$, the condition

$$\lim_{s\to 0} s \int_s^1 \rho(x)\,dx = 0 \tag{1.5}$$

together with (1.4) guarantees [10] discreteness of spectrum of $-(1/\rho)u''$. The result of presented article was announced in [9] for a more general functional differential operator of the form

$$\mathcal{L}u(x) := -\frac{1}{\rho(x)}(p(x)u')' + \int_a^b u(s)r(x,ds), \qquad x \in I = (a,b).$$

For simplicity, we omit the integral term here.

2 Assumptions. Conditions of discreteness

For the operator (1.3) assume that the functions $p(x)$ and $\rho(x)$ are measurable and positive almost everywhere on a finite or infinite interval $I := (a,b)$, $-\infty \le a < b \le \infty$. Assume that $1/p$ and ρ are locally on I integrable, that is, for any s_1, s_2, $a < s_1 < s_2 < b$

$$\int_{s_1}^{s_2} \frac{dx}{p(x)} < \infty, \qquad \int_{s_1}^{s_2} \rho(x)\,dx < \infty.$$

Definition 2.1. If for some $s \in I = (a,b)$

$$\int_a^s \rho(x)\,dx = \infty, \qquad \int_a^s \frac{dx}{p(x)} < \infty \tag{2.1}$$

then $\mathcal{L}$ has singularity at the point $x = a$ by $\rho(x)$. If for some $s \in I = (a,b)$

$$\int_a^s \frac{dx}{p(x)} = \infty, \qquad \int_a^s \rho(x)\,dx < \infty \tag{2.2}$$

say that $\mathcal{L}$ has singularity at the point $x = a$ by $p(x)$. Similarly, we can define the singularity at the right end of the interval.

Only one type of singularity at each end of the interval is allowed. It is clear that the singularity at the right end of the interval can be considered similarly to the left end. Moreover, the singularity at the right end can be reduced to the singularity at the left end by the change of variable $x = -x'$. Therefore, one could consider the singularity only at the left end of the interval. Assuming that

$$\int_s^b \frac{dx}{p(x)} < \infty \quad \text{and} \quad \int_s^b \rho(x)\, dx < \infty \qquad (a < s < b) \tag{2.3}$$

and letting

$$\Phi_1(s) := \int_a^s \frac{dx}{p(x)} \int_s^b \rho(x)\, dx, \qquad \Phi_2(s) := \int_a^s \rho(x)\, dx \int_s^b \frac{dx}{p(x)}$$

we have the following theorem.

Theorem 2.2. *For the spectrum of operator* (1.3) *to be discrete, it is necessary and sufficient that at least one of relations*

$$\lim_{s\to a} \Phi_1(s) = 0 \quad \text{or} \quad \lim_{s\to a} \Phi_2(s) = 0$$

be true.

Remark 2.3. If there is a singularity, then one of the integrals $\Phi_1(s)$ or $\Phi_2(s)$ does not exist. Therefore, only one type of singularity is allowed.

However, it is more convenient to represent Theorem 2.2 in a simpler form (Theorem 2.4 below). For this, we consider *both types of singularities* at different ends of the interval simultaneously. The essence of the content of Theorem 2.2 will not change. So, we assume that

$$\int_s^b \rho(x)\, dx < \infty, \qquad \int_a^s \frac{dx}{p(x)} < \infty, \qquad a < s < b \tag{2.4}$$

but

$$\int_a^s \rho(x)\, dx = \infty, \qquad \int_s^b \frac{dx}{p(x)} = \infty. \tag{2.5}$$

Let

$$\Phi(s) := \int_a^s \frac{dx}{p(x)} \int_s^b \rho(x)\, dx. \tag{2.6}$$

Theorem 2.2 takes the following form.

Theorem 2.4. *For discreteness of the spectrum of the operator* (1.3)*, it is necessary and sufficient that*

$$\lim_{s\to a} \Phi(s) = \lim_{s\to b} \Phi(s) = 0.$$

Proof. It follows from Lemma 5.3 and Section 3. □

To simplify the notation, assume that $a = 0$ and $b = l \le \infty$ (l is the *length of a string*). We use also the boundary condition

$$u(0) = 0. \tag{2.7}$$

Condition (2.7) is not essential for the study of discreteness. It affects the estimate of the first eigenvalue (lower boundary of the spectrum).

3 Variational method

We use the following form of the variational method [8]. In the space $L_2(I,\rho)$ of square integrable functions the scalar product is defined by $(f,g) := \int_I f(x)g(x)\rho(x)\,dx$. Here $I = (a,b) = (0,l)$, $l \le \infty$. The bilinear form

$$[u,v] := \int_0^l p(x)u'(x)v'(x)\,dx \tag{3.1}$$

serves as a scalar product in Hilbert space W of all locally absolutely continuous on $[0,l)$ functions satisfying the boundary condition (2.7). Let $T\colon W \to L_2(I,\rho)$ be defined by the equality $Tu(x) = u(x)$. Note that $T(W)$ is dense in $L_2(I,\rho)$. The equation in variational form

$$[u,v] = (f,Tv) \qquad (\forall v \in W), \tag{3.2}$$

$f \in L_2(I,\rho)$ with respect to u has unique solution $u = T^*f$. Equation (3.2) is equivalent to equation $\mathcal{L}u = f$, where $\mathcal{L} := (T^*)^{-1}$.

If form $[u,v]$ is defined by (3.1), operator $\mathcal{L}$ can be represented by (1.3) under boundary conditions $u(0) = 0$, $pu'|_{x=l} = 0$. Thus, eigenvalue problem

$$\mathcal{L}u = \lambda Tu$$

has the representation

$$-\frac{1}{\rho}(pu')' = \lambda u, \qquad u(0) = 0, \qquad pu'|_{x=l} = 0. \tag{3.3}$$

Discreteness of spectrum of operator $\mathcal{L}$ is equivalent to compactness of the operator T. If T is compact, the eigenvalue problem (3.3) has a system eigenfunctions u_n that forms an orthogonal basis in the space W. The system Tu_n forms an orthogonal basis in $L_2(I,\rho)$.

4 Auxiliary inequalities

Let $u \in W$ and

$$A_u := \int_I \frac{|u(s)u'(s)|}{\omega(s)}\,ds,$$

where the positive parameter function ω will be defined below. By the Cauchy inequality

$$A_u^2 \le \int_I \frac{u(s)^2}{\omega(s)^2}\frac{ds}{p(s)} \cdot \int_I p(s)u'(s)^2\,ds = B_u \cdot [u,u], \tag{4.1}$$

where $B_u := \int_I \frac{u(s)^2}{\omega(s)^2}\frac{ds}{p(s)}$. Hence and since $u(0) = 0$

$$B_u = 2\int_I \frac{ds}{\omega(s)^2 p(s)} \int_0^s u(x)u'(x)\,dx = 2\int_I u(x)u'(x)dx \int_x^l \frac{ds}{\omega(s)^2 p(s)}.$$

Let the function ω be chosen so that

$$\int_x^l \frac{ds}{\omega(s)^2 p(s)} = \frac{1}{\omega(x)} - \frac{1}{\omega(l)} \le \frac{1}{\omega(x)}. \tag{4.2}$$

Then $B_u \le 2\int_I \frac{|u(x)u'(x)|}{\omega(x)}\,dx = 2A_u$. From here and (4.1) $A_u^2 \le 2A_u[u,u]$ and

$$A_u \le 2[u,u]. \tag{4.3}$$

From (4.2) we obtain $-\frac{1}{\omega^2 p} = -\frac{1}{\omega^2}\omega'$ and

$$\omega(s) = \int_0^s \frac{dx}{p(x)}. \tag{4.4}$$

Lemma 4.1. *Let $0 < c < l$, $0 < d < l$. The following inequalities hold:*

$$\sup_{s\in[0,c]}\left(\Phi(s) - \int_0^s \frac{dx}{p(x)}\int_c^l \rho\,dx\right) \le \sup_{\|u\|\le 1}(Tu,Tu)_{[0,c]} \le 4\sup_{s\in[0,c]}\Phi(s), \tag{4.5}$$

$$\sup_{s\in[d,l)}\Phi(s) \le \sup_{\|u\|\le 1}(Tu,Tu)_{[d,l]} \le \Phi(d) + 4\sup_{s\in[d,l)}\Phi(s). \tag{4.6}$$

Proof. The left inequality of (4.5). Let $s \in (0,c]$, $\omega := \int_0^s \frac{dx}{p(x)}$ and

$$u(x) := \begin{cases} \dfrac{1}{\sqrt{\omega}}\displaystyle\int_0^x \frac{dt}{p(t)}, & \text{if } 0 \le x \le s, \\ \sqrt{\omega}, & \text{if } s < x < l. \end{cases}$$

Then $[u,u] = \int_0^s p(x)(u')^2 = \frac{1}{\omega}\int_0^s p(x)\frac{dx}{p(x)^2} = 1$,

$$(Tu,Tu)_{[0,c]} \ge \int_s^c u^2\rho\,dx = \omega\int_s^c \rho\,dx = \Phi(s) - \int_0^s \frac{dx}{p(x)}\int_c^l \rho\,dx.$$

The left inequality of (4.6). Let $s \in [d,l)$, ω and u be defined by the same equalities. Then $[u,u] = 1$,

$$(Tu,Tu)_{[d,l)} \ge \int_s^l u^2\rho\,dx = \omega\int_s^l \rho\,dx = \Phi(s).$$

The right inequality of (4.5). Let $\|u\| \le 1$. By virtue of (4.3) and (4.4)

$$\begin{aligned}\int_0^c (u(x))^2\rho(x)\,dx &= \int_0^c \left(2\int_0^x u(s)u'(s)\,ds\right)\rho(x)\,dx \\ &= 2\int_0^c \frac{u(s)u'(s)}{\omega(s)}\left(\omega(s)\int_s^c \rho(x)\,dx\right)ds \\ &\le 2\sup_{0<s<c}\Phi(s)\int_0^c \frac{u(s)u'(s)}{\omega(s)}\,ds \le 2\sup_{0<s<c}\Phi(s)A_u \le 4\sup_{0<s<c}\Phi(s).\end{aligned}$$

The right inequality of (4.6). Let $\|u\| \le 1$. We have

$$\int_d^l (u(x))^2\rho(x)\,dx = \int_d^l \rho(x)\left((u(d))^2 + 2\int_d^x u(s)u'(s)\,ds\right)dx.$$

Since

$$(u(d))^2 = \left(\int_0^d u'(s)\,ds\right)^2 \le \int_0^d p(s)(u'(s))^2\,ds\int_0^d \frac{ds}{p(s)} \le \int_0^d \frac{ds}{p(s)}$$

we have

$$(u(d))^2\int_d^l \rho\,dx \le \Phi(d).$$

For the second term, in view (4.3)

$$\int_d^l \left(2\int_d^x u(s)u'(s)\,ds\right)\rho(x)\,dx = 2\int_d^l \frac{u(s)u'(s)}{\omega(s)}\left(\omega(s)\int_s^l \rho(x)\,dx\right)ds$$
$$\leq 2\sup_{d<s<l}\Phi(s)\int_0^d \frac{u(s)u'(s)}{\omega(s)}\,ds \leq 2\sup_{d<s<l}\Phi(s)A_u \leq 4\sup_{d<s<l}\Phi(s). \quad \square$$

5 Boundedness and compactness

The boundedness of operator T and its action from space W to space $L_2(I,\rho)$ are necessary for further investigation of the spectrum. The compactness of operator T, as mentioned in Section 3, is equivalent to the discreteness of the spectrum of operator (1.3).

5.1 Boundedness

Since

$$(Tu,Tu) = \int_0^l u^2\rho\,dx = 2\int_0^l \rho(x)\,dx\int_0^x u(s)u'(s)\,ds = 2\int_0^l \frac{u(s)u'(s)}{\omega(s)}\,\omega(s)\int_s^l \rho(x)\,dx\,ds,$$

by virtue of (4.3) and (2.6)

$$(Tu,Tu) \leq 4[u,u]\sup_{s\in(0,l)}\omega(s)\int_s^l \rho(x)\,dx = 4[u,u]\sup_{s\in(0,l)}\Phi(s). \tag{5.1}$$

So, the boundedness of function $\Phi(s)$ guarantees the boundedness of operator T. It seems this is necessary condition. Let λ_0 be the lower boundary of spectrum of $\mathcal{L}$. It satisfies the representation

$$(\lambda_0)^{-1} = \sup_{u\neq 0}\frac{(Tu,Tu)}{[u,u]}.$$

From (5.1) we have the estimate

$$(\lambda_0)^{-1} \leq 4\sup\Phi(s).$$

5.2 Compactness

- Let $(Tu,Tu)_\Delta := \int_\Delta u^2\rho\,dx$. Below we will use $\Delta=[0,c]$ and $\Delta=[d,l)$.

  Below we use the following compactness criterion [4, p. 268], [7, p. 318].

Theorem 5.1 (I. Gelfand). *For the relative compactness of the set A in a Banach space E, it is necessary and sufficient that for any sequence f_n of linear functionals converging on each element of a Banach space E, the convergence is uniform on the set A.*

The following statement is closed to the localization principle [5].

Lemma 5.2. *The condition*

$$\lim_{c\to 0}\sup_{\|u\|\leq 1}(Tu,Tu)_{[0,c]} = 0 \bigwedge \lim_{d\to l}\sup_{\|u\|\leq 1}(Tu,Tu)_{[d,l)} = 0 \tag{5.2}$$

is a necessary and sufficient condition for compactness of T.

Proof. Necessity. Suppose $\exists \sigma > 0$, $\exists c_n \to 0$, $\exists u_n$ such that $\|u_n\| = 1$ and

$$(Tu_n, Tu_n)_{\Delta_n} > \sigma,$$

where $\Delta_n := [0, c_n]$. Let $f_n = \chi_{\Delta_n} \frac{1}{\|Tu_n\|_{\Delta_n}} Tu_n$ (χ_{Δ_n} is the characteristic function of the set Δ_n). Since

$$(f_n, z)^2 \le \frac{1}{\|Tu_n\|^2_{\Delta_n}} \int_0^{c_n} u_n^2 \rho \, dx \int_0^{c_n} z^2 \rho \, dx = \int_0^{c_n} z^2 \rho \, dx \to 0$$

(f_n, z) converges for any $z \in L_2(I, \rho)$. But the following contradicts Theorem 5.1:

$$(f_n, Tu_n) = \frac{1}{\|Tu_n\|_{\Delta_n}} \int_0^{c_n} u_n^2 \rho \, dx = \sqrt{\int_0^{c_n} u_n^2 \rho \, dx} > \sqrt{\sigma}.$$

The necessity of the second condition in (5.2) is proved in exactly the same way.

Sufficiency. Let $f_n \in L_2(I, \rho)$ be a sequence such that $(f_n, z) \to 0$ for any $z \in L_2(I, \rho)$. We have to show that $f_n(Tu) = (f_n, Tu) \to 0$ uniformly on $[u, u] \le 1$. First,

$$\left(\int_0^c f_n(x) u(x) \rho(x) \, dx \right)^2 \le \int_0^c f_n(x)^2 \rho(x) \, dx \int_0^c u(x)^2 \rho(x) \, dx \le C \int_0^c u(x)^2 \rho(x) \, dx.$$

From here and by virtue of (5.2)

$$\lim_{c \to 0} \int_0^c f_n(x) u(x) \rho(x) \, dx = 0$$

uniformly on the set $\{(u, n) \colon [u, u] \le 1,\ n = 1, 2, \ldots\}$. Similarly,

$$\lim_{d \to l} \int_d^l f_n(x) u(x) \rho(x) \, dx = 0$$

uniformly on the set $\{(u, n) \colon [u, u] \le 1,\ n = 1, 2, \ldots\}$.

Therefore, it suffices to establish for any $\alpha, \beta \in (0, l)$ uniform on $[u, u] \le 1$ convergence of the sequence $\int_\alpha^\beta f_n(x) u(x) \rho(x) \, dx$. We have

$$\int_\alpha^\beta f_n(x) u(x) \rho(x) \, dx = \int_\alpha^\beta f_n(x) \left(u(\alpha) + \int_\alpha^x u'(s) \, ds \right) \rho(x) \, dx.$$

The first term converges uniformly since $\int_\alpha^\beta f_n(x) \rho(x) \, dx$ converges and

$$(u(\alpha))^2 = \left(\int_0^\alpha u'(x) \, dx \right)^2 \le \int_0^\alpha p(x) (u'(x))^2 \, dx \int_0^\alpha \frac{dx}{p(x)} \le [u, u] \int_0^\alpha \frac{dx}{p(x)}.$$

Let us estimate the second term:

$$\begin{aligned} \left(\int_\alpha^\beta f_n(x) \left(\int_\alpha^x u'(s) \, ds \right) \rho(x) \, dx \right)^2 &= \left(\int_\alpha^\beta u'(s) \, ds \int_s^\beta f_n(x) \rho(x) \, dx \right)^2 \\ &\le \int_\alpha^\beta p(s) (u'(s))^2 \, ds \int_\alpha^\beta (\varphi_n(s))^2 \, ds \le \int_\alpha^l (\varphi_n(s))^2 \, ds, \end{aligned}$$

where $\varphi_n(s) = (p(s))^{-1/2} \int_s^\beta f_n(x) \rho(x) \, dx$. Note, that $\varphi_n(s) = (f_n, z_s)$ where $z_s(x) = 0$, if $x \notin [s, l]$, and $z_s(x) = (p(s))^{-1/2}$, if $x \in [s, l]$. Thus $\varphi_n(s) = (f_n, z_s) \to 0$ for all $s \in I$.

Since

$$(\varphi_n(s))^2 \le \frac{1}{p(s)} \int_s^\beta \rho(x) \, dx \int_s^\beta (f_n(x))^2 \rho(x) \, dx \le \|f_n\|^2 \frac{1}{p(s)} \int_s^\beta \rho(x) \, dx$$

by virtue of the Lebesgue theorem $\int_\alpha^\beta (\varphi_n(s))^2 \, ds \to 0$. $\square$

Lemma 5.3. *The condition* $\lim_{s\to 0}\Phi(s) = 0$ *and* $\lim_{s\to l}\Phi(s) = 0$ *is a necessary and sufficient condition for compactness of the operator* T.

Proof. It follows from Lemma 5.2 and from inequalities (4.5) and (4.6). For example, consider in detail the proof of the necessity of condition $\lim_{s\to 0}\Phi(s) = 0$. The compactness of operator T implies (5.2). Suppose $\lim_{s\to 0}\Phi(s) = 0$ is not true. Then there are $\varepsilon > 0$ and $s_n \to 0$ such that $\Phi(s_n) \geq \varepsilon$. Let $c > 0$. For some $s_n < c$

$$\Phi(s_n) - \int_0^{s_n} \frac{dx}{p(x)} \int_c^l \rho(x)dx \geq \varepsilon/2.$$

From (4.5) we have $\sup_{\|u\|\leq 1}(Tu, Tu)_{[0,c]} \geq \varepsilon/2$. Since c is arbitrary, this contradicts (5.2).
The other three statements are proved similarly. □

6 Example. Laguerre polynomials

Consider equation $xy'' + (1-x)y' + ny = 0$ generating the Laguerre polynomials. Multiplying by e^{-x}, we get

$$(xe^{-x}y')' + ne^{-x}y = 0.$$

In this case $p(x) = xe^{-x}$, $\rho(x) = e^{-x}$. Let's verify the discreteness conditions for the interval $(0, \infty)$. At the point $x = 0$ it is

$$\int_s^1 \frac{dx}{p(x)} \int_0^s \rho(x)\,dx \to 0$$

when $s \to 0$. It is so since $\int_0^s e^{-x}\,dx = O(s)$ and $\int_s^1 \frac{e^x}{x}\,dx \sim \int_s^1 \frac{dx}{x} = -\ln s$.
At the $x = \infty$ we have to check

$$\int_1^s \frac{dx}{p(x)} \int_s^\infty \rho(x)\,dx \to 0,$$

when $s \to \infty$, that is $\int_1^s \frac{e^x}{x}\,dx \cdot e^{-s} \to 0$. For arbitrary $\varepsilon > 0$ take $A > 0$ such that $1/A < \varepsilon/2$. Then

$$\int_1^s \frac{e^x}{x}\,dx \cdot e^{-s} \leq \int_1^A \frac{e^x}{x}\,dx \cdot e^{-s} + \varepsilon/2.$$

7 Criterion formulation in the article by Krein and Kac

Article [6] discusses equation

$$y'' + \lambda\rho y = 0, \qquad 0 \leq x < L,$$

in which the generalized density is considered to be the derivative dM/dx, $L \leq +\infty$. L is considered the length of the string, and M is its mass.

Spectrum discreteness criterion: for the spectrum of the string to be discrete, it is necessary and sufficient that in case $L = \infty$ condition

$$\lim_{x\to\infty} x(M(\infty) - M(x)) = 0$$

is fulfilled, and in case $M(L) = \infty$ the dual condition

$$\lim_{x\to L} M(x)(L - x).$$

In the first case, it is assumed that $M(L) < \infty$, and in the second $L < \infty$.

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