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Infinitely many weak solutions for a fourth-order equation on the whole space

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Abstract. The existence of infinitely many weak solutions for a fourth-order equation on the whole space with a perturbed nonlinear term is investigated. Our approach is based on variational methods and critical point theory.

Keywords: weak solution, fourth-order equation, critical point theory, variational methods.

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1 Introduction

In this paper we consider the following problem

$$u^{iv}(x) - (q(x)u'(x))' + s(x)u(x) = \lambda f(x, u(x)), \quad \text{a.e. } x \in \mathbb{R}, \tag{1.1}$$

where λ is a positive parameter and $q, s \in L^\infty(\mathbb{R})$ with $q_0 = \operatorname{ess\,inf}_{\mathbb{R}} q > 0$ and $s_0 = \operatorname{ess\,inf}_{\mathbb{R}} s > 0$. Here the function $f : \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function.

As we know, differential equations have many applications in engineering and mechanical science. Many important engineering topics eventually lead to a differential equation. One of the most important and widely used types of such equations is the fourth-order differential equation. These equations play an essential role in describing the large number of elastic deflections in beams. Due to the importance of these equations in applied sciences, many authors have studied different types of these equations and obtained important results. Research on the existence and multiplicity of solutions for different types of these equations can be seen in the work of many authors. For example, to study fourth-order two-point boundary value problems we refer the reader to references [3–5, 8, 10–12].

For instance in [3], the authors researched the following problem:

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(t, u), & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.2}$$

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where A and B are real constants and they achieved multiplicity results using variational methods and critical point theory. It should be noted that in the study of many important problems such as mathematical models of beam deflection, the differential equation is considered at infinite interval. Also, because the operators used to solve equations such as (1.1) on $\mathbb{R}$ are not compact, so the study of such problems is very important. That is why some authors have turned their attention to the whole space. For example in [9], applying the critical point theory the author has studied the existence and multiplicity of solutions for the following problem:

$$u^{iv}(x) + Au''(x) + Bu(x) = \lambda\alpha(x).f(u(x)), \quad \text{a.e. } x \in \mathbb{R}, \tag{1.3}$$

where A is a real negative constant and B is a real positive constant, λ is a positive parameter and $\alpha, f : \mathbb{R} \to \mathbb{R}$ are two functions such that $\alpha \in L^1(\mathbb{R})$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$ and also f is continuous and non-negative.

In this work, using a critical point theorem obtained in [2] which we recall in the next section (Theorem 2.7), we establish the existence of infinitely many weak solutions for the problem (1.1).

2 Preliminaries

Let us recall some basic concepts.

Definition 2.1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be an L^1-Carathéodory function, if

(C_1) the function $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$,

(C_2) the function $t \mapsto f(x,t)$ is continuous for almost every $x \in \mathbb{R}$,

(C_3) for every $\rho > 0$ there exists a function $l_\rho(x) \in L^1(\mathbb{R})$ such that

$$\sup_{|t|\leq\rho} |f(x,t)| \leq l_\rho(x),$$

for a.e. $x \in \mathbb{R}$.

Denote $W_0^{2,2}(\mathbb{R})$ is the closure of $C_0^\infty(\mathbb{R})$ in $W^{2,2}(\mathbb{R})$ and according to the properties of the Sobolev spaces, we know that $W_0^{2,2}(\mathbb{R}) = W^{2,2}(\mathbb{R})$, [1, Corollary 3.19].

We denote by $|\cdot|_t$ the usual norm on $L^t(\mathbb{R})$, for all $t \in [1, +\infty]$ and it is well known that $W^{2,2}(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, [6, Corollary 9.13].

The Sobolev space $W^{2,2}(\mathbb{R})$ is equipped with the following norm

$$\|u\|_{W^{2,2}(\mathbb{R})} = \left(\int_{\mathbb{R}} (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2) dx \right)^{1/2},$$

for all $u \in W^{2,2}(\mathbb{R})$. Also, we consider $W^{2,2}(\mathbb{R})$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}} (|u''(x)|^2 + q(x)|u'(x)|^2 + s(x)|u(x)|^2) dx \right)^{1/2},$$

for all $u \in W^{2,2}(\mathbb{R})$. According to

$$(\min\{1, q_0, s_0\})^{\frac{1}{2}} \|u\|_{W^{2,2}(\mathbb{R})} \leq \|u\| \leq (\max\{1, |q|_\infty, |s|_\infty\})^{\frac{1}{2}} \|u\|_{W^{2,2}(\mathbb{R})}, \tag{2.1}$$

the norm $\|\cdot\|$ is equivalent to the $\|\cdot\|_{W^{2,2}(\mathbb{R})}$ norm. Since the embedding $W^{2,2}(\mathbb{R}) \to L^{\infty}(\mathbb{R})$ is continuous hence there exists a constant $C_{q,s}$ (depending on the functions q and s) such that

$$|u|_{\infty} \leq C_{q,s}\|u\|, \qquad \forall u \in W^{2,2}(\mathbb{R}).$$

In the following proposition, we provide an approximation for this constant.

Proposition 2.2. *We have*

$$|u|_{\infty} \leq \ C_{q,s}\|u\| \tag{2.2}$$

where $C_{q,s} = \left(\frac{1}{4|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{\max\{1,|q|_{\infty},|s|_{\infty}\}}{\min\{1,q_0,s_0\}}\right)^{\frac{1}{2}}$.

Proof. Let $v \in W^{1,1}(\mathbb{R})$, then from [7, p. 138, formula 4.64], one has

$$|v(x)| \leq \frac{1}{2}\int_{\mathbb{R}} |v'(t)|dt. \tag{2.3}$$

Now if $u \in W^{2,2}(\mathbb{R})$ then $v(x) = (|q|_{\infty}|s|_{\infty})^{\frac{1}{2}}|u(x)|^2 \in W^{1,1}(\mathbb{R})$ and thus from (2.3) and Hölder's inequality one has,

$$(|q|_{\infty}|s|_{\infty})^{\frac{1}{2}}|u(x)|^2 \leq \int_{\mathbb{R}}(|q|_{\infty}|s|_{\infty})^{\frac{1}{2}}|u'(t)||u(t)|dt \leq ((|q|_{\infty})^{\frac{1}{2}}|u'|_2)(|s|_{\infty}{}^{\frac{1}{2}}|u|_2)$$

that is ,

$$|u(x)| \leq \left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}((|q|_{\infty})^{\frac{1}{2}}|u'|_2)^{\frac{1}{2}}(|s|_{\infty}{}^{\frac{1}{2}}|u|_2)^{\frac{1}{2}}. \tag{2.4}$$

Now according to $x^a y^{1-a} \leq a^a(1-a)^{1-a}(x+y),\ x,y \geq 0, 0 < a < 1$ [7, p. 130, formula 4.47], and classical inequality $a^{\frac{1}{p}} + b^{\frac{1}{p}} \leq 2^{\frac{(p-1)}{p}}(a+b)^{\frac{1}{p}}$, from (2.1) and (2.4) one has

$$\begin{aligned}
|u(x)| &\leq \left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left[\left(\int_{\mathbb{R}}|q|_{\infty}|u'(t)|^2dt\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}}|s|_{\infty}|u(t)|^2dt\right)^{\frac{1}{2}}\right] \\
&\leq \left(\frac{1}{|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}}(2)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(|q|_{\infty}|u'(t)|^2 + |s|_{\infty}|u(t)|^2)dt\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{4|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}}(|u''(t)|^2 + |q|_{\infty}|u'(t)|^2 + |s|_{\infty}|u(t)|^2)dt\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{4|q|_{\infty}|s|_{\infty}}\right)^{\frac{1}{4}}\left(\frac{\max\{1,|q|_{\infty},|s|_{\infty}\}}{\min\{1,q_0,s_0\}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(|u''(t)|^2 + |u'(t)|^2 + |u(t)|^2)dt\right)^{\frac{1}{2}}
\end{aligned}$$

which means that $|u|_{\infty} \leq \ C_{q,s}\|u\|$. □

Let $\Phi,\ \Psi : W^{2,2}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\|u\|^2 = \frac{1}{2}\int_{\mathbb{R}}(|u''(x)|^2 + q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx \tag{2.5}$$

and

$$\Psi(u) = \int_{\mathbb{R}} F(x, u(x))dx \tag{2.6}$$

for every $u \in W^{2,2}(\mathbb{R})$ where $F(x,\xi) = \int_0^{\xi} f(x,t)dt$ for all $(x,\xi) \in \mathbb{R}^2$. It is well known that Ψ is a sequentially weakly upper semicontinuous whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$\Psi'(u)(v) = \int_{\mathbb{R}} f(x, u(x))v(x)dx.$$

It is clear that Φ is a strongly continuous and coercive functional. Also since the norm $\|\cdot\|$ on Hilbert space $W^{2,2}(\mathbb{R})$ is a weakly sequentially lower semi-continuous functional in $W^{2,2}(\mathbb{R})$ therefore Φ is a sequentially weakly lower semicontinuous functional on $W^{2,2}(\mathbb{R})$. Moreover, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in W^{2,2}(\mathbb{R})$ is

$$\Phi'(u)(v) = \int_{\mathbb{R}} (u''(x)v''(x) + q(x)u'(x)v'(x) + s(x)u(x)v(x))dx$$

for every $v \in W^{2,2}(\mathbb{R})$.

Definition 2.3. Let Φ and Ψ be defined as above. Put $I_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$. We say that $u \in W^{2,2}(\mathbb{R})$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{W^{2,2}(\mathbb{R})^*\}}$, that is, $I'_\lambda(u)(v) = 0$ for all $v \in W^{2,2}(\mathbb{R})$.

Definition 2.4. A function $u : \mathbb{R} \to \mathbb{R}$ is a weak solution to the problem (1.1) if $u \in W^{2,2}(\mathbb{R})$ and

$$\int_{\mathbb{R}} \big(u''(x)v''(x) + q(x)u'(x)v'(x) + s(x)u(x)v(x) - \lambda f(x,u(x))v(x)\big)dx = 0,$$

for all $v \in W^{2,2}(\mathbb{R})$.

Remark 2.5. We clearly observe that the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$.

Lemma 2.6. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative L^1-Carathéodory function. If $u_0 \not\equiv 0$ is a weak solution for problem (1.1) then u_0 is non-negative.*

Proof. From Remark 2.5, one has $\Phi'(u_0)(v) - \lambda\Psi'(u_0)(v) = 0$ for all $v \in W^{2,2}(\mathbb{R})$. Let $v(x) = \bar{u}_0 = \max\{-u_0(x), 0\}$ and we assume that $E = \{x \in \mathbb{R} : u_0(x) < 0\}$. Then we have

$$\int_{E\cup E^c} (u_0''(x)\bar{u}_0''(x) + q(x)u_0'(x)\bar{u}_0'(x) + s(x)u_0(x)\bar{u}_0(x))dx = \int_{\mathbb{R}} \lambda f(x,u_0(x))\bar{u}_0(x)dx,$$

that is

$$\int_E (-|\bar{u}_0''(x)|^2 - q(x)|\bar{u}_0'(x)|^2 - s(x)|\bar{u}_0(x)|^2)dx \geq 0$$

which means that $\|\bar{u}_0\| = 0$ and hence $u_0 \geq 0$ and the proof is complete . $\square$

Our main tool is the following critical point theorem.

Theorem 2.7 ([2, Theorem 2.1])**.** *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*

either

(b_1) *I_λ possesses a global minimum,*

or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:*

either

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3 Main results

Let

$$\tau := \frac{540}{86111\,(\max\{1, |q|_\infty, |s|_\infty\}) {C_{q,s}}^2}, \tag{3.1}$$

$$A := \liminf_{\rho\to+\infty} \frac{\int_{\mathbb{R}} \sup_{|t|\le\rho} F(x,t)dx}{\rho^2}, \tag{3.2}$$

and

$$B := \limsup_{\rho\to+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\rho)dx}{\rho^2}. \tag{3.3}$$

Now we formulate our main result as follows.

Theorem 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function, and assume that*

(i) *$F(x,t) \ge 0$ for every $(x,t) \in \mathbb{R}\times]0, \frac{3}{8}[\cup]\frac{5}{8}, 1[$,*

(ii) *$A < \tau B$, where τ , A and B are given by (3.1) , (3.2) and (3.3) respectively.*

Then for every

$$\lambda \in \left] \frac{(\max\{1, |q|_\infty, |s|_\infty\})}{B} \frac{86111}{1080}, \frac{1}{2A\, {C_{q,s}}^2} \right[$$

the problem (1.1) admits a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Proof. Fix λ as in our conclusion. Our aim is to apply Theorem 2.7, part (b) with $X = W^{2,2}(\mathbb{R})$, and Φ, Ψ are the functionals introduced in section 2. As shown in the previous section, the functionals Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.7. Now, we look on the existence of critical points of the functional I_λ in $W^{2,2}(\mathbb{R})$. To this end, we take $\{\rho_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \rho_n = +\infty$ and

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}} \sup_{|t|\leq\rho_n} F(x,t)dx}{\rho_n^2} = A.$$

Set $r_n := \frac{1}{2}\left(\frac{\rho_n}{C_{q,s}}\right)^2$, for every $n \in \mathbb{N}$.

For each $u \in W^{2,2}(\mathbb{R})$ and bearing (2.2) in mind, we see that

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_n[) &= \{u \in W^{2,2}(\mathbb{R});\, \Phi(u) < r_n\} \\
&= \left\{u \in W^{2,2}(\mathbb{R});\, \frac{1}{2}\|u\|^2 < \frac{1}{2}\left(\frac{\rho_n}{C_{q,s}}\right)^2\right\} \\
&= \left\{u \in X;\, C_{q,s}\|u\| < \rho_n\right\} \subseteq \left\{u \in W^{2,2}(\mathbb{R});\; |u|_\infty \leq \rho_n\right\}.
\end{aligned}$$

Now, since $0 \in \Phi^{-1}(]-\infty, r_n[)$ then we have the following inequalities:

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \int_{\mathbb{R}} F(x,v(x))dx - \int_{\mathbb{R}} F(x,u(x))dx}{r_n - \frac{\|u\|^2}{2}} \\
&\leq \frac{\int_{\mathbb{R}} \sup_{|t|\leq\rho_n} F(x,t)dx}{r_n} = \frac{\int_{\mathbb{R}} \sup_{|t|\leq\rho_n} F(x,t)dx}{\frac{1}{2}\left(\frac{\rho_n}{C_{q,s}}\right)^2} \\
&= 2\, {C_{q,s}}^2\, \frac{\int_{\mathbb{R}} \sup_{|t|\leq\rho_n} F(x,t)dx}{{\rho_n}^2},
\end{aligned}$$

for every $n \in \mathbb{N}$. Hence, it follows that

$$\gamma \leq \liminf_{n\to\infty} \Phi(r_n) \leq 2\, {C_{q,s}}^2 A < +\infty,$$

because condition (ii) shows $A < +\infty$. Now, we prove that the functional I_λ is unbounded from below. For our goal, let $\{\eta_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \eta_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta_n)dx}{{\eta_n}^2} = B. \tag{3.4}$$

Let $\{v_n\}$ be a sequence in $W^{2,2}(\mathbb{R})$ which is defined by

$$v_n(x) := \begin{cases} -\frac{64\,\eta_n}{9}\left(x^2 - \frac{3}{4}x\right), & \text{if } x \in \left[0, \frac{3}{8}\right], \\ \eta_n, & \text{if } x \in \left]\frac{3}{8}, \frac{5}{8}\right], \\ -\frac{64\,\eta_n}{9}\left(x^2 - \frac{5}{4}x + \frac{1}{4}\right), & \text{if } x \in \left]\frac{5}{8}, 1\right], \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

One can compute that

$$\|v_n\|_{W^{2,2}(\mathbb{R})}{}^2 = \frac{86111}{540}\, {\eta_n}^2,$$

and so from (2.1) we have

$$(\min\{1, q_0, s_0\})\frac{86111}{1080}\,\eta_n{}^2 \le \Phi(v_n) \le (\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080}\,\eta_n{}^2. \tag{3.6}$$

Also, by using condition (i), we infer

$$\int_{\mathbb{R}} F(x, v_n(x))dx \ge \int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \eta_n)dx,$$

for every $n \in \mathbb{N}$. Therefore, we have

$$I_\lambda(v_n) = \Phi(v_n) - \lambda\,\Psi(v_n) \le (\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080}\,\eta_n{}^2 - \lambda \int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \eta_n)dx,$$

for every $n \in \mathbb{N}$. If $B < +\infty$, let

$$\epsilon \in \left]\frac{(\max\{1, |q|_\infty, |s|_\infty\})}{\lambda B}\frac{86111}{1080}, 1\right[.$$

By (3.4) there is N_ϵ such that

$$\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \eta_n)dx > \epsilon\, B\, \eta_n{}^2, \qquad (\forall n > N_\epsilon).$$

Consequently, one has

$$\begin{aligned} I_\lambda(v_n) &\le (\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080}\,\eta_n{}^2 - \lambda\epsilon B\eta_n{}^2 \\ &= \eta_n{}^2\left((\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080} - \lambda\epsilon B\right), \end{aligned}$$

for every $n > N_\epsilon$. Thus, it follows that

$$\lim_{n\to\infty} I_\lambda(v_n) = -\infty.$$

If $B = +\infty$, then consider

$$M > \frac{(\max\{1, |q|_\infty, |s|_\infty\})}{\lambda}\frac{86111}{1080}.$$

By (3.4) there is $N(M)$ such that

$$\int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \eta_n)dx > M\eta_n{}^2, \qquad (\forall n > N(M)).$$

So, we have

$$\begin{aligned} I_\lambda(v_n) &\le (\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080}\,\eta_n{}^2 - \lambda M\eta_n{}^2 \\ &= \eta_n{}^2\left((\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080} - \lambda M\right), \end{aligned}$$

for every $n > N(M)$. Taking into account the choice of M, also in this case, one has

$$\lim_{n\to\infty} I_\lambda(v_n) = -\infty.$$

Therefore according to Theorem 2.7, the functional I_λ admits an unbounded sequence $\{u_n\} \subset W^{2,2}(\mathbb{R})$ of critical points. It means that, problem (1.1) admits a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$. $\square$

Now we present the following example to illustrate Theorem 3.1.

Example 3.2. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$F(x,t) := \begin{cases} \frac{t^5\left(1-\cos(\ln|t|)\right)}{1+x^2}, & \text{if } (x,t) \in \mathbb{R}\times\mathbb{R}-\{0\} \\ 0, & \text{if } (x,t) \in \mathbb{R}\times\{0\} \end{cases}$$

and therefore we have

$$f(x,t) := \begin{cases} \frac{5t^4\left(1-\cos(\ln|t|)\right)+t^4\sin(\ln|t|)}{1+x^2}, & \text{if } (x,t) \in \mathbb{R}\times\mathbb{R}-\{0\} \\ 0, & \text{if } (x,t) \in \mathbb{R}\times\{0\}. \end{cases}$$

We observe that

$$A := \liminf_{\rho\to+\infty} \frac{\int_{\mathbb{R}} \sup_{|t|\le\rho} F(x,t)dx}{\rho^2} = 0 \tag{3.7}$$

and

$$B := \limsup_{\rho\to+\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\rho)dx}{\rho^2} = +\infty. \tag{3.8}$$

So, by Theorem 3.1, for every $\lambda \in (0,+\infty)$ the problem

$$\begin{cases} u^{iv}(x) - \left((1+e^{-x^2})u'(x)\right)' + (\pi + \tan^{-1}x)u(x) \\ = \lambda\, \frac{5u(x)^4\left(1-\cos(\ln|u(x)|)\right)+u(x)^4\sin(\ln|u(x)|)}{1+x^2}, \qquad \text{a.e. } x \in \mathbb{R}, \end{cases} \tag{3.9}$$

has a sequence of weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Note that, as in the previous example, under appropriate conditions, the existence of infinitely many weak solutions for problem (1.1) will be guaranteed for any $\lambda \in \mathbb{R}^+$. For this case, the following result is a consequence of Theorem 3.1.

Corollary 3.3. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function. Also, assume that the assumption (i) in Theorem 3.1 holds and $A = \infty$ and $B = 0$ where A and B are given by* (3.2) *and* (3.3) *respectively. Then, for every $\lambda > 0$, the problem* (1.1) *possesses a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.*

A special case of Theorem 3.1 is given in the following corollary.

Corollary 3.4. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function. Also, assume that the assumption (i) in Theorem 3.1 holds and*

(i_1) $(\max\{1, |q|_\infty, |s|_\infty\})\frac{86111}{1080} < B$,

(i_2) $A < \frac{1}{2C_{q,s}{}^2}$.

Then, the problem

$$u^{iv}(x) - (q(x)u'(x))' + s(x)u(x) = f(x,u(x)), \quad \text{a.e. } x \in \mathbb{R}, \tag{3.10}$$

possesses a sequence of many weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Proof. The corollary is an immediate consequence of Theorem 3.1 when $\lambda = 1$. □

Remark 3.5. In Theorem 3.1, we can consider $f(x,t) = \beta(x)g(t)$ where $\beta, g : \mathbb{R} \to \mathbb{R}$ are two functions such that $\beta \in L^1(\mathbb{R})$, $\beta \geq 0$, for a.e. $x \in \mathbb{R}$, $\beta \not\equiv 0$ and also g is continuous and non-negative. We set $G(t) = \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$. Since $G'(t) = g(t) \geq 0$ then G is non-decreasing function. Therefore (3.2) and (3.3) become the following simpler forms:

$$A := \liminf_{\rho \to +\infty} \frac{\int_{\mathbb{R}} \sup_{|t| \leq \rho} F(x,t)dx}{\rho^2} = \liminf_{\rho \to +\infty} \frac{G(\rho)|\beta|_1}{\rho^2} \tag{3.11}$$

and

$$B := \limsup_{\rho \to +\infty} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\rho)dx}{\rho^2} = \limsup_{\rho \to +\infty} \frac{G(\rho) \int_{\frac{3}{8}}^{\frac{5}{8}} \beta(x)dx}{\rho^2}. \tag{3.12}$$

Now if we assume that $A < \tau B$ where τ , A and B are given by (3.1), (3.11) and (3.12) respectively, then according to Theorem 3.1 and Lemma 2.6 for every

$$\lambda \in \left] \frac{(\max\{1, |q|_\infty, |s|_\infty\})}{B} \frac{86111}{1080}, \frac{1}{2A\, {C_{q,s}}^2} \right[$$

the problem

$$u^{iv}(x) - (q(x)u'(x))' + s(x)u(x) = \lambda\, \beta(x)\, g(u(x)), \quad \text{a.e. } x \in \mathbb{R}, \tag{3.13}$$

admits a sequence of many non-negative weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Using the conclusion (c) instead of (b) in Theorem 3.1, can be obtained a sequence of pairwise distinct weak solutions to the problem (1.1) which converges uniformly to zero. In this case, by replacing $\rho \to +\infty$ with $\rho \to 0^+$, A and B will be converted to the following forms:

$$A' := \liminf_{\rho \to 0^+} \frac{\int_{\mathbb{R}} \sup_{|t| \leq \rho} F(x,t)dx}{\rho^2} \tag{3.14}$$

and

$$B' := \limsup_{\rho \to 0^+} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\rho)dx}{\rho^2}. \tag{3.15}$$

Therefore, we can present the other main result of this section as follows.

Corollary 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function, and assume that*

(i) $F(x,t) \geq 0$ for every $(x,t) \in \mathbb{R} \times \left]0, \frac{3}{8}\right[\cup \left]\frac{5}{8}, 1\right[$,

(ii) $A' < \tau B'$, where τ , A' and B' are given by (3.1), (3.14) and (3.15) respectively.

Then for every

$$\lambda \in \left] \frac{(\max\{1, |q|_\infty, |s|_\infty\})}{B'} \frac{86111}{1080}, \frac{1}{2A'\, {C_{q,s}}^2} \right[$$

the problem (1.1) admits a sequence of many weak solutions which strongly converges to zero in $W^{2,2}(\mathbb{R})$.

We present the following example to illustrate Corollary 3.6.

Example 3.7. Let $\alpha > \frac{86111\sqrt{\frac{\pi}{6}}}{120\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx} - 1 \approx 2673$ be a real number and $F : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$F(x,t) := \begin{cases} e^{-x^2}t^2(1+\alpha\cos^2(\frac{1}{t})), & \text{if } (x,t) \in \mathbb{R}\times]0,+\infty[\\ 0, & \text{if } (x,t) \in \mathbb{R}\times]-\infty,0]. \end{cases}$$

From $F(x,t) = \int_0^t f(x,\xi)d\xi$ we have

$$f(x,t) := \begin{cases} e^{-x^2}(2t+2\alpha t\cos^2(\frac{1}{t})+\alpha\sin(\frac{2}{t})), & \text{if } (x,t) \in \mathbb{R}\times]0,+\infty[\\ 0, & \text{if } (x,t) \in \mathbb{R}\times]-\infty,0]. \end{cases}$$

It is clear that $f : \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function.

Let $q(x) = 1 + \frac{1}{1+x^2}$ and $s(x) = 2 + \tanh x$ and therefore $|q|_\infty = 2$, $q_0 = 1$, $|s|_\infty = 3$, $s_0 = 1$ and $\tau = \frac{120\sqrt{6}}{86111}$.

Put $a_n = \frac{1}{\frac{2n+1}{2}\pi}$ and $b_n = \frac{1}{n\pi}$ for every $n \in \mathbb{N}$, one has

$$\begin{aligned} A' &:= \liminf_{\rho\to 0^+} \frac{\sup_{|t|\le\rho} t^2\left(1+\alpha\cos^2\left(\frac{1}{t}\right)\right)\int_{\mathbb{R}} e^{-x^2}dx}{\rho^2} \\ &\le \lim_{n\to\infty} \frac{{a_n}^2\left(1+\alpha\cos^2\left(\frac{1}{a_n}\right)\right)\int_{\mathbb{R}} e^{-x^2}dx}{{a_n}^2} = \sqrt{\pi} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} B' &:= \limsup_{\rho\to 0^+} \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\rho)dx}{\rho^2} \ge \lim_{n\to\infty} \frac{{b_n}^2\left(1+\alpha\cos^2\left(\frac{1}{b_n}\right)\right)\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx}{{b_n}^2} \\ &= (1+a)\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx. \end{aligned} \tag{3.17}$$

Now, since $\alpha > \frac{86111\sqrt{\frac{\pi}{6}}}{120\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx} - 1$, we have

$$A' \le \sqrt{\pi} < \frac{120\sqrt{6}}{86111}(1+a)\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx \le \tau B'$$

and so condition (ii) of the Theorem 3.1 is satisfied. Now, according to the Theorem 3.1 for every

$$\lambda \in \left] \frac{86111}{360(1+a)\left(\int_{\frac{3}{8}}^{\frac{5}{8}} e^{-x^2}dx\right)}, \frac{1}{3}\sqrt{\frac{6}{\pi}} \right[$$

the problem

$$\begin{cases} u^{iv}(x) - \left((1+\frac{1}{1+x^2})u'(x)\right)' + (2+\tanh x)u(x) \\ = \lambda\, e^{-x^2}\left(2u(x)+2\alpha u(x)\cos^2(\frac{1}{u(x)})+\alpha\sin(\frac{2}{u(x)})\right), \quad \text{a.e. } x \in \mathbb{R}, \end{cases} \tag{3.18}$$

admits a sequence of many weak solutions which is converges uniformly to zero.

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