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ALGEBRAIC STRUCTURE OF SPACE AND FIELD

ABSTRACT

We investigate an algebraic structure of the space of solutions of autonomous nonlinear differential equations of certain type. It is shown that for these equations infinitely many binary algebraic laws of addition of solutions exist. We extract commutative and conjugate commutative groups which lead to the conjugate differential equations. Besides one is being able to write down particular form of extended Fourier series for these equations. It is shown that in a space with a moving field, there always exist metrics geodesics of which are the solutions of a given differential equation and its conjugate equation. Connection between the invariant group and the algebraic structure of solution space has also been studied.

Keywords: addition of solutions, characteristic functions, conjugate equation, symmetries, metrics.

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1 Systems of ODE

1. Consider an autonomous system of differential equations of N unknown functions in the complex field

$$\frac{du^k}{dt} = F^k(u^1, ..., u^N), \ (k = 1, ..., N), \tag{1.1}$$

where $F(u)$ are defined and differentiable everywhere in the space of their arguments. t is a real independent variable. We assume that the vector field $F(u)$ is smooth and can only have isolated zeros and infinities. We also assume that $F(u)$ does not have other singularities. Let J be the space of solutions of (1.1). We want to find a binary algebraic operation defined in J. Suppose $u_1 = (u_1^1, ..., u_1^N), u_2 = (u_2^1, ..., u_2^N)$ are elements of J. We want to find a solution of (1.1) in the form

$$u^k = \Phi^k(u_1, u_2). \tag{1.2}$$

Substituting (1.2) into (1.1) and taking into an account that $u_1, u_2 \in J$ we obtain determining equation for Φ^k:

$$\frac{\partial \Phi^k}{\partial u_1^i} F^i(u_1) + \frac{\partial \Phi^k}{\partial u_2^i} F^i(u_2) = F^k(\Phi), (k = 1, ...N), \tag{1.3}$$

where repeating index i means summation from 1 to N. The solution Φ of (1.3), as it follows from (1.2), defines a rule: two arbitrary elements u_1, u_2 of J correspond to the third element in J. But it is a definition of a binary algebraic operation on the set J [1]. Generally speaking there can be defined infinitely many algebraic operations (we consider them in section 3). However our goal is to select, via (1.2-3), only commutative groups. Obviously (1.3) will not change if we replace $u_1$with u_2 and u_2 with u_1. Then we can demand commutativity of Φ

$$\Phi(u_1, u_2) = \Phi(u_2, u_1). \tag{1.4}$$

But if such a function Φ exists we can introduce an algebraic law of addition of the elements in J:

$$u_1^k \dot{+} u_2^k = \Phi^k(u_1, u_2), \quad (k = 1, ..., N). \tag{1.5}$$

(1.4) means commutativity of operation (1.5): $u_1 \dot{+} u_2 = u_2 \dot{+} u_1$.

Let us return to (1.3) and consider its characteristic equations

$$\frac{du_1}{dt} = F(u_1), \frac{du_2}{dt} = F(u_2), \frac{d\Phi}{dt} = F(\Phi). \tag{1.6}$$

In [2] it is shown that the system (1.3) is solvable and the solution can be constructed via characteristic functions. Since the system (1.6) is autonomous, it is easy to see that at least one of the constants can be added with parameter t. Thus the solution of (1.6) in coordinate form can be presented as

$$\varphi^k(u_1) = b^k t + c_1^k, \varphi^k(u_2) = b^k t + c_2^k, \varphi^k(\Phi) = b^k t + c^k, \tag{1.7}$$

where $k = 1, ..., N$, and c_1^k, c_2^k, c^k are the constants of integration. Generally b^k have components $b^1 = ...b^{N-1} = 0$, $b^N = 1$. However such a restriction does not follow from anywhere and in principle b^k can be any nonzero vector. We will always assume that

$$\det \frac{\partial \varphi^k}{\partial u^i} \neq 0$$

everywhere in J except some isolated points. We will see it is equivalent to $F(u)$ having only isolated zeros and infinities.

To construct a solution of (1.3) we consider the system of equalities

$$\exp(c_1^k - c^k) + \exp(c_2^k - c^k) = 1, \quad (k = 1, ..., N) \tag{1.8}$$

and substitute (1.7) in (1.8). Then (1.5) will have the form

$$u_1 \dot{+} u_2 = \varphi^{-1}\left[\ln(\exp\varphi(u_1) + \exp\varphi(u_2))\right], \tag{1.9}$$

where φ^{-1} is the inverse function of φ. From (1.9) we immediately have the condition (1.4). Since φ^{-1} is inverse of φ, associativity of operation (1.9) easily follows. But this means that J space with binary operation (1.9) forms a commutative semigroup.

Example 1 (1.1) *As a simple illustration we consider homogeneous equation* $\dfrac{du}{dt} = u$. *Obviously* $\varphi(u) \equiv \ln u = t + c$. *Inverse to* φ *will be* exp. *Then (1.9) will have a form* $u_1 \dot{+} u_2 = \varphi^{-1}\left[\ln(\exp\varphi(u_1) + \exp\varphi(u_2))\right] = \exp\ln(u_1 + u_2) = u_1 + u_2$. ◀

Example 2 (1.2) *Analogous calculations for the equation*

$$\frac{du}{dt} = u(1-u)$$

give $u_1 \dot{+} u_2 = (u_1 + u_2 - 2u_1u_2)/(1 - u_1u_2)$. ◀

2. Let us consider the following equality instead of (1.8):

$$\exp(c^k - c_1^k) + \exp(c^k - c_1^k) = 1, \quad (k = 1, ..., N). \tag{1.10}$$

We substitute (1.7) into (1.10). Then instead of (1.9) we derive another algebraic operation on J :

$$u_1 \ddot{+} u_2 = \varphi^{-1}\left[-\ln(\exp\left(-\varphi(u_1)\right) + \exp\left(-\varphi(u_2)\right))\right]. \tag{1.11}$$

It is easy to see that J with binary operation (1.11) also forms a commutative semigroup. We will call semigroup (1.9) *the conjugate semigroup of* (1.11), and semigroup (1.11) - the *conjugate semigroup of* (1.9).

Example 3 (1.3) *For the linear equation described above (1.11) will have the form:*

$$\begin{aligned} u_1 \ddot{+} u_2 &= \exp[-\ln\left(\exp(-\ln u_1) + \exp(-\ln u_2)\right)] \\ &= \exp[-\ln\left(\frac{1}{u_1} + \frac{1}{u_2}\right)] = \frac{u_1u_2}{u_1 + u_2}. \end{aligned}$$ ◀

Example 4 (1.4) *For equation* $\dfrac{du}{dt} = u(1-u)$ *the conjugate semigroup will have the form:* $u_1 \ddot{+} u_2 = u_1u_2/\left(u_1 + u_2 - u_1u_2\right)$. ◀

3. Let us introduce the function

$$w^k = \exp\varphi^k(u), \quad (k = 1, ..., N), \tag{1.12}$$

where $\varphi^k(u)$ is the left part of (1.7). If $u \in J$, then it follows from (1.7) that w satisfies system

$$\frac{dw}{dt} = bw, \tag{1.13}$$

where matrix $b = diag(b^1, ..., b^N)$. Obviously (1.12) performs a continuous mapping of the solution space J to the solution space W of the system (1.13) except may be some isolated singular points. The inverse mapping is:

$$u = \varphi^{-1}(\ln w). \tag{1.14}$$

Example 5 (1.5) *For example, for equation* $\dfrac{du}{dt} = \sin u$ *the mapping* $J \to W$ *is* $w = \tan\frac{u}{2}$, *where* W *is the space of solutions of* $\dfrac{dw}{dt} = w$. *The inverse map* $u = 2\arctan w + 2\pi m$ *(m-integer) is not one-to-one in contrast to* $w = u/(1-u)$ *and* $u = w/(1+w)$ *which are derived from* $\dfrac{du}{dt} = u(1-u)$. ◀

4. Back to mapping (1.12) $\exp\varphi : J \to W$. It is clear that the inverse mapping $(\exp\varphi)^{-1} : W \to J$, generally speaking, is not one to one. Let us choose a subspace J^m in J in such a way that the mapping $J^m \to W$ via (1.12) and its inverse via (1.14) are diffeomorphisms ((1.7) is smooth). The map $\exp\varphi : J \to W$ is called *a covering*. W is called *a base of covering* and J - *a space of covering*. We will call J^m *a leaf in* J and m *an index* [3]. m runs through some discrete set M. Notice that due to the existence and uniqueness theorem for (1.1), leaves J^m do not intersect for different m (except for some isolated points). Also $J = \bigcup\limits_{m \in M} J^m$. Since W is connected, the number of leaves does not depend on elements of W [3]. The preimage $\varphi^{-1}(\ln(w))$ where $w \in W$, is *a fiber of discrete elements*. Let D be a group acting on fibers. Since u is an arbitrary solution of (1.1), $d(u) = u, d \in D$, implies d is the identity of D. This means that the group D acts faithfully [3]. Such a covering of J is called *a principal bundle with discrete group D and base W*. Easy to see that each element of a fiber is an element of some leaf.

Let us now consider the algebraic operation (1.9). Let $u_1, u_2 \in J$. $\exp\varphi$ maps them to w_1, w_2 in the base space W. But since W as a space of solutions of linear equation (1.13) is a linear vector space, w_1 and w_2 can be added in a standard way (linear superposition). Then $w_1 + w_2$ is mapped back to J to its preimage. This preimage is a fiber of discrete elements. We will call elements of this fiber *a sum* $u_1 \dot{+} u_2$. Each element of this fiber is a single-valued sum of some leaf of the space of covering J. On the other hand, since (1.13) are N distinct equations, we can introduce *the conjugate sum* $w_1^k \ddot{+} w_2^k = 1/(1/w_1^k + 1/w_2^k)$. Defined algebraic operation in W forms a commutative semigroup. Easy to see that if w_1, w_2 are images of $u_1, u_2 \in J$, then the preimage of $w_1 \ddot{+} w_2$ will be $u_1 \ddot{+} u_2$. Obviously, the preimage $u_1 \ddot{+} u_2$ is a discrete fiber in J.

5. Since (1.13) is linear and homogeneous, W forms a commutative group under addition $w_1 + w_2, w_1, w_2 \in W$. e_0 element with $e_0^k = 0$ is *the identity element of the group*, i.e. for any $w \in W$

$$e_0 + w = w. \tag{1.15}$$

Another definition: if for any q element of a commutative group, $q \neq h$,

$$h + q = h, \tag{1.16}$$

then we call h *a conjugate identity element*. Thus conjugate identity in W is h_0, $h_0^k = \infty, (k = 1, ..., N)$, satisfying $h_0 + w = h_0, \forall w \in W$.

For the algebraic operation $w_1 \ddot{+} w_2$ we have $e_0 \ddot{+} w = e_0$, $h_0 \ddot{+} w = w$.

For $\dfrac{du}{dt} = u(1-u)$ we found $u_1 \dot{+} u_2$ and its conjugate $u_1 \ddot{+} u_2$. Obviously, $u = 0, u = 1$, satisfy $0 \dot{+} u = u, 1 \dot{+} u = 1$. For the conjugate sum we have $0 \ddot{+} u = 0, 1 \ddot{+} u = u$.

6. Let the following limits exist

$$e = \lim_{w \to e_0} \varphi^{-1}(\ln w), \quad h = \lim_{w \to h_0} \varphi^{-1}(\ln w). \tag{1.17}$$

We also admit that generally some coordinates of e and h can be infinite. Let us write (1.17) formally: $e = \varphi^{-1}(\ln e_0)$, $h = \varphi^{-1}(\ln h_0)$. Clearly e and h are preimages of e_0 and h_0 and hence are discrete fibers in J. But since φ^{-1} is inverse of φ then the following must take a place:

$$e_0 = \lim_{u \to e} \exp \varphi(u), \; h_0 = \lim_{u \to h} \exp \varphi(u). \tag{1.18}$$

Using (1.18-1.19) we immediately obtain:

$$(u \dot{+} e) = u, \; (u \dot{+} h) = h, \; u \in J.$$

These equalities show that under the binary operation (1.9) in J elements e_m and h_m of fibers e and h play the role of identity and conjugate identity respectively. Analogous calculations for conjugate binary operation (1.11) give:

$$(h \ddot{+} u) = u, \; (e \ddot{+} u) = e, \; u \in J.$$

Thus when turning from algebra (1.9) to its conjugate algebra (1.11) the identity and the conjugate identity elements change their places.

7. Let us consider now inverse elements w and $-w$ in W, $w + (-w) = e_0$. Let $u = \varphi^{-1}(\ln(w))$ and $u^- = \varphi^{-1}(\ln(-w))$. Then from (1.9) it follows that

$u \dot{+} u^- = \varphi^{-1}[\ln(\exp^{\varphi(u)} + \exp^{\varphi(u^-)})] = \varphi^{-1}[\ln(w + (-w))] = \varphi^{-1}(\ln e_0) = e$.

Analogously

$$u \ddot{+} u^- = h.$$

We showed that u and u^- are inverse elements in both sums (1.9) and (1.11). For example, in the space of solutions of $\dfrac{du}{dt} = u(1-u)$ elements u and $u/(2u-1)$ are inverse.

Elements $e_0, h_0 \in W$ are fixed points hence do not depend on t. But then from (1.17-1.18) we conclude that $e, h \in J$ also do not depend on t. Then e, h are fixed solutions of (1.1) if only they have finite coordinates. Above mentioned examples convince us in that. However inverse is not always true.

Example 6 (1.6) *If* $\dfrac{du}{dt} = \sqrt[3]{u}$, *then it is easy to find the mapping* $w = \exp \sqrt[3]{u^2}$. *In this case* $e = \pm i\infty, h = \pm\infty$. *As for the fixed point* $u = 0$, *it is a ramification point of the map.*◀

2 Characteristic functions

The introduction of algebraic operations brings to our attention the characteristic functions $\varphi^k(u)$ which satisfy

$$\varphi^k(u) = b^k t + c^k. \qquad 2.1$$

In our theory the characteristic functions play the central role building the algebraic and geometric structure of J space. Hence it is important to investigate algebraic properties of $\varphi^k(u)$.

1. Let us differentiate (2.1) with respect to t. Using (1.1), we obtain

$$F^i(u)\frac{\partial \varphi^k}{\partial u^i} = b^k. \qquad (2.2)$$

Multiplying (2.2) by $(\frac{\partial\varphi}{\partial u})^{-1}$, we have

$$F^k(u) = \left[(\frac{\partial\varphi}{\partial u})^{-1}\right]^k_n b^n. \qquad (2.3)$$

From (2.2-3) immediately follows the connection of singularities of $F^k(u)$ and $\det(\frac{\partial\varphi}{\partial u})$.

2. Let us multiply (2.1) by an arbitrary nonsingular constant matrix M^k_n. Then we can write

$$\bar{\varphi}^k = M^k_n \varphi^n, \quad \bar{b}^k = M^k_n b^n. \qquad (2.4)$$

Obviously the set of all transformations (2.4) forms the general linear group $GL(N)$. From (2.3-4) it follows that the group $GL(N)$ leaves invariant $F^k(u)$ and the elements of the space J. But this means that the equation (1.1) is invariant under the transformation (2.4). We call this group *the accompanying group of differential equation.* An infinitesimal notation of elements of the group $GL(N)$ can be written in the form

$$\bar{\varphi}^k = \varphi^k + \sigma^k_n \varphi^n, \quad \bar{b}^k = b^k + \sigma^k_n b^n. \qquad (2.5)$$

From (1.12) it follows that the elements of W are transformed as

$$\bar{w}^k = w^k + \sigma^k_n w^k \ln w^n, \qquad (2.6)$$

where n is the summation index, and k is not. Using (2.5-6), it is easy to prove the invariance of (1.13) under the action of the accompanying group $GL(N)$.

Analogous situation takes place with any group acting in the space J. We will see later, that if some group G acts on J and leaves (1.1) invariant, then it also leaves invariant the characteristic functions $\varphi^k(u)$. Using (1.12) we conclude that the space W of solutions of (1.13) is invariant under the action of group G.

3. In (2.1) b^k and c^k are added as components of N-dimensional vectors. u, $F(u)$, $\varphi(u)$ and w are also represented as N-dimensional vectors. In spite of that, in the equation (1.13) b^k appear as elements of a matrix (although diagonal, but still matrix). Moreover, in contrast to (2.5) there comes up very interesting nonlinear transformation (2.6). This indirectly points that elements of J should probably be interpreted as matrices which is equivalent to the existence of some algebraic structure inside (1.1). But so far we restrict ourselves from further discussion on this subject.

3 Space of Solutions and Algebraic Operations

1. Let us go back to the characteristic functions (1.7). Let $b^{k_0} \neq 0$ for some $k = k_0$. Express t from $\varphi^{k_0}(u_1) = b^{k_0}t + c_1^{k_0}$. Then we can write

$$\varphi^k(u_1) - \tilde{b}^k\varphi^{k_0}(u_1) = \tilde{c}_1^k, \;\; \varphi^k(u_2) - \tilde{b}^k\varphi^{k_0}(u_1) = \tilde{c}_2^k, \tag{3.1}$$

$$\varphi^k(\Phi) - \tilde{b}^k\varphi^{k_0}(u_1) = \tilde{c}^k, \tag{1}$$

where $\tilde{b}^k = b^k/b^{k_0}$. In order to construct the general solution of (1.3) we use the method described in [3]. Let us consider

$$P^k(\tilde{c}_1, \tilde{c}_2, \tilde{c}) = 0, \quad (k = 1, ..., N), \tag{3.2}$$

where P^k are arbitrary smooth functions of their arguments. We demand also

$$\det \frac{\partial P}{\partial \tilde{c}} \neq 0, \tag{3.3}$$

for any $(\tilde{c}_1, \tilde{c}_2, \tilde{c})$. To solve (1.3), we substitute (3.1) into (3.2) and get

$$P^k(\varphi(u_1) - \tilde{b}\varphi^{k_0}(u_1), \varphi(u_2) - \tilde{b}\varphi^{k_0}(u_1), \varphi(\Phi) - \tilde{b}\varphi^{k_0}(u_1)) = 0, \tag{3.4}$$

$$k = 1, ..., N). \tag{2}$$

Because of (3.3), we can solve (3.4) for $\varphi^k(\Phi)$:

$$\varphi^k(\Phi) = L^k(\varphi(u_1), \varphi(u_2)). \tag{3.5}$$

From (3.5) we can easily find solution of (1.3):

$$\Phi = \varphi^{-1}[L(\varphi(u_1), \varphi(u_2))]. \tag{3.6}$$

From an algebraic point of view, (3.6) defines a binary algebraic operation in J, which we denote

$$u_1 * u_2 = \varphi^{-1}[L(\varphi(u_1), \varphi(u_2))]. \tag{3.7}$$

Since L is determined from (3.2) with P being arbitrary with the single condition (3.3), one concludes that there are infinitely many binary algebraic operations in the space J. However we narrow down this set of operations and require in the future that the equality

$$\varphi^{-1}[L(\varphi(u_1), \varphi(u_2))] = u_3$$

was uniquely solvable for each discrete fiber u_1 and u_2 separately.

As an example consider equation $\frac{du}{dt} = u$ with characteristic function $\varphi \equiv \ln u = t + c$. Let (3.2) have the form: $\exp(c_1 - c) - \exp(c_2 - c) = 1$. But then

$$u_1 * u_2 = u_1 - u_2. \tag{3.8}$$

Obviously, the derived algebraic operation is noncommutative and nonassociative. Let us now consider $\exp(q(c_1 - c)) + \exp(q(c_2 - c)) = 1$, with a real number q. Then we find

$$u_1 * u_2 = (u_1^q + u_2^q)^{1/q}. \tag{3.9}$$

It is easy to see that this operation is commutative and associative. The identity elements are $e_0 = 0$, $h_0 = \infty$. Hence the space J with algebraic operation (3.9) is a commutative group. When $q > 0$ this group is isomorphic to a group with operation $u_1 \dot{+} u_2 = u_1 + u_2$, when $q < 0$ it is isomorphic to a group with operation $u_1 \ddot{+} u_2 = u_1 u_2/(u_1 + u_2)$. Using the transformation (2.4): $\varphi \longmapsto q\varphi$, we obtain $(u_1^q + u_2^q)^{1/q} \longmapsto u_1 + u_2$.

This example raises the question of reducibility and irreducibility of binary operations. As it was shown, (3.9) is reducible to $u_1 + u_2$. As for (3.8) and $u_1 + u_2$, they are irreducible.

2. Consider the case when the system (3.2) has the form:

$$P^k(\tilde{c}_1, \tilde{c}) = 0, \ (k = 1, ..., N), \tag{3.10}$$

Substituting (3.1) into (3.10), we have:

$$P^k(\varphi(u_1) - \tilde{b}\varphi^{k_0}(u_1), \varphi(\Phi) - \tilde{b}\varphi^{k_0}(u_1)) = 0.$$

By analogy with (3.6), we find

$$\Phi = \varphi^{-1}[Q(\varphi(u_1))]. \tag{3.11}$$

This is a unary operation in the space J of solutions of (1.1). It can also be interpreted as a mapping of J into itself. In other words, (3.11) is a transformation acting in J and leaving (1.1) invariant. By the assumption made in subsection 3.1 (3.11) is uniquely solvable for u_1 and since the identity transformation $\Phi = u_1$ is contained in (3.11), the set of transformations (3.11) forms a group.

3. Lets perform the transformation (1.12). Then with (2.2), (1.3) is transformed into

$$\frac{\partial m^k}{\partial w_1^i} b_j^i w_1^j + \frac{\partial m^k}{\partial w_2^i} b_j^i w_2^j = b_i^k m^i, \tag{3.12}$$

where $w_1^k = \exp \varphi^k(u_1)$, $w_2^k = \exp \varphi^k(u_2)$, $m^k = \exp \varphi^k(\Phi)$, and matrix $b = diag(b^1, ..., b^N)$. Obviously, (3.12) is the determining equation of binary operations in the space W of solutions of (1.13).

4. It was mentioned in section 2 that one can always find such a representation of algebraic functions $\varphi^k(u)$ that vector b^k has the form

$$b = (0, ..., 0, 1). \tag{3.13}$$

With (3.13), it is easy to find the general solution of (3.12), which has the form

$$m^q = \theta^q(\frac{w_2^N}{w_1^N}, w_1^1, ..., w_1^{N-1}, w_2^1, ..., w_2^{N-1}), \tag{3.14}$$
$$m^N = w_1^N \theta^N(\frac{w_2^N}{w_1^N}, w_1^1, ..., w_1^{N-1}, w_2^1, ..., w_2^{N-1}),$$

where θ^k are arbitrary functions of their arguments and $q = 1, ..., N-1$.

5. Any binary operation which can exist in W can be presented as

$$w_1 \dot{*} w_2 = m(w_1, w_2), \tag{3.15}$$

where $m(w_1, w_2)$ is a solution to (3.12).

In future, like in subsection 3.1, we will narrow down to the binary operations (3.15) with $m(w_1, w_2)$ being smooth and (3.15) being uniquely solvable for w_1 and w_2.

6. Since the matrix b is diagonal it is easy to see that the map

$$w_1^k \mapsto 1/w_1^k, \quad w_2^k \mapsto 1/w_2^k, \quad m^k \mapsto 1/m^k, \tag{3.16}$$

leaves invariant equation (3.12). But then for every operation (3.15) it makes sense to introduce the conjugate binary operation

$$w_1 \ddot{*} w_2 = 1/m(1/w_1, 1/w_2). \tag{3.17}$$

7. In 3.2 we showed that the set of all binary operations contains the subset of unary operations. A unary operation can easily be determined from (3.15). To do that, we pick those solutions m^k of (3.12) which do not contain w_2. Then we will have $m = m(w_1)$. Since w_1 and $m(w_1)$ are solutions of (1.13), we can change $m(w_1)$ for $\bar{w}$. Finally

$$\bar{w} = \bar{w}(w) \tag{3.18}$$

Obviously, (3.18) is a solution of

$$\frac{\partial \bar{w}^k}{\partial w^i} b^i_j w^j = b^k_i \bar{w}^i. \tag{3.19}$$

We have seen that (3.18) is a map from W to W. Using the restrictions on solutions made in 3.5 and since $\bar{w} = w$ is a solution of (3.19), we conclude that the set of transformations (3.18) forms a group.

8. We mentioned that in W space there exist commutative and conjugate commutative groups with identity elements $e_0 = (0, ...0)$, $h_0 = (\infty, ..., \infty)$.

a) Let us substitute e_0 into (3.18). By assumption the solutions (1.3) and (3.12) are sought as smooth functions. Taking into account the action of accompanying group (2.4), from (3.19) we obtain

$$e_0 = \bar{w}(e_0) = e_0. \tag{3.20}$$

For $h_0 \in W$ we suppose that the following takes place

$$\bar{h}_0 = \bar{w}(h_0) = h_0. \tag{3.21}$$

This means that under the action of (3.18) e_0, h_0 are fixed points of the space W.

b) Suppose that $w_2 = e_0$ in (3.12), then

$$\frac{\partial m^k(w_1, e_0)}{\partial w^i_1} b^i_j w^j_1 = b^k_i m^i(w_1, e_0). \tag{3.22}$$

Comparing (3.22) and (3.19) we come to conclusion that

$$m(w_1, e_0) = \bar{w}_1(w_1). \tag{3.23}$$

Analogously, in case of $w_1 = e_0$, we have

$$m(e_0, w_2) = \bar{w}_2(w_2). \tag{3.24}$$

Since unary operations (3.18) are contained in binary ones, we can factorize $m(w_1, w_2)$ with respect to $w_1$or w_2 independently. Obviously, it means the factorization with respect to the group (3.18). Then (3.23-24) will have the form $m(e_0, w) = m(w, e_0) = w$ and we conclude that e_0 is the identity element of the binary operation (3.15)

$$e_0 \dot{*} w = w \dot{*} e_0 = w. \tag{3.25}$$

Using the results of section 1, suppose that for $h_0 \in W$ the following holds

$$h_0 \dot{*} w = w \dot{*} h_0 = h_0. \tag{3.26}$$

From (3.25-26) and (3.17) it is easy to see that

$$h_0 \ddot{*} w = w \ddot{*} h_0 = w, \quad e_0 \ddot{*} w = w \ddot{*} e_0 = e_0. \tag{3.27}$$

9. Now we narrow the set of solutions of (3.12) down one more time and conclude that binary operations (3.15) are associative

$$(w_1 \dot{*} w_2) \dot{*} w_3 = w_1 \dot{*} (w_2 \dot{*} w_3). \tag{3.28}$$

(3.28) and the conditions imposed on $m(w_1, w_2)$ imply immediately that the space W with binary operation (3.15) forms a group. e_0 plays the role of the identity element and h_0 - of its conjugate identity element.

It is easy to see that associativity for the conjugate binary operation (3.17) follows from (3.28)

$$(w_1 \ddot{*} w_2) \ddot{*} w_3 = w_1 \ddot{*} (w_2 \ddot{*} w_3). \tag{3.29}$$

Thus if (3.15) defines a group in W then (3.17) defines its conjugate group. Identity elements are e_0 and h_0. Existence of such groups follows from section 1.

10. We mentioned that the function (1.14) maps W to J. If we consider fiber $u \in J$ as a single element then (1.12) and (1.14) perform a one-to-one mapping. Easy to show that if W is a group with binary operation (3.15) then the space J with the binary operation

$$u_1 \dot{*} u_2 = \varphi^{-1}[\ln m\,(\exp \varphi(u_1), \exp \varphi(u_2))] \tag{3.30}$$

also forms a group. Maps (1.12) and (1.14) establish an isomorphism between these two groups. Discrete fibers $e = \varphi^{-1}(\ln(e_0))$ and $h = \varphi^{-1}(\ln(h_0))$ are the identity and conjugate identity elements in J and satisfy the following, coming from (3.25-26)

$$\begin{aligned} e \dot{*} u &= u \dot{*} e = u, \\ h \dot{*} u &= u \dot{*} h = h. \end{aligned} \tag{3.31}$$

From (3.17) we can easily find the binary operation for the conjugate group

$$u_1 \ddot{*} u_2 = \varphi^{-1}[-\ln m(\exp(-\varphi(u_1)), \exp(-\varphi(u_2)))]. \tag{3.32}$$

In this case the identity and the conjugate identity elements satisfy

$$\begin{aligned} e \ddot{*} u &= u \ddot{*} e = e, \\ h \ddot{*} u &= u \ddot{*} h = u. \end{aligned} \tag{3.33}$$

We come to the conclusion that in the space of solutions J of (1.1) there exist a continuum set of binary operations (3.30). The transformation group which leaves (1.1) invariant forms together with this continuum set one whole entity. The equation (1.1) stays indifferent to commutativity and associativity.

Finally we point that (1.8) and (1.10) do not come from (1.1). The reason we used them was to obtain classic algebraic theory in case of linear differential equations.

4 Systems of PDE

Consider the system

$$a^{\nu k}_{\ n}(u)\frac{\partial u^n}{\partial x^\nu} = F^k(u), \tag{4.1}$$

where the summation is performed by $\nu = 1, ...N_0$ and $n = 1, ..., N$. Elements of matrices $a^\nu(u)$ and vectors $F(u)$ - are smooth functions defined everywhere in the space of their arguments. The vector field $F(u)$ has only isolated zeros and infinities. Independent variables x^ν are real.

1. As before, we introduce the space of solutions J of system (4.1). In order to establish a binary algebraic operation in J we will be looking for the solution of (4.1) of the form

$$u = \Phi(u_1, u_2), \tag{4.2}$$

where u_1, u_2 are arbitrary elements of J. Substitute (4.2) into (4.1). Then we obtain the determining equation for unknown function Φ. In this equation components u_1, u_2 are independent variables. As for $\frac{\partial u_1^i}{\partial x^\nu}$ and $\frac{\partial u_2^i}{\partial x^\nu}$, part of them is determined from (2.1) and another part plays the role of independent variables. In other words, in the determining equation independent variables are u_1, u_2 and some of their partial derivatives if only $\frac{\partial \Phi}{\partial u_1}$ and $\frac{\partial \Phi}{\partial u_2}$ do not commute with the matrix $a_\nu(u)$. But then the determining equation will contain a certain internal contradiction and such a binary operation will not fit J. To avoid this contradiction, Φ in (2.2) must depend not only on u_1, u_2 but also on the points of ∞-jet manifold, generated by (2.1). It is interesting that in order to explain so-called "hidden symmetries" of a differential equation of a field in [4], we had to go beyond the frameworks of standard theories and construct theory of local groups with jet-spaces. It seems that the appearance of objects with infinite number of elements is internally logical for PDE. Recall, for example, that we can expand solutions of linear equations into special functions. The special functions are connected with a group representation, and this group, in its turn, leaves the given equation invariant (symmetry group).

2. To avoid the straightforward introduction of jet-space we do the following. Let us search for the solution of (2.1) in the form of plane waves

$$u_\alpha = u_\alpha(z_\alpha),$$

where $z_\alpha = \alpha_\nu x^\nu, \alpha_\nu$ are independent parameters running through the points of some space Ω of dimension N_0. Substitute (2.3) into (2.1) and obtain

$$a_\alpha(u_\alpha)\frac{du_\alpha}{dz_\alpha} = F(u_\alpha), \tag{4.3}$$

where $a_\alpha = \alpha_\nu a^\nu(u_\alpha)$. Suppose that the following is satisfied

$$\det a_\alpha(u_\alpha) \neq 0, \alpha \neq 0. \tag{4.4}$$

Assume $F(u) \equiv 0$. In order to have a nontrivial solution to (4.3), we demand $\det a_\alpha(u_\alpha) = 0$, [2,5]. This imposes restrictions on α_ν (not all α_ν are independent and generally they are connected with u_α). However at this point we disregard this case and, to simplify the problem, demand the following together with (4.4)

$$\det \frac{\partial F}{\partial u} \neq 0, \tag{4.5}$$

except some isolated points.

Suppose there are commutative and conjugate commutative groups in the space of solutions J_α of the equation (4.3) Assume that the identity elements have finite coordinates, unless mentioned otherwise. Then due to section 1 they are roots of $F(u) = 0$. Thus e and h do not depend on α.

Consider the trivial fiber bundle $P(\Omega, W_\alpha)$ with base space Ω, fibers W_α and projection $\pi(w_\alpha) = \alpha \in \Omega$. In fibers W_α there is a commutative group defined by usual coordinate addition

$$w^k_{\alpha 1} \dot{+} w^k_{\alpha 2} = w^k_{\alpha 1} + w^k_{\alpha 2}$$

with its conjugate group defined by

$$w^k_{\alpha 1} \ddot{+} w^k_{\alpha 2} = 1/(1/w^k_{\alpha 1} + 1/w^k_{\alpha 2}),$$

with $k = 1, ..., N$. The identity elements of these groups are $e_0 = (0, ..., 0)$ and $h_0 = (\infty, ..., \infty)$. Consider another trivial fiber bundle $P(\Omega, J_\alpha)$ with base space Ω, fibers J_α and projection $\pi(J_\alpha) = \alpha \in \Omega$. Let us perform fiberwise mapping $\exp \varphi_\alpha : J_\alpha \to W_\alpha$. The preimages of fibers W_α are J_α and, as described in section 1, J_α are fiber bundles with discrete subfibers, base space W_α and discrete group D_α. The commutative groups J_α are mapped homomorphically to the commutative groups W_α under $\exp \varphi_\alpha : J_\alpha \to W_\alpha$. From 1.3, if the discrete subfiber u_α is considered as one element J_α, then we can establish an isomorphism between groups J_α and W_α.

3. Let us search for the solution of (4.1) in the form

$$u^k = \chi^k(..., w_\alpha, ...), (k = 1, ..., N), \tag{4.6}$$

where χ^k are functions of elements of all fibers W_α of $P(\Omega, W_\alpha)$. Recall that W_α is a space of solutions of

$$\frac{dw_\alpha}{dt} = b_\alpha w_\alpha, \quad b_\alpha = diag(b^1_\alpha, ..., b^N_\alpha). \tag{4.7}$$

But then (4.6) can be interpreted as a certain nonlinear analogy of Fourier expansion of solutions. Substitute

$$w_\alpha = \exp \varphi_\alpha(u_\alpha). \tag{4.8}$$

in (4.6).

After re-denoting, (4.6) has the form

$$u^k = \chi^k(..., u_\alpha, ...), (k = 1, ..., N). \tag{4.9}$$

From (4.8) it follows that if u_α in (4.9) runs through some fixed discrete subfiber of fiber J_α, then (4.8) will not change. This agrees with analogous properties of (1.9) and (1.11).

4. Substitute (4.9) into (4.1) and with (4.3-4) we obtain

$$\sum_{\alpha\in\Omega} a_{\alpha n}^{\ k}(\chi)\frac{\partial\chi^n}{\partial u_\alpha^i} a_{\alpha\ j}^{-1^i}(u_\alpha)F^j(u_\alpha) = F^k(\chi),\ \ (k = 1, ..., N), \tag{4.10}$$

where $a_\alpha^{-1}(u_\alpha)$ is the inverse of $a_\alpha(u_\alpha)$. It is worth mentioning that χ explicitly depends not only on u_α but also on α_ν which are present in matrices a_α. These matrices generally do not commute for different α. This makes impossible to construct a ternary relation based on binary relation. Hence the ternary relation has to be sought from the corresponding equation (4.10). Since Ω is a continuum space, generally speaking, χ must be considered as a function of continuum number of variables.

Using (4.7) by analogy with (4.10) it is easy to write the determining equations for (4.6)

$$\sum_{\alpha\in\Omega} a_{\alpha n}^{\ k}(\chi)\frac{\partial\chi^n}{\partial w_\alpha^i} b_j^i w_\alpha^j = F^k(\chi). \tag{4.11}$$

5. In order to study algebraic construction of the solution χ we consider the case when the equation (4.1) is linear and homogeneous. In this case $e_0 = 0$ and the conjugate identity is $h_0 = \infty$. It is easy to show that one of solutions (4.10) is well-known function

$$\chi = \sum_{\alpha\in\Omega} u_\alpha(z_\alpha). \tag{4.12}$$

Obviously, (4.12) is a symmetric function of its arguments. Also, if all arguments are e_0 (for example, u_α) then $\chi = u_\alpha$. If at least one of arguments in (4.11) is h_0, $\chi = h_0$. Taking this into account, consider (4.10). Clearly this equation is invariant under the permutation of indices $\alpha \to \beta, \beta \to \alpha, \alpha, \beta \in \Omega$. Let us search for solution of (4.10) in the set of symmetric functions of their arguments u_α. We demand from (4.9) that if all arguments, except one, are e, then the following must hold

$$\chi^k(..., e, ..., u_\alpha, ..., e, ...) = u_\alpha^k. \tag{4.13}$$

Besides assume that if one argument in (4.9) is h then the following holds

$$\chi^k(..., u_\beta, ..., h, ..., u_\gamma, ...) = h^k. \tag{4.14}$$

In (4.12-13) α runs through all Ω set. Using (1.18) and (4.8), for (4.6) we obtain from (4.13-14)

$$\chi(..., e_0, ..., w_\alpha, ..., e_0, ...) = u_\alpha, \quad (4.15)$$
$$\chi(..., w_\beta, ..., h_0, ..., w_\gamma, ...) = h.$$

6. As we mentioned in section 1, the set of solutions J_α of (4.3) represents a principal bundle with discrete group D_α and base space W_α. Since $e, h, u_\alpha \in J_\alpha$, it follows from (4.13-14) that the space J must also be discrete fiber space. Same conclusion follows from the fact that J_α is a subspace of J for any $\alpha \in \Omega$. Elements of a discrete fiber cannot be determined immediately from (4.10), but (4.13-14) give the necessary information how to do that. In order to make the following considerations easier, we suppose that the discrete groups D_α are isomorphic for distinct $\alpha \in \Omega$ (this in part is connected to the fact that the identity elements e and h do not depend on α). J becomes a principal bundle with discrete group $D \cong D_\alpha$ and with some base space V which is still to be determined.

7. Let us consider (4.9) in more detail. It is a solution of (4.10). Here $u_\alpha(z_\alpha)$ is a general solution of (4.3) which contains constants of integration c_α. Hence (4.9) contains whole range of constants $\{c_\alpha, \alpha \in \Omega\}$. Obviously, for every collection $\{c_\alpha\}$ we have different solutions of (4.1). An inverse problem appears: can one find a collection $\{c_\alpha\}$ such that (4.9) is equal to a given solution of (4.1)? Not every solution of (4.10) fits into this problem. However for linear equations this problem has one solution when χ is (4.12). Let us consider χ solution of (4.10) satisfying conditions of subsection 5. We assume without proof that the collection of constants $\{c_\alpha\}$ contained in χ can always be chosen uniquely in such a way that the function coincides with a given solution $u(x) \in J$ (here we avoid functional analysis problems of completeness of expansion and normalization inside χ [6]).

Function χ is a map $P(\Omega, J_\alpha) \to J$. Then the determining of unique collection $\{c_\alpha\}$ can be interpreted as an existence of inverse function $\chi^{-1} : J \to P(\Omega, J_\alpha)$. Notice that under these maps the elements of the same fiber cannot be distinguished.

8. Let us consider the linear homogeneous equation with constant coefficients

$$\overset{\circ}{a}{}^{\nu} \frac{\partial v}{\partial x^\nu} = \overset{\circ}{b} v, \quad (4.16)$$

where $\overset{\circ}{a}{}^{\nu}$ and $\overset{\circ}{b}$ are $N \times N$ diagonal matrices satisfying (4.4-5):

$$\det(\alpha_\nu \overset{\circ}{a}{}^{\nu}) \neq 0, \ \det \overset{\circ}{b} = 0,$$

when $\alpha \neq 0$. Write out plane waves equation

$$\frac{dv_\alpha}{dz_\alpha} = \hat{b}_\alpha v_\alpha, \quad (4.17)$$

where

$$\hat{b}_\alpha = \overset{\circ}{a}{}_\alpha^{-1}\overset{\circ}{b}, \quad \overset{\circ}{a}_\alpha = \alpha_\nu a^\nu.$$

Assume α_ν in (4.7) and (4.17) are the same. It is easy to write the characteristic functions of these equations

$$\ln w_\alpha^k = b_\alpha^k z_\alpha + c_\alpha^k, \ln v_\alpha^k = \hat{b}_\alpha^k z_\alpha + \hat{c}_\alpha^k.$$

Clearly there is a unique nonsingular matrix β_α which establishes the following equality

$$\hat{b}_\alpha^k = \beta_{\alpha n}{}^k b_\alpha^n.$$

But then we can write $\ln v_\alpha^k = \beta_{\alpha n}^k \ln w_\alpha^k$. With regard to (4.8) it is easy to find

$$v_\alpha^k = \exp \beta_{\alpha n}{}^k \varphi_\alpha^n(u_\alpha).$$

Then we can write solution of (4.16) in the form

$$v^k = \sum_{\alpha \in \Omega} \exp \beta_{\alpha n}{}^k \varphi_\alpha^n(u_\alpha). \tag{4.18}$$

Thus (4.18) can be uniquely associated with the discrete fiber $u(x)$ of (4.1). The inverse statement is also true, for every solution $v(x)$ to (4.16) which can always be represented as (4.18), there is a unique discrete fiber $u(x) = \chi(..., u_\alpha, ...) \in J$.

We can say that there is a one-to-one correspondence between discrete fibers of J and elements of V of (4.16). But this means that J is discrete fiber space with discrete group D isomorphic to D_α and base space V.

9. Let us denote the right hand side of (4.18) as $\tilde{\chi}(..., w_\alpha, ...)$. Obviously $\tilde{\chi}$ maps trivial fiber bundle $P(\Omega, W_\alpha)$ to V, i.e. $\tilde{\chi} : P(\Omega, W_\alpha) \to V$. But then there is a one-to-one inverse map $\tilde{\chi}^{-1} : V \to P(\Omega, W_\alpha)$. Analogously function (4.6) represents a mapping of $P(\Omega, W_\alpha)$ to J. Elements of J are discrete fibers $u(x)$, i.e. $\chi : P(\Omega, W) \to J$. We have seen in subsection 7, that there is an inverse $\chi^{-1} : J \to P(\Omega, W_\alpha)$. Then the following is true

$$\chi \circ \tilde{\chi}^{-1} : V \to J, \tilde{\chi} \circ \chi^{-1} : J \to V.$$

10. All arguments of the left hand side of (4.14) (solution to (4.10)) can vary arbitrarily. Besides, $a_\alpha(\chi)$ and $F(\chi)$ in (4.10) are smooth functions. Thus the solutions are smooth too. But then it follows from (4.14) that:

$$\frac{\partial \chi^k}{\partial u_\alpha^i} \Big|_{\substack{u_{\alpha_0} = h \\ \alpha \neq \alpha_0}} = 0. \tag{4.19}$$

It is easy to check that (4.19) does not contradict (4.10). If $u_{\alpha_0} = h$ in (4.10), then $a_\alpha(\chi) = a_\alpha(h), F(\chi) = F(h) = 0$ because of (4.14) . (4.19) implies that all terms in the left hand side of (4.10) would be zeros, when $\alpha \neq \alpha_0$. The term with index $\alpha = \alpha_0$ is also zero because of the factor $F(u_{\alpha_0}) = F(h) = 0$. Hence (4.10) is identically true. Notice that in case of (1.1), from (1.3) we can obtain

$$\frac{\partial \Phi(u,h)}{\partial u^i} = 0,$$

assuming that $\Phi(u_1, u_2)$ defines a group $u_1 \dot{+} u_2$ with the identity and the conjugate identity elements e and h.

11. Let us write symbolically the solution to (4.10), (4.13-14) as a formal sum

$$u_m = \underset{\alpha\in\Omega}{\dot{S}_m} u_\alpha(z_\alpha), \tag{4.20}$$

where $m \in M$ and u_m represent elements of a discrete fiber in J.

Together with (4.20) we introduce a conjugate sum. To do that, instead of conditions (4.13-14), we demand that the solution (4.9) of (4.10) satisfies

$$\begin{aligned} \chi^k_m(..., h, ..., u_\alpha, ..., h, ...) &= u_{\alpha m}^{\ k}, \\ \chi^k_m(..., u_\beta, ..., e, ..., u_\gamma, ...) &= e^k_m. \end{aligned} \tag{4.21}$$

We write the solution χ_m of (4.10) under conditions (4.21) as

$$u_m = \underset{\alpha\in\Omega}{\ddot{S}_m} u_\alpha(z_\alpha), \tag{4.22}$$

$\hat{u}_m \in J$. We will call (4.20) and (4.22) conjugate "sums". In the future the solution to (4.10), (4.13-14) we denote as $\chi(..., u_\alpha, ...)$, and the solution to (4.2) as $\overset{\wedge}{\chi}(..., u_\alpha, ...)$. We conclude that the elements of the space J can be represented as (4.20) or as (4.22).

Analogously we can introduce symmetric function $\overset{\wedge}{\chi}(..., w_\alpha, ...)$ satisfying (4.11) and conditions

$$\begin{aligned} \overset{\wedge}{\chi}(..., h_0, ..., w_\alpha, ..., h_0, ...) &= w_\alpha, \\ \overset{\wedge}{\chi}(..., w_\beta, ..., e_0, ..., w_\gamma, ...) &= e. \end{aligned} \tag{4.23}$$

We will call the function $\overset{\wedge}{\chi}$ *the conjugate function of* χ which was introduced in subsection 5. Obviously, the function $\overset{\wedge}{\chi}$, like χ, is a map

$$P(\Omega, W_\alpha) \to J.$$

12. Let us suppose that in fibers J_α there exist commutative groups with the identity elements e and h. Then the sum of two solutions in J represented in the form (4.20) should be defined as

$$u_1 \dot{+} u_2 = \underset{\alpha\in\Omega}{\dot{S}_m} u_{\alpha 1} \dot{+} \underset{\alpha\in\Omega}{\dot{S}_m} u_{\alpha 2} = \underset{\alpha\in\Omega}{\dot{S}_m} (u_{\alpha 1} \dot{+} u_{\alpha 2}). \tag{4.24}$$

Clearly this operation is commutative and associative. We define for the conjugate sum

$$\hat{u}_1 \ddot{+} \hat{u}_2 = \underset{\alpha\in\Omega}{\ddot{S}_m} u_{\alpha 1} \ddot{+} \underset{\alpha\in\Omega}{\ddot{S}_m} u_{\alpha 2} = \underset{\alpha\in\Omega}{\ddot{S}_m} (u_{\alpha 1} \ddot{+} u_{\alpha 2}). \tag{4.25}$$

Suppose $u_\alpha = e$ in (4.13), then

$$e = \underset{\alpha\in\Omega}{\dot{S}_m} e. \tag{4.26}$$

We will always assume summation $\dot{S}$ in some leaf J_m so the index m can be discarded except some special circumstances.

Using (4.24) and (4.26) we obtain

$$u \dot{+} e = \dot{S} u_\alpha \dot{+} \dot{S} e = \dot{S}(u_\alpha \dot{+} e) = \dot{S} u_\alpha = u.$$

Consider $u \dot{+} h$. Using (4.14) we can write

$$\begin{aligned} u \dot{+} h &= \chi(..., u_\beta, ..., u_\alpha, ..., u_\gamma, ...) \dot{+} \chi(..., \tilde{u}_\beta, ..., h, ..., \tilde{u}_\gamma, ...) \\ &= \chi(..., u_\beta \dot{+} \tilde{u}_\beta, ..., u_\alpha \dot{+} h, ..., u_\gamma \dot{+} \tilde{u}_\gamma, ...) \\ &= \chi(..., u_\beta \dot{+} \tilde{u}_\beta, ..., h, ..., u_\gamma \dot{+} \tilde{u}_\gamma, ...) = h. \end{aligned}$$

Analogously it can be proved that

$$u \ddot{+} h = u, u \ddot{+} e = e.$$

Let u_α and u_α^- be inverse elements in J_α so that they satisfy $u_\alpha \dot{+} u_\alpha^- = e$. Then from (4.24) it immediately follows that the elements $u = \dot{S} u_\alpha$ and $u^- = \dot{S} u_\alpha^-$ are also inverse elements i.e. $u \dot{+} u^- = e$. It was shown in section 1 that if $u_\alpha \dot{+} u_\alpha^- = e$ holds, then $u_\alpha \ddot{+} u_\alpha^- = h$ for the conjugate group in J_α. Thus for the conjugate sum (4.25) $\hat{u} \ddot{+} \hat{u}^- = h$, where $\hat{u} = \ddot{S} u_\alpha$, $\hat{u}^- = \ddot{S} u_\alpha^-$.

We come to the conclusion that the algebraic operations (4.24) and (4.25) in J form a commutative group and conjugate commutative group with the identity elements e and h respectively.

13. As a simple example we consider the equation when $N = 1$

$$a^{\nu}\frac{\partial u}{\partial x^{\nu}} = \sin u, \tag{4.27}$$

where a^{ν} are constants. Then equation (4.3) has the form $a_{\alpha}\frac{du_{\alpha}}{dz_{\alpha}} = \sin u_{\alpha}$. Its solution space J_{α} consists of

$$u_{\alpha m} = 2\arctan[c_{\alpha}\exp(z_{\alpha}/a_{\alpha})] + 2\pi m, \tag{4.28}$$

where m is integer. In the leaf J_{α}^{m} we have the following commutative groups

$$\begin{aligned}(u_{\alpha 1}\dot{+}u_{\alpha 2})_m &= 2\arctan\left[\tan(u_{\alpha 1}/2) + \tan(u_{\alpha 2}/2)\right] + 2\pi m, \\ (u_{\alpha 1}\ddot{+}u_{\alpha 2})_m &= 2\,\mathrm{arccot}\left[\cot(u_{\alpha 1}/2) + \cot(u_{\alpha 2}/2)\right] + 2\pi m.\end{aligned} \tag{4.29}$$

In these groups $e_m = 2\pi m$, $h_m = 2\pi m + \pi$. Let us now consider (4.10) for (4.27)

$$\sum_{\alpha\in\Omega}\frac{\partial\chi_m}{\partial u_{\alpha}}\sin u_{\alpha} = \sin\chi_m. \tag{4.30}$$

The solution of this equation under condition (4.13-4.14) has the form

$$\chi_m = 2\arctan(\sum_{\alpha\in\Omega}\tan(u_{\alpha}/2)) + 2\pi m. \tag{4.31}$$

From (4.30) under condition (4.21) we find

$$\overset{\wedge}{\chi}_m = 2\,\mathrm{arccot}(\sum_{\alpha\in\Omega}\cot(u_{\alpha}/2)) + 2\pi m. \tag{4.32}$$

From (4.31-32) we easily write out corresponding operations (4.24) and (4.25).◀

As another example consider the linear equation for $N = 1$

$$a^{\nu}\frac{\partial u}{\partial x^{\nu}} = u, \tag{4.33}$$

where a^{ν} are constants. One can easily find

$$\chi = \sum_{\alpha\in\Omega}u_{\alpha}, \tag{4.34}$$

$$\overset{\wedge}{\chi} = (\sum_{\alpha\in\Omega} u_\alpha^{-1})^{-1}, \tag{4.35}$$

where u_α is a solution to $a_\alpha \frac{du_\alpha}{dz_\alpha} = u_\alpha$, $a_\alpha = \alpha_\nu a^\nu$. Obviously $e = 0, h = \infty$ are the identity elements of the commutative groups of the linear equation. Immediate calculations in these examples convince that (4.19) holds if e and h have finite coordinates.

14. Let us go back to (4.16). Since $\overset{\circ}{a}^\nu$ and $\overset{\circ}{b}$ are diagonal matrices and because of (4.34-35) we can write out conjugate solution which satisfies (4.23):

$$\hat{v}^k = \left[\sum_{\alpha\in\Omega} \exp(-\beta_{\alpha n}^{\ k}\varphi_\alpha^n(u_\alpha))\right]^{-1}, (k = 1, ...N). \tag{4.36}$$

Using the fiberwise map (4.8) one can establish an isomorphism between (4.36) and (4.22) if a discrete fiber $u(x) \in J$ is seen as a single element. Let us denote the right hand side of (4.36) as $\overset{\wedge}{\tilde{\chi}}(..., w_\alpha, ...)$, which maps $P(\Omega, W_\alpha) \to V$. If one introduces a map $\overset{\wedge}{\chi} \circ \overset{\wedge}{\tilde{\chi}}^{-1} : V \to J$ then $\overset{\wedge}{\chi} \circ \overset{\wedge}{\tilde{\chi}}^{-1}$ can be called the conjugate map with respect to $\chi \circ \tilde{\chi}^{-1}$.

15. Let us extend the results of subsection **12** to arbitrary binary operations. In order to do that we introduce an individual binary operation in each fiber J_α of the space $P(\Omega, J_\alpha)$

$$u_{\alpha 1} \dot{*}(\alpha) u_{\alpha 2} = \varphi_\alpha^{-1}[\ln m_\alpha(\exp\varphi_\alpha(u_{\alpha 1}), \exp\varphi_\alpha(u_{\alpha 2}))], \tag{4.37}$$

where m_α defines binary operation in W_α

$$w_{\alpha 1} \dot{*}(\alpha) w_{\alpha 2} = m_\alpha(w_{\alpha 1}, w_{\alpha 2}). \tag{4.38}$$

But then one should define the corresponding binary operation in J in the form of

$$u_1 \dot{*} u_2 = \underset{\alpha\in\Omega}{\dot{S}}(u_{\alpha 1} \dot{*}(\alpha) u_{\alpha 2}), \tag{4.39}$$

where $u_1 = \dot{S} u_{\alpha 1}, u_2 = \dot{S} u_{\alpha 2}$. It is easy to show that if all the binary operations (4.37) are associative then so is (4.39). With our assumptions, the identity elements are e and h. In this case as it follows from section 3, binary operation (4.39) defines a group in J.

Analogously one can define a conjugate binary operation in J

$$\hat{u}_1 \ddot{*} \hat{u}_2 = \underset{\alpha\in\Omega}{\ddot{S}}(u_{\alpha 1} \ddot{*}(\alpha) u_{\alpha 2}), \tag{4.40}$$

where $\hat{u}_1 = \ddot{S}u_{\alpha 1}, \hat{u}_2 = \ddot{S}u_{\alpha 2}$ and

$$u_{\alpha 1}\ddot{*}(\alpha)u_{\alpha 2} = \varphi_\alpha^{-1}[-\ln m_\alpha(\exp(-\varphi_\alpha(u_{\alpha 1})), \exp(-\varphi_\alpha(u_{\alpha 2})))].$$

16. If φ_α^k are characteristic functions of (4.3) then from (2.3) and (4.3) we can write

$$(a_\alpha^{-1}(u_\alpha))_n^k F^n(u_\alpha) = \left[\left(\frac{\partial \varphi_\alpha(u_\alpha)}{\partial u_\alpha}\right)^{-1}\right]_n^k b_\alpha^n. \tag{4.41}$$

From section 2 it follows that one can always find such a representation of $\varphi_\alpha(u_\alpha)$ that b_α^k does not depend on $\alpha \in \Omega$. Moreover, from (2.4) b^k can be defined arbitrarily. On the other hand (4.41) represents an identity with respect to u_α. Because of (4.4), we can write from (4.41)

$$F(u) = \left[\frac{\partial \varphi_\alpha(u)}{\partial u} a_\alpha^{-1}(u)\right]^{-1} b, \tag{4.42}$$

where u^k, $(k = 1, ..., N)$ are arbitrary and not necessarily solutions of (4.1) and (4.3). From this equality we can immediately conclude that matrix $\frac{\partial \varphi_\alpha(u)}{\partial u} a_\alpha^{-1}(u)$ does not depend on parameter α. But then since b is an arbitrary constant vector, we can conclude from (4.42) :

$$\frac{\partial \varphi_\alpha(u)}{\partial u} a_\alpha^{-1}(u) = \frac{\partial \varphi_\beta(u)}{\partial u} a_\beta^{-1}(u). \tag{4.43}$$

Let us substitute (4.42) in the right hand side of (4.10). We obtain

$$\sum_{\alpha \in \Omega} \frac{\partial \varphi_\beta(\chi)}{\partial \chi} a_\beta^{-1}(\chi) a_\alpha(\chi) \frac{\partial \chi}{\partial u_\alpha} a_\alpha^{-1}(u_\alpha) F(u_\alpha) = b.$$

Using (4.41) and (4.43) we can finally write

$$\sum_{\alpha \in \Omega} \frac{\partial \varphi_\alpha^k(\chi)}{\partial u_\alpha^i} \left[\left(\frac{\partial \varphi_\alpha(u_\alpha)}{\partial u_\alpha}\right)^{-1}\right]_n^i b^n = b^k. \tag{4.44}$$

Let us introduce new independent variables

$$r_\alpha^k = \varphi_\alpha^k(u_\alpha). \tag{4.45}$$

Then we can write

$$\frac{\partial \varphi_\alpha^k(\chi)}{\partial u_\alpha^i} = \frac{\partial \varphi_\alpha^k(\chi)}{\partial r_\alpha^n} \frac{\partial \varphi_\alpha^n(u_\alpha)}{\partial u_\alpha^i}.$$

Using this equality and the fact that b^k is an arbitrary element constant vector, (4.44) becomes

$$\sum_{\alpha\in\Omega} \frac{\partial \varphi_\alpha^k(\chi)}{\partial \chi^i} \frac{\partial \chi^i}{\partial r_\alpha^n} = \delta_n^k, \tag{4.46}$$

where δ_n^k is Kronecker's symbol. Thus we have proved that (4.10) can be transformed into matrix equation (4.46).

We now prove that one of solutions of (4.46) has the form

$$\sum_{\alpha\in\Omega} \exp[r_\alpha^k - \varphi_\alpha^k(\chi)] = 1, (k = 1, ..., N). \tag{4.47}$$

Differentiating (4.47) with respect to r_β^n we obtain

$$\left(\sum_{\alpha\in\Omega} \exp(r_\alpha^k - \varphi_\alpha^k(\chi)) \frac{\partial \varphi_\alpha^k(\chi)}{\partial \chi^i}\right) \frac{\partial \chi^i}{\partial r_\beta^n} = \exp(r_\beta^k - \varphi_\beta^k)\delta_n^k.$$

Multiplying this equality by $\frac{\partial \varphi_\beta^n(\chi)}{\partial \chi^l}$ and taking sum with respect to β gives

$$\sum_{\beta\in\Omega}\left(\sum_{\alpha\in\Omega} \exp\left(r_\alpha^k - \varphi_\alpha^k(\chi)\right) \frac{\partial \varphi_\alpha^k(\chi)}{\partial \chi^i}\right) \frac{\partial \chi^i}{\partial r_\beta^n} \frac{\partial \varphi_\beta^n(\chi)}{\partial \chi^l}$$
$$= \sum_{\beta\in\Omega} \exp\left(r_\beta^k - \varphi_\beta^k(\chi)\right) \frac{\partial \varphi_\beta^k(\chi)}{\partial \chi^l},$$

where there is no summation along k index.

If we cancel the last equality by $\sum\limits_{\beta\in\Omega} \exp\left(r_\beta^k - \varphi_\beta^k(\chi)\right) \frac{\partial \varphi_\beta^k(\chi)}{\partial \chi^l}$, we obtain

$$\sum_{\alpha\in\Omega} \frac{\partial \chi^i}{\partial r_\alpha^n} \frac{\partial \varphi_\alpha^n(\chi)}{\partial \chi^l} = \delta_l^i.$$

It is easy to see that taking transpose of this matrix equality gives (4.46). Thus (4.47) is a solution of (4.46).

Using (4.8) and (4.45), algebraic equation (4.47) can be written as

$$\sum_{\alpha\in\Omega} w_\alpha^k \exp\left[-\varphi_\alpha^k(\chi)\right] = 1, (k = 1, ..., N). \tag{4.48}$$

Let all $u_\alpha = e$ on Ω except one fixed $\alpha \in \Omega$. Using (1.18) and (4.8), one obtains from (4.48)

$$w_\alpha^k \exp\left[-\varphi_\alpha^k(\chi)\right] = 1,$$

or, equivalently $\varphi_\alpha^k(u_\alpha) - \varphi_\alpha^k(\chi) = 0$, and (4.13) follows.

Now let us suppose that for some fixed $\alpha \in \Omega$ we have $u_\alpha = h$. Then from (1.18) $w_\alpha = h_0$. Thus we conclude that one term in (4.48) is infinity. But in the right hand side of (4.48) we have 1, so $\chi = h$.

Hence we have shown that the solution χ of equation (4.47) satisfies (4.13-14). Immediately from (4.47) it follows that χ is a symmetric function of its arguments.

Using the above argument one can show that

$$\sum_{\alpha\in\Omega} \exp\left[\varphi_\alpha^k(\hat{\chi}) - \varphi_\alpha^k(u_\alpha)\right] = 1, (k = 1, ..., N) \tag{4.49}$$

is a solution of (4.46). The solution $\hat{\chi}$ of (4.49) satisfies (4.21). Then (4.47) and (4.49) define the conjugate solutions of equation (4.10). Equation (4.30) is a simple example. Since $\varphi_\alpha(u) = \ln\tan(u/2)$, it is easy to show that from (4.47) and (4.49) (4.31) and (4.32) follow respectively.

5 Conjugate Equation

In section 1 we have introduced map (1.12) based on (1.8-9). On the other hand, using (1.10-11) instead of (1.8-9) would give another map

$$\overset{\triangle}{w}^k = \exp(-\varphi^k(u)), (k = 1, ..., N). \tag{5.1}$$

Using equality (2.1) we can easily see that $\overset{\triangle}{w}^k$ satisfies

$$\frac{d\overset{\triangle}{w}^k}{dt} = -b_i^k \overset{\triangle}{w}^i, \tag{5.2}$$

where matrix $b = diag(b^1, ..., b^N)$. Clearly the solutions of (1.13) and (5.2) can be connected by $\overset{\triangle}{w}^k(t) = w^k(-t)$.

Function (5.1) performs mapping $J \to \overset{\triangle}{W}$, where $\overset{\triangle}{W}$ is a solution space of (5.2). As in section 1, J is a space of covering, but we choose $\overset{\triangle}{W}$ as a new base of covering instead of W. Since (5.1) maps J in the new space $\overset{\triangle}{W}$ it follows that we have also to consider map $\varphi^{-1}(-\ln)$ of space W. Let $\overset{\triangle}{u} = \varphi^{-1}(-\ln w)$, or $w = \exp(-\varphi(\overset{\triangle}{u}))$. Let us differentiate the last equality with respect to t. Using (1.13), where matrix b is diagonal, we obtain

$$\frac{\partial \varphi^k}{\partial \overset{\triangle}{u}^i} \frac{d\overset{\triangle}{u}^i}{dt} = -b^k.$$

Let us multiply this equality by matrix $(\frac{\partial \varphi}{\partial \overset{\triangle}{u}})^{-1}$. Because of (2.3), finally we have

$$\frac{d\overset{\triangle}{u}^k}{dt} = -F^k(\overset{\triangle}{u}), (k = 1, ...N), \tag{5.3}$$

where F is the right hand side of (1.1). It is easy to show that for the commutative groups of (1.1) and (5.3) we can establish the following

$$(u_1 \dot{+} u_2)^{\triangle} = \overset{\triangle}{u}_1 \ddot{+} \overset{\triangle}{u}_2, \quad (u_1 \ddot{+} u_2)^{\triangle} = \overset{\triangle}{u}_1 \dot{+} \overset{\triangle}{u}_2,$$
$$\overset{\triangle}{e} = h, \ \overset{\triangle}{h} = e.$$

Let $\overset{\triangle}{J}$ be a space of solutions of (5.3). Then the following take place

$$\varphi^{-1}(\ln) : W \to J, \overset{\triangle}{W} \to \overset{\triangle}{J},$$
$$\varphi^{-1}(-\ln) : W \to \overset{\triangle}{J}, \overset{\triangle}{W} \to J.$$

It follows that (5.3) comes up naturally when one investigates algebraic properties of (1.1). Corresponding commutative groups connect to each other through operation of conjugation. However, in section 7 we will show that in order to correspond the geometrical theory of differential equations, together with the operation of conjugation one should also introduce Hermitian conjugation.

Let us introduce $\overset{\circ}{g}_{kn}$ and $\overset{\circ}{g}{}^{kn}$ satisfying $\overset{\circ}{g}_{kn} \overset{\circ}{g}{}^{nl} = \delta^l_k$, $\overset{\circ}{g}_{kn} = \delta_{kn}$, where δ^k_n, δ_{kn} are Kronecker symbols.

1. We call the equation

$$\frac{du^+_k}{dt} = -F^+_k(u^+), (k = 1, ..., N), \tag{5.4}$$

a conjugate equation of equation (1.1). Here $F^+_k(u^+) = \overset{\circ}{g}_{kn} F^{*n}(\overset{\circ}{g} u^+)$, F^* is the complex conjugate of F, $(\overset{\circ}{g} u^+)^k = \overset{\circ}{g}{}^{kn} u^+_n$. In the future we will do all considerations for some fixed leaf J^m of space of solutions J, so we discard index m. Characteristic functions (2.1) of equation (1.1) in case (5.4) will transform into

$$\varphi^+_k(u^+) = b^+_k t + c^+_k, \tag{5.5}$$

where $b^+_k = \overset{\circ}{g}_{kn} b^{*n}$, $c^+_k = \overset{\circ}{g}_{kn} c^{*n}$,

$$\varphi^+_k(u^+) = -\overset{\circ}{g}_{kn}\varphi^{*n}(\overset{\circ}{g}u^+). \tag{5.6}$$

From this equality we immediately have

$$\begin{aligned}(u_1 \dot{+} u_2)^+ &= u_1^+ \ddot{+} u_2^+,\\ (u_1 \ddot{+} u_2)^+ &= u_1^+ \dot{+} u_2^+.\end{aligned} \tag{5.7}$$

Using (5.7), we obtain: $(u \dot{+} e)^+ = u^+ \ddot{+} (e)^+ = u^+$, $(u \dot{+} h)^+ = u^+ \ddot{+} (h)^+ = (h)^+$. But since $u^+ \ddot{+} (e)^+ = u^+$, $u^+ \ddot{+} (h)^+ = (h)^+$ we conclude that $(e)^+ = h^+, (h)^+ = e^+$. Finally,

$$e^+_k = (h)^+_k = \overset{\circ}{g}_{kn} h^{*n}, h^+_k = (e)^+_k = \overset{\circ}{g}_{kn} e^{*n}. \tag{5.8}$$

If we consider general binary operations discussed in section 3, using (5.6), (5.8) from (3.30) and (3.32) it follows that

$$(u_1 \dot{*} u_2)^+ = u_1^+ \ddot{*} u_2^+, (u_1 \ddot{*} u_2)^+ = u_1^+ \dot{*} u_2^+. \tag{5.9}$$

Functions m^k in (3.30), (3.32) conjugate as

$$m^+_k = \overset{\circ}{g}_{kn} m^{*n}. \tag{5.10}$$

2. Consider the system of equations

$$\frac{\partial u^+_n}{\partial x^\nu} a^{+\nu n}_{\ \ \ k}(u^+) = -F^+_k(u^+), \tag{5.11}$$
$$(k = 1, ..., N),$$

where

$$\begin{aligned} a^{+\nu n}_{\ \ \ k} &= \overset{\circ}{g}{}^{nl}\overset{\circ}{g}_{kp} a^{*\nu p}_{\ \ \ l}(\overset{\circ}{g}u^+), F^+_k(u^+) = \overset{\circ}{g}_{kn} F^{*n}(\overset{\circ}{g}u^+),\\ (\overset{\circ}{g}u^+)^k &= g^{kn} u^+_n.\end{aligned} \tag{5.12}$$

We call the equation (5.11) a *conjugate equation* to (4.1).

The equation of plane waves for (5.11) will have the form

$$\frac{du_\alpha^+}{dz_\alpha} a_\alpha^+(u^+) = -F^+(u_\alpha^+). \tag{5.13}$$

Obviously (4.3) and (5.13) are conjugate equations. The commutative groups in spaces J_α and J_α^+ are connected by the conjugacy relation (5.7).

Analogously for equations (4.10) we derive the conjugate equations

$$\sum_{\alpha\in\Omega} F_j^+(u_\alpha^+)(a_\alpha^+)^{-1j}_{\ \ i}(u_\alpha^+)\frac{\partial \chi_n^+}{\partial u_{\alpha i}^+} a_{\alpha k}^{+n}(\chi^+) = F_k^+(\chi^+). \tag{5.14}$$

Taking conjugation and regarding (5.8), (4.13-14) and (4.21) will change places.

It is easy to show that between solutions of (4.10) and (5.14) the conjugation rule is

$$\begin{aligned} (\dot{S}u_\alpha(z_\alpha))^+ &= \ddot{S}^+ u_\alpha^+(z_\alpha), \\ (\ddot{S}u_\alpha(z_\alpha))^+ &= \dot{S}^+ u_\alpha^+(z_\alpha). \end{aligned}$$

The conjugation rule for commutative groups (4.24) and (4.25) is analogous to (5.7) and (5.8):

$$\begin{aligned} (u_1 \dot{+} u_2)^+ &= u_1^+ \ddot{+} u_2^+, \\ (u_1 \ddot{+} u_2)^+ &= u_1^+ \dot{+} u_2^+, \\ (e)^+ &= h^+, \\ (h)^+ &= e^+, \end{aligned}$$

where $u_1, u_2 \in J$, and $u_1^+, u_2^+ \in J^+$.

One can easily write out the corresponding binary operations for the conjugate equation (5.11). Without repeating the argument we simply point out that

$$\begin{aligned} (u_1 \dot{*} u_2)^+ &= u_1^+ \ddot{*} u_2^+, \\ (u_1 \ddot{*} u_2)^+ &= u_1^+ \dot{*} u_2^+ \end{aligned}$$

take place.

6 Group properties of differential equations

Let Γ^{N_0} be a space with coordinates $x = (x^1, ..., x^{N_0})$. We consider space Γ^N with coordinates $u = (u^1, ..., u^N)$, where $u^k = u^k(x)$ are C^∞ on Γ^{N_0}. Let Γ^∞_N be a jet-space with elements $q = (u, \frac{du}{dx}, \frac{d^2u}{dx^2}, ...), u \in \Gamma^N$. As it was shown in [4], an arbitrary continuous group which acts in the jet-space, has the form

$$
\begin{aligned}
\bar{x}^\nu &= x^\nu + \lambda^\nu(x, q), \\
\bar{u}^k &= u^k + \Lambda^k(x, q), \\
\bar{P}^k_\nu &= P^k_\nu + \frac{d\Lambda^k}{dx^\nu} - \frac{d\lambda^\sigma}{dx^\nu} P^k_\sigma, \\
\bar{P}^k_{\nu\sigma} &= P^k_{\nu\sigma} + \frac{d^2\Lambda^k}{dx^\nu dx^\sigma} - P^k_\mu \frac{d^2\lambda^\mu}{dx^\nu dx^\sigma} - P^k_{\nu\mu}\frac{d\lambda^\mu}{dx^\sigma} - P^k_{\sigma\mu}\frac{d\lambda^\mu}{dx^\nu}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{6.1}
$$

where $P^k_\nu = \frac{du^k}{dx^\nu}, P^k_{\nu\sigma} = \frac{d^2u^k}{dx^\nu dx^\sigma}, ...$ Note: the repeating upper and lower indices assume tensor summation. In (6.1) $\frac{d}{dx^\nu}$ is a total derivative. Group (6.1) is defined by $\lambda^\nu(x, q)$, $\Lambda^k(x, q)$ which are C^∞ and generally depend on infinite number of variables.

Let us consider the system of first order differential equations

$$
I_k(x, u, P) = 0, (k = 1, ..., N), \tag{6.2}
$$

where I_k are C^∞ functions. By differentiation with respect to x^ν we can determine the prolongation of (6.2) up to infinite order. The prolonged equation determines a surface $\pounds$ in $\Gamma^\infty_{N_0N}$, where $\Gamma^\infty_{N_0N}$ is the topological product $\Gamma^{N_0} \times \Gamma^\infty_N$.

The principle of invariance of manifold $\pounds$ with respect to (6.1) [4] asserts that

$$
\begin{gathered}
\varkappa(\lambda, \Lambda) I_k|_\pounds = 0, \\
(k = 1, ..., N),
\end{gathered}
\tag{6.3}
$$

where operator $\varkappa(\lambda, \Lambda)$ has the form

$$
\varkappa(\lambda, \Lambda) = \lambda^\nu \frac{\partial}{\partial x^\nu} + \Lambda^k \frac{\partial}{\partial u^k} + (\frac{d\Lambda^k}{dx^\nu} - \frac{d\lambda^\sigma}{dx^\nu} P^k_\sigma)\frac{\partial}{\partial P^k_\nu} + ... \tag{6.4}
$$

Since $\frac{d}{dx^\nu}$ is a total derivative with respect to x^ν one can easily show that

$$
\varkappa(\lambda, \Lambda) = \varkappa(0, \Lambda - \lambda^\nu P_\nu) + \lambda^\nu \frac{d}{dx^\nu}. \tag{6.5}
$$

If (6.2) holds then $\frac{dI_k}{dx^\nu} = 0$ on manifold $\pounds$. Taking this into account and applying (6.5) to (6.2) we obtain

$$\varkappa(\lambda, \Lambda) I_k \mid_\pounds = \varkappa(0, \Lambda - \lambda^\nu P_\nu) I_k \mid_\pounds . \tag{6.6}$$

From (6.3) and (6.6) it immediately follows that the subgroup

$$\begin{aligned} \bar{x}^\nu &= x^\nu + \lambda^\nu(x, q), \\ \bar{u}^k &= u^k + \lambda^\nu(x, q) P^k_\nu, \end{aligned} \tag{6.7}$$

of group (6.1) acts trivially on any system of differential equations. In (6.7) $\lambda^\nu(x, q)$ is infinitesimal. Some authors discard (6.7) as infinite. However we take it into consideration in order to obtain full picture of algebra of differential equations.

Because of (6.6), we can transfer function λ^ν into the transformation of u^k, i.e.

$$\begin{aligned} \bar{x}^\nu &= x^\nu, \\ \bar{u}^k &= u^k + \tilde{\Lambda}^k(x, q), \\ \bar{P}^k_\nu &= P^k_\nu + \frac{d\tilde{\Lambda}^k(x, q)}{dx^\nu}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{6.8}$$

where $\tilde{\Lambda}^k = \Lambda^k(x, q) - \lambda^\nu(x, q) P^k_\nu$. Obviously, (6.8) is a factorgroup of group (6.1) modulo subgroup (6.7).

Let us consider another subgroup of (6.1)

$$\begin{aligned} \bar{x}^\nu &= x^\nu + \lambda^\nu(x, u), \\ \bar{u}^k &= u^k + \Lambda^k(x, u). \end{aligned} \tag{6.9}$$

This subgroup does not contain subgroup (6.7) in a sense that one cannot determine $\lambda^\nu(x, u) P^k_\nu$ from $\Lambda^k(x, u)$. (except the case of one independent variable on manifold $\pounds$, $N_0 = 1$, when group (6.9) contains a subgroup (6.7). Similar situation takes place when derivatives P^k_ν are expressed in terms of x and u from (6.2)). The principle of invariance (6.3) with given (6.2) imposes very strong restrictions on $\lambda^\nu(x, u)$ and $\Lambda^k(x, u)$ and this allows one to find Lie groups of type (6.9) from the determining equation (6.3).

7 On geometry of space and field

If we think about introducing a metric in differential form then because of the transformation (6.1) it should be considered not only on Γ^{N_0} or Γ^∞_N but on entire $\Gamma^{\infty}_{N_0N}$. This follows from the fact that in (6.1) λ^{ν} and Λ^k depend not only on x but also on q. Generally, if there are no additional restrictions on the group (6.1) (besides (6.3)), there is no guarantee that one can split the metric of $\Gamma^{\infty}_{N_0N}$ into two separate metrics of $\Gamma^{N_0}$and Γ^{∞}_{N}. Mentioning of jet-space Γ^{∞}_{N} points out that the metric may supposedly be infinite-dimensional. On the other side, the algebraic theory of differential equations leads to conjugate equations and hence one has to take them into account to preserve the symmetry when introducing a metric.

1. Let us consider (1.1) with characteristic equations (2.1). We introduce new variables

$$r^k = \varphi^k(u). \tag{7.1}$$

Then for the conjugate equation (5.4-5) we obtain

$$r^+_k = \varphi^+_k(u^+). \tag{7.2}$$

Let the metric be

$$ds^2 = \overset{\circ}{g}dt^2 + 2dr^+_k dr^k, \tag{7.3}$$

where $\overset{\circ}{g}$ is a constant connecting dimensions t and r. Substituting (7.1-2) into (7.3) we obtain:

$$ds^2 = \overset{\circ}{g}dt^2 + 2g^k_n(u^+, u)du^+_k du^n, \tag{7.4}$$

where g^k_n has the form

$$g^k_n = \frac{\partial\varphi^+_l(u^+)}{\partial u^+_k}\frac{\partial\varphi^l(u)}{\partial u^n}. \tag{7.5}$$

In section 5 we introduced the Hermitian conjugation and conjugation $(u)^{\triangle} = \overset{\triangle}{u}$. To justify the simultaneous introduction of these two operations we say that it allows us to write down metric (7.4) which has a real value.

From (7.4) we can write the action integral

$$\int\limits_{t_1}^{t_2}\sqrt{\overset{\circ}{g} + 2g^k_n(u^+, u)\frac{du^+_k}{dt}\frac{du^n}{dt}}dt.$$

After simple transformations, using the variational principle and the fact that $\det\frac{\partial\varphi}{\partial u} \neq 0$, we will have

$$\frac{\partial \varphi^l}{\partial u^k}\frac{du^k}{dt} = \tilde{c}^{\,l}, \frac{\partial \varphi_l^+}{\partial u_k^+}\frac{du_k^+}{dt} = \approx{c}_l,$$

where $\tilde{c}$, $\approx{c}$ are constants of integration. Let us select these constants in such a way that the following is true: $\tilde{c}^{\,k} = b^k$, $\approx{c}_k = \overset{\circ}{g}_{kn} b^{*n}$. Then taking (2.3) and (5.6) into account, we easily derive

$$\frac{du^k}{dt} = F^k(u), \frac{du_k^+}{dt} = -F_k^+(u^+).$$

This means that the solutions of (1.1) and (5.4) are geodesics in space $\Gamma^N \times \Gamma^{+N}$. Hence we obtained the geometry which is generated by the equation (1.1).

It should be mentioned that if (1.1) splits into two separate equations then the metric (7.3) breaks into two terms.

Example 7 (7.1) *Consider* $\frac{du}{dt} = u$. *Here* $\varphi(u) = \ln(u)$, $\varphi^+(u^+) = -\ln(u^+)$. *Then*

$$ds^2 = \overset{\circ}{g} dt^2 - \frac{2}{u^+ u} du^+ du. \blacktriangleleft$$

Example 8 (7.2) *Let us consider free movement of a particle in 3-space. Due to Newton's first law we have*

$$\frac{du^k}{dt} = b^k, (k = 1, 2, 3).$$

Here $\varphi^k(u) = u^k, \varphi_k^+(u^+) = -u_k^+$. *From (7.4) we obtain*

$$ds^2 = \overset{\circ}{g} dt^2 - 2du_k^+ du^k.$$

Let u^k *be independent variables. We introduce notation* $u^k = x^k$. *Taking into account* $u_k^+ = \overset{\circ}{g}_{kn} u^n(-t)$, *we have* $x_k^+ = \overset{\circ}{g}_{kn} x^n$. *Then the metric has the form*

$$ds^2 = \overset{\circ}{g} dt^2 - 2\overset{\circ}{g}_{kn} dx^k dx^n.$$

Assuming $\overset{\circ}{g} = 2c^2$, *where* c *is the speed of light, we arrive to Minkowski metric. Recall that* $\overset{\circ}{g}_{kn} = \delta_{kn}$, *where* δ_{kn} *is Kronecker's symbol.*$\blacktriangleleft$

2. Let us consider equation (4.3) and introduce the metric in $\Gamma_\alpha^N \times \Gamma_\alpha^{+N}$ as

$$ds^2 = \overset{\circ}{g}_\alpha dz_\alpha^2 + 2g_{\alpha n}^{\;\;k}(u_\alpha^+, u_\alpha)du_{\alpha k}^+ du_\alpha^n. \tag{7.6}$$

Metric tensor $g_{\alpha n}^{\;\;k}$ is determined from (7.5) where $\varphi_\alpha^k(u_\alpha)$ and $\varphi_{\alpha k}^+(u_\alpha^+)$ are characteristic functions of equations (4.3) and (5.13).

For different α equations (4.3) are independent. Then from the argument used in subsection 1 we define the total metric as follows

$$ds^2 = \sum_{\alpha\in\Omega} ds_\alpha^2. \tag{7.7}$$

Taking into account that $z_\alpha = \alpha_\nu x^\nu$ and hence $dz_\alpha = \alpha_\nu dx^\nu$, the first term in (7.6) can be represented as $\overset{\circ}{g}_\alpha \alpha_\nu \alpha_\tau dx^\nu dx^\tau$. Then (7.7) will take the form

$$ds^2 = \overset{\circ}{g}_{\nu\tau} dx^\nu dx^\tau + 2\sum_{\alpha\in\Omega} g_{\alpha n}^{\;\;k}(u_\alpha^+, u_\alpha)u_{\alpha k}^+ du_\alpha^n, \tag{7.8}$$

where

$$\overset{\circ}{g}_{\nu\tau} = \sum_{\alpha\in\Omega} \overset{\circ}{g}\alpha_\nu\alpha_\tau. \tag{7.9}$$

One can see that (7.8) is a metric of the trivial bundle $\Gamma^{N_0} \times P(\Omega, \Gamma_\alpha^N, \Gamma_\alpha^{+N})$. Thus we obtain the metric of the geometry which is generated by equation (4.1). The metric is infinite-dimensional and this is consistent with what we mentioned in the beginning of this section.

Example 9 (7.3) *Consider the Dirac equation [7]*

$$\beta\frac{\partial\psi}{\partial x^4} + \beta\vec{\sigma}\frac{\partial\psi}{\partial\vec{x}} = -im\psi, \tag{7.10}$$

where matrices β and $\vec{\sigma}$ are

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \vec{\sigma} = \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix},$$

and τ is the Pauli matrix [7].

Let us write out the equation of plane waves

$$\frac{d\psi_\alpha}{dz_\alpha} = -i\frac{m}{\alpha_\nu^2}(\alpha_4\beta + \beta\vec{\alpha}\vec{\sigma})\psi_\alpha, \tag{7.11}$$

where $\psi_\alpha = \psi_\alpha(z_\alpha)$, $z_\alpha = \alpha_\nu x^\nu = \alpha_4 x^4 + \vec{\alpha}\vec{x}$, $\alpha_\nu^2 = \alpha_4^4 - \vec{\alpha}^2$. *Using the method of Foldy-Wouthusen [7], we will bring the matrix* $\alpha_4\beta + \beta\vec{\alpha}\vec{\sigma}$ *to the diagonal form. In order to do that we introduce the matrix*

$$S_\alpha = \exp\frac{1}{2}q\vec{\alpha}\vec{\sigma}, \tag{7.12}$$

where parameter q *can be defined from*

$$\tanh\alpha q = \frac{\alpha}{\alpha_4},$$

where $\alpha = \sqrt{\vec{\alpha}^2}$. *Let*

$$u_\alpha = S_\alpha\psi_\alpha. \tag{7.13}$$

In this case simple calculation shows that equation (7.11) has the form

$$\frac{du_\alpha}{dz_\alpha} = -i\frac{m}{\sqrt{\alpha_\nu^2}}\beta u_\alpha. \tag{7.14}$$

Since β *is a diagonal matrix the characteristic equations will have the form*

$$\varphi_\alpha^k(u_\alpha) \equiv i\frac{\sqrt{\alpha_\nu^2}}{m}\beta_n^k \ln(u_\alpha^n) = b_\alpha^k z_\alpha + c_\alpha^k, \tag{7.15}$$

where $b_\alpha^k = 1, (k = 1, ..., 4)$, c_α^k *are constants of integration.*

From (7.14) we easily find the conjugate (in a sense of section 5)

$$\frac{du_\alpha^+}{dz_\alpha} = -i\frac{m}{\sqrt{\alpha_\nu^2}}u_\alpha^+\beta. \tag{7.16}$$

The Dirac conjugate spinor is $\tilde{u}_\alpha = u_\alpha^+\beta$. *Multiplying the right hand side of (7.16) by* β *we find the characteristic functions*

$$\varphi_{\alpha k}^+(\tilde{u}_\alpha) \equiv i\frac{\sqrt{\alpha_\nu^2}}{m}\ln(\tilde{u}_{\alpha n})\beta_k^n = b_{\alpha k}z_\alpha + \tilde{c}_{\alpha k}, \tag{7.17}$$

where $b_{\alpha k} = \mathring{g}_{kn}b_\alpha^n$.

Let us introduce diagonal matrices μ_k. *Elements* $[\mu_k]_j^i$ *are identities when* $i = j = k$ *and zeros everywhere else. Obviously* μ_k *commute with* β. *Taking into account* $S_\alpha\beta = \beta S_\alpha^{-1}$, *from (7.13) it follows that* $\tilde{u}_\alpha = \tilde{\psi}_\alpha S_\alpha^{-1}$. *Substituting (7.15) and (7.17) into (7.8) we obtain a metric generated by the Dirac equation*

$$ds^2 = \mathring{g}_{\nu\tau}dx^\nu dx^\tau - 2\sum_{\alpha\in\Omega}\frac{H_\alpha\alpha_\nu^2}{m^2}g_\alpha^k(\tilde{\psi}_\alpha, \psi_\alpha)d\tilde{\psi}_\alpha S_\alpha^{-1}\mu_k S_\alpha d\psi_\alpha, \tag{7.18}$$

where $g_\alpha^k = 1/\tilde{\psi}_\alpha S_\alpha^{-1}\mu_k S_\alpha\psi_\alpha$ *and* H_α *are some constants of dimension of square of length. We assume here that* $\mathring{g}_{\nu\tau}dx^\nu dx^\tau$ *is Minkowski metric.*◄

3. Let us return to metric (7.4) of space $T \times \Gamma^N \times \Gamma^{+N}$, where T is one-dimensional space with coordinate t. It is easy to show that the curvature tensor of space $T \times \Gamma^N \times \Gamma^{+N}$ is identically zero. However the same result immediately follows from (7.3) because of the existence of transformation (7.1). Clearly same is true for space $\Gamma^{N_0} \times P(\Omega, \Gamma^N_\alpha, \Gamma^{+N}_\alpha)$.

Thus autonomous equations lead to the metric geometry where the space is of zero curvature.

8 Symmetry and algebra of ordinary differential equations

One can easily see that the group methods considered in section 6 do not provide a way of determining the algebraic structure of the space of solutions of differential equations. But this does not mean that if we introduce some algebraic structure in the space of solutions, group (6.1) remains indifferent. To investigate this further we consider again autonomous systems.

1. We will conduct our study in a leaf J^m where map $\exp\varphi : J \to W$ is one to one.

Let us introduce the continuous group in infinitesimal form

$$\begin{aligned} \bar{t} &= t + \lambda(t, u), \\ \bar{u}^k &= u^k + \Lambda^k(t, u) \end{aligned} \tag{8.1}$$

From (6.3) we easily find the invariance condition for (1.1) under the action of (8.1)

$$\Lambda^i \frac{\partial F^k}{\partial u^i} - F^i \frac{\partial \Lambda^k}{\partial u^i} = \frac{\partial \Lambda^k}{\partial t} - \frac{\partial \lambda}{\partial t} F^k - F^i \frac{\partial \lambda}{\partial u^i} F^k. \tag{8.2}$$

2. If we differentiate $\Lambda^i \frac{\partial \varphi^k}{\partial u^i}$ with respect to t, we get

$$\frac{d}{dt}(\Lambda^i \frac{\partial \varphi^k}{\partial u^i}) = \frac{\partial \Lambda^i}{\partial t}\frac{\partial \varphi^k}{\partial u^i} + F^j \frac{\partial \Lambda^i}{\partial u^j}\frac{\partial \varphi^k}{\partial u^i} + \Lambda^i F^j \frac{\partial^2 \varphi^k}{\partial u^i \partial u^j}.$$

Obviously

$$\Lambda^i F^j \frac{\partial^2 \varphi^k}{\partial u^i \partial u^j} = \Lambda^i \frac{\partial}{\partial u^i}(F^j \frac{\partial \varphi^k}{\partial u^j}) - \Lambda^i \frac{\partial F^j}{\partial u^i}\frac{\partial \varphi^k}{\partial u^j}.$$

Using (8.2) and (2.2) we have

$$\frac{d}{dt}(\Lambda^i \frac{\partial \varphi^k}{\partial u^i}) = \frac{d\lambda}{dt} b^k.$$

The solution to this equation is

$$\Lambda^i \frac{\partial \varphi^k}{\partial u^i} = \lambda b^k + \eta^k, \tag{8.3}$$

where η^k satisfy $\frac{d\eta^k}{dt} = 0$. Since $\frac{d}{dt} = \frac{\partial}{\partial t} + F^i(u)\frac{\partial}{\partial u^i}$, we can represent η^k along the solutions of (1.1) as

$$\eta^k = \theta^k \left(bt - \varphi(u)\right),$$

where θ^k are arbitrary functions of their arguments. Solving (8.3) for Λ^k we obtain

$$\Lambda^k = \lambda \left[\left(\frac{\partial\varphi}{\partial u}\right)^{-1}\right]_n^k b^n + \left[\left(\frac{\partial\varphi}{\partial u}\right)^{-1}\right]_n^k \theta^n(bt - \varphi(u)),$$

where $\left[(\frac{\partial\varphi}{\partial u})^{-1}\right]_n^k$ is an element of matrix $(\frac{\partial\varphi}{\partial u})^{-1}$. Assuming (2.3) the transformation (8.1) will have form

$$\begin{aligned} \bar{t} &= t + \lambda(t, u), \\ \bar{u}^k &= u^k + \lambda(t, u)F^k(u) + \left[\left(\frac{\partial\varphi}{\partial u}\right)^{-1}\right]_n^k \theta^n(bt - \varphi(u)). \end{aligned} \tag{8.4}$$

This is the most general infinitesimal transformation leaving (1.1) invariant. Note: the second additive term in (8.4) corresponds to the subgroup (6.7).

3. Let us go back to equation (1.3) from which we define binary operations in J. Using the tools developed in section 6 and relation (8.2) we can write the condition of invariance of equation (1.3) under the action of group (8.1) as follows

$$A_1^i \frac{\partial\Phi^k}{\partial u_1^i} + A_2^i \frac{\partial\Phi^k}{\partial u_2^i} = A_0^k, \tag{8.5}$$

where

$$A_\rho^k = \frac{\partial\Lambda^k(t, u_\rho)}{\partial t} - \left[\frac{\partial\lambda(t, u_\rho)}{\partial t} + F^i(u_\rho)\frac{\partial\lambda(t, u_\rho)}{\partial u_\rho^i}\right] F^k(u_\rho), \tag{8.6}$$

$\rho = 0, 1, 2$. Here $u_0^k = \Phi^k$. From (1.3) we express $F^k(\Phi)$ and substitute into the right hand side of (8.5). After grouping and taking into account (8.4) we obtain

$$\left[\frac{\partial\tilde{\Lambda}^i(t, u_1)}{\partial t} + \left(F^j(\Phi)\frac{\partial\lambda(t, \Phi)}{\partial\Phi^j} - F^j(u_1)\frac{\partial\lambda(t, u_1)}{\partial u_1^j}\right) F^i(u_1)\right] \frac{\partial\Phi^k}{\partial u_1^i} + \tag{8.7}$$

$$+ \left[\frac{\partial\tilde{\Lambda}^i(t, u_2)}{\partial t} + \left(F^j(\Phi)\frac{\partial\lambda(t, \Phi)}{\partial\Phi^j} - F^j(u_2)\frac{\partial\lambda(t, u_2)}{\partial u_2^j}\right) F^i(u_2)\right] \frac{\partial\Phi^k}{\partial u_2^i} \tag{3}$$

$$= \frac{\partial\tilde{\Lambda}^k(t, \Phi)}{\partial t}, \tag{4}$$

where

$$\tilde{\Lambda}^k(t,u) = \left[\left(\frac{\partial \varphi}{\partial u} \right)^{-1} \right]_n^k \theta^n \left(bt - \varphi(u) \right). \tag{8.8}$$

From the group properties of differential equations variables u_1, u_2, Φ and some partial derivatives of function Φ on manifold (1.3) would be independent. Taking into account that and the fact that (8.7) must hold identically on manifold (1.3), we conclude

$$\lambda = \lambda(t), \tilde{\Lambda}^k = \tilde{\Lambda}^k(u). \tag{8.9}$$

From (8.8-9) we conclude that the functions θ^k must be solutions of equations $F^i(u)\frac{\partial \theta^k}{\partial u^i} = 0$. Using (3.1) we can write

$$\tilde{\Lambda}^k(u) = \left[\left(\frac{\partial \varphi}{\partial u} \right)^{-1} \right]_n^k \theta^n \left(\varphi(u) - \tilde{b}\varphi^{k_0}(u) \right), \tag{8.10}$$

where θ^k are arbitrary functions of their arguments. Substituting (8.9-10) into (8.4) we obtain

$$\begin{aligned} \bar{t} &= t + \lambda(t), \qquad (8.11)\\ \bar{u}^k &= u^k + \lambda(t)F^k(u) + \left[\left(\frac{\partial \varphi}{\partial u} \right)^{-1} \right]_n^k \theta^n \left(\varphi(u) - \tilde{b}\varphi^{k_0}(u) \right). \end{aligned}$$

Thus the requirement of invariance of (1.3) narrows the group (8.4) down to (8.11).

4. Suppose that when (8.11) acts, the characteristic functions φ^k and components of vector b^k are transformed as follows

$$\bar{\varphi}^k = \varphi^k + Q^k(u,\varphi), \bar{b}^k = b^k + B^k, \tag{8.12}$$

where B^k are constants to be determined.

Using (8.1), (8.2) and (6.1), the invariance condition (6.3) of equations (2.2) leads to the equality

$$\left(\Lambda^j \frac{\partial F^i}{\partial u^j} - F^j \frac{\partial \Lambda^i}{\partial u^j} \right) \frac{\partial \varphi^k}{\partial u^i} + F^i \frac{\partial Q^k}{\partial u^i} + \frac{\partial Q^k}{\partial \varphi^i} b^i - B^k = 0.$$

From (8.2) and (8.11) it follows that

$$\Lambda^i \frac{\partial F^k}{\partial u^i} - F^i \frac{\partial \Lambda^k}{\partial u^i} = 0. \tag{8.13}$$

Then the determining equation for group (8.12) is

$$F^i \frac{\partial Q^k}{\partial u^i} + \frac{\partial Q^k}{\partial \varphi^i} b^i - B^k = 0. \tag{8.14}$$

This equation does not contain parameters $\lambda(t)$ and θ^k of group (8.11). This means groups (8.12) and (8.11) do not intersect. In other words φ^k and b^k are invariant under the action of group (8.11) while t and u^k are invariant under the action of (8.12). This result is consistent with subsection 2.2.

5. Using (8.12), from (7.1) we determine the transformation rule of r^k :

$$\bar{r}^k = r^k + Q^k(u,r). \tag{8.15}$$

Analogously we can write out the transformation for r^+_k.

We demand the metric (7.3) to be invariant under transformations $\bar{t} = t+\lambda(t)$ and (8.15). After short calculations we can see that

$$\begin{aligned} Q^k &= \sigma^k_n r^n + \xi^k, \\ \bar{t} &= t + \xi^0, \end{aligned} \tag{8.16}$$

where ξ^0, ξ^k, σ^k_n are constants. Besides, matrix σ is anti-Hermitian:

$$(\sigma)^+ = -\sigma. \tag{8.17}$$

From (8.14) and (8.16) it immediately follows that

$$B^k = \sigma^k_n b^n. \tag{8.18}$$

Because of invariance of (1.1), (1.3), (2.2) and metric (7.3), infinitesimal transformations (8.1) and (8.12) will finally have form

$$\begin{aligned} \bar{t} &= t + \xi^0, \\ \bar{u}^k &= u^k + \xi^0 F^k(u) + \left[\left(\frac{\partial \varphi}{\partial u}\right)^{-1}\right]^k_n \theta^n(\varphi(u) - \tilde{b}\varphi^{k_0}(u)), \end{aligned} \tag{8.19}$$

and

$$\begin{aligned} \bar{b}^k &= b^k + \sigma^k_n b^n, \\ \bar{\varphi}^k &= \varphi^k + \sigma^k_n \varphi^n + \xi^k. \end{aligned} \tag{8.20}$$

Clearly (8.20) is the accompanying group which we introduced in Section 2. It follows from (8.17) that (8.20) represents a unitary group.

6. In subsection 3.3 we have considered the following equalities

$$w_1 = \exp\varphi(u_1), w_2 = \exp\varphi(u_2),\ m = \exp\varphi(\Phi). \tag{8.21}$$

For these it immediately follows that w_1, w_2 and m are invariant under the action of group (8.19). On the other side, (8.19) is the transformation rule for u_1, u_2 and Φ, the solution of (1.3). But this means that $u_1 \overset{*}{*} u_2$ in (3.30) and $u_1 \ddot{*} u_2$ in (3.32) are transformed via same rule as u_1 and u_2. Analogously, from (8.21) it follows that $w_1 \overset{*}{*} w_2$ in (3.15) and $w_1 \ddot{*} w_2$ in (3.17) are transformed via same rule as w_1 and w_2. Therefore we have shown that the action of groups (8.19-8.20) is compatible with the binary operations.

7. Let e and h be the identity elements of binary operations (3.30) and (3.32). Also suppose that e and h have finite coordinates. Then as we have already seen, $F(e) = 0$, $F(h) = 0$. Since $F(e) = \left(\frac{\partial \varphi}{\partial u}\right)^{-1} b|_{u=e} = 0$, we conclude that

$$\bar{e} = e. \tag{8.22}$$

Same way we obtain

$$\bar{h} = h. \tag{8.23}$$

This implies that if binary operations (3.30) and (3.32) form groups in the space J with the neutral elements e and h then these groups are preserved under transformation (8.19-20).

8. Let us solve (3.10) with respect to $\tilde{c}$ and write

$$\tilde{c}^k = \tilde{c}_1^k + \theta^k(\tilde{c}) + \xi^k. \tag{8.24}$$

where θ^k are arbitrary functions of their arguments and ξ^k are constants. Substituting (3.1) into (8.24) and renaming $\Phi^k = \bar{u}^k$, $u_1^k = u^k$, we have

$$\varphi^k(\bar{u}) = \varphi^k(u) + \theta^k(\varphi(u) - \tilde{b}\varphi^{k_0}(u)) + \xi^k, \tag{8.25}$$

and we immediately have the following

$$\bar{u}^k = \left(\varphi^{-1}\right)^k \left[\varphi(u) + \theta(\varphi(u) - \tilde{b}\varphi^{k_0}(u)) + \xi\right]. \tag{8.26}$$

Suppose now that θ^k and ξ^k are infinitesimal values. Moreover suppose $\xi^k = b^k\xi^0$. Let us expand right hand side of (8.26) in powers of $\theta + b\xi^0$ and disregard quadratic terms. Simple calculations lead to transformation (8.19). Then (8.19) is a collection of unary operations. In particular we conclude that the translation group $t \to t + \xi^0$ calls for unary operations in space J.

We have shown that the largest continuous group consistent with the binary operations and metric (7.3) has the form (8.26).

9. One can interpret the solutions of (1.13) as a collection of one-dimensional representations of translational group $\bar{t} = t + \xi^0$ acting on one-dimensional real space T, with $t \in T$ [7-9]. Then the appearance of spaces W and W^+ is connected with existence of irreducible representations of discrete group of reflections $t \to -t$. This can explain the simultaneous presence of binary and conjugate binary operations in the space of solutions of differential equations.

9 Symmetry of Equations of plane waves

1. Let $\varphi_\alpha(u_\alpha)$ be the characteristic functions of equation (4.3):

$$\varphi_\alpha^k(u_\alpha) = b_\alpha^k z_\alpha + c_\alpha^k. \tag{9.1}$$

Like in (7.1-2) let us introduce the variables

$$r^k_\alpha = \varphi^k_\alpha(u_\alpha), r^+_{\alpha k} = \varphi^+_{\alpha k}(u^+_\alpha). \tag{9.2}$$

Then metric (7.7) generated by equation (4.1) can be represented in the form

$$ds^2 = \sum_{\alpha\in\Omega}(\overset{\circ}{g}_\alpha dz^2_\alpha + dr^+_{\alpha k} dr^k_\alpha). \tag{9.3}$$

We will regard z_α, r^k_α, $r^+_{\alpha k}$ as independent values for distinct $\alpha \in \Omega$. Then the group which leaves invariant the metric (9.3) will have the following (infinitesimal) form:

$$\begin{aligned} \bar{z}_\alpha &= z_\alpha + \sum_{\beta\in\Omega} \tau_{\alpha\beta} z_\beta + \xi_\alpha, \\ \bar{r}^k_\alpha &= r^k_\alpha + \sum_{\beta\in\Omega} [\sigma_{\alpha\beta}]^k_n r^n_\beta + \xi^k_\alpha, \end{aligned} \tag{9.4}$$

where ξ_α and ξ^k_α are the group parameters. Obviously $\tau_{\alpha\beta}$ depend on $\overset{\circ}{g}_\alpha$. Matrices τ and σ are anti-symmetric and anti-Hermitian respectively.

It would be natural to demand that (9.1) under the action of group (9.4) (taking into account (9.2)) would transform into the equality

$$\bar{r}^k_\alpha = \bar{b}^k_\alpha \bar{z}_\alpha + \bar{c}^k_\alpha.$$

Let us substitute (9.4) into this equality. Using (9.1-2) and the infinitesimality of transformation (9.4), we easily find

$$\bar{b}^k_\alpha z_\alpha = b^k_\alpha z_\alpha + \sum_{\beta\in\Omega} \left([\sigma_{\alpha\beta}]^k_n b^n_\beta - \tau_{\alpha\beta} b^k_\alpha \right) z_\beta.$$

But since by our assumption z_α and z_β are independent variables, we conclude that this equality will be identically satisfied with respect to z only if (9.4) has the form

$$\begin{aligned} \bar{z}_\alpha &= z_\alpha + \xi_\alpha, \\ \bar{r}^k_\alpha &= r^k_\alpha + [\sigma_\alpha]^k_n r^n_\alpha + \xi^k_\alpha, \end{aligned} \tag{9.5}$$

where matrix σ_α satisfies

$$\left([\sigma_\alpha]^k_n \right)^+ = - [\sigma_\alpha]^k_n . \tag{9.6}$$

From $\bar{r}^k_\alpha = \bar{b}^k_\alpha \bar{z}_\alpha + \bar{c}^k_\alpha$ and (9.5) it is easy to find

$$\bar{b}^k_\alpha = b^k_\alpha + [\sigma_\alpha]^k_n b^n_\alpha. \tag{9.7}$$

Comparing (9.5-7) to (8.20) we can see that (9.5-7) is the accompanying group of equations (4.3), which leaves invariant metric $ds^2_\alpha = g_\alpha dz^2_\alpha + dr^+_{\alpha k} dr^k_\alpha$. Thus we have shown that geometrically the differential equation narrows group (9.4) down to group (9.5).

2. From the equality $z_\alpha = \alpha_\nu x^\nu$ and from (9.5) it follows that x^ν must be transformed linearly. If we assume that the group parameters depend on α we will have the transformation $\bar{x}^\nu = x^\nu + [\xi_\alpha]^\nu_\sigma x^\sigma + \xi^\nu_\alpha$. This leads to the conclusion that x^ν should be equipped with index α and therefore in the theory coordinates x^ν_α arise for every equation (4.3). Then it follows that the space Γ^{N_0} is a trivial fiber bundle with base Ω and fibers $\Gamma^{N_0}_\alpha$. In this case from (4.9) we have $\frac{\partial \chi^n}{\partial x^\nu_\alpha} = \alpha_\nu \frac{\partial \chi^n}{\partial u^i_\alpha} \left[a^{-1}_\alpha\right]^i_j (u_\alpha) F^j(u_\alpha)$, where $\left[a^{-1}_\alpha\right]^i_j (u_\alpha)$ is (i, j) element of matrix a^{-1}_αdepending on u_α. Then (4.10) will have the form $\sum\limits_{\alpha\in\Omega} a^{\nu k}_{\ n}(\chi) \frac{\partial \chi^n}{\partial x^\nu_\alpha} = F^k(\chi)$. This equation is much wider than the initial (4.1). Therefore in the transformation of x^ν group parameters ξ^ν_σ, ξ^ν should be independent of α. Then we can write infinitesimal transformation

$$\bar{x}^\nu = x^\nu + \xi^\nu_\sigma x^\sigma + \xi^\nu. \tag{9.8}$$

We assume that (9.8) leaves invariant (7.8). Then

$$\overset{\circ}{g}_{\nu\tau} \xi^\tau_\sigma + \overset{\circ}{g}_{\sigma\tau} \xi^\tau_\nu = 0. \tag{9.9}$$

Let α_ν is transformed as

$$\bar{\alpha}_\nu = \alpha_\nu + l_\nu(\alpha), \tag{9.10}$$

where $l_\nu(\alpha)$ are unknown functions of α. Substitute (9.8) and (9.10) in $\bar{z}_\alpha = \bar{\alpha}_\nu \bar{x}^\nu$. Using (9.5) we obtain $x^\sigma(l_\sigma(\alpha) + \xi^\nu_\sigma \alpha_\nu) + \xi^\sigma \alpha_\sigma = \xi_\alpha$. Taking into account the independence of x^ν and α_ν we derive $l_\nu = -\xi^\sigma_\nu \alpha_\sigma$ and

$$\xi_\alpha = \alpha_\nu \xi^\nu. \tag{9.11}$$

But then (9.10) will eventually be

$$\bar{\alpha}_\nu = \alpha_\nu - \xi^\sigma_\nu \alpha_\sigma. \tag{9.12}$$

We notice that the rise of groups (9.8-9) and (9.12) is caused by the conditions (4.4) and (4.5) applied to equation (4.1).

3. Let us write the equation of plane waves in the form

$$\frac{du_\alpha}{dz_\alpha} = F_\alpha(u_\alpha), \tag{9.13}$$

where $F_\alpha = a^{-1}_\alpha F$. Suppose now that under the action of groups (9.8-9), (9.12) we have the following

$$\bar{u}^k_{\bar{\alpha}} = u^k_\alpha + \Lambda^k_\alpha(u_\alpha). \tag{9.14}$$

Also, the invariance condition (6.3) of equation (9.13) leads to

$$F^i_\alpha \frac{\partial \Lambda^k_\alpha}{\partial u^i_\alpha} - \Lambda^i_\alpha \frac{\partial F^k_\alpha}{\partial u^i_\alpha} = -\xi^\nu_\sigma \alpha_\nu \frac{\partial F^k_\alpha}{\partial \alpha_\sigma}. \tag{9.15}$$

Here we have taken into account the fact that $F^k_\alpha(u_\alpha)$ explicitly depend on α_ν. The logic of the determining equation (1.3) allows us to conclude that α_ν and u_α are independent variables.

As an example we consider equation (7.11) which appeared on the base of Dirac equation (7.10). We are looking for the solution of equation (9.15) in the form

$$\Lambda^k_\alpha = A^k_{\alpha n}\psi^n_\alpha.$$

Simple calculations result in

$$\Lambda^k_\alpha = \frac{1}{8}\xi^\nu_\sigma[\gamma_\nu, \gamma^\sigma]^k_n\psi^n_\alpha, \text{ where } \gamma^4 = \beta, \gamma^1 = \beta\sigma^1, \gamma^2 = \beta\sigma^2, \gamma^3 = \beta\sigma^3.$$

Using (9.14) we obtain well-known transformation of Dirac spinor

$$\bar{\psi}^k_\alpha = \psi^k_\alpha + \frac{1}{8}\xi^\nu_\sigma[\gamma_\nu, \gamma^\sigma]^k_n\psi^n_\alpha.$$

Let us consider expression $\Lambda^i_\alpha \frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha}$. The following take place in the space J_α of the solutions of equation (9.13)

$$\frac{d\varphi^k_\alpha}{dz_\alpha} = b^k_\alpha, \quad \frac{d}{dz_\alpha} = \frac{\partial}{\partial z_\alpha} + F^i_\alpha(u_\alpha)\frac{\partial}{\partial u^i_\alpha}. \tag{9.16}$$

Then one can verify that the following is true

$$\frac{d}{dz_\alpha}\left(\Lambda^i_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha}\right) = \left(F^i_\alpha\frac{\partial\Lambda^j_\alpha}{\partial u^i_\alpha} - \Lambda^i_\alpha\frac{\partial F^j_\alpha}{\partial u^i_\alpha}\right)\frac{\partial\varphi^k_\alpha}{\partial u^j_\alpha} + \Lambda^i_\alpha\frac{\partial}{\partial u^i_\alpha}\left(F^j_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^j_\alpha}\right).$$

Let us transform the right hand side of this equality. Using (9.15) and taking into account that α_ν and u^k_α are independent variables, we derive

$$\frac{d}{dz_\alpha}\left(\Lambda^i_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha}\right) \tag{5}$$

$$= -\xi^\nu_\sigma\alpha_\nu\frac{\partial}{\partial\alpha_\sigma}\left(F^i_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha}\right) + F^i_\alpha\frac{\partial}{\partial u^i_\alpha}\left(\xi^\nu_\sigma\alpha_\nu\frac{\partial\varphi^k_\alpha}{\partial\alpha_\sigma}\right) + \Lambda^i_\alpha\frac{\partial}{\partial u^i_\alpha}\left(F^j_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^j_\alpha}\right). \tag{6}$$

We can divide (9.1) by nonzero b^k_α and obtain the representation of the characteristic functions φ^k_α with b^k_α either 1 or 0. But then taking (9.16) into account we derive the final form of (9.17)

$$\frac{d}{dz_\alpha}\left(\Lambda^i_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha} - \xi^\nu_\sigma\alpha_\nu\frac{\partial\varphi^k_\alpha}{\partial\alpha_\sigma}\right) = 0. \tag{9.18}$$

Without the repetition of the argument presented in section 8, we note that the general solution of (9.18) on J_α has form

$$\Lambda^i_\alpha\frac{\partial\varphi^k_\alpha}{\partial u^i_\alpha} - \xi^\nu_\sigma\alpha_\nu\frac{\partial\varphi^k_\alpha}{\partial\alpha_\sigma} = \theta^k_\alpha\left(\varphi_\alpha(u_\alpha) - \tilde{b}_\alpha\varphi^{k_0}_\alpha(u_\alpha)\right),$$

where θ_α^k are arbitrary functions of their arguments. Solving this equality for Λ_α^k, we obtain

$$\Lambda_\alpha^k = \left[\left(\frac{\partial\varphi_\alpha}{\partial u_\alpha}\right)^{-1}\right]_n^k \left[\xi_\sigma^\nu \alpha_\nu \frac{\partial\varphi_\alpha^k}{\partial\alpha_\sigma} + \theta_\alpha^k \left(\varphi_\alpha - \tilde{b}_\alpha \varphi_\alpha^{k_0}\right)\right]. \tag{9.19}$$

Using equality $F_\alpha^k = \left[\left(\frac{\partial\varphi_\alpha}{\partial u_\alpha}\right)^{-1}\right]_n^k b_\alpha^n$, direct substitution gives that (9.19) identically satisfies (9.15).

Functions θ_α^k in (9.19) are arbitrary and hence one can always express constants $\xi^\nu \alpha_\nu b_\alpha^k$ from (9.19). Let us substitute (9.19) into (9.14). Finally we obtain the infinitesimal transformation

$$\begin{aligned} \bar{\alpha}_\nu &= \alpha_\nu - \xi_\nu^\sigma \alpha_\sigma, \\ \bar{z}_\alpha &= z_\alpha + \xi^\nu \alpha_\nu, \\ \bar{u}_{\bar{\alpha}}^k &= u_\alpha^k + \left[\left(\frac{\partial\varphi_\alpha}{\partial u_\alpha}\right)^{-1}\right]_n^k \left[\xi^\nu \alpha_\nu b_\alpha^n + \xi_\sigma^\nu \alpha_\nu \frac{\partial\varphi_\alpha^n}{\partial\alpha_\sigma} + \theta_\alpha^n \left(\varphi_\alpha - \tilde{b}_\alpha \varphi_\alpha^{k_0}\right)\right]. \end{aligned} \tag{9.20}$$

It is easy to check that group (9.20) which identically satisfies (9.15), leaves invariant equation (1.3) (if we assign indices α to the functions in (1.3)).

The action of (9.12) gives $\bar{F}_{\bar{\alpha}}^k = F_\alpha^k - \xi_\sigma^\nu \alpha_\nu \frac{\partial F_\alpha^k}{\partial\alpha_\sigma}$. But then the characteristic functions φ_α are invariant under the action of (9.20) which is consistent with Section 8.

On the basis of subsection 8.7 we come to the conclusion that the neutral elements e and h do not change under the action of (9.20)

$$\bar{e} = e, \quad \bar{h} = h.$$

As we expected the algebraic structure of trivial fiber bundle $P(\Omega, J_\alpha)$ under the action of group (9.20) does not change.

10 Symmetry of field equations

We want to investigate the group properties of differential equation (4.1). In the previous section we have seen that the rotation group (9.8) (with parameters ξ_σ^ν) and the translation group (with parameters ξ^ν) act in space Γ^{N_0}. Let us study the correlation of these groups with the algebraic structure of space J of solutions.

1. Let

$$\bar{x}^\nu = x^\nu + \xi^\nu. \tag{10.1}$$

Then $\bar{\alpha}^\nu = \alpha^\nu$, and phase variable z_α is transformed as follows

$$\bar{z}_\alpha = z_\alpha + \xi^\nu \alpha_\nu, \tag{10.2}$$

as it follows from (9.5) and (9.11).

Let us consider the group which leaves invariant (4.3) and preserves the algebraic structure of the solution space J_α of equations (4.3). As we have shown in section 8, such a group has the following infinitesimal form

$$\begin{aligned}
&\bar{z}_\alpha = z_\alpha + \xi^\nu \alpha_\nu, \\
&\bar{u}^k_\alpha = u^k_\alpha + \xi^\nu \alpha_\nu F^k_\alpha(u_\alpha) + \left[\left(\frac{\partial \varphi_\alpha}{\partial u_\alpha} \right)^{-1} \right]^k_n \theta^n_\alpha \left(\varphi(u_\alpha) - \tilde{b}_\alpha \varphi^{k_0}_\alpha(u_\alpha) \right),
\end{aligned} \tag{10.3}$$

where

$$F^k_\alpha(u_\alpha) = \left[(a_\alpha)^{-1} \right]^k_n (u_\alpha) F^n(u_\alpha).$$

Thus we have shown that (10.3) is contained in (9.20).

Besides (10.3) there is also the accompanying group

$$\begin{aligned}
\bar{b}^k_\alpha &= b^k_\alpha + [\sigma_\alpha]^k_n b^n, \\
\bar{\varphi}^k_\alpha &= \varphi^k_\alpha + [\sigma_\alpha]^k_n \varphi^n_\alpha + \xi^n_\alpha,
\end{aligned} \tag{10.4}$$

where matrix σ_α satisfies conditions (9.6).

Let us now study equation (4.10) under the action of group (10.3). We seek for transformation of function χ in (4.10) in the class of the following transformations

$$\bar{\chi}^k = \chi^k + Q^k(..., u_\alpha, ...; \chi), \tag{10.5}$$

where Q^k are symmetric functions of u_α. It follows from section 6 that in general case Q must also be a function of all derivatives of χ with respect to u_α. However we restrict ourselves with transformation (10.5) in order to understand better the group properties of algebraic structure of solution space of the initial differential equation (4.1).

Since u_α in (4.10) play the role of independent variables, from (6.1) we have

$$\frac{\partial \bar{\chi}^k}{\partial \bar{u}^n_\alpha} = \frac{\partial \chi^k}{\partial u^n_\alpha} + \frac{\partial Q^k}{\partial u^n_\alpha} + \frac{\partial Q^k}{\partial \chi^l} \frac{\partial \chi^l}{\partial u^n_\alpha} - \frac{\partial \Lambda^l_\alpha}{\partial u^n_\alpha} \frac{\partial \chi^k}{\partial u^l_\alpha}, \tag{10.6}$$

where Λ^k_α is present in (10.3) and has the form

$$\Lambda^k_\alpha = \xi^\nu \alpha_\nu F_\alpha(u_\alpha) + \left[\left(\frac{\partial \varphi_\alpha}{\partial u_\alpha} \right)^{-1} \right]^k_n \theta^n_\alpha \left(\varphi_\alpha(u_\alpha) - \tilde{b}_\alpha \varphi^{k_0}_\alpha(u_\alpha) \right). \tag{10.7}$$

From (8.2) and (10.3) we can find that

$$\Lambda^i_\alpha \frac{\partial F^k_\alpha}{\partial u^i_\alpha} - F^i_\alpha \frac{\partial \Lambda^k_\alpha}{\partial u^i_\alpha} = 0. \tag{10.8}$$

After simple calculations, using (10.6) and (10.8), the invariance condition (6.3) for equation (4.10) becomes

$$\sum_{\alpha\in\Omega}\left[\begin{array}{c} Q^j \frac{\partial[a_\alpha]_l^k(\chi)}{\partial\chi^j}\frac{\partial\chi^l}{\partial u_\alpha^i}F_\alpha^i(u_\alpha)+[a_\alpha]_l^k(\chi)\frac{\partial Q^l}{\partial u_\alpha^i}F_\alpha^i(u_\alpha)+ \\ [a_\alpha]_l^k(\chi)\frac{\partial Q^l}{\partial\chi^n}\frac{\partial\chi^n}{\partial u_\alpha^i}F_\alpha^i(u_\alpha) \end{array}\right]=Q^l\frac{\partial F^k(\chi)}{\partial\chi^l}. \tag{7}$$

This equation is defined on the manifold (4.10).

We see that equation (10.9) does not contain the parameters of group (10.3). But this means that the action of group (10.3) does not depend on group (10.5). Obviously (10.9) contains the trivial solution

$$Q^k=0. \tag{10.10}$$

In this case the arguments of function χ are transformed under the action of (10.3), but the form of the function does not change. On this stage we will consider only cases (10.10) and (10.3).

In subsection 4.2 we have pointed out that the identity elements e and h do not depend on α. But then from (8.22-23) we can see that these elements are invariant with respect to group (10.3). Hence the following is true

$$\Lambda_\alpha^k(e)=0,\Lambda_\alpha^k(h)=0. \tag{10.11}$$

We show now that conditions (4.13-15) are preserved with transformation (10.3). Indeed, it follows from (10.11) that

$$\chi(...,\bar{e},...,\bar{u}_\alpha,...,\bar{e},...)=\chi(...,e,...,\bar{u}_\alpha,...,e,...)=\bar{u}_\alpha.$$

Using (4.19) and (10.11) we easily find that

$$\begin{aligned}\chi^k(...,\bar{u}_\beta,...,\bar{h},...,\bar{u}_\gamma,...)&=\chi^k(...,u_\beta+\Lambda_\beta,...,h,...,u_\gamma+\Lambda_\gamma,...)\\ &=\chi^k(...,u_\beta,...,h,...,u_\gamma,...)+\sum_{\alpha\in\Omega\backslash\alpha_0}\frac{\partial\chi^k}{\partial u_\alpha^i}|_{u_{\alpha_0}=h}\Lambda^i(u_\alpha)=h.\end{aligned}$$

Binary operations that could exist in the space of solutions J of the equation (4.1) are defined in the form (4.39). We use the results of section 8 and take into account (10.10) to write the following formula

$$\begin{aligned}\overline{u_1\dot{*}u_2}&=\chi(...,\overline{u_{\alpha_1}\dot{*}(\alpha)u_{\alpha_2}},...)\\ &=\chi(...,\bar{u}_{\alpha_1}\dot{*}(\alpha)\bar{u}_{\alpha_2},...)=\chi(...,\bar{u}_{\alpha_1},...)\dot{*}\chi(...,\bar{u}_{\alpha_2},...)=\bar{u}_1\dot{*}\bar{u}_2.\end{aligned}$$

One can easily show that if (4.39) defines a group with the identity elements e and h then $\bar{u}\dot{*}\bar{e}=\bar{u}\dot{*}e=\bar{u}=\overline{u\dot{*}e}$, $\bar{u}\dot{*}h=h=\bar{h}=\overline{u\dot{*}h}$.

By the same argument for the conjugate binary operation (4.40) we see that (10.3) also preserves it: $\overline{u_1\ddot{*}u_2}=\bar{u}_1\ddot{*}\bar{u}_2$.

2. Let us consider equation (4.33) for $N=1$. In order for (4.33) to be invariant under (9.8), we introduce the transformation $\overline{u}=u$, $\overline{a}^\nu=a^\nu+\xi_\sigma^\nu a^\sigma$.

Let us go back to system (4.16). It represents N first order linear equations which can be written in the form (4.33). But then the action of the group (9.8) leads to the transformation of the form

$$\begin{gathered}\bar{v}^k = v^k, \overline{\overset{\circ}{b}}_n = \overset{\circ}{b}{}^k_n, \\ \overline{\overset{\circ}{a}}{}^{\nu k}_{\ \ n} = \overset{\circ}{a}{}^{\nu k}_{\ \ n} + \xi^{\nu}_{\sigma} \overset{\circ}{a}{}^{\sigma k}_{\ \ n},\end{gathered} \tag{10.12}$$

where $\overset{\circ}{a}{}^{\nu}$, $\overset{\circ}{b}$ are diagonal matrices presented in (4.16). From the obvious equality

$$\overline{\overset{\circ}{a}}_{\alpha} = \overset{\circ}{a}_{\alpha},$$

where $\overset{\circ}{a}_{\alpha} = \alpha_{\nu} \overset{\circ}{a}{}^{\nu}$, it follows that transformation (10.12) preserves the algebraic structure of equation (4.16). Indeed, the equality $v^k_{\alpha} = \exp \beta_{\alpha n}^{\ \ k} \varphi^n_{\alpha}(u_{\alpha})$ from subsection 4.8 points out that the characteristic functions φ^k_{α} must be invariant under the action of (9.8). But then, taking into account (4.8), we conclude that the binary operations (3.15) and (3.17) do not change under transformation (9.8).

3. Let us investigate equation (4.10) under the action of groups (9.8) and (9.20). We will be seeking the transformation of functions χ in (4.10) in the class

$$\bar{\chi}^k = \chi^k + Q^k(..., u_{\alpha}, ...; \chi), \tag{10.13}$$

where Q^k are assumed to be symmetric functions of their arguments u_{α}. Taking into account (10.6) and (9.5), the condition of invariance (6.3) of equation (4.10) takes form

$$\sum_{\alpha \in \Omega} \left[\begin{array}{c} -\xi^{\nu}_{\sigma} \alpha_{\nu} a^{\sigma k}_{\ \ n}(\chi) \frac{\partial \chi}{\partial u_{\alpha^i}} + Q^l \frac{\partial a_{\alpha n}^{\ \ k}(\chi)}{\partial \chi^l} \frac{\partial \chi^n}{\partial u_{\alpha^i}} + \\ a_{\alpha n}^{\ \ k}(\chi) \frac{\partial Q^n}{\partial u^i_{\alpha}} + a_{\alpha l}^{\ \ k}(\chi) \frac{\partial Q^l}{\partial \chi^n} \frac{\partial \chi^n}{\partial u_{\alpha^i}} \end{array} \right] F^i_{\alpha}(u_{\alpha}) = Q^n \frac{\partial F^k(\chi)}{\partial \chi^n}. \tag{8}$$

Equation (10.14) is defined on manifold (4.10).

It should be mentioned that in (10.14), like in (10.9), functions (9.19) are not explicitly present. This means that the transformation of u_{α} does not affect the form of the function χ^k.

One can easily see that equation (10.14) is symmetric in arguments u_{α}. But since, like in section 4, we build our theory in the class of symmetric functions, it is reasonable to seek the solution of (10.14) in this class. Then (9.20) and (10.13) transform the symmetric function $u = \chi(..., u_{\alpha}, ...)$ into symmetric function $\overline{u} = \overline{\chi}(..., \overline{u}_{\overline{\alpha}}, ...)$. In section 4 we defined a binary operation in the form of (4.39). Then we suppose that $\overline{u} = \overline{\chi}(..., \overline{u}_{\overline{\alpha}}, ...)$ defines a binary operation which can be represented as

$$\bar{u}_1 \dot{*} \bar{u}_2 = \bar{\chi}(..., \bar{u}_{\bar{\alpha}1}, ...) \dot{*} \bar{\chi}(..., \bar{u}_{\bar{\alpha}2}, ...) = \bar{\chi}(..., \bar{u}_{\bar{\alpha}1} \dot{*} u_{\bar{\alpha}2}, ...).$$

In the preceeding subsections we proved that the transformation (9.20) preserves the binary operations, therefore we can write

$$\overline{\chi}(..., \overline{u}_{\overline{\alpha}_1} \dot{*} \overline{u}_{\overline{\alpha}_2}, ...) = \overline{\chi}(..., \overline{u_{\alpha_1} \dot{*} u_{\alpha_2}}, ...) = \overline{u_1 \dot{*} u_2}.$$

Thus

$$\overline{u_1 \dot{*} u_2} = \overline{u}_1 \dot{*} \overline{u}_2. \tag{10.15}$$

Let us continue our argument for the case when (4.37) defines a group in the space J_α. Let the identity elements e and h be transformed into the solutions $\overline{e}$ and $\overline{h}$ of the transformed equation (4.1) under the action of (10.13). From equality $\overline{u}*\overline{e} = \overline{u*e} = \overline{u}$, $\overline{u}*\overline{h} = \overline{u*h} = \overline{h}$ it follows immediately that $\overline{e}$ and $\overline{h}$ must be the identity elements. By assumption, e and h have finite coordinates, consequently $F(e) = 0$, $F(h) = 0$. On the other hand we consider only those $F(u)$ that can only have isolated zeros. The parameters of the infinitesimal group can always be chosen so small that $\overline{e}$ belongs to some neighborhood of e, in which $F(u)$ does not have other zeros than e. Since $\overline{e}$ is the identity element and $F(\overline{e}) = 0$ we conclude $\overline{e} = e$. Analogously, $\overline{h} = h$. So

$$\overline{e} = e, \overline{h} = h \tag{10.16}$$

In order for $\overline{u} = \overline{\chi}(..., \overline{u}_{\overline{\alpha}}, ...)$ to define a group (4.39) in the space of solutions of the transformed equation (4.1) function χ must satisfy conditions analogous to (4.13-14). We have $\overline{e} = e$ under the action of group (4.20). Then from transformation (10.13) we obtain

$$\overline{\chi}(..., e, ..., \overline{u}_{\overline{\alpha}}, ...e, ...) = \chi(..., e, ...u_\alpha, ...e, ...) + Q(..., e, ..., u_\alpha, ..., e, ...; u_\alpha).$$

By analogy with (4.13) the left hand side of the derived equality must equal to $\overline{u}_{\bar{\alpha}} = u_\alpha + \Lambda_\alpha(u_\alpha)$. But then we finally obtain

$$Q^k(..., e, ..., u_\alpha, ..., e, ...; u_\alpha) = \Lambda^k_\alpha(u_\alpha). \tag{10.17}$$

Using (4.14), the same argument leads to

$$Q^k(..., u_\beta, ..., h, ..., u_\gamma, ...; h) = 0. \tag{10.18}$$

In subsection 4.10 we have shown that equality (4.19) follows from (4.14). Let us consider (10.6) and assume that one of the arguments u_{α_0} on the right hand side is h and on the left hand side $\overline{u}_{\overline{\alpha}_0} = h$, where $\overline{\alpha}_{0\nu} = \alpha_{0\nu} - \xi^\sigma_\nu \alpha_{0\sigma}$. By the remark made above the following must be true

$$\frac{\partial \overline{\chi}}{\partial \overline{u}_{\overline{\alpha}}}|_{\overline{u}_{\overline{\alpha}_0} = h, \overline{\alpha} \neq \overline{\alpha}_0} = 0.$$

Then using (4.19), from (10.6) we derive

$$\frac{\partial Q^k}{\partial u^i_\alpha}|_{u_{\alpha_0} = h, \alpha \neq \alpha_0} = 0. \tag{10.19}$$

It is easy to check that (10.17-19) do not contradict to equation (10.14).

We have studied how the group (9.8) acts on the binary operation (4.39). As for the conjugate binary operation (4.40) the determining equation for Q

will still be (10.14) if we look for the transformation of this operation in the frame of (10.13). The conditions (4.21) lead to equalities that are obtained from (10.17-19) through the exchange of the identity elements e and h.

We mentioned that the equation (10.14) does not connect the transformations (9.20) and (10.13) and in principle they can be independent. However the invariance of algebraic structure of space J with respect to the action of groups (9.20) and (10.13) leads to conditions (10.17-19). Obviously these conditions connect the transformations (9.20) and (10.13). Moreover the choice of transformation (10.13) depends on which binary operation we want to transform, (4.39) or (4.40).

A simple example would be equation (4.27). Let the following be the infinitesimal form of the group which leaves invariant the equations $a_\alpha \frac{du_\alpha}{dz_\alpha} = \sin u_\alpha$ of plane waves

$$\overline{u}_\alpha = u_\alpha + \xi_\alpha \sin u_\alpha, \tag{10.20}$$

where ξ_α are group parameters. It is easy to see that the invariant group of equation (4.30) is

$$\overline{\chi} = \chi + \xi \sin \chi, \tag{10.21}$$

with the group parameter ξ.

In section 4.13 it was shown that the identity elements are $e = 2\pi m$, $h = 2\pi m + \pi$, where m is an integer. We have also shown that the function χ satisfies conditions (4.13-14) and (4.19).

From (10.21) it follows that $Q = \xi \sin \chi$. Since function χ of (4.31) satisfies (4.14) and (4.19) then (10.18) and (10.19) are satisfied identically. As for the condition (10.17), it imposes restrictions on the group (10.20) and we have $\xi_\alpha = \xi$.

4. Back to the solution (4.47) of equation (4.10). The action of groups (9.20) and (10.13) gives

$$\sum_{\alpha\in\Omega} \exp\left[\varphi^k_\alpha(u_\alpha + \Lambda_\alpha) - \varphi^k_\alpha(\chi + Q)\right] = 1, (k = 1, ..., N), \tag{10.22}$$

where Λ^k_α is (9.19). Since Λ_α and Q are infinitely small functions, from (10.22) it is easy to write

$$\sum_{\alpha\in\Omega} \exp\left[\varphi^k_\alpha(u_\alpha) - \varphi^k_\alpha(\chi)\right] \left[\frac{\partial\varphi^k_\alpha(u_\alpha)}{\partial u^n}\Lambda^n_\alpha - \frac{\partial\varphi^k_\alpha(\chi)}{\partial u^n}Q^n\right] = 0.$$

Here the summation is assumed along the index n but not k. From this equality it is easy to express Q via Λ_α

$$Q^k = \left[\left(\sum_{\beta\in\Omega} \exp\left[\varphi^k_\beta(u_\beta) - \varphi^k_\beta(\chi)\right] \frac{\partial\varphi_\beta(\chi)}{\partial u}\right)^{-1}\right]^k_n \times \tag{10.23}$$

$$\times \sum_{\alpha\in\Omega} \exp\left[\varphi^n_\alpha(u_\alpha) - \varphi^n_\beta(\chi)\right] \frac{\partial\varphi^n_\alpha(u_\alpha)}{\partial u^i}\Lambda^i_\alpha. \tag{9}$$

An immediate verification shows that (10.23) satisfies (10.17). Let us now substitute (9.19) into (10.23) and finally obtain

$$Q^k = \left[\left(\sum_{\beta\in\Omega} \exp\left[\varphi^k_\beta(u_\beta) - \varphi^k_\beta(\chi)\right] \frac{\partial \varphi_\beta(\chi)}{\partial u} \right)^{-1} \right]^k_n \times \tag{10.24}$$

$$\times \sum_{\alpha\in\Omega} \exp\left[\varphi^n_\alpha(u_\alpha) - \varphi^n_\beta(\chi)\right] \left[\xi^\nu_\sigma \alpha_\nu \frac{\partial \varphi^n_\alpha(u_\alpha)}{\partial \alpha_\sigma} + \theta^n_\alpha\right], \tag{10}$$

where $\theta^n_\alpha = \theta^n_\alpha \left(\varphi_\alpha(u_\alpha) - \widetilde{b}_\alpha \varphi^{k_0}_\alpha(u_\alpha)\right)$.

If we disregard functions θ^n_α then obtain

$$Q^k = \xi^\nu_\sigma T^{k\sigma}_{\ \nu},$$

where $T^{k\sigma}_{\nu}$ can be found from (10.24). Let us construct the tangent vector

$$X_a = \sum_{\alpha\in\Omega} \Lambda^k_{a\alpha} \frac{\partial}{\partial u^k_\alpha} + T^k_a \frac{\partial}{\partial \chi^k}, \tag{10.25}$$

where a is a double index $\binom{\sigma}{\nu}$ and the function θ^n_α is not present in $\Lambda^k_{a\alpha}$. Since (10.25) is the generator of the rotation group, we have the following equality

$$[X_a, X_b] = C^c_{ab} X_c,$$

where C^c_{ab} are the structural constants of the rotation group. Hence we have approached the problem of irreducible representations of the rotation group for nonlinear equations (4.1).

By analogy with (4.47) we can consider the solution (4.49) of equation (4.10). Then instead of the transformation (10.13) we will have

$$\overline{\overset{\wedge}{\chi}{}^k} = \overset{\wedge}{\chi}{}^k + \hat{Q}{}^k,$$

where $\hat{Q}{}^k$ is derived from (10.24) with the simple change in exponents: $\varphi_\alpha(u_\alpha) - \varphi_\alpha(\chi) \to \varphi_\alpha(\overset{\wedge}{\chi}) - \varphi_\alpha(u_\alpha)$.

We have rather extensively studied algebraic properties of the solution space of the differential equations (4.1). The revealed algebro-geometrical properties of these equations can equally be extended to the conjugate space of solutions of equations (5.11).

11 Homomorphic and isomorphic relations

1. Together with (1.1) we consider the system of the same dimension N:

$$\frac{dv^k}{dt} = f^k(v^1, ...v^N), \tag{11.1}$$

where functions $f^k(v)$ (by analogy with $F^k(u)$) are defined and smooth for all points in Γ_ν^N except may be some isolated points. As in the case of (1.1), equation $f(v) = 0$ has isolated solutions. Let the characteristic equations of (11.1) be

$$\psi^k(v) = l^k t + A^k. \tag{11.2}$$

Since both systems (1.1) and (11.1) have the same dimension, there exists such a space W of dimension N that the following maps take place:

$$\exp\varphi : J_u \to W, \quad \exp\psi : J_v \to W, \tag{11.3}$$

where J_u and J_v are the spaces of the solutions of (1.1) and (11.1) respectively. Let us consider the mapping: $J_u \to W \to J_v$. Then we can write $\exp\varphi(u) = \exp\psi(v)$. Taking the principal value of logarithm we obtain

$$\varphi(u) = \psi(v). \tag{11.4}$$

Immediately follows

$$v = \mu(u), \tag{11.5}$$

where $\mu = \psi^{-1} \circ \varphi$. Hence we have obtained the map $\mu : J_u \to J_v$.

As it was shown in section 1, J_u and J_v are fiber spaces. Discrete groups D_u and D_v act in discrete fibers of J_u and J_v respectively. If one can establish homomorphic relation between groups D_u and D_v then the spaces J_u and J_v should be called homomorphic. If D_u and D_v are isomorphic then we call the spaces J_u and J_v isomorphic. Note that these relations are regulated by the map $\mu = \psi^{-1} \circ \varphi$ from where it follows that $J_u \xrightarrow{\varphi} W \xrightarrow{\psi^{-1}} J_v$. Also if the binary operations (3.15) and (3.17) take place in the space W then it is easy to see that

$$\mu(u_1 * u_2) = \mu(u_1) * \mu(u_2). \tag{11.6}$$

If e_u, h_u and e_v, h_v are the identity elements in J_u and J_v respectively then there is the following connection between them

$$e_v = \mu(e_u), h_v = \mu(h_u). \tag{11.7}$$

We conclude that the spaces of solutions of $\dfrac{du}{dt} = u(1-u)$ and $\dfrac{du}{dt} = \sin u$ are homomorphic. The spaces of solutions of $\dfrac{du}{dt} = u$ and $\dfrac{du}{dt} = u(1-u)$ are isomorphic. Finally we note that the conjugate spaces of solutions J and J^+ of equations (1.1) and (5.4) are isomorphic.

2. Let us consider the collection of equations (4.3) with α running through the set Ω. Recall that we assumed that the discrete groups D_α are isomorphic for different α. Then it is easy to show that $J_\alpha, \alpha \in \Omega$ are isomorphic too.

Consider the following equation

$$\tilde{a}^{\nu k}_{n}(v)\frac{\partial v^n}{\partial x^\nu} = \tilde{F}^k(v), \tag{11.8}$$

where $k, n = 1, ..., N; \nu = 1, ..., N_0$. We assume that the summation is performed along the repeating indices. Restrictions imposed on $\tilde{a}^{\nu k}_{n}(v)$ and $\tilde{F}^k(v)$ are the same as those on functions in (4.1). Therefore the following takes place

$$\det\left[\alpha_\nu \tilde{a}^\nu(v)\right] \neq 0, \quad \det \frac{\partial \tilde{F}(v)}{\partial v} \neq 0. \tag{11.9}$$

Suppose that the characteristic functions of plane waves of the equation (11.8) are $\psi_\alpha(v_\alpha)$. Let $\tilde{J}_\alpha$ are the spaces of solutions of equations of plane waves of (11.8). Similar to section 4 we suppose that discrete groups $\tilde{D}_\alpha$ acting in discrete fibers of spaces $\tilde{J}_\alpha$ are isomorphic. Using the argument of the previous subsection we can construct the maps $\mu_\alpha = \psi_\alpha^{-1} \circ \varphi_\alpha : J_\alpha \to \tilde{J}_\alpha$. The spaces J_α and $\tilde{J}_\alpha$ will be homomorphic or isomorphic depending on whether discrete groups D_α and $\tilde{D}_\alpha$ are homomorphic or isomorphic.

Since J_α and $\tilde{J}_\alpha$ are fibers in the spaces of solutions J_u and J_v of equations (4.1) and (11.8) respectively, we can perform fiberwise mapping

$$\mu_\alpha : J_\alpha \to \tilde{J}_\alpha, \tag{11.10}$$

so that the following holds

$$\varphi_\alpha(u_\alpha) = \psi_\alpha(v_\alpha). \tag{11.11}$$

Using (4.47), an arbitrary solution $v \in \tilde{J}_v$ can be presented in the implicit form

$$\sum_{\alpha\in\Omega} \exp\left[\psi^k_\alpha(v_\alpha) - \psi^k_\alpha(v)\right] = 1. \tag{11.12}$$

With (11.4) and the map (11.10) we have $\sum\limits_{\alpha\in\Omega} \exp\left[\varphi^k_\alpha(u_\alpha) - \psi^k_\alpha(v)\right] = 1$. Taking into account (4.8), we finally have

$$\sum_{\alpha\in\Omega} w^k_\alpha \exp\left[-\psi^k_\alpha(v)\right] = 1. \tag{11.13}$$

Let $u \in J_u$ be some solution. Using (4.18) by $\chi = u$ we define the coefficients of expansion $w^k_\alpha(z_\alpha)$. Let us substitute $w^k_\alpha(z_\alpha)$ in (11.13). Solving the algebraic equation (11.13) for v we obtain the rule of mapping of the space J_u into space J_v.

Therefore if the discrete groups D_α and $\tilde{D}_\alpha$ are homomorphic then the spaces J_u and J_v are homomorphic. Also the homomorphic relations are established between the binary operations in the spaces J_u and J_v. Analogous argument is used when D_α and $\tilde{D}_\alpha$ are isomorphic, i.e. J_u and J_v are isomorphic.

3. As we mentioned, the metric (7.8) corresponds to equation (4.1). Let us introduce the fiberwise mapping (11.10). From (11.11) it follows that the second terms of the arising metrics satisfy the following equality:

$$\sum_{\alpha\in\Omega} d\varphi^{+}_{\alpha k} d\varphi^{k}_{\alpha} = \sum_{\alpha\in\Omega} d\psi^{+}_{\alpha k} d\psi^{k}_{\alpha}.$$

As for the first terms $\overset{\circ}{g}_{\nu\tau} dx^{\nu} dx^{\tau}$ and $\widetilde{\overset{\circ}{g}}_{\nu\tau} dx^{\nu} dx^{\tau}$, they can have different signatures. If the signatures are the same then the metric

$$ds^2 = \overset{\circ}{g}_{\nu\tau} dx^{\nu} dx^{\tau} + \sum_{\alpha\in\Omega} dr^{+}_{\alpha k} dr^{k}_{\alpha},$$

with $r^k_\alpha = \varphi^k_\alpha(u_\alpha) = \psi^k_\alpha(v_\alpha)$ is common for both equations (4.1) and (11.8).

4. The form of the function $\varphi^k_\alpha(v_\alpha)$ does not change under the action of group (9.20). Let us consider $\bar v^k_{\bar\alpha} = \mu^k_\alpha(\bar u_{\bar\alpha})$. Since (9.20) is an infinitesimal transformation, we can write

$$\bar v^k_{\bar\alpha} = \mu^k_\alpha(u_\alpha) + \frac{\partial \mu^k_\alpha(u_\alpha)}{\partial u^i_\alpha} \left[\left(\frac{\partial \varphi_\alpha}{\partial u_\alpha} \right)^{-1} \right]^i_n \left[\xi^\nu \alpha_\nu b^n_\alpha + \xi^\nu_\sigma \alpha_\nu \frac{\partial \varphi^n_\alpha}{\partial \alpha_\sigma} + \theta^n_\alpha \right].$$

But on the other hand, taking into account (11.11), we can write

$$\frac{\partial \mu^k_\alpha(u_\alpha)}{\partial u^i_\alpha} = \frac{\partial \psi^{-1}{}^k_\alpha(\varphi_\alpha)}{\partial \varphi^j_\alpha} \frac{\partial \varphi^j_\alpha(u_\alpha)}{\partial u^i_\alpha} = \frac{\partial \psi^{-1}{}^k_\alpha(\psi_\alpha)}{\partial \psi^j_\alpha} \frac{\partial \varphi^j_\alpha(u_\alpha)}{\partial u^i_\alpha},$$

and using

$$\frac{\partial \psi^{-1}{}^k_\alpha}{\partial \psi^j_\alpha} \frac{\partial \psi^j_\alpha}{\partial v^n_\alpha} = \delta^k_n,$$

we derive

$$\bar v^k_{\bar\alpha} = v^k_\alpha + \left[\left(\frac{\partial \psi_\alpha(v_\alpha)}{\partial v_\alpha} \right)^{-1} \right]^k_n \left[\xi^\nu \alpha_\nu b^n_\alpha + \xi^\nu_\sigma \alpha_\nu \frac{\partial \varphi^n_\alpha}{\partial \alpha_\sigma} + \theta^n_\alpha \right]. \tag{11.4}$$

From (9.1), (11.11) and $\psi^k_\alpha(v_\alpha) = l^k_\alpha z_\alpha + A^k_\alpha$ we can conclude that $l^k_\alpha = b^k_\alpha$, $A^k_\alpha = c^k_\alpha$. From these equalities we write

$$\theta^k_\alpha \left(\varphi_\alpha(u_\alpha) - \tilde b_\alpha \varphi^{k_0}_\alpha(u_\alpha) \right) = \theta^k_\alpha \left(\psi_\alpha(v_\alpha) - \tilde b_\alpha \psi^{k_0}_\alpha(v_\alpha) \right).$$

Taking partial derivatives of (11.11) with respect to α_ν (α_ν and u_α in (9.15) are considered to independent variables) we obtain

$$\frac{\partial \varphi^n_\alpha(u_\alpha)}{\partial \alpha_\nu} = \frac{\partial \psi^n_\alpha(v_\alpha)}{\partial \alpha_\nu}.$$

Summarizing this argument, (11.4) will have the form

$$\bar{v}^k_{\bar{\alpha}} = v^k_\alpha + \left[\left(\frac{\partial\psi_\alpha(v_\alpha)}{\partial v_\alpha}\right)^{-1}\right]^k_n \left[\xi^\nu\alpha_\nu b^n_\alpha + \xi^\nu_\sigma\alpha_\nu\frac{\partial\psi^n_\alpha(v_\alpha)}{\partial\alpha_\sigma} + \theta^n_\alpha\right], \qquad (11.15)$$

where $\theta^k_\alpha = \theta^k_\alpha\left(\psi_\alpha(v_\alpha) - \tilde{b}_\alpha\psi^{k_0}_\alpha(v_\alpha)\right)$.

Therefore, the group (9.20) is isomorphically mapped under the fiberwise mapping (11.10) into the group (11.15).

Analogously it can be shown that the accompanying group (10.4) is also isomorphically mapped under the map (11.10-11) into the accompanying group

$$\bar{b}^k_\alpha = b^k_\alpha + \sigma_{\alpha n}{}^k b^n_\alpha,$$
$$\bar{\psi}^k_\alpha = \psi^k_\alpha + \sigma_{\alpha n}{}^k\psi^n_\alpha.$$

Conclusion

It is the nature of differential equations that they bind independent variables (space-time) and field variables. Our goal was to set a path on the way of studying algebro-geometrical patterns of space and field to which the theory of differential equations eventually brings.

We have showed that the superposition principle of quantum mechanics [6,7] is not only the case of linear differential equations. It seems that this principle can be extended to nonlinear equations (at least autonomous). Turns out that the group of reflections plays the fundamental role in algebro-geometric properties of space and field. This group is connected with the appearance of the binary and conjugate binary operations, existence of two mutually conjugate identity elements of the group and also mutually conjugate expansion of the quazilinear equation into plane waves. We should also mention the conjugate equations. The discovery of these objects allowed us to introduce the metric of space and field (space of field variables). It was shown that every equation together with its conjugate has its own metric so that geodesics of this geometry are the solutions of the equations. The curvature of this geometry is zero. The introduction of the metric allows one to revise the groups of symmetries of differential equations. Simultaneously, the existence of the accompanying group has been detected. This group leaves the metrics invariant. The variables of differential equation are indifferent with respect to the accompanying group. It has also been shown that invariance groups of differential equations are the collection of unary operations acting in the space of solutions. The unary operations are the special form of the binary operations. Hence the differential equation generates a single ensemble of interconnected algebro-geometrical objects.

In the construction of algebraic structures in the space of solutions of differential equations the central role is played by the characteristic functions which are solutions to ordinary differential equations. The entire construction of algebro-geometric structure of differential equations is based on the characteristic functions. We showed that for partial differential equations there exists so-called extended Fourier series which is expressed in terms of the characteristic functions of plain waves. For nonlinear differential equations under consideration we also have managed to find the particular form (4.47) and (4.49) of these series.

We have investigated wide range of differential equations. We want to point out that the study of the general differential equations was not our goal. Our approach allowed us simply to detect and magnify essential algebro-geometric properties of phenomena described by differential equations. The investigated class of equations is restricted because of conditions (4.4-5) imposed on (4.1). We can see from (4.16-17) that we have equations of the field with mass. Then the equations of Maxwell and Einstein of gravitational field are left beyond the scope of our study. In particular, if the right hand side of (4.1) is absent then $\det a_\alpha = 0$ is the condition that must be imposed on in order for the equation (4.3) to have a nontrivial solution. This condition imposes a restriction on α_ν which can be interpreted as characteristic directions [2,5]. Hence such a class of equations needs independent study which may expand and complete our view of algebro-geometric properties of the objects associated with space and field.

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