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Uniqueness and nonuniqueness of fronts for degenerate diffusion-convection reaction equations

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Abstract. We consider a scalar parabolic equation in one spatial dimension. The equation is constituted by a convective term, a reaction term with one or two equilibria, and a positive diffusivity which can however vanish. We prove the existence and several properties of traveling-wave solutions to such an equation. In particular, we provide a sharp estimate for the minimal speed of the profiles and improve previous results about the regularity of wavefronts. Moreover, we show the existence of an infinite number of semi-wavefronts with the same speed.

Keywords: degenerate and doubly degenerate diffusivity, diffusion-convection-reaction equations, traveling-wave solutions, sharp profiles, semi-wavefronts.

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1 Introduction

We study the existence and qualitative properties of traveling-wave solutions to the scalar diffusion-convection-reaction equation

$$\rho_t + f(\rho)_x = (D(\rho)\rho_x)_x + g(\rho), \qquad t \geq 0,\ x \in \mathbb{R}. \tag{1.1}$$

Here $\rho = \rho(t,x)$ is the unknown variable and takes values in the interval $[0,1]$. The convective term f satisfies the condition

(f) $f \in C^1[0,1]$, $f(0) = 0$.

The requirement $f(0) = 0$ is not a real assumption, since f is defined up to an additive constant; we denote $h(\rho) = \dot{f}(\rho)$, where with a dot we intend the derivative with respect to the variable ρ (or φ later on). About the diffusivity D and the reaction term g we consider two different scenarios, where the assumptions are made on the pair D, g; we assume either

(D1) $D \in C^1[0,1]$, $D > 0$ in $(0,1)$ and $D(1) = 0$,

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(g0) $g \in C^0[0,1]$, $g > 0$ in $(0,1]$, $g(0) = 0$,

or else

(D0) $D \in C^1[0,1]$, $D > 0$ in $(0,1)$ and $D(0) = 0$,

(g01) $g \in C^0[0,1]$, $g > 0$ in $(0,1)$, $g(0) = g(1) = 0$.

In the above notation, the numbers suggest where it is *mandatory* that the corresponding function vanishes. Notice that (D1) leaves open the possibility for D to vanish or not at 0, and (D0) for D at 1. We refer to Figure 1.1 for a graphical illustration of these assumptions. Notice that the product Dg always vanishes at both 0 and 1 under both set of assumptions.

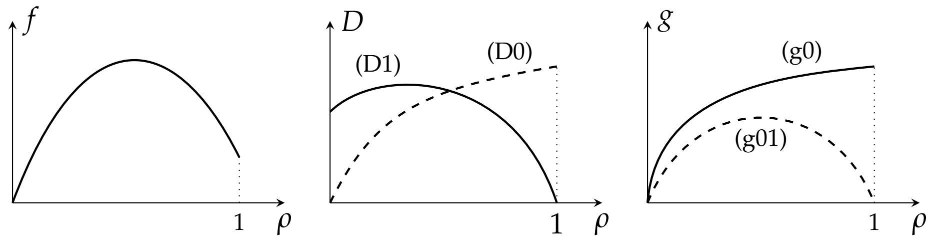

Figure 1.1: Typical plots of the functions f, D and g. In the plots of D and g, solid or dashed lines depict pairs of functions D and g that are considered together in the following. The possibility that D vanishes at the other extremum is left open.

We also require the following condition on the product of D and g:

$$\limsup_{\varphi \to 0^+} \frac{D(\varphi)g(\varphi)}{\varphi} < +\infty, \tag{1.2}$$

which is equivalent to $D(\varphi)g(\varphi) \leq L\varphi$, for some $L > 0$ and φ in a right neighborhood of 0.

In (1.1), the notation $\rho = \rho(t,x)$ suggests a density; this is indeed the case. Recently, the modeling of collective movements has attracted the interest of several mathematicians [9,10,22]. This paper is partly motivated by such a research stream and carries on the analysis of a scalar parabolic model begun in [5–7]. Indeed, if $f(\rho) = \rho v(\rho)$, where the velocity v is an assigned function, then equation (1.1) can be understood as a simplified model for a crowd walking with velocity v along a straight path with side entries for other pedestrians, which are modeled by g; here ρ is understood as the crowd normalized density. Assumption (g01), for instance, means that pedestrians do not enter if the road is empty ($g(0) = 0$, modeling an aggregative behavior) or if it is fully occupied ($g(1) = 0$, because of lack of space). If the diffusivity is small, then the diffusion term accounts for some "chaotic" behavior, which is common in crowds movements. In this framework, D may degenerate at the extrema of the interval where it is defined [2, 4, 20]; for more details we refer to [6]. The assumption (g0) is better motivated by population dynamics. In this case g is a growth term which, for instance, increases with the population density ρ. We refer to [19] for analogous modelings in biology. Anyhow, apart from the above possible applications, equation (1.1) is a quite general diffusion-convection-reaction equation that deserves to be fully understood.

A *traveling-wave solution* is, roughly speaking, a solution to (1.1) of the form $\rho(t,x) = \varphi(x-ct)$, for some profile $\varphi = \varphi(\xi)$ and constant wave speed c, see [11] for general information. In this case the profile must satisfy, in some sense, the equation

$$\left(D(\varphi)\varphi'\right)' + (c - h(\varphi))\,\varphi' + g(\varphi) = 0, \tag{1.3}$$

where $'$ denotes the derivative with respect to ξ. We consider in this paper non-constant, monotone profiles, and focus on the case they are decreasing. As a consequence, we aim at determining solutions to (1.3) whose values at $\pm\infty$ are the zeroes of the function g and then satisfy either

$$\varphi(-\infty) = 1, \qquad \varphi(+\infty) = 0, \tag{1.4}$$

or simply

$$\varphi(+\infty) = 0, \tag{1.5}$$

according to we make assumption (g01) or (g0). The former profiles are called *wavefronts*, the latter are *semi-wavefronts*; precise definitions are provided in Definition 2.1. Notice that in both cases the equilibria may be reached for a finite value of the variable ξ as a consequence of the degeneracy of D at those points. These solutions represent single-shape smooth transitions between the two constant densities 0 and 1. The interest of wavefronts lies in the fact that they are viscous approximations of shock waves to the inviscid version of equation (1.1), i.e., when $D = 0$. Semi-wavefronts lack of this motivation but are nevertheless meaningful for applications [6]; moreover, wavefronts connecting “nonstandard” end states can be constructed by pasting semi-wavefronts [7]. At last, we point out that assumption (1.2) is usual in this framework, when looking for decreasing profiles, see e.g. [1].

If $D(\rho) \geq 0$, the existence of solutions to the initial-value problem for (1.1) is more or less classical [24]; however, the *fine structure* of traveling waves reveals a variety of different patterns. We refer to [15,16], respectively, for the cases where D is non degenerate, i.e., $D > 0$, and for the degenerate case, where D can vanish at either 0 or 1. The main results of those papers is that there is a critical threshold c^*, depending on both f and the product Dg, such that traveling waves satisfying (1.4) exist if and only if $c \geq c^*$. The smoothness of the profiles depend on f, D and c but not on g. In both papers the source term satisfies (g01); see [5,6] for the case when g has only one zero.

The case when D changes sign, which is not studied in this paper, also has strong motivations: we quote [13,21] for biological models and [7] for applications to collective movements. Several results about traveling waves have been obtained in [7,8,12–14].

In this paper we study semi-wavefronts and wavefronts for (1.1), thus completing the analysis of [5, 6]. We prove that in both cases there is a threshold c^* such that profiles only exists for $c \geq c^*$; we also study their regularity and strict monotonicity, namely whether they are *classical* (i.e., C^1) or *sharp* (and then reach an equilibrium at a finite ξ in a no more than continuous way). We strongly rely on [15, 16] and exploit some recent results obtained in [18]. Several examples are scattered throughout the paper to show that our assumptions are necessary in most cases.

This research has some important novelties. First, we give a refined estimate for c^*, which allows to better understand the meaning of this threshold. Second, we improve a result obtained in [16] about the appearance of wavefronts with a sharp profile. Third, in the case of semi-wavefronts, we show that for any speed $c \geq c^*$ there exists a family of profiles with speed c. This phenomenon does not show up in [5,6].

The main tool to investigate (1.3) is the analysis of singular first-order problems as

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = 0. \end{cases} \tag{1.6}$$

Problem (1.6) is deduced by problem (1.3)–(1.5) by the singular change of variables $z(\varphi) := D(\varphi)\varphi'$, where the right-hand side is understood to be computed at $\varphi^{-1}(\varphi)$, see e.g. [6, 15]. Notice that φ^{-1} exists by the assumption of monotony of φ.

On the other hand, the analysis of problem (1.6) is fully exploited in the forthcoming paper [3], which deals with the case in which D changes sign. In that paper we show that there still exist wavefronts joining 1 with 0, which travel across the region where D is negative; they are constructed by pasting two semi-wavefronts obtained in the current paper. Similar results in the case $g = 0$ are proved in [7].

Here is an account of the paper. In Section 2 we provide some basic definitions and state our main results. The analysis of problem (1.6) and of other related singular problems occupies Sections 3 to 8. Then, in Sections 9 and 10 we exploit such results to construct semi-wavefronts and wavefronts, respectively; there, we prove our main results.

2 Main results

We give some definitions on traveling waves and their profiles. Let $I \subseteq \mathbb{R}$ be an open interval.

Definition 2.1. Assume $f, D, g \in C[0,1]$. Consider a function $\varphi \in C(I)$ with values in $[0,1]$, which is differentiable a.e. and such that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(I)$; let c be a real constant. The function $\rho(x,t) := \varphi(x - ct)$, for (x,t) with $x - ct \in I$, is a *traveling-wave* solution of equation (1.1) with wave speed c and wave profile φ if, for every $\psi \in C_0^\infty(I)$,

$$\int_I \left(D\left(\varphi(\xi)\right)\varphi'(\xi) - f\left(\varphi(\xi)\right) + c\varphi(\xi)\right)\psi'(\xi) - g\left(\varphi(\xi)\right)\psi(\xi)\, d\xi = 0. \tag{2.1}$$

Definition 2.1 can be made more precise. Below, *monotonic* means that $\varphi(\xi_1) \le \varphi(\xi_2)$ (or $\varphi(\xi_1) \ge \varphi(\xi_2)$) for every $\xi_1 < \xi_2$ in the domain of φ; in *(iii)* we assume $g(0) = g(1) = 0$, while in *(iv)* we only require that g vanishes at the point which is specified by the semi-wavefront. A traveling-wave solution is

(i) *global* if $I = \mathbb{R}$ and *strict* if $I \neq \mathbb{R}$ and φ is not extendible to $\mathbb{R}$;

(ii) *classical* if φ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (1.3) holds a.e.; *sharp at* ℓ if there exists $\xi_\ell \in I$ such that $\varphi(\xi_\ell) = \ell$, with φ classical in $I \setminus \{\xi_\ell\}$ and not differentiable at ξ_ℓ;

(iii) *a wavefront* if it is global, with a monotonic, non-constant profile φ satisfying either (1.4) or the converse condition;

(iv) *a semi-wavefront to* 1 (or *to* 0) if $I = (a, \infty)$ for $a \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to \infty$; *a semi-wavefront from* 1 (or *from* 0) if $I = (-\infty, b)$ for $b \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to -\infty$.

In *(iv)* we say that φ connects $\varphi(a^+)$ (1 or 0) with 1 or 0 (resp., with $\varphi(b^-)$).

The smoothness of a profile depends on the degeneracy of D, see [11]. More precisely, assume (f), and either (D1), (g0) or (D0), (g01); let ρ be any traveling-wave solution of (1.1) with profile φ defined in I and speed c. Then φ is classical in each interval $J \subset I$ where $D(\varphi(\xi)) > 0$ for $\xi \in J$, and $\varphi \in C^2(J)$. Profiles are determined up to a space shift.

Our first main result concerns *semi-wavefronts*.

Theorem 2.2. *Assume* (f), (D1), (g0) *and* (1.2). *Then, there exists* $c^* \in \mathbb{R}$, *which satisfies*

$$\max\left\{\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}, h(0)+2\sqrt{\liminf_{\varphi\to 0^+}\frac{D(\varphi)g(\varphi)}{\varphi}}\right\} \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]}\frac{D(\varphi)g(\varphi)}{\varphi}}+\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}, \tag{2.2}$$

such that (1.1) *has strict semi-wavefronts to* 0*, connecting* 1 *to* 0*, if and only if* $c \ge c^*$.

Moreover, if φ *is the profile of one of such semi-wavefronts, then it holds that*

$$\varphi'(\xi) < 0 \ \textit{ for any } \ 0 < \varphi(\xi) < 1. \tag{2.3}$$

For a fixed $c > c^*$, the profiles of Theorem 2.2 are *not* unique. This lack of uniqueness is not due only to the action of space shifts but, more intimately, to the non-uniqueness of solutions to problem (1.6) that is proved in Proposition 5.1 below. Roughly speaking, these profiles depend on a parameter b ranging in the interval $[\beta(c), 0]$, for a suitable threshold $\beta(c) \le 0$. As a conclusion, the family of profiles can be precisely written as

$$\varphi_b = \varphi_b(\xi), \qquad \text{for } b \in [\beta(c), 0]. \tag{2.4}$$

Moreover, $\beta(c) < 0$ if $c > c^*$ and $\beta(c) \to -\infty$ as $c \to +\infty$. The threshold $\beta(c)$ essentially corresponds to the minimum value that the quantity $D(\varphi_b)\varphi_b'$ achieves when φ_b reaches 1, for $b \in [\beta(c), 0]$. This loss of uniqueness is a novelty if we compare Theorem 2.2 with analogous results in [5, 6]. In particular, in [6, Theorem 2.7] the assumptions on the functions D and g are *reversed*: both of them are positive in $(0, 1)$ with $D(0) = 0 < g(0)$, $D(1) > 0 = g(1)$; in [5, Theorem 2.3] D and g are still positive in $(0, 1)$ but the vanishing conditions are $D(1) = 0 = g(1)$. In both cases the profiles exist for every $c \in \mathbb{R}$ and are unique. The different results are due to the nature of the equilibria of the dynamical systems of (1.3).

The estimates (2.2) deserve some comments. The left estimate improves analogous bounds (see [18] for a comprehensive list) by including the term $\sup_{\varphi\in(0,1]} f(\varphi)/\varphi \ge h(0)$ on the left-hand side. This improvement looks more significative if we also assume $(\dot{Dg})(0) = 0$, as we do in the Theorem 2.3. In this case (2.2) reduces to

$$\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi} \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]}\frac{D(\varphi)g(\varphi)}{\varphi}}+\sup_{\varphi\in(0,1]}\frac{f(\varphi)}{\varphi}. \tag{2.5}$$

which can be written with obvious notation as

$$c_{con} \le c^* \le c_{dr} + c_{con},$$

where the indexes label velocities related to the convection or diffusion-reaction components. In (2.5) the *same* term, accounting for the dependence on f, occurs in *both* the lower and upper bound. This symmetry, which shows the shift of the critical threshold as a consequence of the convective term f, occurs in *none* of the previous estimates.

The meaning of c_{dr} is known since [1]; we comment on c_{con}. In the diffusion-convection case (i.e., when $g = 0$), there exist profiles connecting $\ell \in (0,1]$ to 0 if and only if

$$s_\ell(\varphi) := \frac{f(\ell)}{\ell}\varphi > f(\varphi), \quad \text{for } \varphi \in (0,\ell), \tag{2.6}$$

see [11, Theorem 9.1]. The quantity c_{con} then represents the *maximal speed* that can be reached by the profiles connecting ℓ to 0, for $\ell \in (0,1]$. Condition (2.6) is also necessary and sufficient in the purely hyperbolic case (i.e., when also $D = 0$) in order that the equation $u_t + f(u)_x = 0$ admits a shock wave of speed $f(\ell)/\ell$ with ℓ as left state and 0 as right state. This is not surprising since the viscous profiles approximate the shock wave and converge to it in the vanishing viscosity limit. Indeed, condition (2.6) does not depend on D.

The presence of the positive reaction term g satisfying (g01) (if (g0) holds we only have semi-wavefronts, but the same bounds still hold) does not allow profile speeds to be less than c_{con}: assuming that z satisfies (1.6), by the positivity of both D and g we deduce

$$c \geq \sup_{\varphi\in(0,1]} \left(\frac{f(\varphi)}{\varphi} - \frac{z(\varphi)}{\varphi}\right) \geq c_{con}. \tag{2.7}$$

Then, c_{con} now becomes a bound for the *minimal speed* of the profiles. The bound (2.7) is strict (i.e., there is a gap between c_{con} and c^*) if $(\dot{Dg})(0) > 0$; this occurs for instance if $D(0) > 0$ and $\dot{g}(0) > 0$ and follows by integrating $(1.6)_1$ from 0 to φ and (2.2), see Remark 5.6. If $f = 0$, then the corresponding strict bound $c^* > 0$ occurs for any positive and continuous D and g: if $c^* = 0$ then z should be an increasing function by (3.11), a contradiction.

In some cases, semi-wavefronts are sharp at 0. We refer to Corollary 9.4 for a detailed account of the behavior of the profiles when they reach the equilibrium.

We now present our result on *wavefronts*; we assume that D and g satisfiy (D0) and (g01). The goal is to extend results contained in [16, Theorems 2.1 and 6.1] regarding the existence and, more importantly, the regularity of wavefronts of Equation (1.1). In particular, the next theorem has the merit to derive the classification of wavefronts under (D0), merely, without additional assumptions (which were instead required in [16, Theorems 2.1 and 6.1]). Notice that in the following result we require that D vanishes at 0; this assumption leads to improve not only the left-hand bound (2.2) on c^* by (2.5), but also the right-hand bound, by means of a recent integral estimate provided in [18].

Theorem 2.3. *Assume* (f), (D0) *and* (g01) *and* (1.2). *Then there exists c^*, satisfying*

$$\sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} \leq c^* \leq \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi\in(0,1]} \frac{1}{\varphi}\int_0^\varphi \frac{D(\sigma)g(\sigma)}{\sigma}\, d\sigma}, \tag{2.8}$$

such that Equation (1.1) *admits a (unique up to space shifts) wavefront, whose wave profile φ satisfies* (1.4), *if and only if $c \geq c^*$. Moreover, we have $\varphi'(\xi) < 0$, for $0 < \varphi(\xi) < 1$, and*

(i) if $c > c^*$, then φ is classical at 0;

(ii) if $c = c^*$ and $c^* > h(0)$, then φ is sharp at 0 and if it reaches 0 at $\xi_0 \in \mathbb{R}$ then

$$\lim_{\xi\to\xi_0^-} \varphi'(\xi) = \begin{cases} \frac{h(0)-c^*}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) > 0, \\ -\infty & \text{if } \dot{D}(0) = 0. \end{cases}$$

As in analogous cases [6], Theorem 2.3 provides no information about the smoothness of the profiles when $c = c^* = h(0)$. We show in Remark 10.1 that in such a case profiles may be either sharp or classical.

3 Singular first-order problems

Here we begin the analysis of problem (1.6). First, we consider, for $c \in \mathbb{R}$, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \end{cases} \tag{3.1}$$

where we assume

$$q \in C^0[0,1] \quad \text{and} \quad q > 0 \text{ in } (0,1). \tag{3.2}$$

We point out that the differential equation $(3.1)_1$ generalizes $(1.6)_1$ since the assumptions on q are a bit less strict than the ones on Dg, under (D1)–(g0) or (D1)–(g01). In the following lemma we prove that a solution of (3.1) can be extended continuously up to the boundary.

Lemma 3.1. *Assume* (3.2)*. If $z \in C^1(0,1)$ is a solution of* (3.1)*, then it can be extended continuously to the interval* $[0,1]$.

Proof. Since $q/z < 0$ in $(0,1)$, then for any $0 < \varphi < \varphi_1 < 1$ the function

$$\varphi \to \int_\varphi^{\varphi_1} \frac{q(\sigma)}{z(\sigma)}\, d\sigma$$

is strictly increasing. Hence, we can pass to the limit as $\varphi \to 0^+$ in the expression

$$z(\varphi) = z(\varphi_1) - \int_\varphi^{\varphi_1} (h(\sigma) - c)\, d\sigma + \int_\varphi^{\varphi_1} \frac{q(\sigma)}{z(\sigma)}\, d\sigma, \tag{3.3}$$

which is obtained by integrating $(3.1)_1$ in (φ, φ_1). Then $z(0^+)$ exists and necessarily lies in $[-\infty, 0]$ because of $(3.1)_2$. If $z(0^+) = -\infty$, then by passing to the limit for $\varphi \to 0^+$ in (3.3) we find a contradiction, since the last integral converges as $\varphi \to 0^+$. Hence, $z(0^+) \in (-\infty, 0]$.

For $z(1^-)$ the proof is even simpler: by integrating $(3.1)_1$ in (φ_2, φ), for $0 < \varphi_2 < \varphi < 1$, we obtain (3.3) with φ_2 replacing φ_1. As before, we deduce that $z(1^-)$ exists. Also, since the last integral in (3.3) is now positive, we get $z(\varphi) > z(\varphi_2) + \int_{\varphi_2}^{\varphi} (h(\sigma) - c)\, d\sigma$, for any $\varphi \in (\varphi_2, 1)$. This directly rules out the alternative $z(1^-) = -\infty$ and concludes the proof. □

We now summarize [6, Lemmas 4.1 and 4.3] in a version for our purposes, by also exploiting Lemma 3.1. These tools were obtained in [6] under stricter assumptions on q, but it is easy to verify that they also apply to the current case, in virtue of (3.2). For $\mu < 0$ and $\sigma \in (0,1]$ or $\sigma \in [0,1)$, they deal with the systems

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \ \varphi < \sigma, \\ z(\sigma) = \mu, \end{cases} \qquad \begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \ \varphi > \sigma, \\ z(\sigma) = \mu. \end{cases} \tag{3.4}$$

A function $\eta \in C^1(\sigma_1, \sigma_2)$, for $0 \le \sigma_1 < \sigma_2 \le 1$, is an upper-solution of $(3.1)_1$ in (σ_1, σ_2) if

$$\dot{\eta}(\varphi) \ge h(\varphi) - c - \frac{q(\varphi)}{\eta(\varphi)} \quad \text{for any} \ \ \sigma_1 < \varphi < \sigma_2. \tag{3.5}$$

The upper-solution η is said strict if the inequality in (3.5) is strict. A function $\omega \in C^1(\sigma_1, \sigma_2)$ is a (strict) lower-solution of $(3.1)_1$ in (σ_1, σ_2) if the (strict) inequality in (3.5) is reversed.

Lemma 3.2. *Assume (3.2) and consider equation $(3.1)_1$; the following results hold.*

1. *Set $\mu < 0$. Then,*
   (a) *let $\sigma \in (0,1]$; then problem $(3.4)_1$ admits a unique solution $z \in C^0[0,\sigma] \cap C^1(0,\sigma)$;*
   (b) *let $\sigma \in [0,1)$; then problem $(3.4)_2$ admits a unique solution $z \in C^0[\sigma,\delta] \cap C^1(\sigma,\delta)$, for some maximal $\sigma < \delta \le 1$. Moreover, either $\delta = 1$ or $z(\delta) = 0$.*

2. *Set $0 \le \sigma_1 < \sigma_2 \le 1$; let z be a solution of (3.1) in (σ_1,σ_2). It holds that:*
   (a) *if η is a strict upper-solution of $(3.1)_1$ in (σ_1,σ_2), then*
      (i) *if $\eta(\sigma_2) \le z(\sigma_2) < 0$, then $\eta < z$ in (σ_1,σ_2);*
      (ii) *if $0 > \eta(\sigma_1) \ge z(\sigma_1)$ then $\eta > z$ in (σ_1,σ_2); moreover, if η is defined in $[0,1]$, then z must be defined in $[\sigma_1,1]$ and $\eta > z$ in $(\sigma_1,1)$;*
   (b) *if ω is a strict lower-solution of $(3.1)_1$ in (σ_1,σ_2), then*
      (i) *if $0 > \omega(\sigma_2) \ge z(\sigma_2)$, then $\omega > z$ in (σ_1,σ_2); moreover, if ω is defined in $[0,1]$, then z must be defined in $[0,\sigma_2]$ and $\omega > z$ in $(0,\sigma_2)$;*
      (ii) *if $\omega(\sigma_1) \le z(\sigma_1) < 0$ then $\omega < z$ in (σ_1,σ_2).*

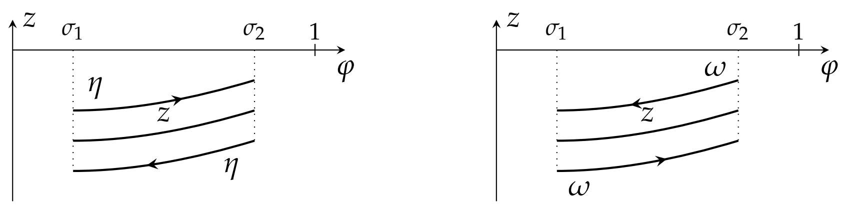

Figure 3.1: An illustration of Lemma 3.2 *(2)*. Left: supersolutions η; right: subsolutions ω.

In the context of equations as $(3.1)_1$, proper limit arguments are often needed.

Lemma 3.3. *Assume (3.2). Let $\{c_n\}_n$ be a sequence of real numbers and $c \in \mathbb{R}$ such that $c_n \to c$ as $n \to \infty$. Let $z_n \in C^0[0,1] \cap C^1(0,1)$ satisfy (3.1) corresponding to c_n. If $\{z_n\}_n$ is increasing and there exists $v \in C^0[0,1]$ such that*

$$z_n(\varphi) \le v(\varphi) < 0 \quad \text{for any} \quad n \in \mathbb{N} \quad \text{and} \quad \varphi \in (0,1), \tag{3.6}$$

then z_n converges (uniformly on $[0,1]$) to a solution $\bar{z} \in C^0[0,1] \cap C^1(0,1)$ of (3.1).

The same conclusion holds if $\{z_n\}_n$ is decreasing and there exists $w \in C^0[0,1]$ such that

$$z_n(\varphi) \ge w(\varphi) \quad \text{for any} \quad n \in \mathbb{N} \quad \text{and} \quad \varphi \in (0,1).$$

Proof. Take first $\{z_n\}_n$ increasing. From (3.6), we can define $\bar{z} = \bar{z}(\varphi)$ as

$$\lim_{n\to\infty} z_n(\varphi) =: \bar{z}(\varphi), \qquad \varphi \in (0,1).$$

It is obvious that $z_1 \le \bar{z} \le v < 0$ in $(0,1)$. By integrating $(3.1)_1$, we have

$$z_n(\varphi) - z_n(\varphi_0) = \int_{\varphi_0}^{\varphi} \left\{ h(\sigma) - c_n + \frac{q(\sigma)}{-z_n(\sigma)} \right\} d\sigma \quad \text{for any} \quad \varphi_0, \varphi \in (0,1).$$

Since, for every $\sigma \in (0,1)$, the sequence $\{q(\sigma)/(-z_n(\sigma))\}_n$ is increasing, then the Monotone Convergence Theorem implies that

$$\bar{z}(\varphi) - \bar{z}(\varphi_0) = \int_{\varphi_0}^{\varphi} \left\{ h(\sigma) - c - \frac{q(\sigma)}{\bar{z}(\sigma)} \right\} d\sigma \quad \text{for any} \quad \varphi_0, \varphi \in (0,1),$$

where all the involved quantities are finite. This tells us that $\bar{z}$ is absolutely continuous in every compact interval $[a,b] \subset (0,1)$. By differentiating, we then obtain that $\bar{z} \in C^1(0,1)$ satisfies (3.1). From Lemma 3.1, we also have that $\bar{z} \in C^0[0,1]$. To conclude that z_n converges to $\bar{z}$ uniformly on $[0,1]$, it only remains to prove that

$$\bar{z}(0^+) = \lim_{n\to\infty} z_n(0) \quad \text{and} \quad \bar{z}(1^-) = \lim_{n\to\infty} z_n(1). \tag{3.7}$$

Indeed, if (3.7) holds, then $\{z_n\}_n$ turns out to be a monotone sequence of continuous functions converging pointwise to $\bar{z} \in C^0[0,1]$ on a compact set. Then, by *Dini's monotone convergence theorem* (see [23, Theorem 7.13]), z_n must converge uniformly to $\bar{z}$ on $[0,1]$. We prove only $(3.7)_1$ since $(3.7)_2$ follows as well. If $z_n(0) \to 0$, as $n \to \infty$, then $\bar{z}(0^+) = 0$, because $z_n \le \bar{z} < 0$ in $(0,1)$. Hence $(3.7)_1$ is verified. If instead $z_n(0) \to \mu < 0$, we argue as follows. Consider $\delta \in \mathbb{R}$ such that $c_n > \delta$, for any $n \in \mathbb{N}$, and let $\eta = \eta(\varphi)$ satisfy

$$\begin{cases} \dot{\eta}(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta(\varphi)}, & \varphi > 0, \\ \eta(0) = \mu. \end{cases} \tag{3.8}$$

By Lemma 3.2 *(1.b)* such an η exists, in its maximal-existence interval $[0,\sigma)$, for some $\sigma \in (0,1]$. Moreover, we have

$$\dot{\eta}(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta(\varphi)}, \qquad \varphi \in (0,\sigma).$$

Hence, in $(0,\sigma)$, η is a strict upper-solution of $(3.1)_1$ with $c = c_n$ and $z_n(0) \le \eta(0) < 0$. Thus, Lemma 3.2 *(2.a.ii)* implies that $z_n \le \eta$ in $(0,\sigma)$. By passing to the pointwise limit, for $n \to \infty$, it is clear that $\bar{z} \le \eta$ in $(0,\sigma)$. Since $\bar{z}, \eta$ are continuous up to $\varphi = 0$, then $\bar{z}(0^+) \le \mu$. On the other hand we have $\bar{z}(0^+) \ge \mu$ because $z_n \le \bar{z}$ in $(0,1)$ and $z_n, \bar{z} \in C^0[0,1]$. Then $\bar{z}(0^+) = \mu$ and this concludes the proof of $(3.7)_1$.

Consider $\{z_n\}_n$ decreasing. By adapting the arguments used in the first part of this proof, we can show that z_n converges pointwise in $(0,1)$ to $\bar{z} \in C^0[0,1] \cap C^1(0,1)$ satisfying (3.1). As before we need (3.7) to conclude. To this end, we again observe that similarly to the case of $\{z_n\}_n$ increasing, we have (3.7) if both $z_n(0) \to \mu < 0$ and $z_n(1) \to \nu < 0$. Instead, the proofs of either $(3.7)_1$ when $z_n(0) \to 0$ and $(3.7)_2$ when $z_n(0) \to 0$ are now more subtle. We provide them both. First, since $z_n < 0$ in $(0,1)$, observe that requiring that $z_n(0) \to 0$ (or $z_n(1) \to 0$) corresponds to have $z_n(0) = 0$ (or $z_n(1) = 0$), for every $n \in \mathbb{N}$.

Take $z_n(0) = 0$, for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and for $\varphi \in (0,1)$, let $\sigma_\varphi \in (0,\varphi)$ be defined by

$$\dot{z}_n(\sigma_\varphi) = \frac{z_n(\varphi)}{\varphi}.$$

Take $\delta_1 \in \mathbb{R}$ such that $\delta_1 > c_n$, for each $n \in \mathbb{N}$. By using $(3.1)_1$ and the fact that $q/z_n < 0$ in $(0,1)$, we deduce, for any $\varphi \in (0,1)$,

$$\frac{z_n(\varphi)}{\varphi} = \dot{z}_n(\sigma_\varphi) > h(\sigma_\varphi) - c_n > \inf_{\varphi\in(0,1)} h(\varphi) - \delta_1 =: C < 0. \tag{3.9}$$

The sign of C is due to $c_n \geq h(0)$, for $n \in \mathbb{N}$; otherwise, it would not be possible to have z_n satisfying (3.1) and $z_n(0) = 0$. Inequality (3.9) implies that $z_n(\varphi) > C\varphi$ for $\varphi \in (0,1)$. Hence, letting $n \to \infty$, this leads to $\bar{z}(\varphi) \geq C\varphi$, for $\varphi \in (0,1)$. Passing to the limit as $\varphi \to 0^+$ gives $\bar{z}(0^+) \geq 0$, which in turn implies that $\bar{z}(0^+) = 0$. Thus, $(3.7)_1$ is verified.

Lastly, let $z_n(1) = 0$, for any $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and consider $\eta_2 = \eta_2(\varphi)$ such that

$$\begin{cases} \dot{\eta}_2(\varphi) = h(\varphi) - \delta - \frac{q(\varphi)}{\eta_2(\varphi)}, & \varphi > 0, \\ \eta_2(1) = -\varepsilon < 0, \end{cases} \tag{3.10}$$

where $\delta \in \mathbb{R}$ is such that $\delta < c_n$, for any $n \in \mathbb{N}$. Such an η_2 exists and is defined and continuous in $[0,1]$, because of Lemma 3.2 *(1.a)* and Lemma 3.1. Take an arbitrary $n \in \mathbb{N}$. From $0 = z_n(1) > \eta_2(1)$, it follows that $\eta_2 < z_n$ in $[\sigma_n, 1]$, for some $\sigma_n > 0$, with $z_n(\sigma_n) < 0$. Thus, since

$$\dot{\eta}_2(\varphi) > h(\varphi) - c_n - \frac{q(\varphi)}{\eta_2(\varphi)}, \qquad \varphi \in (0,1),$$

then η_2 is a strict upper-solution of $(3.1)_1$ with $c = c_n$ in $(0, \sigma_n)$ and $\eta_2(\sigma_n) < z_n(\sigma_n) < 0$. An application of Lemma 3.2 *(2.a.i)* implies that $\eta_2 < z_n$ in $(0, \sigma_n)$. Thus, $z_n > \eta_2$ in $(0,1)$, for any $n \in \mathbb{N}$. By passing to the pointwise limit, as $n \to \infty$, we then have $\bar{z}(\varphi) \geq \eta_2(\varphi)$, for $\varphi \in (0,1)$. By the continuity of both $\bar{z}$ and η_2 at $\varphi = 1$, we obtain $0 \geq \bar{z}(1^-) \geq -\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that necessarily $\bar{z}(1^-) = 0$. □

Because of Lemmas 3.1 and 3.3, in the following we always mean solutions z to problem (3.1), and analogous ones, in the class $C[0,1] \cap C^1(0,1)$, *without any further mention*.

Motivated by Lemma 3.1, in the next sections we focus the following problem, where the boundary condition is given on the *left* extremum of the interval of definition:

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = 0. \end{cases} \tag{3.11}$$

Problem (3.11) is exploited for *semi-wavefronts*. The value of $z(1)$ is not prescribed; from $(3.11)_2$, we have $z(1) \leq 0$. The extremal case $z(1) = 0$ is needed in the study of *wavefronts*:

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi) < 0, & \varphi \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{3.12}$$

4 The singular problem with two boundary conditions

Problems (3.11) and (3.12) have solutions only when c is larger than a critical threshold c^*. In this section we first give a new estimate to c^* under mild conditions on q; then, we obtain a result of existence and uniqueness of solutions to (3.12) if $c \geq c^*$. Recalling (D1), (g0) and (1.2) and (D0)–(g01), throughout the next sections we need to strengthen the assumptions (3.2) of Section 3; for commodity we gather them all here below. We assume

(q) $q \in C^0[0,1]$, $q > 0$ in $(0,1)$, $q(0) = q(1) = 0$ and $\limsup_{\varphi \to 0^+} \dfrac{q(\varphi)}{\varphi} < +\infty$.

We improve, as in [18, Theorem 3.1], a well-known result [1,11,15]. If q is differentiable at 0, in [18, Theorem 3.1] it is proved that Problem (3.11) has a solution if

$$c > \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi\in(0,1]} \frac{1}{\varphi}\int_0^\varphi \frac{q(\sigma)}{\sigma}\,d\sigma}. \tag{4.1}$$

The last assumption in (q) is weaker than the differentiability of q at 0 and our result below is less stronger than the one in [18]. It is an open problem whether the existence of solutions to Problem (3.12) under (4.1) can be achieved by only assuming $\limsup_{\varphi\to 0^+} q(\varphi)/\varphi < +\infty$.

Lemma 4.1. *Assume* (q). *Then Problem* (3.12) *admits a solution if*

$$c > \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi}}. \tag{4.2}$$

Proof. We follow [18, Theorem 3.1]. By (4.2) we see that there exists $K > 0$, $\varepsilon > 0$ so that

$$K^2 + \left(\sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} - c\right) K + \sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi} < -\varepsilon K < 0 \quad \text{for} \quad \varphi \in (0,1].$$

For every $\tau > 0$, we get, for any $\varphi > \tau$,

$$\frac{1}{\varphi-\tau}\int_\tau^\varphi \frac{q(s)}{s}\,ds = \frac{q(s_{\varphi,\tau})}{s_{\varphi,\tau}} \le \sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi},$$

where $s_{\varphi,\tau} \in (\tau,\varphi)$ is given by the Mean Value Theorem. As a consequence, for any $\tau > 0$,

$$K^2 + \left(\sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon - c\right) K + \frac{1}{\varphi-\tau}\int_\tau^\varphi \frac{q(s)}{s}\,ds < 0 \quad \text{for every} \quad \varphi \in (\tau,1].$$

A continuity argument in [18] implies that there exists $\overline{\tau}$ such that for any $\tau < \overline{\tau}$ we have

$$\frac{f(\varphi)-f(\tau)}{\varphi-\tau} \le \frac{f(\varphi)}{\varphi} + \varepsilon \le \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} + \varepsilon, \qquad \varphi \in (\tau,1],$$

and thus, for such values of τ, it must hold

$$K^2 + \left(\frac{f(\varphi)-f(\tau)}{\varphi-\tau} - c\right) K + \frac{1}{\varphi-\tau}\int_\tau^\varphi \frac{q(s)}{s}\,ds < 0 \quad \text{for every} \quad \varphi \in (\tau,1].$$

This implies that the function $\eta_\tau = \eta_\tau(\varphi)$, defined for $\varphi \in [\tau,1]$ by

$$\eta_\tau(\varphi) := -K\tau + \int_\tau^\varphi \left\{h(\sigma) - c - \frac{q(\sigma)}{-K\sigma}\right\} d\sigma,$$

is an upper-solution of $(3.11)_1$ such that $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau,1]$, and $\eta_\tau(\tau) = -K\tau < 0$. Arguments based on Lemma 3.2 *(2.a.ii)* imply that it results defined in $[\tau,1]$ a function z_τ which solves $(3.4)_2$ with $\mu = -K\tau$; we extend continuously z_τ to $[0,\tau]$ by $z_\tau(\varphi) = -K\varphi$, for $\varphi \in [0,\tau]$. This gives a family $\{z_\tau\}_{\tau>0}$ of decreasing functions as $\tau \to 0^+$ (in the sense that $z_{\tau_1} \le z_{\tau_2}$ in $[0,1]$ for $0 < \tau_1 < \tau_2$). After some manipulations of the differential equation in $(3.4)_2$, based on the sign of q/z_τ and on $\eta_\tau(\varphi) < -K\varphi$, for $\varphi \in (\tau,1]$, we deduce that

$$f(\varphi) - c\varphi \le z_\tau(\varphi) \le -K\varphi, \qquad \varphi \in [0,1].$$

Hence, applying Lemma 3.3 in each interval $(a,b) \subset [0,1]$ we finally deduce that $\bar{z}$, the limit of z_τ for $\tau \to 0^+$, solves $(3.11)_1$, $\bar{z} < 0$ in $(0,1)$ and $\bar{z}(0) = 0$. Hence, $\bar{z}$ is a solution of (3.11). Finally, as observed in [18], an application of [17, Lemma 2.1] implies the conclusion. □

We now give a result about solutions to (3.12); see Figure 5.1 on the left.

Proposition 4.2. *Assume* (q). *Then, there exists* c^* *satisfying*

$$h(0) + 2\sqrt{\liminf_{\varphi\to 0^+} \frac{q(\varphi)}{\varphi}} \le c^* \le 2\sqrt{\sup_{\varphi\in(0,1]} \frac{q(\varphi)}{\varphi}} + \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi}, \tag{4.3}$$

such that there exists a unique z *satisfying* (3.12) *if and only if* $c \ge c^*$.

Proof. The result, apart from the refined estimate (4.3) is proved in [17, Proposition 1]. Estimate (4.3) follows from Lemma 4.1 and $\sup_{\varphi\in(0,1]} f(\varphi)/\varphi \le \max_{\varphi\in[0,1]} h(\varphi)$. □

5 The singular problem with left boundary condition

Now we face problem (3.11). We always assume (q) and refer to the threshold c^* introduced in Proposition 4.2; we denote by z^* the corresponding unique solution to (3.12). See Figure 5.1 on the left for an illustration of Proposition 5.1.

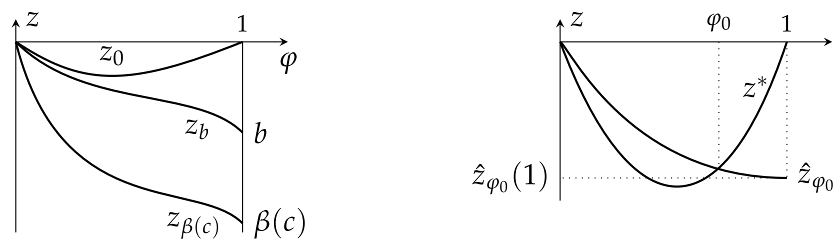

Figure 5.1: Left: an illustration of Propositions 4.2 and 5.1, for fixed $c > c^*$. Solutions to (3.11) are labelled according to their right-hand limit: z_0 occurs in the former proposition, z_b in the latter. Right: the functions $\hat z_{\varphi_0}$ and z^* in *Step (i)* of Proposition 5.1.

Proposition 5.1. *Assume* (q). *For every* $c > c^*$, *there exists* $\beta = \beta(c) < 0$ *satisfying*

$$\beta \ge f(1) - c, \tag{5.1}$$

such that problem (3.11) *with the additional condition* $z(1) = b < 0$ *admits a unique solution* z *if and only if* $b \ge \beta$.

In the above proposition, the threshold case $c = c^*$ is a bit more technical; we shall prove in Proposition 6.3 that $\beta(c^*) = 0$ under some further assumptions.

Proof of Proposition 5.1. For any $c > c^*$, we define the set $\mathcal{A}_c$ as

$$\mathcal{A}_c := \{b < 0 : (3.11) \text{ admits a solution with } z(1) = b\}.$$

We show that $\mathcal{A}_c = [\beta, 0)$, for some $\beta = \beta(c) < 0$, by dividing the proof into four steps.

Step (i): $\mathcal{A}_c \neq \varnothing$. We claim that there exists $\hat z$ which satisfies (3.11) and $\hat z(1) < 0$. Take $\varphi_0 \in (0,1)$ and consider the following problem, see Figure 5.1 on the right,

$$\begin{cases} \dot z(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, \\ z(\varphi_0) = z^*(\varphi_0). \end{cases} \tag{5.2}$$

Lemma 3.2 *(1)* implies the existence of a solution $\hat{z}_{\varphi_0}$ of (5.2) defined in its maximal-existence interval $(0,\delta)$, for some $\varphi_0 < \delta \le 1$. Since $\hat{z}_{\varphi_0}$ satisfies $(5.2)_1$ and $c > c^*$, then

$$\dot{\hat{z}}_{\varphi_0}(\varphi) = h(\varphi) - c^* - \frac{q(\varphi)}{\hat{z}_{\varphi_0}(\varphi)} + (c^* - c) < h(\varphi) - c^* - \frac{q(\varphi)}{\hat{z}_{\varphi_0}(\varphi)}, \qquad \varphi \in (0,\delta).$$

This implies that $\hat{z}_{\varphi_0}$ is a strict lower-solution of $(3.11)_1$ with $c = c^*$. From Lemma 3.2 *(2.b)*, this and $\hat{z}_{\varphi_0}(\varphi_0) = z^*(\varphi_0) < 0$ imply that

$$z^* < \hat{z}_{\varphi_0} \ \text{ in } \ (0,\varphi_0) \quad \text{ and } \quad \hat{z}_{\varphi_0} < z^* \ \text{ in } \ (\varphi_0,\delta). \tag{5.3}$$

Since $z^* < \hat{z}_{\varphi_0} < 0$ in $(0,\varphi_0)$, we get $\hat{z}_{\varphi_0}(0^+) = 0$. Since $\hat{z}_{\varphi_0} < z^*$ in (φ_0,δ), we obtain that $\hat{z}_{\varphi_0}(\delta^-) \le z^*(\delta^-)$. Thus $\delta = 1$, otherwise $\hat{z}_{\varphi_0}(\delta) < 0$, in contradiction with the fact that $(0,\delta)$ is the maximal-existence interval of $\hat{z}_{\varphi_0}$.

From Lemma 3.1, $\hat{z}_{\varphi_0}(1) \in \mathbb{R}$. It remains to prove that $\hat{z}_{\varphi_0}(1) < 0$. From what we observed above, it follows that $z^* > \hat{z}_{\varphi_0}$ in $(\varphi_0, 1)$. Hence, for any $\varphi \in (\varphi_0,1)$, we have

$$\dot{z^*}(\varphi) - \dot{\hat{z}}_{\varphi_0}(\varphi) = c - c^* + \frac{q(\varphi)}{z^*(\varphi)\hat{z}_{\varphi_0}(\varphi)}\left(z^* - \hat{z}_{\varphi_0}\right)(\varphi) > \frac{q(\varphi)}{z^*(\varphi)\hat{z}_{\varphi_0}(\varphi)}\left(z^* - \hat{z}_{\varphi_0}\right)(\varphi) > 0.$$

This implies that $(z^* - \hat{z}_{\varphi_0})$ is strictly increasing in $(\varphi_0,1)$ and hence

$$-\hat{z}_{\varphi_0}(1) = z^*(1) - \hat{z}_{\varphi_0}(1) > z^*(\varphi_0) - \hat{z}_{\varphi_0}(\varphi_0) = 0,$$

which means $\hat{z}_{\varphi_0}(1) < 0$. Thus, $\hat{z}_{\varphi_0}(1) \in \mathcal{A}_c$.

Step (ii): if $b \in \mathcal{A}_c$ then $[b,0) \subset \mathcal{A}_c$. Suppose that there exists $b \in \mathcal{A}_c$ and let z_b be the solution of (3.11) and $z_b(1) = b$. Take $b < b_1 < 0$. For Lemma 3.2 *(1.a)* there exists z_{b_1} defined in $(0,1)$ satisfying $(3.11)_1$ and $z_{b_1}(1) = b_1 < 0$.

We claim that $z_b < z_{b_1}$ in $(0,1)$. If not, then $z_b(\varphi_0) = z_{b_1}(\varphi_0) =: y_0 < 0$, for some $\varphi_0 \in (0,1)$. Without loss of generality we can assume $z_b < z_{b_1}$ in $(\varphi_0,1]$. We denote by $f_c(\varphi,y) = h(\varphi) - c - q(\varphi)/y$ the right-hand side of the differential equation in (3.11); the function f_c is continuous in $[0,1] \times (-\infty,0)$ and locally Lipschitz-continuous in y. Hence, z_b and z_{b_1} are two different solutions of

$$\begin{cases} y' = f_c(\varphi,y), & \varphi \in (\varphi_0,1), \\ y(\varphi_0) = y_0, \end{cases}$$

which contradicts the uniqueness of the Cauchy problem. Thus, $z_b < z_{b_1} < 0$ in $(0,1)$. Since z_b satisfies $(3.11)_3$ then $z_{b_1}(0^+) = 0$ and hence $b_1 \in \mathcal{A}_c$.

Step (iii): $\inf \mathcal{A}_c \in \mathbb{R}$. Suppose that z satisfies Equation $(3.11)_1$. As already observed, this implies $\dot{z}(\varphi) > h(\varphi) - c$, $\varphi \in (0,1)$. Thus, for any $\varphi \in (0,1)$,

$$z(\varphi) = z(\varphi) - z(0) \ge \int_0^{\varphi} h(\sigma) - c \, d\sigma = f(\varphi) - c\varphi. \tag{5.4}$$

This implies that $z(1) \ge f(1) - c$. Define $\beta = \beta(c)$ by

$$\beta := \inf \mathcal{A}_c.$$

Thus, $\beta \ge f(1) - c > -\infty$, which also proves (5.1).

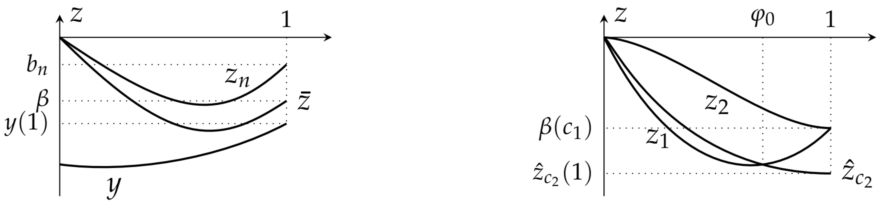

Figure 5.2: Left: the functions z_n, y and $\bar{z}$ in *Step (iv)* of Proposition 5.1. Right: the functions z_1, z_2 and $\hat{z}_{c_2}$ in the proof of *(i)* of Corollary 5.3.

Step (iv): $\beta \in \mathcal{A}_c$. Let $\{b_n\}_n \subset \mathcal{A}_c$ be a strictly decreasing sequence such that $b_n \to \beta^+$. Since $b_n \in \mathcal{A}_c$, each b_n is associated with a solution z_n of (3.11) and $z_n(1) = b_n$. From the uniqueness of the solution of Cauchy problem for $(3.11)_1$, the sequence z_n is decreasing.

For any given $\delta < \beta$, let y be defined by

$$\begin{cases} \dot{y}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{y(\varphi)}, & \varphi < 1 \\ y(1) = \delta < \beta. \end{cases}$$

Such a y exists and is defined in $[0,1]$ from Lemma 3.2 *(1.a)*. Also, $b_n > \delta$, for any $n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, $z_n \geq y$ in $[0,1]$. Lemma 3.3 implies that there exists $\bar{z}$ satisfying (3.1) such that $z_n \to \bar{z}$ uniformly in $[0,1]$ (see Figure 5.2 on the left). In particular, we deduce that $\bar{z}(0) = 0$ and $\bar{z}(1) = \beta$. Hence, we conclude that $\beta \in \mathcal{A}_c$.

Putting together *Steps (i)–(iv)*, we conclude that $\mathcal{A}_c = [\beta, 0)$. □

The monotonicity of solutions of (3.11) now follows. We omit the proof since it is quite standard, once that Lemma 3.2 *(2)* is given. (See [6, Lemma 5.1].)

Corollary 5.2 (Monotonicity of solutions)**.** *Assume* (q)*. Let $c_2 > c_1 \geq c^*$ and assume that z_1 and z_2 satisfy* (3.11) *with $c = c_1$ and $c = c_2$, respectively. Then, if $z_1(1) \leq z_2(1)$ it occurs that $z_1 < z_2$ in $(0,1)$.*

A monotony property of $\beta(c)$ now follows.

Corollary 5.3. *Under* (q) *we have:*

(i) $\beta(c_2) < \beta(c_1)$ *for every* $c_2 > c_1 > c^*$*;*

(ii) $\beta(c) \to -\infty$ *as* $c \to +\infty$*.*

Proof. To prove *(i)*, let z_1 be a solution of (3.11) corresponding to $c = c_1$ and such that $z_1(1) = b_1 \in \mathcal{A}_{c_1}$. As a consequence of Lemma 3.2 *(1.a)*, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(1) = b_1 < 0, \end{cases}$$

admits a (unique) solution z_2 defined in $[0,1]$. From the monotonicity of solutions given by Corollary 5.2, we have $z_1 < z_2 < 0$ in $(0,1)$. Since $z_1(0) = 0$, then we have $z_2(0) = 0$. Thus, $\mathcal{A}_{c_1} \subseteq \mathcal{A}_{c_2}$ and hence $\beta(c_1) \geq \beta(c_2)$. To prove $\beta(c_1) > \beta(c_2)$ we argue as follows.

For any $\varphi_0 \in (0,1)$ we can repeat the same arguments as in *Step (i)* of Proposition 5.1, by replacing c with c_2 and z^* with z_1 in (5.2). Thus, the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c_2 - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi_0) = z_1(\varphi_0) < 0, \end{cases}$$

admits a unique solution $\hat{z}_{c_2}$ defined in $[0,1]$, because necessarily any solution of the last problem must be bounded from above by z_2, see Figure 5.2 on the right. Moreover, by applying Lemma 3.2 *(2.b.ii)*, $\hat{z}_{c_2} < z_1$ in $(\varphi_0, 1)$, which implies that $\hat{z}_{c_2}(1) < z_1(1)$, since

$$\dot{\hat{z}}_{c_2}(\varphi) - \dot{z_1}(\varphi) = c_1 - c_2 + \frac{q(\varphi)}{z_1(\varphi)\hat{z}_{c_2}(\varphi)} \left(\hat{z}_{c_2}(\varphi) - z_1(\varphi)\right) < 0 \quad \text{for any} \quad \varphi \in (\varphi_0, 1).$$

Since $\beta(c_2) \le \hat{z}_{c_2}(1) < z_1(1) = b_1$ then we proved *(i)* since b_1 is arbitrary in $\mathcal{A}_{c_1}$.

Finally, we prove *(ii)*. For $c > c^*$, let z_c be the solution of (3.11) such that $z_c(1) = \beta(c)$. For any fixed $c_1 > c^*$, we have $z_c < z_{c_1}$ in $(0,1)$, if $c > c_1$. Thus, for any $c > c_1$,

$$\dot{z}_c(\varphi) = h(\varphi) - c + \frac{q(\varphi)}{-z_c(\varphi)} < h(\varphi) - c + \frac{q(\varphi)}{-z_{c_1}(\varphi)}, \qquad \varphi \in (0,1).$$

In particular, since $z_{c_1} < 0$ in $(0,1]$, then, for any $0 < \delta < 1$, there exists $M > 0$ such that $q(\varphi)/(-z_{c_1}(\varphi)) \le M$ for any $\varphi \in (\delta, 1]$. Thus, for any $\varphi \in (\delta, 1)$,

$$z_c(\varphi) \le z_c(\delta) + f(\varphi) - f(\delta) + (M - c)(\varphi - \delta) < f(\varphi) - f(\delta) + (M - c)(\varphi - \delta),$$

which implies $\beta(c) = z_c(1) \le f(1) - f(\delta) + (M - c)(1 - \delta)$. This proves *(ii)*. □

We now collect some consequences of (5.4) and Lemma 4.1, concerning a sharper estimate to c^*. To the best of our knowledge these estimates are new and we provide some comments.

Corollary 5.4. *Assume* (q). *It holds that*

$$c^* \ge \max\left\{ \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}, h(0) + 2\sqrt{\liminf_{\varphi \to 0^+} \frac{q(\varphi)}{\varphi}} \right\}. \tag{5.5}$$

Proof. Formula (5.4) in *Step (iii)* implies that $f(\varphi) < c\varphi$, for $\varphi \in (0,1)$. Thus, $f(\varphi) \le c^*\varphi$, for $\varphi \in (0,1)$. This implies $c^* \ge \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi}$, which, together with (4.3) implies (5.5). □

Remark 5.5. Lemma 4.1 and Corollary 5.4 imply that, under (q), the threshold c^* verifies (2.2). Moreover, make the assumption $\dot{q}(0) = 0$, which is valid if $q = Dg$ under (D1), with $D(0) = 0$, (g0) or under (D0) and (g01). In this case, the estimates in (2.8) hold true. Indeed, the assumptions on q are covered by [18, Theorem 3.1] and hence it follows that

$$c^* \le \sup_{\varphi \in (0,1]} \frac{f(\varphi)}{\varphi} + 2\sqrt{\sup_{\varphi \in (0,1]} \frac{1}{\varphi} \int_0^{\varphi} \frac{q(\sigma)}{\sigma}\, d\sigma}.$$

The bound from above in (2.8) is then proved. The bound from below in (2.8) is instead due directly to (5.5), because of $\dot{q}(0) = 0$.

Remark 5.6. We can now make precise the statement following formula (2.7) about the gap between c_{con} and c^*. If c_{con} is obtained at some $\varphi \in (0,1]$, then the sup in the right-hand side of (2.7) is strictly larger than c_{con} because $z < 0$ in $(0,1)$. Then $c^* > c_{con}$. Otherwise, if $\sup_{\varphi \in (0,1]} f(\varphi)(\varphi) = h(0)$, then $c_{con} = h(0)$ and by (5.5) we still deduce $c^* > c_{con}$.

6 Further existence and non-existence results

Propositions 4.2 and 5.1 completely treat the existence of solutions of (3.12) and (3.11), respectively, in the cases $c \geq c^*$ and $c > c^*$. In this section, we investigate the remaining cases and show that such propositions are somehow optimal.

We now deal with the following problem, where $c \in \mathbb{R}$ but, differently from (3.11), the boundary condition is imposed on the *right* extremum of the interval of definition:

$$\begin{cases} \dot{\zeta}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{\zeta(\varphi)}, & \varphi \in (0,1), \\ \zeta(\varphi) < 0, & \varphi \in (0,1), \\ \zeta(1) = 0. \end{cases} \tag{6.1}$$

The differential equation in (3.11) and (6.1) is the same; it inherits the properties of the dynamical system underlying (1.3). For slightly more regular functions g, the dynamical system has a center or a node at $(0,0)$ and a saddle at $(1,0)$. The corresponding results, Proposition 5.1 and Lemma 6.1, differ as in Lemma 3.2 *(1)*.

Moreover, while in problem (3.11) the threshold c^* discriminated the existence of solutions, for problem (6.1) solutions will be proved to exist *for every* $c \in \mathbb{R}$; instead, the threshold c^* enters into the problem to discriminate whether solutions reach 0 or not (see Figure 6.1). A related behavior was pointed out in [6, Theorem 2.6]. On the contrary, the monotonicity properties stated in Corollary 5.2 and in Lemma 6.1 are the same.

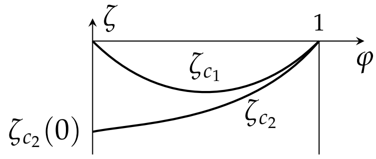

Figure 6.1: An illustration of Lemma 6.1. Here, $c_1 \geq c^*$ while $c_2 < c^*$ and $\zeta_{c_2}(0) < 0$.

Lemma 6.1. *Assume* (q). *For any* $c \in \mathbb{R}$, *Problem* (6.1) *admits a unique solution* ζ_c. *If* $c \geq c^*$ *then* $\zeta_c(0) = 0$ *and if* $c < c^*$ *then* $\zeta_c(0) < 0$. *Moreover, we have:*

(i) if $c_2 > c_1$ *then* $\zeta_{c_2} > \zeta_{c_1}$ *in* $(0,1)$;

(ii) it holds that $z^*(\varphi) = \lim_{c \to c^*} \zeta_c(\varphi)$ *for any* $\varphi \in [0,1]$.

Proof. The existence and uniqueness was proved in [6, Theorem 2.6], while the monotonicity as stated in *(i)* was given in [6, Lemma 5.1]. It remains to prove *(ii)*. We show that

$$\lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi) = z^*(\varphi) \quad \text{for } \varphi \in [0,1].$$

For any $\varphi \in [0,1]$, by *(i)* we have

$$\zeta_{c^*-\delta_2}(\varphi) < \zeta_{c^*-\delta_1}(\varphi) < z^*(\varphi) < \zeta_{c^*+\delta_1}(\varphi) < \zeta_{c^*+\delta_2}(\varphi) \text{ for any } 0 < \delta_1 < \delta_2. \tag{6.2}$$

Lemma 3.3 and (6.2) imply that there exist two functions $\overline{w}, \underline{w} \in C^0[0,1] \cap C^1(0,1)$ so that $\overline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*+\delta}(\varphi)$ and $\underline{w}(\varphi) = \lim_{\delta \to 0^+} \zeta_{c^*-\delta}(\varphi)$, $\varphi \in [0,1]$, and that both $\underline{w}$ and $\overline{w}$ satisfy (3.1) with $c = c^*$. Since $\underline{w}(1) = \overline{w}(1) = 0$, both of them then solve (6.1). By the uniqueness of solutions of (6.1) it follows that $\underline{w} = \overline{w} = z^*$. $\square$

Remark 6.2. Note that, because of the uniqueness stated in Lemma 6.1, it follows that, for any $c \geq c^*$, the solution z given by Proposition 4.2 corresponds to ζ_c of Lemma 6.1. Moreover, for $c < c^*$ fixed, there exists a bound from below for $\zeta_c(0) < 0$. We have

$$\zeta_c(0) \geq -1 - A_c, \qquad \text{for } A_c := \max\{\max_{\varphi\in[0,1]} h(\varphi) - c, 0\} + \max_{\varphi\in[0,1]} q(\varphi) > 0.$$

Indeed, the function $\eta(\varphi) := A_c\,(\varphi - 1) - 1$, for $\varphi \in [0,1]$, is a strict upper-solution of $(6.1)_1$. Therefore, if $\zeta_c(\varphi_0) \leq \eta(\varphi_0)$, for some $\varphi_0 \in (0,1)$, then $\zeta_c < \eta$ in $(\varphi_0, 1)$ by Lemma 3.2 *(2.a.ii)*, which is in contradiction with $\zeta_c(1) = 0 > \eta(1)$. Thus, $\zeta_c(0) \geq \eta(0) = -A_c - 1$. Notice that, for $c \geq \max h$, $A_c = \max q$ does not depend on c, while $A_c \to \infty$, as $c \to -\infty$.

We now show that $\beta(c^*) = 0$ under some additional conditions. First, we assume (also for future reference) that $\dot{q}(0)$ exists:

$$\dot{q}(0) = \lim_{\varphi\to 0^+} \frac{q(\varphi)}{\varphi} \in [0,\infty). \tag{6.3}$$

Proposition 6.3. *Assume* (q), (6.3) *and also*

$$\int_0 \frac{q(\sigma)}{\sigma^2}\, d\sigma < +\infty \quad \text{and} \quad c^* > h(0). \tag{6.4}$$

Then Problem (3.11) *with $c = c^*$ admits a unique solution z, which satisfies $z(1) = 0$.*

Notice that $(6.4)_1$ above strengthens the last condition in (q) and is satisfied if $\dot{q}(\varphi) = O(\varphi^\alpha)$ for $\varphi \to 0^+$, for some $\alpha > 0$; in any case it implies $\dot{q}(0) = 0$ by (6.3).

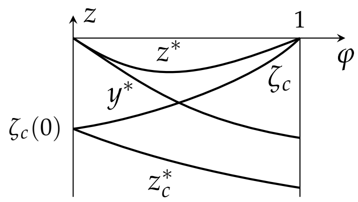

Figure 6.2: The functions z^*, ζ_c, y^* and z_c^*, for $c < c^*$.

Proof of Proposition 6.3. Suppose, by contradiction, that there exists y^* which solves (3.11) with $c = c^*$ and $y^*(1) < 0$; observe that

$$z^* > y^* \quad \text{in } (0,1]. \tag{6.5}$$

We show that y^* is an upper bound for the family of functions $\{z_c^*\}_{c<c^*}$ defined as follows, see Figure 6.2. For any $c < c^*$, let ζ_c be the solution of (6.1), given in Lemma 6.1. Consider the initial-value problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c^* - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(0) = \zeta_c(0) < 0. \end{cases} \tag{6.6}$$

By Lemma 3.2 *(1.b)*, problem (6.6) admits a unique solution z_c^* in $[0,\delta]$ for some $\delta \leq 1$. Moreover, since $z_c^*(0) < 0$ and z_c^* satisfies (6.6), then $z_c^* < y^*$ in $[0,\delta)$. Thus, if $\delta < 1$ then we have $-\infty < z_c^*(\delta) < y^*(\delta) < 0$; again by Lemma 3.2 *(1.b)* we deduce $\delta = 1$. Then

$$y^* > z_c^* \quad \text{in } [0,1). \tag{6.7}$$

By both Lemma 6.1 *(ii)* and (6.7) we now find a contradiction, which implies that such a y^* cannot exist. For this, for any $c < c^*$, define η_c by

$$\eta_c(\varphi) = \zeta_c(\varphi) - z_c^*(\varphi), \qquad \varphi \in [0,1].$$

Since z_c^* is a strict lower-solution of $(3.11)_1$, then Lemma 3.2 *(2.b.ii)* implies $\eta_c > 0$ in $(0,1)$. We claim that, for any fixed $\varphi_0 \in (0,1]$, $\eta_c(\varphi_0)$ is uniformly bounded from below for c close to c^*. Indeed, for any $0 < \delta < (z^* - y^*)(\varphi_0)$, we clearly have, by (6.7) and (6.5),

$$\eta_c(\varphi_0) > \zeta_c(\varphi_0) - y^*(\varphi_0) = (\zeta_c - z^*)\,(\varphi_0) + (z^* - y^*)\,(\varphi_0) > (\zeta_c - z^*)\,(\varphi_0) + \delta.$$

Thus, in virtue of Lemma 6.1 *(ii)*, for any c sufficiently close to c^*, we have

$$\eta_c(\varphi_0) \geq \frac{\delta}{2} > 0, \tag{6.8}$$

which proves our claim. On the other hand, define $k = k(\varphi) > 0$ by

$$k(\varphi) := \frac{q(\varphi)}{(z^* y^*)\,(\varphi)}, \qquad \varphi \in (0,1).$$

From assumptions (6.3) and $(6.4)_2$ we deduce $\dot{z}^*(0) = h(0) - c^* < 0$ because of [6, Proposition 5.2]. Also, by (6.5) we deduce that $y^* z^* > z^{*2}$ in $(0,1]$. Thus,

$$k(\varphi) < \frac{q(\varphi)}{\varphi^2}\left(\frac{\varphi}{z^*(\varphi)}\right)^2 = \frac{q(\varphi)}{\varphi^2}\left\{\frac{1}{(c^* - h(0))^2} + o(1)\right\} \quad \text{for } \ \varphi \to 0^+.$$

This leads to

$$\int_0^{\varphi_0} k(\sigma)\, d\sigma =: M < +\infty$$

by means of $(6.4)_1$. Since ζ_c and z_c^* satisfy $(3.11)_1$ with $c < c^*$ and $c = c^*$, respectively, and since $\zeta_c z_c^* > z^* y^*$ by the monotonicity stated in Lemma 6.1 and (6.7), then

$$\dot{\eta}_c(\varphi) = c^* - c - \frac{q(\varphi)}{\zeta_c(\varphi) z_c^*(\varphi)}\,(z_c^*(\varphi) - \zeta_c(\varphi)) < c^* - c + k(\varphi)\eta_c(\varphi),$$

for $\varphi \in (0,1)$. After some straightforward manipulations, this gives

$$\frac{d}{d\varphi}\left(\eta_c(\varphi) e^{-\int_0^{\varphi} k(\sigma)\, d\sigma}\right) \leq (c^* - c)\, e^{-\int_0^{\varphi} k(\sigma)\, d\sigma}, \qquad \varphi \in (0,1).$$

By integrating in $(0, \varphi_0)$ (where φ_0 is the point for which (6.8) holds) we obtain

$$0 < \eta_c(\varphi_0) \leq (c^* - c)\, e^{\int_0^{\varphi_0} k(\sigma)\, d\sigma} \int_0^{\varphi_0} e^{-\int_0^{\sigma} k(s)\, ds}\, d\sigma \leq (c^* - c)\, e^M \varphi_0, \tag{6.9}$$

since $e^{-\int_0^{\sigma} k(s)\, ds} \leq 1$, for any $0 < \sigma < \varphi_0$, because of $k > 0$. Since M does not depend on c, from (6.9), we conclude that $\eta_c(\varphi_0) \to 0$, for $c \to c^*$. This contradicts (6.8). □

We notice that if $q = Dg$, with $D \in C^1[0,1]$, then $(6.4)_1$ follows if we have both $D(0) = 0$ and there exists $L \geq 0$ such that $g(\varphi) \leq L\varphi^\alpha$ for any φ in a right neighborhood of 0 and some $\alpha > 0$. The next remark deals with $(6.4)_2$.

Remark 6.4. First, from (4.3), we have $c^* \geq \sup_{\varphi\in(0,1]} \frac{f(\varphi)}{\varphi} \geq h(0)$. We show that the case $c^* = h(0)$ can indeed occur and then $(6.4)_2$ is a real assumption. Set, for $\varphi \in (0,1)$,

$$q(\varphi) = \varphi^3 (1-\varphi), \qquad h(\varphi) = 3\varphi(\varphi - 1), \tag{6.10}$$

and $z(\varphi) = \varphi^2(\varphi - 1)$. Direct computations show that z satisfies (3.11) with $c = 0 = h(0)$. Hence, $c^* = h(0)$, because of $c^* \geq h(0)$.

Second, in the spirit of [16, Theorems 1.2 and 1.3], which concerns a similar case, we claim that $(6.4)_2$ occurs if there exists $\delta > 0$ such that $h(\varphi) \geq h(0)$ for all $\varphi \in [0,\delta]$. Indeed, if z is a solution of (3.11) with $c = c^*$, then from $(3.11)_1$ we have $\dot{z}(\varphi) > h(\varphi) - c^* \geq h(0) - c^*$, for $\varphi \in (0,\delta)$. This implies $h(0) - c^* \leq \inf_{\varphi\in(0,\delta)} \dot{z}(\varphi) < 0$, because of $(3.11)_2$ and $(3.11)_3$, which proves our claim.

Lastly, we show by a counter-example that the conclusion of Proposition 6.3 fails when $(6.4)_1$ holds but $(6.4)_2$ *does not*. Consider, for $\varphi \in [0,1]$, $q(\varphi) = \varphi^4(1-\varphi)$ and $y^*(\varphi) = -\varphi^2$. Clearly, $y^* < 0$ in $(0,1)$ and $y^*(0) = 0$. Furthermore, we have

$$\dot{y}^*(\varphi) + \frac{q(\varphi)}{y^*(\varphi)} = -2\varphi - \varphi^2(1-\varphi), \qquad \varphi \in (0,1).$$

This implies that y^* satisfies $(3.11)_1$ with $h(\varphi) = -2\varphi - \varphi^2(1-\varphi)$ and $c = 0$. As a consequence, by $c^* \geq h(0) = 0$, we deduce $c^* = h(0) = 0$. Thus, we proved that there exists q satisfying $(6.4)_1$ such that (3.11) with $c = c^* = h(0)$ admits a solution $y^* \neq z^*$.

Proposition 6.5. *Assume* (q). *For no $c < c^*$ problem* (3.11) *admits solutions.*

Proof. Take $c < c^*$ and assume by contradiction that problem (3.11) has a solution z. If $\zeta = \zeta_c$ is the solution to (6.1) given by Lemma 6.1, then $\zeta(0) < 0$, by Proposition 4.2. Then $\zeta(\varphi_0) = z(\varphi_0) =: y_0 < 0$, for some $\varphi_0 \in (0,1)$; see Figure 6.3. This contradicts the uniqueness of the Cauchy problem associated to $(6.1)_1$. The proof is concluded.

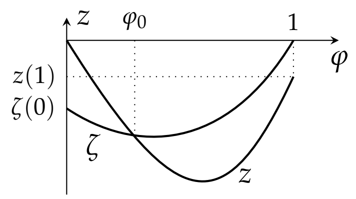

Figure 6.3: The functions z and ζ.

□

7 The behavior of z near 1

In this section and in the next one we investigate the behavior of the solutions z to (3.11) at 1 and 0. We now deal with the former case. We suppose that, analogously to (6.3),

$$\dot{q}(1) \in (-\infty, 0]. \tag{7.1}$$

Proposition 7.1. *Assume* (q) *and* (7.1)*; consider* $c \geq c^*$ *and let* z *be a solution of* (3.11)*. Then,* $\dot{z}(1)$ *exists and it holds that*

(i) if $z(1) \in [\beta, 0)$*, then* $\dot{z}(1) = h(1) - c$*;*

(ii) if $z(1) = 0$*, then*

$$\dot{z}(1) = \begin{cases} \frac{1}{2}\left[h(1) - c + \sqrt{(h(1) - c)^2 - 4\dot{q}(1)}\right] & \text{if } \dot{q}(1) < 0, \\ \max\{0, h(1) - c\} & \text{if } \dot{q}(1) = 0. \end{cases} \tag{7.2}$$

Proof. In case *(i)*, we only need to take the limit for $\varphi \to 1^-$ in $(3.11)_1$.

In case *(ii)*, the proof of the existence of $\dot{z}(1)$ is analogous to the proof of [16, Lemma 2.1], even if in that paper there is the further assumption $\dot{q}(1) = 0$. In the current case, $\dot{z}(1)$ must coincide with one of the roots of the equation $\gamma^2 - (h(1) - c)\,\gamma + \dot{q}(1) = 0$, which are

$$r_{\pm} := \frac{h(1) - c \pm \sqrt{(h(1) - c)^2 - 4\dot{q}(1)}}{2}.$$

A direct check shows that the right-hand side of (7.2) corresponds exactly to r_+. Thus, if we prove that $\dot{z}(1) = r_+$ then we conclude the proof.

If $\dot{q}(1) < 0$, the fact that $r_- < 0$ implies necessarily that $\mu = r_+$, because of $\dot{z}(1) \geq 0$.

Let $\dot{q}(1) = 0$. Since we do not yet know whether $\dot{z}$ is continuous at 1 (see Remark 9.2), we argue as follows. For any $\varphi \in (0, 1)$, by the Mean Value Theorem there exists $\sigma_\varphi \in (\varphi, 1)$ satisfying $\dot{z}(\sigma_\varphi) = \frac{z(\varphi)}{\varphi - 1}$. By the definition of $\dot{z}(1)$ it then follows that

$$\lim_{\varphi \to 1^-} \dot{z}(\sigma_\varphi) = \dot{z}(1) \quad \text{and} \quad \lim_{\varphi \to 1^-} \frac{z(\sigma_\varphi)}{\sigma_\varphi - 1} = \dot{z}(1). \tag{7.3}$$

From $(3.11)_1$, the sign conditions in $(3.2)_2$ and $(3.11)_2$ imply that

$$\dot{z}(\sigma_\varphi) > h(\sigma_\varphi) - c, \qquad \varphi \in (0, 1). \tag{7.4}$$

By (7.3), passing to the limit as $\varphi \to 1^-$ gives $\dot{z}(1) \geq h(1) - c$, because of the continuity of h at 1. Moreover, since $\dot{z}(1) \geq 0$ it holds that $\dot{z}(1) \geq \max\{0, h(1) - c\} = r_+$. This concludes the proof, since it necessarily follows that $\dot{z}(1) = r_+$ also in this case. □

Remark 7.2. We prove in Remark 9.2 that $z \in C^1(0, 1]$ under the assumptions of Proposition 7.1. We now show that (7.1) is necessary for the existence of $\dot{z}(1)$. We define

$$q(\varphi) = \varphi^3\,(1 - \varphi)\left[(\sin(\log(1 - \varphi)) + 2)^2 + 2\cos(\log(1 - \varphi)) + \frac{1}{2}\sin(2\log(1 - \varphi))\right],$$

for $\varphi \in [0, 1]$. The function q satisfies (q), while $\dot{q}(1)$ does not exist. Direct computations show that the function $z = z(\varphi)$ defined by $z(\varphi) = -\left(2 + \sin(\log(1 - \varphi))\right)(1 - \varphi)\,\varphi^2$ satisfies (3.11) with $c = 0$ and $h(\varphi) = \varphi(\varphi - 1)\left[\cos(\log(1 - \varphi)) + 3\sin(\log(1 - \varphi)) + 6\right]$. It is easy to verify that $\dot{z}(1)$ does not exist.

8 The behavior of z near 0

For $\varphi_0 \in (0,1)$ we consider the problem, see Figure 8.1 on the left,

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z(\varphi)}, & \varphi \in (0,1), \\ z(\varphi_0) = z^*(\varphi_0). \end{cases} \tag{8.1}$$

Lemma 8.1. *Assume* (q)*. Fix $c > c^*$. For every $\varphi_0 \in (0,1)$ there is a unique solution $\hat{z}_{\varphi_0} \in C[0,1] \cap C^1(0,1)$ to problem* (8.1)*. We have $\hat{z}_{\varphi_0}(0) = 0$, and also*

$$\hat{z}_{\varphi_0} < z^* \quad \textit{in } (\varphi_0, 1] \quad \textit{and} \quad \hat{z}_{\varphi_0} \geq z_\beta \quad \textit{in } (0,1], \tag{8.2}$$

where z_β is the solution to (3.11) *with $z_\beta(1) = \beta$. If $0 < \varphi_1 < \varphi_0$ then $\hat{z}_{\varphi_1} < \hat{z}_{\varphi_0}$ in $(0,1]$.*

Proof. The existence and uniqueness of solutions is proved by *Step (i)* in the proof of Proposition 5.1. Inequality $(8.2)_1$ follows from the arguments contained in *Step (i)* of the proof of Proposition 5.1, while $(8.2)_2$ is obvious.

If $0 < \varphi_1 < \varphi_0$ then $\hat{z}_{\varphi_1}(\varphi_0) < \hat{z}_{\varphi_0}(\varphi_0)$, because $\hat{z}_{\varphi_1} < z^*$ in $(\varphi_1, 1]$ and $\varphi_0 \in (\varphi_1, 1]$. The monotony follows by the uniqueness of solutions to the Cauchy problem associated to $(3.11)_1$. The regularity of $\hat{z}_{\varphi_0}$ follows from both $(8.1)_1$ and Lemma 3.1; directly from $(8.2)_2$, we deduce $\hat{z}_{\varphi_0}(0) = 0$. □

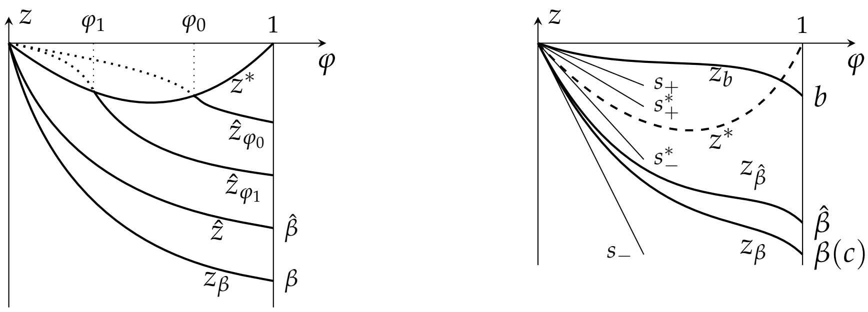

Figure 8.1: Left: the functions $\hat{z}_{\varphi_0}$, $\hat{z}$ and z_β in Lemma 8.1. Right: an illustration of Proposition 8.2 for fixed $c > c^*$. Solutions are labelled according to their right-hand limit; $s_\pm$ denote the slope of the tangent of z at 0. The dashed curve is the plot of z^*.

For every $c > c^*$, by the monotonicity of $\{\hat{z}_{\varphi_0}\}_{\varphi_0}$ and $(8.2)_2$, Lemma 3.3 implies that there exists $\hat{z} \in C^0[0,1] \cap C^1(0,1)$ which solves (3.1) such that

$$\hat{z}(\varphi) = \lim_{\varphi_0 \to 0^+} \hat{z}_{\varphi_0}(\varphi), \qquad \varphi \in [0,1]. \tag{8.3}$$

Such a $\hat{z}$ satisfies $z_\beta \leq \hat{z} \leq z^*$ in $(0,1)$ by (8.2) and then (3.11). Define $\hat{\beta} \in [\beta, 0)$ by

$$\hat{\beta} := \hat{z}(1). \tag{8.4}$$

In the following result we assume again (6.3). We shall prove in Remark 8.3 that such a condition is necessary for the existence of $\dot{z}(0)$. From (4.3) and (6.3) we deduce $(h(0)-c)^2 - 4\dot{q}(0) \geq 0$ for any $c \geq c^*$; we can then denote

$$s_\pm(c) := \frac{h(0)-c}{2} \pm \frac{\sqrt{(h(0)-c)^2 - 4\dot{q}(0)}}{2}, \quad \text{for } c \geq c^*.$$

The next proposition generalizes [6, Proposition 5.2] to the case of a more generic q, and, more deeply, to the case $z(1) < 0$. It is worth noting that this latter case reveals the behavior detected by (8.6), and shown in Figure 8.1 on the right, which was not contained in [6].

Proposition 8.2. *Assume* (q) *and* (6.3). *If $c \geq c^*$ and z is a solution of* (3.11), *then, $\dot z(0)$ exists. Moreover, it holds that*

$$\dot z(0) = \begin{cases} s_+(c) & \text{if } c > c^* \text{ and } z(1) > \hat\beta, \\ s_-(c^*) & \text{if } c = c^*, \end{cases} \tag{8.5}$$

and, if $c^* > h(0)$,

$$\dot z(0) = s_-(c) \quad \text{if } c > c^* \ \text{ and } \ z(1) \in [\beta, \hat\beta]. \tag{8.6}$$

Proof. Arguing as in the proof of [6, Proposition 5.2], we deduce that $\dot z(0)$ exists for $c \geq c^*$ and is one of the root of the equation $\gamma^2 - (h(0) - c)\,\gamma + \dot q(0) = 0$. Then $\dot z(0) \in \{s_-(c), s_+(c)\}$ for every $c \geq c^*$. Straightforward computations give

$$s_-(c) < s_-(c^*) \leq s_+(c^*) \leq s_+(c) \leq 0 \quad \text{for any } \ c > c^* \tag{8.7}$$

and $h(0) - c \leq s_-(c)$, for any $c \geq c^*$. We denote $s^*_\pm := s_\pm(c^*)$.

Take $c > c^*$. Let $\hat z_{\varphi_0}$ and $\hat z$ be defined as in the beginning of Section 8, see Figure 8.1 on the left. If $z(1) > \hat\beta$ then $z(1) > \hat z_{\varphi_1}(1)$, for some $\varphi_1 \in (0,1)$, because of (8.3). Thus, $z > \hat z_{\varphi_1}$ in $(0,1]$. We observed in (5.3) that $\hat z_{\varphi_1} > z^*$ in $(0,\varphi_1)$. Thus, $z > z^*$ in $(0,\varphi_1)$ and hence $\dot z(0) \geq \dot z^*(0)$. Since $s_-(c) < s^*_- \leq 0$ by (8.7), we deduce $\dot z(0) = s_+(c)$. This proves $(8.5)_1$.

Now, we prove $(8.5)_2$. If $z = z^*$, then $(8.5)_2$ was obtained in [6, Proposition 5.2] under some specific assumptions on q. Since the relevant ones were (3.2) and (6.3), we deduce that $(8.5)_2$ occurs also in the current case. If $z = y^*$ is a solution of (3.11), different from z^* (such a y^* can exist, as we proved in Remark 6.4, since (6.4) does not necessarily follow), then $y^* < z^*$ in $(0,1]$ by Proposition 4.2. Since $\dot y^*(0) \in \{s^*_-, s^*_+\}$ and $\dot z^*(0) = s^*_-$ then we have $\dot y^*(0) = s^*_-$. Hence, $(8.5)_2$ holds.

It remains to prove (8.6) under the additional condition $h(0) - c^* < 0$. By $\beta \leq z(1) \leq \hat\beta$ we have $z \leq \hat z$ and hence $z < z^*$, which implies $\dot z(0) \leq \dot z^*(0)$. Since, under the additional condition $h(0) - c^* < 0$, we have $s^*_- < s^*_+$ and since we proved that $\dot z^*(0) = s^*_-$, we conclude that $\dot z(0) = s_-(c)$, which is (8.6). This concludes the proof. □

Remark 8.3. Now, we prove that (6.3) is necessary for the existence of $\dot z(0)$. For $\varphi \in [0,1]$ define $q(\varphi) = \varphi(1-\varphi)^4\,(2 + \sin(\log\varphi))\,(3 - \cos(\log\varphi) - \sin(\log\varphi))$. The function q satisfies (q), while $\dot q(0)$ does not exist, since $\liminf_{\varphi\to 0^+} q(\varphi)/\varphi < \limsup_{\varphi\to 0^+} q(\varphi)/\varphi$. Direct computations show that the function $z(\varphi) = -\,(2 + \sin(\log\varphi))\,(1-\varphi)^2\,\varphi$ solves (3.11) with $c = 0$ and $h(\varphi) = 2\,(2 + \sin(\log\varphi))\,(1-\varphi)\,\varphi - 5\,(1-\varphi)^2$. Clearly, $\dot z(0)$ does not exists.

We now show that, under the assumptions of Proposition 6.3, the threshold $\hat\beta(c)$ defined in (8.4) and occurring in Proposition 8.2 coincides with the threshold $\beta(c)$ introduced in Proposition 5.1. It is an open problem whether the two thresholds differ without assuming (6.3) and (6.4).

Proposition 8.4. *Assume* (q), (6.3), (6.4) *and $c > c^*$. Then $\beta(c) = \hat\beta(c)$.*

Proof. Consider $\varepsilon > 0$ and let z_ε be the solution of

$$\begin{cases} \dot z_\varepsilon(\varphi) = h(\varphi) - c - \frac{q(\varphi)}{z_\varepsilon(\varphi)}, & \varphi > 0, \\ z_\varepsilon(0) = -\varepsilon < 0. \end{cases}$$

Lemma 3.2 *(1.b)* implies that z_ε exists and it is defined in its maximal-existence interval $[0,\delta]$, for some $0 < \delta \le 1$. By the uniqueness of solutions of the Cauchy problem associated to $(3.1)_1$, we have necessarily $z_\varepsilon < z_\beta$ in $[0,\delta]$, where z_β was defined in the statement of Lemma 8.1. Since $z_\beta(\delta) < 0$ then $\delta = 1$.

We claim that z_ε converges for $\varepsilon \to 0^+$ to both $\hat{z}$ and z_β, where $\hat{z}$ is defined in (8.3), see Figure 8.1 on the left. From the uniqueness of the limit, it follows that $\hat{z}$ and z_β coincides and hence that $\beta = \hat{\beta}$. To prove the claim, consider

$$\eta_\varepsilon(\varphi) := \hat{z}(\varphi) - z_\varepsilon(\varphi), \qquad \varphi \in [0,1].$$

Since $\hat{z} \ge z_\beta > z_\varepsilon$ in $[0,1]$, then $\eta_\varepsilon > 0$ in $[0,1]$. Moreover, $\eta_\varepsilon(0) = \varepsilon$. We have

$$\dot{\eta}_\varepsilon(\varphi) = \frac{q(\varphi)}{z_\varepsilon(\varphi)\hat{z}(\varphi)}\eta_\varepsilon(\varphi), \qquad \varphi \in (0,1).$$

Thus,

$$\frac{\dot{\eta}_\varepsilon(\varphi)}{\eta_\varepsilon(\varphi)} = \frac{q(\varphi)}{z_\varepsilon(\varphi)\hat{z}(\varphi)}, \qquad \varphi \in (0,1)$$

and hence, for any $0 < \tau < \varphi$,

$$\log\left(\eta_\varepsilon(\varphi)\right) - \log\left(\eta_\varepsilon(\tau)\right) = \int_\tau^\varphi \frac{q(s)}{z_\varepsilon(s)\hat{z}(s)}\,ds \le \int_\tau^1 \frac{q(s)}{z_\beta(s)\hat{z}(s)}\,ds. \tag{8.8}$$

Notice, from $(6.4)_2$ it follows that we can apply (8.6) with $\dot{q}(0) = 0$ (because of (6.3)) and obtain $z_\beta(s)\hat{z}(s) = \left(h(0) - c\right)^2 s^2 + o(s^2)$, as $s \to 0^+$. Hence, from $(6.4)_1$,

$$\sup_{\tau>0} \int_\tau^1 \frac{q(s)}{z_\beta(s)\hat{z}(s)}\,ds =: C < +\infty.$$

From (8.8), by taking the limit as $\tau \to 0^+$ we deduce $\eta_\varepsilon(\varphi) \le \varepsilon e^C$, $\varphi \in [0,1)$, and then

$$\lim_{\varepsilon\to 0^+} z_\varepsilon(\varphi) = \hat{z}(\varphi), \qquad \varphi \in [0,1). \tag{8.9}$$

We now apply Lemma 3.3 to deduce that z_ε converges (uniformly on $[0,1]$) to a solution $\bar{z}$ of $(3.1)_1$ in $(0,1)$ such that $\bar{z} < 0$ in $(0,1)$ and $\bar{z}(0) = 0$. Since $z_\varepsilon < z_\beta$ and z_β lies below every solution of (3.1), by the very definition of z_β, we conclude that $\bar{z}$ coincides with z_β, that is $\lim_{\varepsilon\to 0^+} z_\varepsilon(\varphi) = z_\beta(\varphi)$, $\varphi \in [0,1]$. From this formula and (8.9) we clearly have $z_\beta = \hat{z}$. □

9 Strongly non-unique strict semi-wavefronts

We now apply the previous results to study semi-wavefronts of Equation (1.1) when D and g satisfy (D1), (g0) and (1.2); in particular, we prove Theorem 2.2 and Corollary 9.4. Indeed, all the results obtained in Sections 4–8 apply when we set

$$q := Dg, \tag{9.1}$$

since such q fulfills (q). Throughout this section, by c^* we always intend the threshold given by Proposition 4.2 for q as in (9.1), for which it holds (2.2), as observed in Remark 5.5.

Lemma 9.1. *Assume* (D1), (g0) *and* (1.2). *Consider* $c \geq c^*$ *and let* z *be the solution of* (3.12) *when* (9.1) *occurs. Then, it holds that*

$$\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \begin{cases} \frac{h(1)-c-\sqrt{(h(1)-c)^2-4\dot{D}(1)g(1)}}{2g(1)} & \text{if } \dot{D}(1) < 0, \\ \min\left\{0, \frac{h(1)-c}{g(1)}\right\} & \text{if } \dot{D}(1) = 0. \end{cases} \tag{9.2}$$

Proof. First, observe that Proposition 7.1 applies to the current case.

If either $\dot{D}(1) < 0$ or $\dot{D}(1) = 0$ and $c < h(1)$, then $\dot{z}(1) > 0$, by (7.2), because $\dot{q}(1) = \dot{D}(1)g(1)$. As a consequence, we have

$$\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \lim_{\varphi \to 1^-} \frac{\frac{D(\varphi)}{\varphi - 1}}{\frac{z(\varphi)}{\varphi - 1}} = \frac{\dot{D}(1)}{\dot{z}(1)},$$

which, together with (7.2), implies both $(9.2)_1$ and the first half of $(9.2)_2$.

If $\dot{D}(1) = 0$ and $c \geq h(1)$, we need a refined argument based on strict upper- and lower-solutions of $(3.11)_1$. We split the proof in two subcases.

(i) Assume first $\dot{D}(1) = 0$ and $c > h(1)$. Fix $\varepsilon > 0$ and define $\omega = \omega(\varphi)$ by

$$\omega(\varphi) := -\frac{g(1)}{c - h(1) + \varepsilon g(1)} D(\varphi), \quad \text{for } \varphi \in (0,1). \tag{9.3}$$

First, we observe that $\omega < 0$ in $(0,1)$. Moreover, we get

$$\dot{\omega}(\varphi) = -\frac{g(1)}{c - h(1) + \varepsilon g(1)} \dot{D}(\varphi),$$

which in turn implies $\dot{\omega}(1) = 0$, since $\dot{D}(1) = 0$. Now, if we compute the right-hand side of $(3.11)_1$ applied to ω, we obtain

$$h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)} = h(\varphi) - c + \frac{g(\varphi)\left[c - h(1) + \varepsilon g(1)\right]}{g(1)}, \quad \text{for } \varphi \in (0,1),$$

which tends to $\varepsilon g(1) > 0$ as $\varphi \to 1^-$. Hence, there exists $\sigma \in (0,1)$ such that

$$\dot{\omega}(\varphi) < h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{\omega(\varphi)}, \qquad \varphi \in [\sigma, 1), \tag{9.4}$$

that is, ω is a (strict) lower-solution of $(3.11)_1$ in $[\sigma, 1)$.

Since $\dot{z}(1) = 0$, we can take a sequence $\{\varphi_n\}_n \subset (\sigma, 1)$, with $\varphi_n \to 1$ as $n \to \infty$, such that $\dot{z}(\varphi_n) \to 0$ as follows. Let $\{\sigma_n\}_n \subset (\sigma, 1)$ be such that $\sigma_n \to 1$. For any $n \in \mathbb{N}$, the Mean Value Theorem implies that there exists $\varphi_n \in (\sigma_n, 1)$ for which it holds $\dot{z}(\varphi_n) = \frac{z(\sigma_n)}{\sigma_n - 1}$. Since the sequence in the right-hand side of this last identity tends to $\dot{z}(1) = 0$, as $n \to \infty$, we obtained the desired $\{\varphi_n\}_n$. With this in mind, from $(3.11)_1$, we obtain

$$\lim_{n \to \infty} \frac{D(\varphi_n)g(\varphi_n)}{z(\varphi_n)} = h(1) - c, \tag{9.5}$$

and then

$$\lim_{n \to \infty} \frac{\omega(\varphi_n)}{z(\varphi_n)} = \frac{c - h(1)}{c - h(1) + \varepsilon g(1)} = 1 - \frac{\varepsilon g(1)}{c - h(1) + \varepsilon g(1)} < 1.$$

Hence, there exists $\overline{n}$ such that $\omega(\varphi_n) > z(\varphi_n)$ for $n \geq \overline{n}$. Without loss of generality we assume that $\overline{n} = 1$. We claim that

$$\omega(\varphi) > z(\varphi), \quad \text{for } \varphi \in (\varphi_1, 1). \tag{9.6}$$

We reason by contradiction, see Figure 9.1. Suppose that there exists $\tilde{\varphi} \in (\varphi_1, 1)$ such that $\omega(\tilde{\varphi}) \leq z(\tilde{\varphi})$. There exists $n \in \mathbb{N}$ for which $\tilde{\varphi} \in (\varphi_n, \varphi_{n+1})$. Since $\omega(\varphi_n) > z(\varphi_n)$ and $\omega(\varphi_{n+1}) > z(\varphi_{n+1})$, the existence of such a $\tilde{\varphi}$ implies that the function $(\omega - z)$ in $(\varphi_n, \varphi_{n+1})$ admits a non-positive minimum at $\tilde{\varphi}_2 \in (\varphi_n, \varphi_{n+1})$, that is $\dot{\omega}(\tilde{\varphi}_2) = \dot{z}(\tilde{\varphi}_2)$ and $\omega(\tilde{\varphi}_2) \leq z(\tilde{\varphi}_2)$. Thus, from $(3.11)_1$ and (9.4) we have that

$$h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{z(\tilde{\varphi}_2)} = \dot{z}(\tilde{\varphi}_2) = \dot{\omega}(\tilde{\varphi}_2) < h(\tilde{\varphi}_2) - c - \frac{(Dg)(\tilde{\varphi}_2)}{\omega(\tilde{\varphi}_2)},$$

which in turn implies $1/z(\tilde{\varphi}_2) > 1/\omega(\tilde{\varphi}_2)$ because of $(Dg)(\tilde{\varphi}_2) > 0$. Hence, $z(\tilde{\varphi}_2) < \omega(\tilde{\varphi}_2)$ which contradicts the existence of $\tilde{\varphi}_2$. Then (9.6) is proved. At last, we have

$$\frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\frac{c - h(1)}{g(1)} - \varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.7}$$

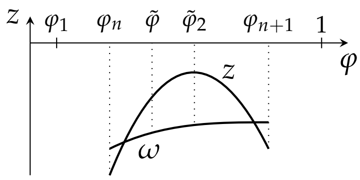

Figure 9.1: A detail of the plots of functions ω and z in case *(i)*.

Analogously, for $\varepsilon > 0$ small enough to satisfy $c > h(1) + \varepsilon g(1)$, we define $\eta = \eta(\varphi)$ by

$$\eta(\varphi) := -\frac{g(1)}{c - h(1) - \varepsilon g(1)} D(\varphi), \qquad \varphi \in (0, 1).$$

By arguing as above when we considered ω in (9.3), we deduce that η is a (strict) upper-solution of $(3.11)_1$ in $[\sigma_2, 1)$ for some $\sigma_2 \in (0, 1)$. Proceeding as we did to obtain (9.7), we now get $\eta(\varphi) < z(\varphi)$ for $\varphi \in (\varphi_1, 1)$, for some $\varphi_1 > \sigma_2$. Thus,

$$\frac{D(\varphi)}{z(\varphi)} < \frac{D(\varphi)}{\eta(\varphi)} = -\frac{c - h(1)}{g(1)} + \varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.8}$$

Finally, putting together (9.7) and (9.8), since $\varepsilon > 0$ is arbitrary, we deduce

$$\lim_{\varphi \to 1^-} \frac{D(\varphi)}{z(\varphi)} = \frac{h(1) - c}{g(1)}. \tag{9.9}$$

Thus, we proved $(9.2)_2$ with $c > h(1)$.

(ii) Now, we consider the case $\dot{D}(1) = 0$ and $c = h(1)$. Fix $\varepsilon > 0$. Set

$$\omega(\varphi) := -\frac{D(\varphi)}{\varepsilon}, \qquad \varphi \in (0, 1), \tag{9.10}$$

which coincides with (9.3) in the current case. By proceeding exactly as in the case *(ii)*, we obtain (9.3) for ω defined as in (9.10), namely $0 > \omega(\varphi) > z(\varphi)$, for $\varphi \in (\varphi_1, 1)$, for some $\varphi_1 \in (0,1)$. This implies, as in (9.7),

$$0 > \frac{D(\varphi)}{z(\varphi)} > \frac{D(\varphi)}{\omega(\varphi)} = -\varepsilon, \qquad \varphi \in (\varphi_1, 1). \tag{9.11}$$

Then (9.11) implies $D(\varphi)/z(\varphi) \to 0^-$ as $\varphi \to 1^-$, which is $(9.2)_2$ in the case $c = h(1)$. □

Remark 9.2. Let $c \geq c^*$ and z be any solution of (3.11). We infer that $z \in C^1(0,1]$. In fact, if $z(1) = b < 0$, in the proof of case *(i)* of Proposition 7.1 we already checked that this is true, since $\lim_{\varphi \to 1^-} \dot{z}(\varphi) = \dot{z}(1)$. If $z(1) = 0$, from (9.2) it follows that the right-hand side of $(3.11)_1$ still has a finite limit, as $\varphi \to 1^-$. As observed, this means that $z \in C^1(0,1]$.

We now prove Theorem 2.2.

Proof of Theorem 2.2. We first prove that there exists a semi-wavefront to 0 of (1.1) *if* $c \geq c^*$. For $q = Dg$, consider one of the solutions $z = z(\varphi)$ of (3.11), provided by Propositions 4.2 and 5.1. Consider the Cauchy problem

$$\begin{cases} \varphi' = \frac{z(\varphi)}{D(\varphi)}, \\ \varphi(0) = \frac{1}{2}. \end{cases} \tag{9.12}$$

The right-hand side of $(9.12)_1$ is of class C^1 in a neighborhood of $\frac{1}{2}$, and then there exists a unique solution φ in its maximal-existence interval (a, ξ_0), for $-\infty \leq a < \xi_0 \leq \infty$. Since $z(\varphi)/D(\varphi) < 0$ for $\varphi \in (0,1)$, we deduce that φ is decreasing and then $\lim_{\xi \to a^+} \varphi(\xi) = 1$, $\lim_{\xi \to \xi_0^-} \varphi(\xi) = 0$. By $(9.12)_1$, the profile φ satisfies (1.3) in (a, ξ_0). We show that, if $\xi_0 \in \mathbb{R}$, we can extend φ and obtain a solution of (1.3), in the sense of Definition 2.1, defined in the half-line $(a, +\infty)$.

Assume $\xi_0 \in \mathbb{R}$ and set $\varphi(\xi) = 0$, for any $\xi \geq \xi_0$. The new function (which without any ambiguity we still call φ) is clearly of class $C^0(a, +\infty) \cap C^2((a, +\infty) \setminus \{\xi_0\})$ and is a classical solution of (1.3) in $(a, +\infty) \setminus \{\xi_0\}$. Moreover, observe that, as a consequence of both the fact that z satisfies $(3.11)_3$, and $(9.12)_1$, we have

$$\lim_{\xi \to \xi_0^-} D(\varphi(\xi))\, \varphi'(\xi) = 0. \tag{9.13}$$

This implies that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(a, +\infty)$.

To show that φ is a solution of (1.3) according to Definition 2.1, it remains to prove (2.1). For this purpose, consider $\psi \in C_0^\infty(a, +\infty)$, and let $a < \xi_1 < \xi_2 < \infty$ be such that $\psi(\xi) = 0$, for any $\xi \geq \xi_2$ or $\xi \leq \xi_1$. Our goal is then to prove the following:

$$\int_{\xi_1}^{\xi_2} \left(D(\varphi)\, \varphi' - f(\varphi) + c\varphi \right) \psi' - g(\varphi)\psi \, d\xi = 0. \tag{9.14}$$

Identity (9.14) is obvious if $\xi_2 < \xi_0$, since φ solves (1.3) in (a, ξ_0). Assume $\xi_2 \geq \xi_0$. In the interval (ξ_0, ξ_2) we have $\varphi = 0$, and since $g(0) = f(0) = 0$ we deduce

$$\int_{\xi_0}^{\xi_2} \left(D(\varphi)\, \varphi' - f(\varphi) + c\varphi \right) \psi' - g(\varphi)\psi \, d\xi = 0. \tag{9.15}$$

In the interval (ξ_1, ξ_0) we have, by (9.13),

$$
\begin{aligned}
&\int_{\xi_1}^{\xi_0} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi\, d\xi \\
&\qquad = \lim_{\varepsilon\to 0^+} \int_{\xi_1}^{\xi_0-\varepsilon} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\psi' - g(\varphi)\psi\, d\xi \\
&\qquad = \lim_{\varepsilon\to 0^+} \left(\left(D\left(\varphi\right)\varphi' - f(\varphi) + c\varphi\right)\psi\right)\left(\xi_0 - \varepsilon\right) = 0.
\end{aligned}
\tag{9.16}
$$

Thus, identities (9.15) and (9.16) imply (9.14).

At last, we claim that $a \in \mathbb{R}$, i.e., that φ is *strict*. For this, it is sufficient to prove

$$
\lim_{\xi\to a^+} \varphi'(\xi) < 0. \tag{9.17}
$$

We stress that the case $\lim_{\xi\to a^+} \varphi'(\xi) \to -\infty$, for short $\varphi'(a^+) = -\infty$, is included in (9.17). To prove (9.17), we notice that, from (9.12),

$$
\lim_{\xi\to a^+} \varphi'(\xi) = \lim_{\varphi\to 1^-} \frac{z(\varphi)}{D(\varphi)}.
$$

Thus, (9.17) easily follows from either a direct check, in the case $z(1) < 0$, or the application of Lemma 9.1, in the case $z(1) = 0$. This concludes the first part of the proof.

Conversely, we prove that if there exists a semi-wavefront φ to 0 defined in $(a, +\infty)$, then $c \geq c^*$. Let $\bar{b}$ be defined by

$$
\bar{b} := \sup\left\{\xi > a : \varphi(\xi) > 0\right\} \in (a, +\infty]. \tag{9.18}
$$

We have $0 < \varphi < 1$ in $\left(a, \bar{b}\right)$ and so φ is a classical solution of (1.3) in $\left(a, \bar{b}\right)$. We claim that

$$
\lim_{\xi\to \bar{b}^-} D\left(\varphi(\xi)\right)\varphi'(\xi) = 0. \tag{9.19}
$$

Suppose $\bar{b} \in \mathbb{R}$. Take $\xi_1 > a$ and $\xi_2 > \bar{b}$. By choosing, in Definition 2.1, $\psi \in C_0^\infty(a, +\infty)$ with support in (ξ_1, ξ_2) such that $\psi(\bar{b}) \neq 0$, (2.1) reads as (passing to the limit in the integral as in (9.16))

$$
\begin{aligned}
0 &= \int_{\xi_1}^{\xi_2} \left(D\left(\varphi\right)\varphi' + c\varphi - f\left(\varphi\right)\right)\psi' - g\left(\varphi\right)\psi\, d\xi \\
&= \int_{\xi_1}^{\bar{b}} \left(D\left(\varphi\right)\varphi' + c\varphi - f\left(\varphi\right)\right)\psi' - g\left(\varphi\right)\psi\, d\xi = \left(D(\varphi)\varphi'\right)\left(\bar{b}^-\right)\psi(\bar{b}).
\end{aligned}
$$

Then we got (9.19) in this case. If $\bar{b} = +\infty$, by integrating (1.3) in $[\eta, \xi] \subset (\bar{a}, +\infty)$, we have

$$
\begin{aligned}
&D\left(\varphi(\xi)\right)\varphi'(\xi) \\
&\qquad = D\left(\varphi(\eta)\right)\varphi'(\eta) - c\left(\varphi(\xi) - \varphi(\eta)\right) + \left(f(\varphi(\xi)) - f(\varphi(\eta))\right) - \int_\eta^\xi g\left(\varphi(\sigma)\right)\, d\sigma.
\end{aligned}
\tag{9.20}
$$

Since the function

$$
\xi \mapsto \int_\eta^\xi g(\varphi(\sigma))\, d\sigma
$$

is increasing (because $g > 0$ in $(0,1)$), then $\lim_{\xi\to\infty} D(\varphi(\xi))\,\varphi'(\xi) = \ell$ for some $\ell \in [-\infty, 0]$. If $\ell < 0$, then, $\varphi'(\xi)$ tends either to some negative value or to $-\infty$ as $\xi \to +\infty$. In both cases, this contradicts the boundedness of φ, and so (9.19) is proved.

We show now (2.3). Suppose by contradiction that (2.3) does not occur, there exists $\xi_0 \in (a, \bar{b})$, with $0 < \varphi(\xi_0) < 1$, such that $\varphi'(\xi_0) = 0$. Then (1.3) implies $\varphi''(\xi_0) = -g(\varphi(\xi_0))/D(\varphi(\xi_0)) < 0$ and hence ξ_0 is a local maximum point of φ. It is plain to see that, in turn, this implies that there exists $a < \xi_1 < \xi_0$ which is a local minimum point of φ. From what we said about ξ_0, we necessarily have $\varphi(\xi_1) = \varphi'(\xi_1) = 0$.

Take $\xi \in (\xi_1, \bar{b})$. Integrating (1.3) in $[\xi_1, \xi]$ gives (9.20) with ξ_1 replacing η. By passing to the limit for $\xi \to \bar{b}^-$, from (9.19) we obtain the contradiction $0 < 0$. This proves (2.3).

From (2.3), we can define the function $z = z(\varphi)$, for $\varphi \in (0,1)$, by

$$z(\varphi) := D(\varphi)\varphi'(\xi(\varphi)), \tag{9.21}$$

where $\xi = \xi(\varphi)$ is the inverse function of φ. Again by (2.3), it follows also that $z < 0$ in $(0,1)$. From (9.19), we clearly have $z(0^+) = 0$; furthermore, a direct computation shows that z solves equation $(1.6)_1$. Thus, z solves problem (1.6), which is (3.11) with $q = Dg$. At last, Proposition 6.5 implies $c \geq c^*$. □

Remark 9.3. The proof of Theorem 2.2 provides a formula for $\varphi'(a^+)$. If $z(1) < 0$, then $\varphi'(a^+) = -\infty$. If $z(1) = 0$, Lemma 9.1 leads to

$$\lim_{\xi\to a^+} \varphi'(\xi) = \begin{cases} \frac{2g(1)}{h(1)-c-\sqrt{(h(1)-c)^2-4\dot{D}(1)g(1)}} & \text{if } \dot{D}(1) < 0, \\ \frac{g(1)}{h(1)-c} & \text{if } \dot{D}(1) = 0 \text{ and } c > h(1), \\ -\infty & \text{if } \dot{D}(1) = 0 \text{ and } c \leq h(1). \end{cases} \tag{9.22}$$

We now investigate the qualitative properties of the profiles when they reach the equilibrium 0. The classification is complete, apart from some cases corresponding to $c^* = h(0)$, when further assumptions are needed, see Remark 10.1. Below the existence of the $\lim_{\xi\to a^+} D(\varphi(\xi))\,\varphi'(\xi)$ is a consequence of the definition (9.21) and Lemma 3.1.

Corollary 9.4. *Under the assumptions of Theorem 2.2, let $c \geq c^*$ and φ be a strict semi-wavefront to 0 of (1.1), connecting 1 to 0, defined in its maximal-existence interval $(a, +\infty)$. Then, for $c > c^*$, there exists $\hat{\beta}(c) \in [\beta(c), 0]$ such that the following results hold.*

(i) $D(0) > 0$ *implies that φ is classical and strictly decreasing.*

(ii) $D(0) = 0$, $c > c^*$ *and*

$$\lim_{\xi\to a^+} D(\varphi(\xi))\,\varphi'(\xi) > \hat{\beta}(c), \tag{9.23}$$

imply that φ is classical; moreover, φ reaches 0 at some $\xi_0 > a$ if

$$c > h(0) + \limsup_{\varphi\to 0^+} \frac{g(\varphi)}{\varphi}. \tag{9.24}$$

(iii) $D(0) = 0$, $c^* > h(0)$ *and*

$$\text{either } \ c = c^* \ \text{ or } \ \lim_{\xi\to a^+} D(\varphi(\xi))\,\varphi'(\xi) \leq \hat{\beta}(c) \tag{9.25}$$

imply that φ is sharp at 0 (reached at some $\xi_0 > a$) with

$$\lim_{\xi\to\xi_0^-} \varphi'(\xi) = \begin{cases} \frac{h(0)-c}{\dot D(0)} < 0 & \text{if } \dot D(0) > 0, \\ -\infty & \text{if } \dot D(0) = 0. \end{cases} \tag{9.26}$$

Notice that β is related to the existence of the semi-wavefronts while $\hat\beta$ deals with their smoothness (see Figure 9.2). The two thresholds coincide under the assumptions of Proposition 8.4.

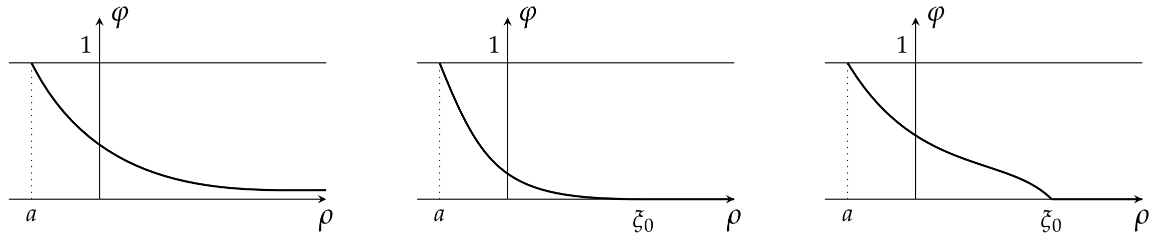

Figure 9.2: Examples of profiles occurring in Corollary 9.4. From the left to the right, they depict, respectively, what stated in Parts *(i)*, *(ii)* and *(iii)*.

Proof of Corollary 9.4. Define $\xi_0 := \sup\{\xi > a : \varphi(\xi) > 0\} \in (a, +\infty]$. We assume without loss of generality that $a < 0 < \xi_0$ and $\varphi(0) = 1/2$. Let z be the function defined in (9.21). Notice, $1 = D(\varphi)\varphi'/z(\varphi)$ if $\varphi \in (0,1)$. Thus, for any $\xi > 0$, it follows that

$$\xi = \int_0^\xi \frac{D(\varphi(s))}{z(\varphi(s))}\varphi'(s)ds = \int_{1/2}^{\varphi(\xi)} \frac{D(\sigma)}{z(\sigma)}d\sigma = \int_{\varphi(\xi)}^{1/2} \frac{D(\sigma)}{-z(\sigma)}d\sigma.$$

Therefore, $\xi_0 \in \mathbb{R}$ if and only if it holds that

$$\int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)}d\sigma := \lim_{\varphi\to 0^+}\int_\varphi^{1/2} \frac{D(\sigma)}{-z(\sigma)}d\sigma < +\infty. \tag{9.27}$$

For $c > c^*$, let $\hat\beta(c)$ be given by (8.4).

We prove *(i)*. By Proposition 8.2 we know that $\dot z(0)$ exists and it is finite; since $D(0) > 0$ we deduce that (9.27) does not hold. Then, $\xi_0 = +\infty$ and so φ is strictly decreasing. This, and the fact that φ is of class C^2 when $\varphi \in (0,1)$, imply $\varphi \in C^2(a, +\infty)$, hence φ is classical. Part *(i)* is hence showed.

Assume $D(0) = 0$. In this case, Formula (6.3) holds with $\dot q(0) = 0$ and $\dot z(0)$ exists by Proposition 8.2.

We show *(ii)*. Since (9.23) holds then (8.5) reads as $\dot z(0) = 0$. We treat separately the cases $\dot D(0) > 0$ or $\dot D(0) = 0$. Suppose that $\dot D(0) > 0$. Therefore,

$$\lim_{\xi\to\xi_0^-} \varphi'(\xi) = \frac{\dot z(0)}{\dot D(0)} = 0 \tag{9.28}$$

and hence φ (not necessarily strictly monotone) is classical. Suppose then $D(0) = \dot D(0) = 0$. Fix $\varepsilon > 0$ and define $\eta(\varphi) := -\varepsilon D(\varphi)$, $\varphi \in (0,1)$. We have

$$\dot\eta(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} \to -h(0) + c > 0, \quad \text{as } \varphi \to 0^+.$$

Therefore η is a strict upper-solution of $(1.6)_1$ in $(0,\delta]$, for some $\delta > 0$. Also, since $\dot z(0) = 0$, there exists a sequence $\{\varphi_n\}_n$, with $\delta \geq \varphi_n \to 0^+$, such that $\dot z(\varphi_n) \to 0$. From $(1.6)_1$, this implies that

$$\lim_{n\to\infty} \frac{\varepsilon D(\varphi_n)}{-z(\varphi_n)} = \varepsilon \lim_{n\to\infty} \frac{\dot z(\varphi_n) + c - h(\varphi_n)}{g(\varphi_n)} = \infty.$$

Hence, $-\eta(\delta_1) = \varepsilon D(\delta_1) > -z(\delta_1)$, for some $0 < \delta_1 \leq \delta$ small enough. An application of Lemma 3.2 *(2.a.i)* then gives

$$z(\varphi) > -\varepsilon D(\varphi), \qquad \varphi \in (0,\delta_1]. \tag{9.29}$$

This clearly implies that

$$0 > \frac{z(\varphi)}{D(\varphi)} > -\varepsilon, \qquad \varphi \in (0,\delta_1].$$

Since $\varepsilon > 0$ is arbitrary, then we have $\varphi'(\xi) \to 0$ for $\xi \to \xi_0^-$ and hence φ is classical, that is we showed the first part of *(ii)*. Define $\eta(\varphi) := -\varphi D(\varphi)$. We have, for any $\varphi \in (0,1)$,

$$\dot\eta(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)} = -\dot D(\varphi)\varphi - D(\varphi) - h(\varphi) + c - \frac{g(\varphi)}{\varphi}.$$

Thus, by means of (9.24), we get

$$\liminf_{\varphi\to 0^+} \left[\dot\eta(\varphi) - h(\varphi) + c + \frac{D(\varphi)g(\varphi)}{\eta(\varphi)}\right] = c - h(0) - \limsup_{\varphi\to 0^+} \frac{g(\varphi)}{\varphi} > 0.$$

Therefore, η is a strict upper-solution of $(1.6)_1$ in $(0,\delta]$, for some $\delta > 0$. Furthermore, taking the same sequence $\varphi_n \to 0^+$ as above such that $\dot z(\varphi_n) \to 0$, as $n \to \infty$, then we have

$$\liminf_{n\to\infty} \frac{D(\varphi_n)\varphi_n}{-z(\varphi_n)} = \liminf_{n\to\infty} \frac{\dot z(\varphi_n) + c - h(\varphi_n)}{g(\varphi_n)/\varphi_n} = \frac{c - h(0)}{\limsup_{n\to\infty} g(\varphi_n)/\varphi_n} > 1,$$

since (9.24) holds. Thus, as in (9.29), we deduce that $D(\varphi)\varphi > -z(\varphi)$ in $(0,\delta]$, after choosing $0 < \delta \leq 1/2$ small enough. Hence,

$$\int_0^{1/2} \frac{D(\sigma)}{-z(\sigma)}\, d\sigma > \int_0^{\delta} \frac{d\sigma}{\sigma} = +\infty,$$

which concludes the proof of *(ii)*, by means of (9.27).

We show *(iii)*. By (8.5), (8.6), $c^* > h(0)$ and (9.25) we obtain $\dot z(0) = h(0) - c < 0$. Then,

$$\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot D(0) + o(1)}{c - h(0) + o(1)} \quad \text{as } \sigma \to 0^+,$$

and consequently (9.27) is verified. Thus, $\xi_0 \in \mathbb{R}$. Furthermore, from (9.21),

$$\lim_{\xi\to\xi_0^-} \varphi'(\xi) = \lim_{\varphi\to 0^+} \frac{z(\varphi)/\varphi}{D(\varphi)/\varphi} = \frac{h(0) - c}{\dot D(0)} \in [-\infty, 0),$$

which implies that φ is sharp at 0 and that (9.26) holds. $\square$

10 New regularity classification of wavefronts

In this section we prove Theorem 2.3. Analogously to Section 9, but now thanks to assumptions (D0)–(g01), we apply results of Sections 4–8 to the case $q = Dg$.

Proof of Theorem 2.3. We first show that wavefronts are allowed if and only if $c \geq c^*$ for c^* satisfying (2.8); the proof is mostly contained in the proof of Theorem 2.2. Then, we prove *(i)* and *(ii)*, by exploiting some of the arguments in the proof of Corollary 9.4.

Set $q = Dg$. Clearly, q satisfies (q), with in particular $\dot{q}(0) = 0$. By Proposition 4.2, Problem (3.12) admits a unique solution z if and only if $c \geq c^*$ where for c^* it holds (4.3). As observed in Remark 5.5, since (D0) and (g01) hold true, in this case c^* satisfies (2.8).

To the solution z there is associated the solution $\varphi = \varphi(\xi)$ of the problem

$$\begin{cases} \varphi' = \frac{z(\varphi)}{D(\varphi)}, \\ \varphi(0) = \frac{1}{2}. \end{cases} \tag{10.1}$$

Such a φ exists and satisfies $(10.1)_1$ in some maximal interval (ξ_1, ξ_0), so that

$$\lim_{\xi \to \xi_1^+} \varphi(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to \xi_0^-} \varphi(\xi) = 0.$$

Also, φ satisfies (1.3) in (ξ_1, ξ_0). As discussed in the proof of Theorem 2.2, if $\xi_0 \in \mathbb{R}$, then φ can be extended continuously to a solution of (1.3) in $(\xi_0, +\infty)$, by setting $\varphi(\xi) = 0$, for $\xi \geq \xi_0$. Since $g(1) = 0$, it also holds that if $\xi_1 \in \mathbb{R}$ then we can extend φ to a solution of (1.3) in $(-\infty, \xi_1)$, by setting $\varphi(\xi) = 1$ for $\xi \leq \xi_1$. Thus, we can always consider φ satisfying weakly (1.3) in $\mathbb{R}$; moreover φ solves $(10.1)_1$ in (ξ_1, ξ_0) with

$$\xi_1 = \inf\{\xi \in \mathbb{R} : \varphi(\xi) < 1\} \in [-\infty, 0), \qquad \xi_0 = \sup\{\xi \in \mathbb{R} : \varphi(\xi) > 0\} \in (0, +\infty],$$

and it is constant in $\mathbb{R} \setminus (\xi_1, \xi_0)$. Thus, we showed that if $c \geq c^*$ then there exists a wavefront φ whose profile satisfies (1.4).

By reasoning as in the proof of Theorem 2.2, also the converse implication holds. Indeed, if φ is a profile of a wavefront satisfying (1.4), then the function z defined by $z(\varphi) := D(\varphi)\varphi'\left(\varphi^{-1}(\varphi)\right)$, $0 < \varphi < 1$, is a solution of (3.12). Thus, $c \geq c^*$.

We prove *(i)*. Assume $c > c^*$. From (8.5) in Proposition 8.2, we have $\dot{z}(0) = 0$. Hence, if $\dot{D}(0) \neq 0$ then it holds

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} = 0. \tag{10.2}$$

If $\dot{D}(0) = 0$, then we argue as in the proof of Corollary 9.4, see (9.29), to show that, for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that $z(\varphi) > -\varepsilon D(\varphi)$, $\varphi \in (0, \delta]$. Hence,

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)}{D(\varphi)} \geq -\varepsilon.$$

Since $\varphi' < 0$ in (ξ_1, ξ_0) and ε is arbitrarily small, it follows again (10.2).

We prove now *(ii)*. By $(8.5)_2$, from $c = c^* > h(0)$ we have $\dot{z}(0) = h(0) - c^* < 0$. Then,

$$\frac{D(\sigma)}{-z(\sigma)} = \frac{\dot{D}(0) + o(1)}{c - h(0) + o(1)} \quad \text{as} \quad \sigma \to 0^+,$$

and consequently (9.27) is verified. Thus, $\xi_0 \in \mathbb{R}$. Furthermore, from (9.21),

$$\lim_{\xi \to \xi_0^-} \varphi'(\xi) = \lim_{\varphi \to 0^+} \frac{z(\varphi)/\varphi}{D(\varphi)/\varphi} = \frac{h(0) - c^*}{\dot{D}(0)} \in [-\infty, 0),$$

and thus the conclusions hold. □

Remark 10.1 (Case $c = c^* = h(0)$)**.** Part *(i)* and *(ii)* of Theorem 2.3 do not cover the case $c = c^* = h(0)$. The following discussion shows that, to classify the behavior in that case, further assumptions are needed. More precisely, either a classical and a sharp wavefront can indeed occur under (D0) and (g01). Take q and h as in (6.10) in Remark 6.4. There, we proved that in this case it holds $c^* = h(0) = 0$. Consider

$$\begin{cases} D_1(\varphi) = \varphi^2, \\ g_1(\varphi) = \varphi(1-\varphi), \end{cases} \qquad \begin{cases} D_2(\varphi) = \varphi, \\ g_2(\varphi) = \varphi^2(1-\varphi). \end{cases}$$

Clearly, D_1 and g_1 satisfy (D0) and (g01) and so D_2 and g_2. Also, since $D_1 g_1 = q = D_2 g_2$, then $c_1^* = c_2^* = h(0) = 0$, where c_1^* and c_2^* are the thresholds given by Proposition 4.2 associated with $D_1 g_1$ and $D_2 g_2$, respectively. Define, for $\xi \in \mathbb{R}$,

$$\varphi_1(\xi) := \begin{cases} 1 - \frac{e^\xi}{2}, & \xi < \log(2), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \varphi_2(\xi) := \frac{1}{1+e^\xi}.$$

Direct computations show that φ_1 and φ_2 are two wave profiles defining two wavefronts, both of them associated with $c = h(0)$. Plainly, φ_1 is sharp at $\xi = \log(2)$ while φ_2 is classical.

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