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Oscillation and spectral properties of some classes of higher order differential operators and weighted nth order differential inequalities

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Abstract. In this paper, we obtain strong oscillation and non-oscillation conditions for a class of higher order differential equations in dependence on an integral behavior of its coefficients in a neighborhood of infinity. Moreover, we establish some spectral properties of the corresponding higher order differential operator. In order to prove these we establish a certain weighted differential inequality of independent interest.

Keywords: higher order differential operator, oscillation, non-oscillation, variational principle, weighted inequality, eigenvalues, spectrum discreteness, spectrum positive definiteness, nuclear operator.

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1 Introduction

Let $I = (0, \infty)$ and u be a continuous and nonnegative function. Suppose that v is a positive function such that it is sufficiently times continuously differentiable on the interval I and for any $a > 0$ the function v^{-1} is integrable on the interval $(0, a)$.

Let $T \geq 0$, $I_T = (T, \infty)$ and $W^n_{2,v} \equiv W^n_{2,v}(I_T)$ be the space of functions $f : I_T \to \mathbb{R}$ having generalized derivatives up to nth order on the interval I_T, for which $\|f^{(n)}\|_{2,v} < \infty$, where $\|g\|_{2,v} = \left(\int_T^\infty v(t)|g(t)|^2 dt\right)^{\frac{1}{2}}$ is the standard norm of the weighted space $L_{2,v}(I) \equiv L_{2,v}$. From the conditions on the function v it easily follows the existence of the finite limit $\lim_{t\to T^+} f^{(i)}(t) \equiv f^{(i)}(T)$, $i = 0, 1, \ldots, n-1$, for any $f \in W^n_{2,v}$. Therefore, the space $W^n_{2,v}$ is a normalized space with the norm

$$\|f\|_{W^n_{2,v}} = \|f^{(n)}\|_{2,v} + \sum_{i=0}^{n-1} |f^{(i)}(T)|. \tag{1.1}$$

Let $\mathring{M}_2(I_T) = \{f \in W^n_2(I_T) \colon \operatorname{supp} f \subset I_T \text{ and } \operatorname{supp} f \text{ is compact}\}$.

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By the assumptions on the function v we have that $\mathring{M}_2(I_T) \subset W^n_{2,v}$. Denote by $\mathring{W}^n_{2,v} = \mathring{W}^n_{2,v}(I_T)$ the closure of the set $\mathring{M}_2$ with respect to norm (1.1).

In the paper we investigate three related problems.

Problem 1. Establish criteria of strong oscillation and non-oscillation of the $2n$th order differential equation

$$(-1)^n(v(t)y^{(n)}(t))^{(n)} - \lambda u(t)y(t) = 0, \qquad t \in I, \tag{1.2}$$

where $n > 1$ and $\lambda > 0$.

A solution of equation (1.2) is a function $y : I \to \mathbb{R}$ that is n times differentiable together with the function $v(t)y^{(n)}(t)$ on the interval I, satisfying equation (1.2) for all $t \in I$.

Equation (1.2) is called [9, p. 6] oscillatory, if for any $T > 0$ there exists a (non-trivial) solution of this equation, having more than one zero with multiplicity n to the right of T. Otherwise equation (1.2) is called non-oscillatory. In the sequel, the expression "solution of equation" will mean "non-trivial solution of equation" unless the opposite is specified.

Equation (1.2) is called strong non-oscillatory (oscillatory), if it is non-oscillatory (oscillatory) for all values $\lambda > 0$.

In the mathematical literature, the most number of works is devoted to the oscillatory properties of linear, semilinear and nonlinear second-order differential equations (see, e.g., [5] and references given there). However, such studies for a higher order equation are relatively rare due to the fact that not all methods of studying a second order equation are extended to a higher order equation (see [6]). One of the more universal methods to study the oscillatory properties of symmetric differential equations is the variational method. However, in the variational method, the problem is reduced to solving Problem 3, which has not yet been completely studied. Another method of studying an equation in the form (1.2) is to transfer from equation (1.2) to the system of Hamilton's equations, but even here it is difficult to find the fundamental solutions of the Hamiltonian system, especially when the coefficients of equation (1.2) are arbitrary functions. Therefore, in the works devoted to the problem of oscillation or strong oscillation of higher order equations in the form (1.2), all or one of the coefficients are power functions (see, [6–8] and references given there). In a more general case, in terms of the coefficients of the equation, criteria for its strong oscillation and non-oscillation are given in [20].

The oscillatory and non-oscillatory properties of higher order differential equations and their relation to the spectral characteristics of the corresponding differential operators are well presented in monograph [9].

Problem 2. Investigate the spectral properties of the self-adjoint differential operator L generated by the differential expression

$$l(y) = (-1)^n \frac{1}{u(t)}(v(t)y^{(n)})^{(n)}, \tag{1.3}$$

in the Hilbert space $L_{2,u} \equiv L_{2,u}(I)$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$, where $u > 0$.

The investigation of the spectral characteristics of the operator L is the subject of many works (see, e.g., [2,3], [9, Chapters 29 and 34] , [10,14,21] and references given there).

Problem 3. Find necessary and sufficient conditions for the validity of the inequality

$$\int_T^\infty u(t)|f(t)|^2dt \le C_T \int_T^\infty v(t)|f^{(n)}(t)|^2dt, \qquad f \in \mathring{W}^n_{2,v} \tag{1.4}$$

and the sharp estimate of the constant C_T.

The inequality of the type (1.4) was considered in many works (see, e.g., [1,11,17,18] and references given there). The history of the problem and the main achievements are shortly presented in monographs [12] and [13]. Let us note that in [13, Chapter 4] the corresponding comments are given wider than in [12].

We study all these three problems depending on an integral behavior of the function v in a neighborhood of infinity. Problems 1 and 2 have been already investigated in the strong singular case

$$\int_T^\infty v^{-1}(t)dt = \infty. \tag{1.5}$$

Here we assume that

$$\int_T^\infty v^{-1}(t)dt < \infty \quad \text{and} \quad \int_T^\infty v^{-1}(t)t^2dt = \infty \tag{1.6}$$

for any $T \ge 0$.

The work is organized as follows. In Section 2 we give necessary and sufficient conditions for the validity of inequality (1.4). In Section 3 on the basis of the results on inequality (1.4) we find necessary and sufficient conditions for the functions u and v, under which equation (1.2) is strong oscillatory or non-oscillatory. In Section 4, some spectral characteristics of the operator L are obtained.

The symbol $A \ll B$ means $A \le CB$ with some constant C. If $A \ll B \ll A$, then we write $A \approx B$. Moreover, χ_M stands for the characteristic function of the set M.

2 Validity of inequality (1.4)

We investigate (1.4) under condition (1.6). First, we present the known results required for the proof of the validity of inequality (1.4).

Let $0 \le a < b \le \infty$. From the paper [13, p. 6 and 7], the following theorem follows.

Theorem A.
(i) The inequality

$$\left(\int_a^b u(x)\left(\int_a^x f(t)dt\right)^2 dx\right)^{\frac{1}{2}} \le C\left(\int_a^b v(t)f^2(t)dt\right)^{\frac{1}{2}}, \qquad f \ge 0, \tag{2.1}$$

holds if and only if

$$A^+ = \sup_{a<z<b}\left(\int_z^b u(x)dx\right)^{\frac{1}{2}}\left(\int_a^z v^{-1}(t)dt\right)^{\frac{1}{2}} < \infty.$$

Moreover, $A^+ \le C \le 2A^+$, where C is the best constant in (2.1).
(ii) The inequality

$$\left(\int_a^b u(x)\left(\int_x^b f(t)dt\right)^2 dx\right)^{\frac{1}{2}} \le C\left(\int_a^b v(t)f^2(t)dt\right)^{\frac{1}{2}}, \qquad f \ge 0, \tag{2.2}$$

holds if and only if

$$A^- = \sup_{a<z<b} \left(\int_a^z u(x)dx\right)^{\frac{1}{2}} \left(\int_z^b v^{-1}(t)dt\right)^{\frac{1}{2}} < \infty.$$

Moreover, $A^- \le C \le 2A^-$, *where* C *is the best constant in* (2.2).

Let

$$A_1 = \sup_{a<z<b} \left(\int_z^b u(x)dx\right)^{\frac{1}{2}} \left(\int_a^z (z-t)^{2(n-1)} v^{-1}(t)dt\right)^{\frac{1}{2}},$$

$$A_2 = \sup_{a<z<b} \left(\int_z^b (x-z)^{2(n-1)} u(x)dx\right)^{\frac{1}{2}} \left(\int_a^z v^{-1}(t)dt\right)^{\frac{1}{2}}.$$

The next statement follows from the results in the work [21].

Theorem B. *The inequality*

$$\int_a^b u(z) \left(\int_a^z (z-t)^{n-1} f(t)dt\right)^2 dz \le C \int_a^b v(t) f^2(t)dt, \qquad f \ge 0, \tag{2.3}$$

holds if and only if $\max\{A_1, A_2\} < \infty$. *Moreover,*

$$C \approx \max\{A_1, A_2\}, \tag{2.4}$$

where C *is the best constant in* (2.3).

Assume that $\lim_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$ and

$$LR^{(n-1)}W^n_{2,v} = \left\{f \in W^n_{2,v} : f^{(i)}(T) = 0,\ i = 0, 1, \dots, n-1;\ f^{(n-1)}(\infty) = 0\right\},$$
$$LW^n_{2,v} = \left\{f \in W^n_{2,v} : f^{(i)}(T) = 0, i = 0, 1, \dots, n-1\right\}.$$

From Theorems 1 and 2 in [15] in view of the conditions on v^{-1} in a neighborhood of zero, it follows the next statement.

Theorem C.

(i) If (1.5) *holds, then*

$$\mathring{W}^n_{2,v} \equiv LW^n_{2,v}; \tag{2.5}$$

(ii) if (1.6) *holds, then*

$$\mathring{W}^n_{2,v} \equiv LR^{(n-1)}W^n_{2,v} \quad \textit{and} \quad LW^n_{2,v}(I_{T+1}) \equiv LR^{(n-1)}W^n_{2,v}(I_{T+1}) \oplus P_\infty, \tag{2.6}$$

where $P_\infty = \{P(t) = c\chi_{I_{T+1}}(t)t^{n-1} : c \in R\}$.

Assume that $J(f) = \frac{\int_T^\infty u(t)|f(t)|^2dt}{\int_T^\infty v(t)|f^{(n)}(t)|^2dt}$, $C_L = \sup_{f\in LW^n_{2,v}} J(f)$ and $C_{LR} \equiv C_T = \sup_{f\in LR^{(n-1)}W^n_{2,v}} J(f)$. It is obvious that $C_{LR} \le C_L$. We investigate the estimate of the value C_{LR} under the assumption $C_L = \infty$, that in view of (2.6) is equivalent to the condition

$$\int_\alpha^\infty u(x)x^{2(n-1)}dx = \infty \tag{2.7}$$

for any $\alpha > T$.

Let τ be an arbitrary point of the interval I_T. Assume

$$A_{1,1}(T,\tau) = \sup_{T<z<\tau} \int_z^\tau u(x)dx \int_T^z (z-t)^{2(n-1)} v^{-1}(t)dt,$$

$$A_{1,2}(T,\tau) = \sup_{T<z<\tau} \int_z^\tau u(x)(x-z)^{2(n-1)}dx \int_T^z v^{-1}(t)dt,$$

$$A_{1,3}(T,\tau) = \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx \int_T^\tau (\tau-t)^2 v^{-1}(t)dt,$$

$$A_{1,4}(T,\tau) = \int_\tau^\infty u(x)dx \int_T^\tau (\tau-t)^{2(n-1)} v^{-1}(t)dt,$$

$$A_{2,1}(T,\tau) = \sup_{z>\tau} \int_z^\infty u(x)(x-\tau)^{2(n-2)}dx \int_\tau^z (t-\tau)^2 v^{-1}(t)dt,$$

$$A_{2,2}(T,\tau) = \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt,$$

$$A(T,\tau) = \max\left\{A_{1,1}(T,\tau), A_{1,2}(T,\tau), A_{1,3}(T,\tau), A_{1,4}(T,\tau), A_{2,1}(T,\tau), A_{2,2}(T,\tau)\right\}.$$

Due to (2.6) inequality (1.4) can be written in the form

$$\int_T^\infty u(t)|f(t)|^2dt \le C_T \int_T^\infty v(t)|f^{(n)}(t)|^2dt, \qquad f \in LR^{(n-1)}W^n_{2,v}.$$

In work [18] it is obtained that $A(T,\tau) < \infty$ is necessary and sufficient condition for the validity of this inequality, where $\int_T^\tau v^{-1}(t)dt = \int_\tau^\infty v^{-1}(t)dt$. Here we obtain a simpler criterion that is usable for the application to Problem 1 and 2.

Theorem 2.1. *Let $T \ge 0$. Let (1.6) and (2.7) hold. Inequality (1.4) holds if and only if*

$$\lim_{z\to\infty} \int_z^\infty u(x)(x-\tau)^{2(n-2)}dx \int_\tau^z (t-\tau)^2 v^{-1}(t)dt < \infty \tag{2.8}$$

and

$$\lim_{z\to\infty} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt < \infty. \tag{2.9}$$

Moreover, there exists a point $\tau_T : T < \tau_T < \infty$ such that

$$C_T \approx \mathcal{A}(T,\tau_T) = \max\{A_{2,1}(T,\tau_T), A_{2,2}(T,\tau_T)\}, \tag{2.10}$$

where C_T is the best constant in (1.4).

Proof. Sufficiency. Let (2.8) and (2.9) hold. Then, due to the conditions on the weight functions u and v, we get that $A(T,\tau) < \infty$ for any $\tau \in I_T$. Therefore, on the basis of the results in [18], inequality (1.4) holds. Now, let us estimate the constant C_T from above. From (2.6) it follows that $f^{(i)}(T) = 0$, $i = 0,1,\dots,n-1$, $f^{(n-1)}(\infty) = 0$ for any $f \in \mathring{W}^n_{2,v}$. Hence, we present $f \in \mathring{W}^n_{2,v}$ in the form $f(x) = \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2} f^{(n-1)}(s)ds$, $x > T$, where $f^{(n-1)}(s) = \int_T^s f^{(n)}(t)dt = -\int_s^\infty f^{(n)}(t)dt$, $s > T$. Let $\tau \in I_T$. Next, for $T < s < \tau$ we assume that $f^{(n-1)}(s) = \int_T^s f^{(n)}(t)dt$, and for $s > \tau$ we assume that $f^{(n-1)}(s) = -\int_s^\infty f^{(n)}(t)dt$. Then $f(x) = \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds$ for $T < x < \tau$ and

$$\begin{aligned} f(x) &= \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2} f^{(n-1)}(s)ds \\ &= \frac{1}{(n-2)!}\left[\int_T^\tau (x-s)^{n-2} f^{(n-1)}(s)ds + \int_\tau^x (x-s)^{n-2} f^{(n-1)}(s)ds\right] \\ &= \frac{1}{(n-2)!}\left[\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right] \end{aligned}$$

for $x > \tau$. Therefore, we have

$$\begin{aligned}\int_T^\infty u(x)|f(x)|^2dx &= \int_T^\tau u(x)|f(x)|^2dx + \int_\tau^\infty u(x)|f(x)|^2dx\\ &= \frac{1}{[(n-2)!]^2}\int_T^\tau u(x)\left|\int_T^x (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds\right|^2 dx\\ &\quad + \frac{1}{[(n-2)!]^2}\int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right|^2 dx\\ &= \frac{1}{[(n-2)!]^2}\left[F_1(f^{(n)}) + F_2(f^{(n)})\right], \end{aligned} \tag{2.11}$$

where

$$\begin{aligned}F_1(f^{(n)}) &= \int_T^\tau u(x)\left|\int_T^x (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds\right|^2 dx\\ &= \frac{1}{(n-1)^2}\int_T^\tau u(x)\left|\int_T^x (x-t)^{n-1}f^{(n)}(t)dt\right|^2 dx,\end{aligned}$$

$$\begin{aligned}F_2(f^{(n)}) &= \int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right|^2 dx\\ &= \int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^x f^{(n)}(t)dtds\right.\\ &\qquad \left. - \int_\tau^x (x-s)^{n-2}dx\int_x^\infty f^{(n)}(t)dtds\right|^2 dx.\end{aligned}$$

Assume that $f^{(n)} = g$, then $\int_T^\infty g(t)dt = 0$ and the condition $f \in \mathring{W}^n_{2,v}$ is equivalent to the condition $g \in \widetilde{L}_2(I_T) \equiv \{g \in L_2(I_T) : \int_T^\infty g(t)dt = 0\}$. Therefore, from (2.11) it follows that inequality (1.4) is equivalent to the inequality

$$\frac{1}{[(n-2)!]^2}\left[F_1(g) + F_2(g)\right] \le C_T\int_T^\infty v(t)|g(t)|^2dt, \qquad g \in \widetilde{L}_2(I_T). \tag{2.12}$$

Moreover, the best constants in inequalities (1.4) and (2.12) coincide.

On the basis of Theorem B we have

$$\begin{aligned}F_1(g) &= \frac{1}{(n-1)^2}\int_T^\tau u(x)\left|\int_T^x (x-t)^{n-1}g(t)dt\right|^2 dx\\ &\ll \max\{A_{1,1}(T,\tau), A_{1,2}(T,\tau)\}\int_T^\tau v(t)|g(t)|^2dt.\end{aligned} \tag{2.13}$$

Now, we estimate $F_2(g)$.

$$\begin{aligned}F_2(g) \le \int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s |g(t)|dtds + \int_\tau^x (x-s)^{n-2}\int_s^x |g(t)|dtds\right.\\ \left. + \int_\tau^x (x-s)^{n-2}ds\int_x^\infty |g(t)|dt\right|^2 dx\\ = \int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt + \int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right.\\ \left. + \frac{1}{n-1}(x-\tau)^{n-1}\int_x^\infty |g(t)|dt\right|^2 dx\end{aligned}$$

$$\leq 3\left[\int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt\right|^2 dx\right.$$
$$+\int_\tau^\infty u(x)\left|\int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right|^2 dx$$
$$\left.+\frac{1}{(n-1)^2}\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\left(\int_x^\infty |g(t)|dt\right)^2 dx\right]$$
$$=3\left[J_0+J_1+\frac{J_2}{(n-1)^2}\right], \tag{2.14}$$

where

$$J_0=\int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt\right|^2 dx,$$
$$J_1=\int_\tau^\infty u(x)\left|\int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right|^2 dx,$$
$$J_2=\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\left(\int_x^\infty |g(t)|dt\right)^2 dx.$$

Let us estimate J_0, J_1 and J_2 separately. For the estimate of J_0 using $(x-s)^{n-2} = (x-\tau+\tau-s)^{n-2} \approx (x-\tau)^{n-2}+(\tau-s)^{n-2}$ and Hölder's inequality, we get

$$J_0 \approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx\left(\int_T^\tau (\tau-t)|g(t)|dt\right)^2+\int_\tau^\infty u(x)dx\left(\int_T^\tau (\tau-t)^{n-1}|g(t)|dt\right)^2$$
$$\ll \max\{A_{1,3}(T,\tau),A_{1,4}(T,\tau)\}\int_T^\tau v(t)|g(t)|^2dt. \tag{2.15}$$

For the estimate of J_1 using $\int_\tau^t (x-s)^{n-2}ds=\frac{1}{n-1}\left((x-\tau)^{n-1}-(x-t)^{n-1}\right)\approx(x-\tau)^{n-2}(t-\tau)$ and Theorem A, we get

$$J_1 \approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}\left(\int_\tau^x (t-\tau)|g(t)|dt\right)^2 dx \ll A_{2,1}(T,\tau)\int_\tau^\infty v(t)|g(t)|^2dt. \tag{2.16}$$

By Theorem A we have

$$J_2 \ll A_{2,2}(T,\tau)\int_\tau^\infty v(t)|g(t)|^2dt. \tag{2.17}$$

From (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17) it follows that there exist positive numbers α and β such that the estimate

$$\int_T^\infty u(x)|f(x)|^2dx \leq \beta\mathcal{A}_0(T,\tau)\int_T^\tau v(t)|f^{(n)}(t)|^2dt+\alpha\mathcal{A}(T,\tau)\int_\tau^\infty v(t)|f^{(n)}(t)|^2dt \tag{2.18}$$

holds, where $\mathcal{A}_0(T,\tau)=\max\{A_{1,1}(T,\tau),\ A_{1,2}(T,\tau),\ A_{1,3}(T,\tau),A_{1,4}(T,\tau)\}$ and $\mathcal{A}(T,\tau)=\max\{A_{2,1}(T,\tau),A_{2,2}(T,\tau)\}$.

In view of (2.8) and (2.9), we have that the value $\mathcal{A}_0(T,\tau)$ satisfies the properties $\lim_{\tau\to T}\mathcal{A}_0(T,\tau)=0$ and $\lim_{\tau\to\infty}\mathcal{A}_0(T,\tau)=\infty$, and the value $\mathcal{A}(T,\tau)$ is non-increasing in τ and $\lim_{\tau\to\infty}\mathcal{A}(T,\tau)<\infty$. Therefore, the following point

$$\tau_T=\sup\{\tau\in I_T:\beta\mathcal{A}_0(T,\tau)\leq\alpha\mathcal{A}(T,\tau)\} \tag{2.19}$$

is defined. Then from (2.18) we have

$$\int_T^\infty u(t)|f(t)|^2dt \ll \mathcal{A}(T,\tau_T)\int_T^\infty v(t)|f^{(n)}(t)|^2dt, \tag{2.20}$$

i.e., inequality (1.4) holds with the estimate

$$C_T \ll \mathcal{A}(T,\tau_T) \tag{2.21}$$

for the best constant C_T in (1.4).

Necessity. Let us use the technique used in works [17] and [18]. Let inequality (1.4) hold with the best constant $C_T > 0$. By condition (1.6) we have that $\int_T^\infty v^{-1}(t)dt < \infty$. Suppose that $\gamma_{\tau_T} = \gamma(\tau_T) > 0$ and the function $\rho : (T,\tau_T) \to (\tau_T,\infty)$ is such that

$$\int_T^{\tau_T} v^{-1}(t)dt = \gamma_{\tau_T}\int_{\tau_T}^\infty v^{-1}(t)dt$$

and

$$\int_T^s v^{-1}(t)dt = \gamma_{\tau_T}\int_{\rho(s)}^\infty v^{-1}(t)dt,\ s \in (T,\tau_T). \tag{2.22}$$

It is obvious that the decreasing function ρ is locally absolutely continuous on the interval (T,τ_T) and $\lim\limits_{s\to T^+}\rho(s) = \infty$, $\lim\limits_{s\to\tau_T}\rho(s) = \tau_T$. The differentiation of the both sides of (2.22) gives

$$v^{-1}(s) = -\gamma_{\tau_T}v^{-1}(\rho(s))\rho'(s) = \gamma_{\tau_T}v^{-1}(\rho(s))|\rho'(s)|$$

or

$$\frac{1}{\gamma_{\tau_T}} = \frac{v^{-1}(\rho(s))|\rho'(s)|}{v^{-1}(s)} \tag{2.23}$$

for almost all $s \in (T,\tau_T)$. Let

$$K^+(T,\tau_T) = \{g \in L_1(T,\tau_T) \cap L_{2,v}(T,\tau_T) :\ g \geq 0, g \not\equiv 0\},$$
$$K^-(\tau_T,\infty) = \{g \in L_1(\tau_T,\infty) \cap L_{2,v}(\tau_T,\infty) :\ g \leq 0, g \not\equiv 0\}.$$

Let us show that for every $g_2 \in K^-(\tau_T,\infty)$ there exists $g_{1,2} \in K^+(T,\tau_T)$ such that for the functions $g(t) = g_{1,2}(t)$, $t \in (T,\tau_T)$ and $g(t) = g_2(t)$, $t \in (\tau_T,\infty)$ we have that $g \in \widetilde{L}_{2,v}(T,\infty)$.

For $g_2 \in K^-(\tau_T,\infty)$ we assume that $g_{1,2}(x) = -\gamma_{\tau_T}g_2(\rho^{-1}(x))\frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))}$. Then $g_{1,2} \geq 0$. Changing the variables $\rho^{-1}(x) = t$ and using (2.23), we have

$$\int_T^{\tau_T} g_{1,2}(x)dx = \gamma_{\tau_T}\int_T^{\tau_T}|g_2(\rho^{-1}(x))|\frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))}dx = -\gamma_{\tau_T}\int_{\tau_T}^\infty |g_2(t)|\frac{v^{-1}(\rho(t))}{v^{-1}(t)}\rho'(t)dt$$
$$= \gamma_{\tau_T}\int_{\tau_T}^\infty |g_2(t)|\frac{v^{-1}(\rho(t))}{v^{-1}(t)}|\rho'(t)|dt = \int_{\tau_T}^\infty |g_2(t)|dt < \infty. \tag{2.24}$$

From (2.24) it follows that $\int_T^{\tau_T} g_{1,2}(x)dx < \infty$ and

$$\int_T^{\tau_T} g_{1,2}(x)dx + \int_{\tau_T}^\infty g_2(t)dt = \int_T^\infty g(t)dt = 0. \tag{2.25}$$

Again, changing the variables $\rho^{-1}(x) = t$ and using (2.23), we have

$$\begin{aligned}\int_T^{\tau_T} |g_{1,2}(t)|^2 v(t)dt &= \gamma_{\tau_T}^2 \int_T^{\tau_T} \left| g_2(\rho^{-1}(x)) \frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))} \right|^2 v(x)dx \\ &= \gamma_{\tau_T}^2 \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t) \frac{v^{-1}(\rho(t))}{v^{-1}(t)} |\rho'(t)|dt \\ &= \gamma_{\tau_T} \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt < \infty.\end{aligned}$$

Hence,

$$\begin{aligned}\int_T^{\infty} |g(t)|^2 v(t)dt = \int_T^{\tau_T} |g_{1,2}(t)|^2 v(t)dt + \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt \\ = (1+\gamma_{\tau_T}) \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt < \infty, \quad (2.26)\end{aligned}$$

i.e., $g \in L_{2,v}(I_T)$. The last and (2.25) give that $g \in \widetilde{L}_{2,v}(I_T)$.

Let $g_2 \in K^-(\tau_T, \infty)$ and $g_{1,2} \in K^+(T, \tau_T)$ be a function defined by g_2. Then $g \in \widetilde{L}_{2,v}(I_T)$, where $g(t) = g_{1,2}(t)$, $t \in (T, \tau_T)$ and $g(t) = g_2(t)$, $t \in (\tau_T, \infty)$. Since $g \in \widetilde{L}_2(I_T)$, then replacing the function g in (2.12) for $\tau = \tau_T$ and taking into account that $g_{1,2} \geq 0$, $g_2 \leq 0$, we have

$$\begin{aligned}\frac{1}{[(n-2)!]^2} \Bigg[F_1(g_{1,2}) + \int_{\tau_T}^{\infty} u(x) \Bigg(\int_T^{\tau_T} (x-s)^{n-2} \int_T^s g_{1,2}(t)dtds \\ + \int_{\tau_T}^x (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \Bigg)^2 dx \Bigg] \leq C_T \int_T^{\infty} v(t)|g(t)|^2 dt,\end{aligned}$$

that together with (2.26) gives

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x) \left(\int_{\tau_T}^x (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \right)^2 dx \\ \ll (1+\gamma_{\tau_T}) C_T \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty). \quad (2.27)\end{aligned}$$

Since

$$\int_{\tau_T}^x (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \geq (x-\tau_T)^{n-2} \int_{\tau_T}^x (t-\tau_T)|g_2(t)|dt + \frac{1}{n-1}(x-\tau_T)^{n-1} \int_x^{\infty} |g_2(t)|dt,$$

then from (2.27) we have

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x)(x-\tau_T)^{2(n-2)} \left(\int_{\tau_T}^x (t-\tau_T)|g_2(t)|dt \right)^2 dx \\ \leq (1+\gamma_{\tau_T}) C_T \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty), \quad (2.28)\end{aligned}$$

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x)(x-\tau_T)^{2(n-1)} \left(\int_x^{\infty} |g_2(t)|dt \right)^2 dx \\ \leq C_T (1+\gamma_{\tau_T}) \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty). \quad (2.29)\end{aligned}$$

For any $\tau_T < z < \infty$ the functions $g_2^+(t) = -\chi_{(\tau_T,z)}(t)(t-\tau_T)v^{-1}(t)$, $g_2^-(t) = -\chi_{(z,\infty)}(t)v^{-1}(t)$ belong to the set $K^-(\tau_T,\infty)$. Replacing the functions g_2^+ and g_2^- into (2.28) and (2.29), respectively, we get

$$\mathcal{A}(T,\tau_T) \ll C_T. \tag{2.30}$$

This relation together with (2.21) gives (2.10). From the finiteness of the value $\mathcal{A}(T,\tau_T) = \max\{A_{2,1}(T,\tau_T), A_{2,2}(T,\tau_T)\}$ we have (2.8) and (2.9). The proof of Theorem 2.1 is complete. □

3 Oscillatory properties of equation (1.2)

The main aim of this Section is the investigation of strong oscillation and non-oscillation of differential equation (1.2) in a neighborhood of infinity. Oscillatory properties of (1.2) we investigate under conditions (1.6) and (2.7). Case (1.5) has been investigated in paper [20].

We consider the inequality

$$\int_T^\infty \lambda u(t)|f(t)|^2dt \le \lambda C_T \int_T^\infty v(t)|f^{(n)}(t)|^2dt, \qquad f \in \mathring{W}^n_{2,v}, \tag{3.1}$$

with a constant λC_T, where C_T is the best constant in (1.4).

We investigate the oscillatory properties of equation (1.2) by the variation method, i.e., on the basis of the known variational statement.

Lemma A ([9, Theorem 28]). *Equation (1.2) is non-oscillatory if and only if there exists $T > 0$ such that*

$$\int_T^\infty \left[v(t)|f^{(n)}(t)|^2 - \lambda u(t)|f(t)|^2\right]dt \ge 0 \tag{3.2}$$

for all $f \in \mathring{M}_2(I_T)$.

Due to the compactness of the set $suppf$ for $f \in \mathring{M}_2(I_T)$, inequality (3.2) coincide with the inequality

$$\int_T^\infty \lambda u(t)|f(t)|^2dt \le \int_T^\infty v(t)|f^{(n)}(t)|^2dt, \ \ \forall f \in \mathring{M}_2(I_T). \tag{3.3}$$

Lemma 3.1. *Equation (1.2)*

(i) is non-oscillatory if and only if there exists $T > 0$ such that inequality (3.1) holds with the best constant $\lambda C_T : 0 < \lambda C_T \le 1$;

(ii) is oscillatory if and only if for any $T > 0$ the best constant is such that $\lambda C_T > 1$ in (3.1).

Proof. Let us prove the statement (i), the statement (ii) is the opposite of the statement (i). If equation (1.2) is non-oscillatory, then for some $T > 0$ inequality (3.3) holds, which means that inequality (3.1) holds with the best constant $0 < \lambda C_T \le 1$. Inversely, if for some $T > 0$ inequality (3.1) holds with the best constant $0 < \lambda C_T \le 1$, then inequality (3.3) holds and by Lemma A equation (1.2) is non-oscillatory. The proof of Lemma 3.1 is complete. □

On the basis of Lemma 3.1 and Theorem 2.1, we have the following statement.

Theorem 3.2. *Let (1.6) and (2.7) hold. Then equation (1.2) is strong non-oscillatory if and only if*

$$\lim_{z\to\infty} \int_z^\infty u(x)(x-T)^{2(n-2)}dx \int_T^z (t-T)^2 v^{-1}(t)dt = 0 \tag{3.4}$$

and

$$\lim_{z\to\infty} \int_T^z u(x)(x-T)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt = 0. \tag{3.5}$$

Proof. Let equation (1.2) be strong non-oscillatory. Then by Lemma 3.1 for each $\lambda > 0$ there exists $T_\lambda = T(\lambda) > 0$ such that $\lambda C_{T_\lambda} \le 1$ in (3.1). This gives that $\lim_{\lambda\to\infty} C_{T_\lambda} = 0$, and from (2.10) we have

$$\lim_{\lambda\to\infty} \mathcal{A}(T_\lambda, \tau_{T(\lambda)}) = 0. \tag{3.6}$$

From $\lambda_2 C_{T(\lambda_2)} \le 1$ it follows that $\lambda_1 C_{T(\lambda_2)} \le 1$ for $0 < \lambda_1 \le \lambda_2$. Therefore, $T(\lambda_2) \ge T(\lambda_1)$, $\tau_{T(\lambda_2)} \ge \tau_{T(\lambda_1)}$ and $\lim_{\lambda\to\infty} T(\lambda) = \lim_{\lambda\to\infty} \tau_{T(\lambda)} = \infty$.

Since the value $\mathcal{A}(T,\tau)$ does not increase in $\tau > 0$, from (3.6) we have $\lim_{\tau\to\infty} \mathcal{A}(T,\tau) = 0$, i.e.,

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_z^\infty u(x)(x-\tau)^{2(n-2)}dx \int_\tau^z (t-\tau)^2 v^{-1}(t)dt = 0, \tag{3.7}$$

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt = 0. \tag{3.8}$$

By the definition of the limit (3.7) for any $\varepsilon > 0$ there exists $T_\varepsilon = T(\varepsilon) > T$ such that

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)}dx \int_{T_\varepsilon}^z (t-T_\varepsilon)^2 v^{-1}(t)dt \le \frac{\varepsilon}{5\cdot 2^{2n-3}} \tag{3.9}$$

for all $z \ge T_\varepsilon$. Then there exists $T_1(\varepsilon) \ge T_\varepsilon$ such that

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)}dx \int_T^{T_\varepsilon} (T_\varepsilon - T)^2 v^{-1}(t)dt \le \frac{\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \ge T_1(\varepsilon). \tag{3.10}$$

From (3.9) and (3.10) we get

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)}dx \int_T^z (t-T)^2 v^{-1}(t)dt \le \frac{4\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \ge T_1(\varepsilon). \tag{3.11}$$

Further, there exists $T_2(\varepsilon) \ge T_1(\varepsilon)$ and

$$\int_z^\infty (T_\varepsilon - T)^{2(n-2)}u(x)dx \int_T^z (t-T)^2 v^{-1}(t)dt \le \frac{\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \ge T_2(\varepsilon). \tag{3.12}$$

Then from (3.11) and (3.12) we have

$$\int_z^\infty u(x)(x-T)^{2(n-2)}dx \int_T^z (t-T)^2 v^{-1}(t)dt \le \varepsilon$$

for all $z \ge T_2(\varepsilon)$. It means that (3.4) holds. Similarly, we can prove that from (3.8) it follows (3.5).

Sufficiency. Let (3.4) and (3.5) hold. From (3.5) we have

$$\lim_{z\to\infty} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt = 0$$

for any $\tau \ge T$. Thus,

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt = \lim_{\tau\to\infty} A_{2,2}(T,\tau) = 0.$$

Similarly, from (3.4) we have that $\lim_{\tau\to\infty} A_{2,1}(T,\tau) = 0$. Then $\lim_{\tau\to\infty} \mathcal{A}(T,\tau) = 0$.

Since $\lim_{T\to\infty} \tau_T = \infty$, then $\lim_{T\to\infty} \mathcal{A}(T,\tau_T) = 0$. Hence, from (2.10) we have $\lim_{T\to\infty} C_T = 0$. Therefore, for any $\lambda > 0$ there exists $T_\lambda \ge T$ such that $\lambda C_{T_\lambda} \le 1$ and by Lemma 3.1 equation (1.2) is non-oscillatory for any $\lambda > 0$. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let* (1.6) *and* (2.7) *hold. Then equation* (1.2) *is strong oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty u(x)(x-T)^{2(n-2)}dx\int_T^z(t-T)^2v^{-1}(t)dt=\infty \tag{3.13}$$

or

$$\lim_{z\to\infty}\int_T^z u(x)(x-T)^{2(n-1)}dx\int_z^\infty v^{-1}(t)dt=\infty. \tag{3.14}$$

Proof. Necessity. Let equation (1.2) be strong oscillatory. Then by Lemma 3.1 $\lambda C_T>1$ for any $T\geq 0$ and $\lambda>0$. It means that $C_T>\frac{1}{\lambda}$ and for $\lambda\to 0^+$ it follows that $C_T=\infty$ for any $T>0$. Then from (2.10) we have that $\mathcal{A}(T,\tau_T)=\infty$, i.e., $A_{2,1}(T,\tau_T)=\infty$ or $A_{2,2}(T,\tau_T)=\infty$ for all $T\geq 0$. Therefore, (3.13) or (3.14) holds, respectively.

Sufficiency. Let (3.13) or (3.14) hold. Then $A_{2,1}(T,\tau_T)=\infty$ or $A_{2,2}(T,\tau_T)=\infty$, respectively, i.e., $\mathcal{A}(T,\tau_T)=\infty$ for any $T\geq 0$. Then $\lambda\mathcal{A}_(T,\tau_T)=\infty$ for any $\lambda>0$ and $T\geq 0$. Hence, from (2.10) we have $\lambda C_T>1$ for any $\lambda>0$ and $T\geq 0$. It means that equation (1.2) is oscillatory for any $\lambda>0$. The proof of Theorem 3.3 is complete. □

Next, we suppose that functions v and u are positive and n times continuously differentiable on I. Then on the basis of the reciprocity principle [4] equation (1.2) and the reciprocal equation

$$(-1)^n\big(u^{-1}(t)y^{(n)}\big)^{(n)}-\lambda v^{-1}(t)y=0 \tag{3.15}$$

are simultaneously oscillatory or non-oscillatory.

Suppose that for equation (3.15) the following conditions

$$\int_T^\infty u(t)dt<\infty,\qquad \int_T^\infty u(t)t^2dt=\infty\quad\text{and}\quad\int_\alpha^\infty v^{-1}(t)t^{2(n-1)}dt=\infty \tag{3.16}$$

hold for any $\alpha\geq T$.

Applying the reciprocity principle, on the basis of Theorems 3.2 and 3.3 we get the following theorems.

Theorem 3.4. *Let* $T\geq 0$ *and* (3.16) *hold. Then equation* (1.2) *is strong non-oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty v^{-1}(x)(x-T)^{2(n-2)}dx\int_T^z(t-T)^2u(t)dt=0, \tag{3.17}$$

$$\lim_{z\to\infty}\int_T^z v^{-1}(x)(x-T)^{2(n-1)}dx\int_z^\infty u(t)dt=0. \tag{3.18}$$

Theorem 3.5. *Let* $T\geq 0$ *and* (3.16) *hold. Then equation* (1.2) *is strong oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty v^{-1}(x)(x-T)^{2(n-2)}dx\int_T^z(t-T)^2u(t)dt=\infty$$

or

$$\lim_{z\to\infty}\int_T^z v^{-1}(x)(x-T)^{2(n-1)}dx\int_z^\infty u(t)dt=\infty.$$

4 Spectral characteristics of L

Let the minimal differential operator $L_{\min}$ be generated by differential expression (1.3) in the space $L_{2,u}$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$. It means that $L_{\min}(y) = l(y)$ is an operator with the domain

$$D(L_{\min}) = \Big\{ y : I \to R : y^{(i)} \in AC^{\mathrm{loc}}(I),\ \operatorname{supp} y^{(i)} \subset I,$$
$$\operatorname{supp} y^{(i)} \ \text{ is compact},\ i = 0,1,\dots,n-1,\ l(y) \in L_{2,u} \Big\}.$$

It is known that all self-adjoint extensions of the minimal differential operator L have the same spectrums (see [9]).

Let us consider the problem of boundedness from below and discreteness of the operator L in case (1.6). Case (1.5) was considered in [21].

The relation between the oscillatory properties of equation (1.2) and spectral properties of the operator L is given in the following statement.

Lemma 4.1 ([9]). *The operator L is bounded from below and has the discrete spectrum if and only if equation (1.2) is strong non-oscillatory.*

On the basis of Lemma 4.1, by Theorems 3.2 and 3.4 we have the following theorem.

Theorem 4.2.

(i) If conditions (1.6) and (2.7) hold, then the operator L is bounded from below and has the discrete spectrum if and only if (3.4) and (3.5) hold;

(ii) If condition (3.16) holds, then the operator L is bounded from below and has the discrete spectrum if and only if (3.17) and (3.18) hold.

The operator $L_{\min}$ is non-negative. Therefore, it has the Friedrichs extension L_F. By Theorem 4.2 the operator L_F has the discrete spectrum if and only if (i) (3.4) and (3.5) hold in case (1.6) and (2.7); (ii) (3.17) and (3.18) hold in case (3.16).

From Theorem 2.1 we can state Theorem 4.3.

Theorem 4.3. *Let (1.6) and (2.7) hold. Then the operator L_F is positive defined if and only if $\mathcal{A}(0,\tau_0) = \max\{A_{2,1}(0,\tau_0), A_{2,2}(0,\tau_0)\} < \infty$. Moreover, there exist constants $\alpha, \beta : 0 < \alpha < \beta$ such that the estimate $\alpha\mathcal{A}(0,\tau_0) \le \lambda_1^{-1/2} \le \beta\mathcal{A}(0,\tau_0)$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

On the basis of Rellih's Lemma [16, p. 183] and Theorem 2.1 it follows one more theorem.

Theorem 4.4. *Let (1.6) and (2.7) hold. Then*

(i) the embedding $\mathring{W}^n_{2,v}(I) \hookrightarrow L_{2,u}(I)$ is compact if and only if (3.4) and (3.5) hold;

(ii) the operator L_F^{-1} is completely continuous on $L_{2,u}$ if and only if (3.4) and (3.5) hold.

The next statement is presented in [3].

Lemma B. *Let $H = H(I)$ be a certain Hilbert function space and $C[0,\infty) \cap H$ be dense in it. For any point $x_0 \in I$ we introduce the operator $F_{x_0} f = f(x_0)$ defined on $C[0,\infty) \cap H$, which acts in the space of complex numbers. Let us assume that F_{x_0} is a closure operator. Then the norm of this operator is equal to the value $\left(\sum_{n=1}^\infty |\varphi_n(x_0)|^2\right)^{1/2}$ (finite or infinite), where $\{\varphi_n(\cdot)\}_{n=1}^\infty$ is any complete orthonormal system of continuous functions in H.*

Lemma 4.5. *Let (1.6) and (2.7) hold. Then for $x \in I$*

$$\frac{\sup_{\tau\in I} D(x,\tau)}{(n-1)!} \le \sup_{f\in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \le \sqrt{2}\frac{\inf_{\tau\in I} D(x,\tau)}{(n-1)!}, \tag{4.1}$$

where $\tau \in I$ and

$$\begin{aligned} D(x,\tau) = \Bigg[&\chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(n-1)^2\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(n-1)^2\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds\Bigg]^{1/2}. \end{aligned}$$

Proof. Let $f \in \mathring{W}^n_{2,v}$. Then due to (1.6) we have $f \in LR^{(n-1)}W^n_{2,v}$. Let $\tau \in I$. Similarly as in the proof of sufficiency of Theorem 2.1, we get

$$f(x) = \frac{1}{(n-1)!}\begin{cases} \int_0^x (x-s)^{n-1}f^{(n)}(s)ds & \text{if } 0<x<\tau; \\ (n-1)\int_0^\tau (x-t)^{n-2}\int_0^t f^{(n)}(s)dsdt & \\ -(n-1)\int_\tau^x (x-t)^{n-2}\int_t^\infty f^{(n)}(s)dsdt & \text{if } x>\tau \end{cases}$$

or

$$f(x) = \frac{1}{(n-1)!}\begin{cases} \int_0^x (x-s)^{n-1}f^{(n)}(s)ds & \text{if } 0<x<\tau; \\ (n-1)\int_0^\tau f^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds & \\ -(n-1)\int_\tau^x f^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds & \\ -(x-\tau)^{n-1}\int_x^\infty f^{(n)}(s)ds & \text{if } x>\tau, \end{cases}$$

for all $\tau \in I$. The last expression can be rewritten in the form

$$\begin{aligned} f(x) = \frac{1}{(n-1)!}\Bigg[&\chi_{(0,\tau)}(x)\int_0^x (x-s)^{n-1}f^{(n)}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(n-1)\int_0^\tau f^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds\Bigg] \\ &-\chi_{(\tau,\infty)}(x)\left[(n-1)\int_\tau^x f^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds + (x-\tau)^{n-1}\int_x^\infty f^{(n)}(s)ds\right]. \end{aligned} \tag{4.2}$$

Using Hölder's inequality, we have

$$\begin{aligned} |f(x)| \le \frac{1}{(n-1)!}\Bigg\{&\Bigg[\chi_{(0,\tau)}(x)\left(\int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds\right)^{1/2} \\ &+\chi_{(\tau,\infty)}(x)(n-1)\left(\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds\right)^{1/2}\Bigg]\left(\int_0^\tau v(t)|f^{(n)}(t)|^2dt\right)^{1/2} \\ &+\chi_{(\tau,\infty)}(x)\Bigg[(n-1)\left(\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds\right)^{1/2} \\ &+(x-\tau)^{n-1}\left(\int_x^\infty v^{-1}(s)ds\right)^{1/2}\Bigg]\left(\int_\tau^\infty v(t)|f^{(n)}(t)|^2dt\right)^{1/2}\Bigg\}. \end{aligned} \tag{4.3}$$

One more time using Hölder's inequality for sums in (4.3), we obtain

$$
\begin{aligned}
|f(x)| \le \frac{1}{(n-1)!} \Bigg\{ & \Bigg[\chi_{(0,\tau)}(x) \left(\int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds \right)^{1/2} \\
& + \chi_{(\tau,\infty)}(x)(n-1) \left(\int_0^\tau \left(\int_s^\tau (x-t)^{n-2} dt \right)^2 v^{-1}(s)ds \right)^{1/2} \Bigg]^2 \\
& + \chi_{(\tau,\infty)}(x) \Bigg[(n-1) \left(\int_\tau^x \left(\int_\tau^s (x-t)^{n-2} dt \right)^2 v^{-1}(s)ds \right)^{1/2} \\
& + (x-\tau)^{n-1} \left(\int_x^\infty v^{-1}(s)ds \right)^{1/2} \Bigg]^2 \Bigg\}^{\frac{1}{2}} \\
& \times \left(\int_0^\tau v(t)|f^{(n)}(t)|^2 dt + \int_\tau^\infty v(t)|f^{(n)}(t)|^2 dt \right)^{\frac{1}{2}} \\
\le \frac{1}{(n-1)!} & \Bigg[\chi_{(0,\tau)}(x) \int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds \\
& + \chi_{(\tau,\infty)}(x)(n-1)^2 \int_0^\tau \left(\int_s^\tau (x-t)^{n-2} dt \right)^2 v^{-1}(s)ds \\
& + 2\chi_{(\tau,\infty)}(x)(n-1)^2 \int_\tau^x \left(\int_\tau^s (x-t)^{n-2} dt \right)^2 v^{-1}(s)ds \\
& + 2\chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)} \int_x^\infty v^{-1}(s)ds \Bigg]^{\frac{1}{2}} \left(\int_0^\infty v(t)|f^{(n)}(t)|^2 dt \right)^{\frac{1}{2}}
\end{aligned}
$$

for any $\tau \in I$. Therefore,

$$
|f(x)| \le \frac{\sqrt{2}}{(n-1)!} \inf_{\tau \in I} D(x,\tau) \left(\int_0^\infty v(t)|f^{(n)}(t)|^2 dt \right)^{1/2}.
$$

Then

$$
\sup_{f \in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \le \frac{\sqrt{2}}{(n-1)!} \inf_{\tau \in I} D(x,\tau). \tag{4.4}
$$

Now, we estimate the value $\sup_{f \in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}}$ from below. In (4.2) we fix $x \in I$, so that we choose a function $f^{(n)}$, depending on x, as follows

$$
f_x^{(n)}(s) = \begin{cases}
\chi_{(0,x)}(s)(x-s)^{n-1} v^{-1}(s) & \text{if } 0 < x < \tau; \\
\chi_{(0,\tau)}(s)(n-1) \int_s^\tau (x-t)^{n-2} dt v^{-1}(s) & \\
-\chi_{(\tau,x)}(s)(n-1) \int_\tau^s (x-t)^{n-2} dt v^{-1}(s) & \\
-\chi_{(x,\infty)}(s)(x-\tau)^{n-1} v^{-1}(s) & \text{if } x > \tau.
\end{cases}
$$

Replacing this function in (4.2), we get the value of the function $f(f_x^{(n)})(t)$ at the point $t = x$:

$$f_x(x) = \frac{1}{(n-1)!}\Bigg(\chi_{(0,\tau)}(x)\int_0^x (x-s)^{n-1} f_x^{(n)}(s)ds + \chi_{(\tau,\infty)}(x)(n-1)\int_0^\tau f_x^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds - \chi_{(\tau,\infty)}(x)(n-1)\int_\tau^x f_x^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds - \chi_{(\tau,\infty)}(x)(x-\tau)^{n-1}\int_x^\infty f_x^{(n)}(s)ds\Bigg).$$

If $0 < x < \tau$, then $\chi_{(\tau,\infty)}(x) = 0$. Hence, all terms of $f_x(x)$, except the first one, are equal to zero. For the first term the variable s changes from 0 to x, i.e., $\chi_{(0,x)}(s) \neq 0$ and we replace $f_x^{(n)}(s)$ with $(x-s)^{n-1}v^{-1}(s)$. If $x > \tau$, then $\chi_{(0,\tau)}(x) = 0$. It means that the first term is equal to zero, so $f_x(x)$ is defined by the other three terms. In this case, we replace $f_x^{(n)}(s)$ with its values in the intervals $(0,\tau)$, (τ,x) and (x,∞), respectively. Thus, we get

$$f_x(x) = \frac{1}{(n-1)!}\Bigg(\chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(n-1)^2\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(n-1)^2\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds\Bigg) = \frac{D^2(x,\tau)}{(n-1)!} \quad (4.5)$$

for any $\tau \in I$.

Let us calculate the norm $L_{2,v}$ of the function $f_x^{(n)}$. For $0 < x < \tau$ we take $f_x^{(n)}(s) = \chi_{(0,x)}(s)(x-s)^{n-1}v^{-1}(s)$ and have

$$\int_0^\infty v(s)|f_x^{(n)}(s)|^2 ds = \int_0^x v(s)\big((x-s)^{n-1}v^{-1}(s)\big)^2 ds = \int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds. \quad (4.6)$$

For $x > \tau$ we take the values of $f_x^{(n)}$ on the intervals $(0,\tau)$, (τ,x) and (x,∞), respectively, and get

$$\begin{aligned}\int_0^\infty v(s)|f_x^{(n)}(s)|^2 ds &= \int_0^\tau v(s)|f_x^{(n)}(s)|^2 ds + \int_\tau^x v(s)|f_x^{(n)}(s)|^2 ds + \int_x^\infty v(s)|f_x^{(n)}(s)|^2 ds \\ &= (n-1)^2\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &\quad + (n-1)^2\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &\quad + (x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds. \end{aligned} \quad (4.7)$$

Then using the functions $\chi_{(0,\tau)}(x)$ and $\chi_{(\tau,\infty)}(x)$, we combine (4.6) and (4.7) and obtain

$$\left(\int_0^\infty v(t)|f_x^{(n)}(t)|^2dt\right)^{1/2} = \Bigg[\chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(n-1)^2\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(n-1)^2\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds + \chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds\Bigg]^{1/2} = D(x,\tau) \tag{4.8}$$

for any $\tau \in I$.

From (4.5) and (4.8) we get

$$\sup_{f\in \mathring{W}_{2,v}^n}\frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \geq \frac{|f_x(x)|}{\|f_x^{(n)}\|_{2,v}} = \frac{\sup_{\tau\in I} D(x,\tau)}{(n-1)!}.$$

This relation together with (4.4) gives (4.1). The proof of Lemma 4.5 is complete. □

Let the operator L_F^{-1} be completely continuous on $L_{2,u}$. Let $\{\lambda_k\}_{k=1}^\infty$ be eigenvalues and $\{\varphi_k\}_{k=1}^\infty$ be a corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}.

Theorem 4.6. *Let* (1.6), (2.7), (3.4) *and* (3.5) *hold. Then*

(i)
$$\frac{\sup_{\tau\in I} D^2(x,\tau)}{[(n-1)!]^2} \leq \sum_{k=1}^\infty \frac{|\varphi_k(x)|^2}{\lambda_k} \leq \sqrt{2}\frac{\inf_{\tau\in I} D^2(x,\tau)}{[(n-1)!]^2}; \tag{4.9}$$

(ii) the operator L_F^{-1} is nuclear if and only if $\inf_{\tau\in I}\int_0^\infty u(x)D^2(x,\tau)dx < \infty$. Moreover, there exists $\tau = \mu \in I$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} the relation

$$\frac{2}{[(n-1)!]^2}D_1(\mu) \leq \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^\infty \frac{1}{\lambda_k} \leq \frac{2\sqrt{2}}{[(n-1)!]^2}D_1(\mu) \tag{4.10}$$

holds, where

$$D_1(\mu) = (n-1)^2\int_\mu^\infty u(x)\int_\mu^x\left(\int_\mu^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx + \int_\mu^\infty u(x)(x-\mu)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx.$$

Proof. By the condition of Theorem 4.4 we have that the operator L_F^{-1} is completely continuous on $L_{2,u}$. In Lemma B we take $\mathring{W}_{2,v}^n(I)$ with the norm $\left(\int_0^\infty v(t)|f^{(n)}(t)|^2\right)^{1/2}$ as the space $H(I)$. Since the system of functions $\{\lambda_k^{-1/2}\varphi_k\}_{k=1}^\infty$ is complete orthonormal system in the space $\mathring{W}_{2,v}^n(I)$, then by Lemma B we get

$$\|F_x\|^2 = \left(\sup_{f\in \mathring{W}_{2,v}^n}\frac{|f(x)|}{\|f^{(n)}\|_{2,v}}\right)^2 = \sum_{k=1}^\infty \frac{|\varphi_k(x)|^2}{\lambda_k},$$

where $F_x f = f(x)$. This and (4.1) give (4.9).

Since $\inf_{s\in I} D^2(x,s) \le D^2(x,\tau) \le \sup_{s\in I} D^2(x,s)$ for any $\tau \in I$, multiplying both sides of (4.9) by u and integrating them with respect to x from zero to infinity, we get

$$\frac{1}{[(n-1)!]^2}\int_0^\infty u(x)D^2(x,\tau)dx \le \sum_{k=1}^{\infty}\frac{1}{\lambda_k} \le \frac{\sqrt{2}}{[(n-1)!]^2}\int_0^\infty u(x)D^2(x,\tau)dx \tag{4.11}$$

for all $\tau \in I$. Let us present the integral $\int_0^\infty u(x)D^2(x,\tau)dx$ in the following way

$$\begin{aligned}\int_0^\infty u(x)D^2(x,\tau)dx = &\int_0^\tau u(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx = D_0(\tau)+D_1(\tau),\end{aligned}$$

where

$$\begin{aligned}D_0(\tau) = &\int_0^\tau u(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx,\end{aligned}$$

$$\begin{aligned}D_1(\tau) = &(n-1)^2\int_\tau^\infty u(x)\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx.\end{aligned}$$

The functions $D_0(\tau)$, $D_1(\tau)$ are continuous and the function $D_1(\tau)$ is decreasing on the interval I and $\lim_{\tau\to\infty} D_1(\tau) = 0$. Since

$$\begin{aligned}&\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &\qquad\approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx\int_0^\tau(\tau-s)^2 v^{-1}(s)ds\\ &\qquad\quad+\int_\tau^\infty u(x)dx\int_0^\tau(\tau-s)^{2(n-1)}v^{-1}(s)ds,\end{aligned}$$

then we get $\lim_{\tau\to 0^+} D_0(\tau) = 0$. Therefore, there exists a point $\tau = \mu$ such that $D_0(\mu) = D_1(\mu)$. Hence, from (4.11) we have (4.10). The proof of Theorem 4.6 is complete. □

Remark 4.7. In Theorems 4.3, 4.4 and 4.6 and in their proofs replacing v^{-1} by u, u by v^{-1} and conditions (3.4) and (3.5) by (3.17) and (3.18) in the required places, we get the similar statements but under condition (3.16).

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