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Volterra–Stieltjes integral equations and impulsive Volterra–Stieltjes integral equations

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Abstract. In this paper, we prove existence and uniqueness of solutions of Volterra–Stieltjes integral equations using the Henstock–Kurzweil integral. Also, we prove that these equations encompass impulsive Volterra–Stieltjes integral equations and prove the existence and uniqueness for these equations. Finally, we present some examples to illustrate our results.

Keywords: Volterra–Stieltjes integral, impulsive equations, existence and uniqueness, Henstock–Kurzweil–Stieltjes integral

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1 Introduction

In this paper, we are interested in the study of integral equations that can be modeled in the form

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\, dg(s), \qquad t \in [t_0, t_0+\sigma], \tag{1.1}$$

where the integral on the right-hand side is in the sense of Henstock–Kurzweil–Stieltjes [22]. This class of equations plays an important role from the theoretical point of view as well as for applications, since they subsumes many types of well known mathematical models. As a matter of fact, they can be used to model different problems such as anomalous diffusion processes, heat conduction with memory and diffusion of fluids in porous media, among others. See [3, 5, 7, 20, 21] for instance. On the other hand, the subject of Volterra integral equations has been attracting the attention by several researchers, since they represent a powerful tool for applications. See, for instance, [1, 4, 6, 8, 9, 14, 17].

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It is worth noticing that depending on the choice of the kernel $a : [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma] \to \mathbb{R}$, we can study in an unified way a very general class of problems. For instance, if $a(t,s) = 1$ for all $(t,s) \in [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma]$, then equation (1.1) reduces to the classical measure differential equation, which is very well-developed in the literature (see [12]). On the other hand, if $a(t,s) = k(t - s)$ for all $(t,s) \in [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma]$, then the integral equation (1.1) reduces to a Volterra integral equation which have many applications to the study of heat flow in the materials of fading memory type (see [7,20,21]), among others.

In the present paper, our goal is to prove existence and uniqueness results for the integral equation (1.1) under very weak conditions for the functions f, a and g. These results are more general than the ones presented in the literature, since the required conditions allow that either the function f in (1.1) be highly oscillating, or the functions a, f and g that appear in (1.1) may have a countable number of discontinuities. Also, we present three examples to illustrate our results.

Further, we prove that under certain assumptions the integral equation given by (1.1) can be regarded as an impulsive Volterra–Stieltjes integral equation described by

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < t}} I_k(x(t_k)). \tag{1.2}$$

These last equations can also be regarded as an impulsive Volterra Δ-integral equation on time scales given by

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x(s),s) \Delta s + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < t}} I_k(x(t_k)), \tag{1.3}$$

when $g(t) = \inf\{s \in \mathbb{T} : s \geq t\}$. We only illustrate the first correspondence in this paper, since it brings more complexity due to the kernel from Volterra–Stieltjes integral equation. On the other hand, we have omitted the second one to turn the paper simpler and shorter, but following similar steps from [12], it is possible to prove such correspondence.

This paper is organized as follows. In the second section, we present the basic concepts and properties concerning the Henstock–Kurzweil–Stieltjes integral which is the main tool to prove our results. In the third section, we investigate the Volterra–Stieltjes integral equations and we prove a result concerning the existence and uniqueness of solutions of these equations. The last section is devoted to present a correspondence between Volterra–Stieltjes integral equations and impulsive Volterra–Stieltjes equations and also, to prove a result concerning existence and uniqueness of solutions for these last equations.

2 Henstock–Kurzweil–Stieltjes integral

In this section, we recall some properties concerning the Henstock–Kurzweil–Stieltjes integral. See [22] for more details.

Let $[a,b]$ be an interval of $\mathbb{R}$, $-\infty < a < b < +\infty$. A *tagged division* of $[a,b]$ is a finite collection of point-interval pairs $D = (\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_{|D|} = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, |D|$, where the symbol $|D|$ denotes the number of subintervals in which $[a,b]$ is divided.

A *gauge* on a set $B \subset [a,b]$ is any function $\delta : B \to (0,\infty)$. Given a gauge δ on $[a,b]$, we say that a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is *δ-fine* if for every $i \in \{1,2,\ldots,|D|\}$, we have

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

A function $f : [a,b] \to \mathbb{R}$ is called *Henstock–Kurzweil–Stieltjes* integrable on $[a,b]$ with respect to a function $g : [a,b] \to \mathbb{R}$, if there is an element $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$\left| \sum_{i=1}^{|D|} f(\tau_i) \left(g(s_i) - g(s_{i-1}) \right) - I \right| < \varepsilon,$$

for all δ–fine tagged partition of $[a,b]$. In this case, I is called *Henstock–Kurzweil–Stieltjes* integral of f with respect to g over $[a,b]$ and it will be denoted by $\int_a^b f(s)\,\mathrm{d}g(s)$, or simply $\int_a^b f\,\mathrm{d}g$.

The Henstock–Kurzweil–Stieltjes integral has the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals (see [22]).

We recall the reader that a function $f : [a,b] \to \mathbb{R}$ is called *regulated* if the lateral limits

$$f(t-) = \lim_{s \to t-} f(s), \quad t \in (a,b] \quad \text{and} \quad f(t+) = \lim_{s \to t+} f(s), \quad t \in [a,b)$$

exist. The space of all regulated functions $f : [a,b] \to \mathbb{R}$ will be denoted by $G([a,b],\mathbb{R})$, which is a Banach space when endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{s \in [a,b]} |f(s)|\,.$$

Given a regulated function $f : [a,b] \to \mathbb{R}$, we will use the notations $\Delta^+ f(t)$ and $\Delta^- f(t)$ throughout this paper to denote

$$\Delta^+ f(t) := f(t+) - f(t) \quad \text{and} \quad \Delta^- f(t) := f(t) - f(t-),$$

respectively.

The next result ensures the existence of the Henstock–Kurzweil–Stieltjes integral. We observe that the inequalities follow from the definition of the Henstock–Kurzweil–Stieltjes integral. This result can be found in [22, Corollary 1.34].

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a regulated function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be a nondecreasing function. Then the following conditions hold.*

(i) *The integral $\int_a^b f(s)\mathrm{d}g(s)$ exists;*

(ii) $\left| \int_a^b f(s)\mathrm{d}g(s) \right| \leqslant \int_a^b |f(s)|\,\mathrm{d}g(s) \leqslant \|f\|_\infty \,(g(b) - g(a))$.

The following inequalities follow directly from the definition of the Henstock–Kurzweil–Stieltjes integral. A similar version was proved in [2, Theorem 7.20] for the case of the Riemann–Stieltjes integral. We omit its proof here, since it is similar to the proof of [2].

Theorem 2.2. *Let $f_1,\ f_2 : [a,b] \to \mathbb{R}$ be Henstock–Kurzweil–Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$ and such that $f_1(t) \leqslant f_2(t)$, for $t \in [a,b]$. Then*

$$\int_a^b f_1(s)\mathrm{d}g(s) \leqslant \int_a^b f_2(s)\mathrm{d}g(s).$$

The next result is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $f : [a,b] \to \mathbb{R}$ be Henstock–Kurzweil–Stieltjes integrable function on the interval $[a,b]$ with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$ and such that $f(t) \geqslant 0$, for $t \in [a,b]$. Then*

(i) $\int_a^b f(s)\mathrm{d}g(s) \geqslant 0$.

(ii) *The function $[a,b] \ni t \mapsto \int_a^t f(s)\mathrm{d}g(s)$ is nondecreasing.*

In the sequel, we present a Gronwall–type inequality. See [22, Corollary, 1.43].

Lemma 2.4. *Let $g : [a,b] \to [0,\infty)$ be a nondecreasing and left-continuous function, $k > 0$ and $l \geqslant 0$. Assume that $\psi : [a,b] \to [0,\infty)$ is bounded and satisfies*

$$\psi(\xi) \leqslant k + l \int_a^\xi \psi(s)\mathrm{d}g(s), \qquad \xi \in [a,b].$$

Then $\psi(\xi) \leqslant k e^{l(g(\xi)-g(a))}$ for all $\xi \in [a,b]$.

The following result, which describes some properties of the indefinite Henstock–Kurzweil–Stieltjes integral, is a special case of [22, Theorem 1.16].

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(s)\mathrm{d}g(s)$ exists. Then the function*

$$h(t) = \int_a^t f(s)\mathrm{d}g(s), \qquad t \in [a,b]$$

is regulated on $[a,b]$ and satisfy

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \qquad t \in [a,b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \qquad t \in (a,b]. \end{aligned}$$

The following assertion is a Substitution Theorem for the Henstock–Kurzweil–Stieltjes integral. It can be found in [19, Theorem 2.19].

Theorem 2.6. *Assume the function $h : [a,b] \to \mathbb{R}$ is bounded and that the integral $\int_a^b f(s)\mathrm{d}g(s)$ exists. If one of the integrals*

$$\int_a^b h(t)\mathrm{d}\left(\int_a^t f(\xi)\mathrm{d}g(\xi)\right), \qquad \int_a^b h(t)f(t)\mathrm{d}g(t),$$

exists, then the other one exists as well, in which case the equality below holds

$$\int_a^b h(t)\mathrm{d}\left(\int_a^t f(\xi)\mathrm{d}g(\xi)\right) = \int_a^b h(t)f(t)\mathrm{d}g(t).$$

Now we present a result which is a type of the Dominated Convergence Theorem for Henstock–Kurzweil–Stieltjes integrals. See [22, Corollary 1.32].

Theorem 2.7. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing function on $[a,b]$. Assume that $\varphi_n : [a,b] \to \mathbb{R}$ are functions such that the integral $\int_a^b \varphi_n(s)\,\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Suppose that for all $s \in [a,b]$, we have $\lim_{n\to\infty} \varphi_n(s) = \varphi(s)$ and that for $n \in \mathbb{N}$, $s \in [a,b]$ the inequalities $\kappa(s) \leqslant \varphi_n(s) \leqslant \omega(s)$ hold, where $\omega, \kappa : [a,b] \to \mathbb{R}$ are functions such that the integrals $\int_a^b \kappa(s)\,\mathrm{d}g(s)$ and $\int_a^b \omega(s)\,\mathrm{d}g(s)$ exist. Then the integral $\int_a^b \varphi(s)\,\mathrm{d}g(s)$ exists and*

$$\lim_{n\to\infty} \int_a^b \varphi_n(s)\,\mathrm{d}g(s) = \int_a^b \varphi(s)\,\mathrm{d}g(s).$$

The following lemma is a direct consequence of $G([a,b],\mathbb{R}^n)$ being a Banach space.

Lemma 2.8. *If a sequence $\{x_k\}_{k=1}^{\infty}$ of regulated functions (from $[a,b]$ to $\mathbb{R}$) converges uniformly on the interval $[a,b]$ to a function $x : [a,b] \to \mathbb{R}$, then this function is also regulated on $[a,b]$.*

We recall the reader that a set $\mathcal{A} \subset G([a,b],\mathbb{R})$ is called *equiregulated*, if it has the following property: for every $\varepsilon > 0$ and $t_0 \in [a,b]$, there is a $\delta > 0$ such that

(1) if $x \in \mathcal{A}$, $s \in [a,b]$ and $t_0 - \delta < s < t_0$, then $|x(t_0-) - x(s)| < \varepsilon$,

(2) if $x \in \mathcal{A}$, $s \in [a,b]$ and $t_0 < s < t_0 + \delta$, then $|x(t_0+) - x(s)| < \varepsilon$.

The next result describes a necessary and sufficient condition for a subset of $G([a,b],\mathbb{R})$ to be relatively compact, which is an immediate consequence of [15, Theorem 2.18]. We remark that even though the result in [15] requires v to be an increasing function, it is enough to assume that v is nondecreasing and let $\vartheta(t) := v(t) + t$, $t \in [a,b]$, to see that the original assumption is satisfied.

Theorem 2.9. *The following conditions are equivalent.*

(i) *$\mathcal{A} \subset G([a,b],\mathbb{R})$ is relatively compact.*

(ii) *The set $\{x(a) : x \in \mathcal{A}\}$ is bounded and there is a nondecreasing function $v : [a,b] \to \mathbb{R}$ such that*

$$|x(\tau_2) - x(\tau_1)| \leqslant v(\tau_2) - v(\tau_1),$$

for all $x \in \mathcal{A}$ and all $a \leqslant \tau_1 \leqslant \tau_2 \leqslant b$.

The following lemma will be crucial to prove that an impulsive Volterra integral equation can always be transformed to a Volterra integral equation without impulses. This result can be found in [12, Lemma 2.4].

Lemma 2.10. *Let $m \in \mathbb{N}$, $a \leqslant t_1 < t_2 < \cdots < t_m \leqslant b$. Consider a pair of functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, where g is regulated, left-continuous on $[a,b]$, and continuous at $t_1, \ldots, t_m$. Let $\tilde{f} : [a,b] \to \mathbb{R}$ and $\tilde{g} : [a,b] \to \mathbb{R}$ be such that $\tilde{f}(t) = f(t)$ for every $t \in [a,b] \backslash \{t_1, \ldots, t_m\}$ and $\tilde{g} - g$ is constant on each of the intervals $[a,t_1]$, $(t_1,t_2], \ldots, (t_{m-1},t_m]$, $(t_m,b]$. Then the integral $\int_a^b \tilde{f}\,\mathrm{d}\tilde{g}$ exists if and only if the integral $\int_a^b f\,\mathrm{d}g$ exists; in that case, we have*

$$\int_a^b \tilde{f}(s)\,\mathrm{d}\tilde{g}(s) = \int_a^b f(s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

The next result will be essential to prove the existence of solution of Volterra–Stieltjes integral equations. It is a classical result of fixed point.

Theorem 2.11 (Schauder Fixed-Point Theorem)**.** *Let $(E, \|\cdot\|)$ be a normed vector space, S a nonempty convex and closed subset of E and $T : S \to S$ is a continuous function such that $T(S)$ is relatively compact. Then T has a fixed point in S.*

3 Volterra–Stieltjes integral equations

In this section, our goal is to study the following type of equation

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0+\sigma], \quad t_0 \in \mathbb{R},$$

where the Henstock–Kurzweil–Stieltjes integral on the right–hand side is taken with respect to a nondecreasing function $g : [t_0, t_0+\sigma] \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$, $\sigma > 0$, $x_0 \in \mathbb{R}$, and $a : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, where $[t_0, t_0+\sigma]^2 = [t_0, t_0+\sigma] \times [t_0, t_0+\sigma]$.

Throughout this paper, we will use the symbol $G_2([t_0, t_0+\sigma]^2, \mathbb{R})$ to denote the set of all functions $b : [t_0, t_0+\sigma]^2 \to \mathbb{R}$ such that b is regulated with respect to the second variable, that is, for any fixed $t \in [t_0, t_0+\sigma]$, the function

$$b(t,\cdot) : s \in [t_0, t_0+\sigma] \longmapsto b(t,s) \in \mathbb{R}$$

is regulated.

In what follows, we say that $c : [t_0, t_0+\sigma]^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable if for any fixed $s \in [t_0, t_0+\sigma]$, the function

$$c(\cdot,s) : t \in [t_0, t_0+\sigma] \longmapsto c(t,s) \in \mathbb{R}$$

is nondecreasing.

We assume the following conditions are satisfied.

(A1) The function $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is nondecreasing and left-continuous on $(t_0, t_0+\sigma]$.

(A2) The function $a \in G_2([t_0, t_0+\sigma]^2, \mathbb{R})$ is nondecreasing with respect to the first variable.

(A3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{t_0}^{t_0+\sigma} a(t,s) f(x(s),s) \mathrm{d}g(s)$$

exists, for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$ and all $t \in [t_0, t_0+\sigma]$.

(A4) There exists a Henstock–Kurzweil–Stieltjes integrable function $M : [t_0, t_0+\sigma] \to \mathbb{R}^+$ with respect to g such that

$$\left| \int_{\tau_1}^{\tau_2} (c_2 a(\tau_2,s) + c_1 a(\tau_1,s)) f(x(s),s) \mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |c_2 a(\tau_2,s) + c_1 a(\tau_1,s)|\, M(s) \mathrm{d}g(s),$$

for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$ and all $[\tau_1, \tau_2] \subset [t_0, t_0+\sigma]$. In particular, we have that

$$\left| \int_{\tau_1}^{\tau_2} a(\tau,s) f(x(s),s) \mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau,s)| M(s) \mathrm{d}g(s),$$

and

$$\left| \int_{\tau_1}^{\tau_2} (a(\tau_2,s) - a(\tau_1,s)) f(x(s),s) \mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau_2,s) - a(\tau_1,s)| M(s) \mathrm{d}g(s)$$

for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$, and all $\tau, \tau_1, \tau_2 \in [t_0, t_0+\sigma]$.

(A5) There exists a regulated function $L : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x(s), s) - f(z(s), s)]\mathrm{d}g(s) \right| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, L(s)|x(s) - z(s)|\mathrm{d}g(s),$$

for all $x, z \in G([t_0, t_0 + \sigma], \mathbb{R})$ and all $[\tau_1, \tau_2] \subset [t_0, t_0 + \sigma]$.

Remark 3.1. Note that $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, M(s)\mathrm{d}g(s)$ and $\int_{t_0}^{t_0+\sigma} |a(t, s)|\, L(s)|x(s) - z(s)|\mathrm{d}g(s)$ exist. Indeed, by Corollary 2.3, $[t_0, t_0 + \sigma] \ni t \mapsto \int_{t_0}^{t} M(s)\mathrm{d}g(s)$ is a nondecreasing function. On the other hand, the function $[t_0, t_0 + \sigma] \ni s \mapsto c_2 a(\tau_2, s) + c_1 a(\tau_1, s)$ is regulated. Then, by Theorem 2.1, the integral $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, \mathrm{d}\left(\int_{t_0}^{s} M(\xi)\mathrm{d}g(\xi)\right)$ exists. Using this fact, the boundedness of $c_2 a(\tau_2, \cdot) + c_1 a(\tau_1, \cdot)$ and Theorem 2.6, we have that the integral $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, M(s)\mathrm{d}g(s)$ exists. For the second integral, note that the function $s \mapsto |a(t, s)|\, L(s)|x(s) - z(s)|$ is regulated.

Remark 3.2. Note that when $s \mapsto a(\tau, s) f(x(s), s)$ is a regulated function on $[t_0, t_0 + \sigma]$ for $t_0 \leq \tau \leq t_0 + \sigma$ and g is nondecreasing, then (A4) holds by Theorem 2.1.

Remark 3.3. Suppose that g is a nondecreasing function. Then, the condition (A4) is true whenever the function f is bounded in x. Moreover, we observe that condition (A5) holds whenever the following Lipschitz type condition is satisfied:

$$|f(x(s), s) - f(z(s), s)| \leqslant L(s)|x(s) - z(s)|, \qquad t_0 \leqslant s \leqslant t_0 + \sigma,$$

where $L : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ is a regulated function.

Remark 3.4. Suppose that a satisfies condition (A2). Since $a(t_0, y) \leqslant a(x, y) \leqslant a(t_0 + \sigma, y)$ for all $x, y \in [t_0, t_0 + \sigma]$ and the functions $a(t_0, y), a(t_0 + \sigma, y)$ are regulated in y, we have that a is bounded in $[t_0, t_0 + \sigma]^2$.

Next, we present the main result of this section. It ensures the existence and uniqueness of solution of Volterra–Stieltjes integral equations. In order to prove it, we employ the Schauder Fixed Point Theorem and Gronwall's inequality for Stieltjes integral.

Theorem 3.5. *Assume $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies the conditions (A3), (A4) and (A5), $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ satisfies condition (A2) and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies condition (A1). Then there exists a unique solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ of*

$$x(t) = x_0 + \int_{t_0}^{t} a(t, s) f(x(s), s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma]. \tag{3.1}$$

Proof. Let us define the following constants:

$$c := \sup_{(t,s) \in [t_0, t_0+\sigma]^2} |a(t, s)|, \tag{3.2}$$

$$\beta := \int_{t_0}^{t_0+\sigma} c M(s)\, \mathrm{d}g(s). \tag{3.3}$$

Notice that all these constants are finite and well-defined in view of conditions (A2), (A4) and Remark 3.4.

Existence. Consider the set

$$H := \{\varphi \in G([t_0, t_0 + \sigma], \mathbb{R}) : \varphi(t_0) = x_0 \text{ and } |\varphi(t) - x_0| \leqslant \beta,\ t \in [t_0, t_0 + \sigma]\}.$$

The set H is nonempty, since

$$\begin{aligned} \varphi : [t_0, t_0+\sigma] &\to \mathbb{R} \\ s &\mapsto \varphi(s) := x_0, \end{aligned}$$

belongs to H. Define $T : H \to H$ given by

$$(Tx)(t) := x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s), \qquad x \in H. \tag{3.4}$$

Taking into account the condition (A3), we infer that the integral on the right-hand side of (3.4) is well-defined. Now, given $x \in H$ and $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$, by conditions (A2), (A3), (A4), Theorem 2.2 and Corollary 2.3, we have

$$\begin{aligned} &|(Tx)(\tau_2) - (Tx)(\tau_1)| \\ &= \left| \int_{t_0}^{\tau_2} a(\tau_2,s) f(x(s),s)\, \mathrm{d}g(s) - \int_{t_0}^{\tau_1} a(\tau_1,s) f(x(s),s)\, \mathrm{d}g(s) \right| \\ &= \left| \int_{t_0}^{\tau_1} a(\tau_2,s) f(x(s),s)\, \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} a(\tau_2,s) f(x(s),s)\, \mathrm{d}g(s) - \int_{t_0}^{\tau_1} a(\tau_1,s) f(x(s),s)\, \mathrm{d}g(s) \right| \\ &\leqslant \left| \int_{\tau_1}^{\tau_2} a(\tau_2,s) f(x(s),s)\, \mathrm{d}g(s) \right| + \left| \int_{t_0}^{\tau_1} (a(\tau_2,s) - a(\tau_1,s))\, f(x(s),s)\, \mathrm{d}g(s) \right| \\ &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\, M(s)\, \mathrm{d}g(s) + \int_{t_0}^{\tau_1} |a(\tau_2,s) - a(\tau_1,s)|\, M(s)\, \mathrm{d}g(s) \\ &\overset{\text{Thm 2.2, (A2), (A4) and (3.2)}}{\leqslant} \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{\tau_1} (a(\tau_2,s) - a(\tau_1,s)) M(s)\, \mathrm{d}g(s) \\ &\leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2,s) - a(\tau_1,s)) M(s)\, \mathrm{d}g(s), \end{aligned}$$

that is,

$$|(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2,s) - a(\tau_1,s)) M(s)\, \mathrm{d}g(s). \tag{3.5}$$

Define $v : [t_0, t_0+\sigma] \to \mathbb{R}$ by

$$v(t) := \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s), \tag{3.6}$$

for every $t \in [t_0, t_0+\sigma]$. Since M is a Henstock–Kurzweil–Stieltjes integrable function, $\int_{t_0}^{t} cM(s)\, \mathrm{d}g(s)$ exists for all $t \in [t_0, t_0+\sigma]$. On the other hand, using the same arguments as in the Remark 3.1, we ensure the existence of $\int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s)$ for all $t \in [t_0, t_0+\sigma]$. Then v is well-defined. Also, it is easy to check that v is a nondecreasing function. Using (3.5) and (3.6), we have

$$|(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant v(\tau_2) - v(\tau_1), \tag{3.7}$$

for all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$. Note that the limits $(Tx)(t+)$ for $t \in [t_0, t_0+\sigma)$ and $(Tx)(t-)$ for $t \in (t_0, t_0+\sigma]$ exist. Indeed, since v is a nondecreasing function, then the limits $v(t+)$ for $t \in [t_0, t_0+\sigma)$ and $v(t-)$ for $t \in (t_0, t_0+\sigma]$ exist and, therefore, (3.7) ensures the Cauchy condition is satisfied, which implies the existence of the corresponding limits $(Tx)(t+)$ and

$(Tx)(t-)$. From this, we get that $Tx \in G([t_0, t_0+\sigma], \mathbb{R})$. Also, for $t_0 \leqslant t \leqslant t_0+\sigma$, by condition (A4), Theorem 2.2 and Corollary 2.3, we obtain

$$\begin{aligned} |(Tx)(t) - x_0| &= \left| \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s) \right| \\ &\leqslant \int_{t_0}^{t} |a(t,s)| M(s)\, \mathrm{d}g(s) \\ &\leqslant \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s) \\ &\leqslant \int_{t_0}^{t_0+\sigma} cM(s)\, \mathrm{d}g(s) \\ &\overset{\overset{(3.3)}{\downarrow}}{=} \beta. \end{aligned}$$

Clearly, $(Tx)(t_0) = x_0$. It implies that $Tx \in H$ for all $x \in H$. Hence, T is well-defined.

Assertion 1. H is convex and closed.
Let $\varphi, \phi \in H$. Then for all $\theta \in [0,1]$, we have $(1-\theta)\phi + \theta\varphi \in G([t_0, t_0+\sigma])$ and

$$\begin{aligned} |(1-\theta)\phi(t) + \theta\varphi(t) - x_0| &= |(1-\theta)\phi(t) + \theta\varphi(t) - ((1-\theta)x_0 + \theta x_0)| \\ &\leqslant (1-\theta)|\phi(t) - x_0| + \theta|\varphi(t) - x_0| \\ &\leqslant (1-\theta)\beta + \theta\beta = \beta. \end{aligned}$$

This proves that H is convex.

On the other hand, let $\{\varphi_k\}_{k\in\mathbb{N}} \subset H$ be such that $\varphi_k \overset{\|\cdot\|_\infty}{\longrightarrow} \varphi$ (on $[t_0, t_0+\sigma]$) as $k \to \infty$. Since each φ_k is regulated and φ_k converges uniformly to φ on $[t_0, t_0+\sigma]$, Lemma 2.8 guarantees that φ is regulated on $[t_0, t_0+\sigma]$ and, therefore, $\varphi \in G([t_0, t_0+\sigma], \mathbb{R})$. Also, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|\varphi(t) - x_0| \leqslant |\varphi_k(t) - \varphi(t)| + |\varphi_k(t) - x_0| \leqslant \varepsilon + \beta,$$

for all $t \in [t_0, t_0+\sigma]$ and $k \geqslant N$. Since $\varepsilon > 0$ is arbitrary, we get $|\varphi(t) - x_0| \leqslant \beta$ for all $t \in [t_0, t_0+\sigma]$. It implies that H is closed.

Assertion 2. $\mathcal{A} := T(H) = \{Tx : x \in H\}$ is relatively compact.
Note that the set $\{y(t_0) : y \in \mathcal{A}\} = \{\underbrace{(Tx)(t_0)}_{x_0} : x \in H\}$ is bounded. On the other hand, for an arbitrary $y = Tx$, $x \in H$ and $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$, by (3.7), we have

$$|y(\tau_2) - y(\tau_1)| = |(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant v(\tau_2) - v(\tau_1). \tag{3.8}$$

Hence, by Theorem 2.9, $\mathcal{A} = T(H)$ is relatively compact.

Assertion 3. T is continuous.
By condition (A5), Theorem 2.2 and Corollary 2.3, we have that for $x, z \in H$ and for $t_0 \leqslant t \leqslant t_0 + \sigma$,

$$
\begin{aligned}
|(Tx)(t) - (Tz)(t)| &= \left| \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s) - \int_{t_0}^{t} a(t,s) f(z(s),s)\, \mathrm{d}g(s) \right| \\
&= \left| \int_{t_0}^{t} a(t,s) (f(x(s),s) - f(z(s),s))\, \mathrm{d}g(s) \right| \\
&\leqslant \int_{t_0}^{t} |a(t,s)| L(s) |x(s) - z(s)|\, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t} |x(s) - z(s)| c L(s)\, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t_0+\sigma} |x(s) - z(s)| c L(s)\, \mathrm{d}g(s) \\
&\leqslant \|x - z\|_\infty \left(\int_{t_0}^{t_0+\sigma} c L(s)\, \mathrm{d}g(s) \right).
\end{aligned}
$$

From the above estimate, we conclude that T is continuous.

Therefore, all the hypotheses of the Schauder Fixed-Point Theorem (Theorem 2.11) are satisfied, which implies that T has a fixed point in H. Thus, we conclude that the equation (3.1) possesses a solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$.

It remains to prove the uniqueness of the solution of (3.1).

Uniqueness: Assume that $x, z : [t_0, t_0 + \sigma] \to \mathbb{R}$ are two solutions of Volterra–Stieltjes integral equation (3.1). Fix arbitrarily $t \in [t_0, t_0 + \sigma]$. Then, keeping in mind condition (A5) and Theorem 2.2, we infer the following estimates

$$
\begin{aligned}
|x(t) - z(t)| &= \left| \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s) - \int_{t_0}^{t} a(t,s) f(z(s),s)\, \mathrm{d}g(s) \right| \\
&= \left| \int_{t_0}^{t} a(t,s) (f(x(s),s) - f(z(s),s))\, \mathrm{d}g(s) \right| \\
&\leqslant \int_{t_0}^{t} |a(t,s)| L(s) |x(s) - z(s)|\, \mathrm{d}g(s) \\
&\leqslant c \|L\|_\infty \int_{t_0}^{t} |x(s) - z(s)|\, \mathrm{d}g(s) \\
&< \varepsilon + c \|L\|_\infty \int_{t_0}^{t} |x(s) - z(s)|\, \mathrm{d}g(s),
\end{aligned}
$$

for every $\varepsilon > 0$. Hence, in view of Lemma 2.4, we have

$$|x(t) - z(t)| \leqslant \varepsilon e^{c\|L\|_\infty (g(t) - g(t_0))}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $x(t) = z(t)$ for all $t \in [t_0, t_0 + \sigma]$, that is, $x = z$. □

Remark 3.6. If $a(t,s) = a_1(t) b_1(s)$ where a_1 is nondecreasing on $[t_0, t_0 + \sigma]$ and b_1 is regulated and positive on $[t_0, t_0 + \sigma]$, then it is clear that a satisfies condition (A2).

Example 3.7. Consider the Volterra–Stieltjes integral equation given by

$$x(t) = x_0 + \int_{t_0}^{t} k(t - s) f(x(s), s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma],$$

where $t_0, x_0 \in \mathbb{R}$, $\sigma > 0$, $k : [-\sigma, \sigma] \to \mathbb{R}$ is a nondecreasing function, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies condition (A1) and $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies conditions (A3)–(A5) from Theorem 3.5.

Define $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ given by

$$a(t,s) := k(t-s), \qquad (t,s) \in [t_0, t_0 + \sigma]^2.$$

In the sequel, we show that a satisfies condition (A2) from Theorem 3.5. Indeed, notice that given $t, s \in [t_0, t_0 + \sigma]$, we have $t - s \in [-\sigma, \sigma] = \mathrm{Dom}(k)$ and, therefore, a is well-defined over $[t_0, t_0 + \sigma]^2$.

Obviously, $a(\cdot, s)$ is nondecreasing for any $s \in [t_0, t_0 + \sigma]$ and $a(t, \cdot)$ is nonincreasing for any $t \in [t_0, t_0 + \sigma]$, getting (A2).

We will present an example of a Volterra–Stieltjes integral equation of the form (3.1) which satisfies all the hypotheses of the previous theorem.

Example 3.8. Consider the Volterra–Stieltjes integral equation given by

$$x(t) = x_0 + \int_0^t a(t,s) f(x(s), s)\, \mathrm{d}g(s), \qquad t \in [0, 3/\delta],$$

where $x_0 \in \mathbb{R}$, $\delta > 0$, $g : [0, \frac{3}{\delta}] \to \mathbb{R}$ is a nondecreasing function, $a : [0, \frac{3}{\delta}]^2 \to \mathbb{R}$ and $f : \mathbb{R} \times [0, \frac{3}{\delta}] \to \mathbb{R}$ are given, respectively, by

$$a(t,s) = st^3 e^{-\delta t}, \qquad (t,s) \in [0, 3/\delta]^2$$

and

$$f(x,t) = \frac{\{t+2\} \cos(2x)}{4^t + [t]}, \qquad (x,t) \in \mathbb{R} \times [0, 3/\delta],$$

where the symbol $[t]$ denotes the integer part of t, and the symbol $\{t\} := t - [t]$ denotes the fractional part of t. We will verify the conditions (A1)–(A5). Indeed, clearly g satisfies condition (A1).

Note that for any fixed $t \in [0, \frac{3}{\delta}]$, the function $[0, \frac{3}{\delta}] \ni s \mapsto a(t,s)$ is regulated on $[0, \frac{3}{\delta}]$. Since $a(t,s) = st^3 e^{-\delta t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t,s) = st^2 e^{-\delta t}(3 - \delta t) \geqslant 0,$$

for all $t \in [0, \frac{3}{\delta}]$. Thus, a is a nondecreasing function with respect to the first variable, proving condition (A2).

Let $x \in G([0, \frac{3}{\delta}], \mathbb{R})$ and $t \in [0, \frac{3}{\delta}]$ be given. Notice that $[0, \frac{3}{\delta}] \ni s \mapsto a(t,s) f(x(s), s)$ is a regulated function on $[0, \frac{3}{\delta}]$. Thus by Theorem 2.1 (item (i)), $\int_0^{\frac{3}{\delta}} a(t,s) f(x(s), s) \mathrm{d}g(s)$ exists, obtaining condition (A3).

Define $M : [0, \frac{3}{\delta}] \to \mathbb{R}^+$ by $M(s) = \{s+2\}$, for $s \in [0, \frac{3}{\delta}]$. Evidently, M is a Henstock–Kurzweil–Stieltjes integrable function with respect to g and for $x \in G([0, \frac{3}{\delta}], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$,

$[\tau_1, \tau_2] \subset [0, \frac{3}{\delta}]$ and $b_{\tau_2,\tau_1}(s) := c_1 a(\tau_2, s) + c_1 a(\tau_1, s)$, we have

$$\begin{aligned}
\left| \int_{\tau_1}^{\tau_2} b_{\tau_2,\tau_1}(s) f(x(s), s) \mathrm{d}g(s) \right| &\overset{\text{Thm 2.1}}{\leqslant} \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)| \, |f(x(s), s)| \, \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)| \left| \frac{\{s+2\} \cos(2x(s))}{4^s + [s]} \right| \mathrm{d}g(s) \\
&\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)| \, \{s+2\} \, \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)| \, M(s) \mathrm{d}g(s),
\end{aligned}$$

proving the condition (A4).

On the other hand, define $L : [0, \frac{3}{\delta}] \to \mathbb{R}^+$ by $L(t) = 2$, for $t \in [0, \frac{3}{\delta}]$. Note that L is a regulated function and for $x,\ y \in G([0, \frac{3}{\delta}], \mathbb{R})$ and $\tau_1,\ \tau_2 \in [0, \frac{3}{\delta}]$, $\tau_1 \leqslant \tau_2$, we get

$$\begin{aligned}
\left| \int_{\tau_1}^{\tau_2} a(\tau_2, s) \left[f(x(s), s) - f(y(s), s) \right] \mathrm{d}g(s) \right| &\overset{\text{Thm 2.1}}{\leqslant} \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, |f(x(s), s) - f(y(s), s)| \, \mathrm{d}g(s) \\
&\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, \Big| \cos(2x(s)) - \cos(2y(s)) \Big| \mathrm{d}g(s) \\
&\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, |2x(s) - 2y(s)| \, \mathrm{d}g(s) \\
&= 2 \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, |x(s) - y(s)| \, \mathrm{d}g(s),
\end{aligned}$$

getting the condition (A5). Hence f, a and g fulfill all the hypotheses of Theorem 3.5.

The next example is an adaptation of [18, Example 7.8]. It is a modified version of a model of a single artificial effective neuron with dissipation. See [10,16].

Example 3.9. Consider the equation

$$x(t) = x_0 + \int_0^t k(s) \tanh(x(s)) \, ds, \qquad t \in [0, 1]$$

where k is a nondecreasing function on $[0, 1]$. Define $a(t, s) := k(s)$ for all $(t, s) \in [0, 1]^2$, $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ by $f(x, t) := \tanh(x)$ for all $(x, t) \in \mathbb{R} \times [0, 1]$, and $g(s) = s$ for all $s \in [0, 1]$.

Observe that, by definition, the function g is left-continuous on $(0, 1]$ and increasing on $[0, 1]$.

Notice that the function a is constant with relation to the first variable. Thus, a is a nondecreasing function with respect to the first variable. Also, since k is a nondecreasing function, we have that for any fixed $t \in [0, 1]$, the function $[0, 1] \ni s \mapsto a(t, s) = k(s)$ is regulated on $[0, 1]$, obtaining the condition (A2). Moreover, $a(t, s) f(x(s), s)$ is a regulated function on $[0, 1]$, for all $x \in G([0, 1], \mathbb{R})$, and all $t \in [0, 1]$. Hence, the integral $\int_0^1 a(t, s) f(x(s), s) \mathrm{d}g(s)$ exists, getting (A3).

On the other hand, define $M : [0, 1] \to \mathbb{R}^+$ by $M(t) = 1$, for $t \in [0, 1]$. By Theorem 2.1, we have

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} b_{\tau_2,\tau_1}(s) f(x(s),s)\mathrm{d}g(s)\right| &\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\,|f(x(s),s)|\,\mathrm{d}g(s)\\ &= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\,|\tanh(x(s))|\,\mathrm{d}g(s)\\ &\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\, M(s)\mathrm{d}g(s),\end{aligned}$$

for $x \in G([0,1],\mathbb{R})$, $c_1, c_2 \in \mathbb{R}$, $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$ and $b_{\tau_2,\tau_1}(s) := c_1 a(\tau_2, s) + c_2 a(\tau_1, s)$, where the third inequality follows of the fact that $-1 < \tanh(x) < 1$ for all $x \in \mathbb{R}$.

Finally, define $L : [0,1] \to \mathbb{R}^+$ by $L(t) = 1$, for $t \in [0,1]$. Evidently, L is a regulated function and

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} a(\tau_2,s)\,[f(x(s),s) - f(y(s),s)]\,\mathrm{d}g(s)\right| &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,|f(x(s),s) - f(y(s),s)|\,\mathrm{d}g(s)\\ &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,|x(s) - y(s)|\,\mathrm{d}g(s),\end{aligned}$$

for $x,\ y \in G([0,1],\mathbb{R})$ and all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$, obtaining the condition (A5). Notice that $|\tanh(v) - \tanh(u)| \leqslant |v - u|$ for all $v, u \in \mathbb{R}$. Hence f, a and g fulfill all the hypotheses of Theorem 3.5.

4 Impulsive Volterra–Stieltjes integral equations

In this section, we are interested in the study of impulsive Volterra–Stieltjes integral equations.

Consider a Volterra–Stieltjes integral equation given by:

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\,\mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma],$$

where the Henstock–Kurzweil–Stieltjes integral on the right-hand side is taken with respect to a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$.

Let the set $D = \{t_1, \ldots, t_m\} \subset [t_0, t_0 + \sigma]$ be such that $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$ and let the functions $I_k : \mathbb{R} \to \mathbb{R}$ be given for $k \in \{1, \ldots, m\}$. Assume that $a(\cdot, s)$ and g are continuous at each $\tau \in D$ and consider the problem to determine a function x satisfying the given Volterra–Stieltjes integral equation and impulse conditions $\Delta^+ x(t_k) = I_k(x(t_k))$ for $k \in \{1, \ldots, m\}$. Using this, we achieve the following formulation of the problem:

$$\begin{aligned} x(v) - x(u) &= \int_{t_0}^{v} a(v,s) f(x(s),s)\,\mathrm{d}g(s)\\ &\quad - \int_{t_0}^{u} a(u,s) f(x(s),s)\,\mathrm{d}g(s) \quad \text{for } u, v \in J_k,\ k \in \{0, \ldots, m\},\\ \Delta^+ x(t_k) &= I_k(x(t_k)), \qquad k \in \{1, \ldots, m\},\\ x(t_0) &= x_0,\end{aligned}$$

where $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1, \ldots, m-1\}$, and $J_m = (t_m, t_0 + \sigma]$.

The value of the following integrals

$$\int_{t_0}^{v} a(v,s) f(x(s),s)\,\mathrm{d}g(s) \quad \text{and} \quad \int_{t_0}^{u} a(u,s) f(x(s),s)\,\mathrm{d}g(s),$$

where $u, v \in J_k$, are the same if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k. This follows from the properties of Henstock–Kurzweil–Stieltjes integral. Also, let us assume g is a left-continuous function which is continuous at t_k, for each $k = 1, \ldots, m$. Therefore, it implies that $\Delta^+ g(t_k) = 0$ for every $k \in \{1, \ldots, m\}$. Moreover, we assume a is continuous with respect to first variable at $t_1, \ldots, t_m$ and also, a satisfies condition (A2) presented in Section 3. Further suppose that f and g satisfy conditions (A1), (A3) and (A4) presented in Section 3. Under these assumptions, our problem can be rewritten as

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\}, \\ t_k < t}} I_k(x(t_k)). \tag{4.1}$$

It is not difficult to see that by the assumptions above, the function

$$t \mapsto \int_{t_0}^{t} a(t,s) f(x(s), s) \mathrm{d}g(s)$$

is continuous at $t_1, \ldots, t_m$ (see Remark 4.1 below) and, therefore, $\Delta^+ x(t_k) = I_k(x(t_k))$ for every $k \in \{1, \ldots, m\}$.

Remark 4.1. We assume that f, g and a satisfy the assumptions above. Using the same arguments as in the proof of Theorem 3.5, we can prove the following inequality

$$\left| \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) - \int_{t_0}^{\tau} a(\tau,s) f(x(s), s)\, \mathrm{d}g(s) \right| \leqslant |v(t) - v(\tau)|\,, \tag{4.2}$$

for all $t, \tau \in [t_0, t_0 + \sigma]$, where v is given by

$$v(t) := \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma]. \tag{4.3}$$

Here $c := \sup_{(t,s) \in [t_0, t_0+\sigma]^2} |a(t,s)|$. Notice that every point in $[t_0, t_0 + \sigma]$ at which the function v is continuous, is a continuity point of the function $t \mapsto \int_{t_0}^{t} a(t,s) f(x(s), s) \mathrm{d}g(s)$. Next, let us prove that v given by (4.3) is a continuous function at $t_1, \ldots, t_m$. Clearly, $v_1(t) = \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s)$, $t \in [t_0, t_0 + \sigma]$, is continuous at $t_1, \ldots, t_m$.

Assertion 1. $v_2(t) = \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s)$, $t \in [t_0, t_0 + \sigma]$, is continuous at $t_1, \ldots, t_m$.
Let $i \in \{1, \ldots, m\}$ and $(\tau_n)_{n \in \mathbb{N}} \subset [t_0, t_0 + \sigma]$ such that $\tau_n \overset{n\to\infty}{\to} t_i$.

Define the sequence of functions

$$\varphi_n(s) := a(\tau_n, s) M(s), \qquad s \in [t_0, t_0 + \sigma], \tag{4.4}$$

and $\varphi : [t_0, t_0 + \sigma] \to \mathbb{R}$ by $\varphi(s) := a(t_i, s) M(s), s \in [t_0, t_0 + \sigma]$. As $a(\cdot, s)$ is continuous at t_i and $(\tau_n)_{n \in \mathbb{N}} \subset [t_0, t_0 + \sigma]$ is such that $\tau_n \overset{n\to\infty}{\to} t_i$, we have $\lim_{n\to\infty} a(\tau_n, s) = a(t_i, s)$, and therefore,

$$\lim_{n\to\infty} \varphi_n(s) = \lim_{n\to\infty} a(\tau_n, s) M(s) = a(t_i, s) M(s) = \varphi(s).$$

According to condition (A3), $\int_{t_0}^{t_0+\sigma} a(\tau_n, s) M(s)\, \mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Using this fact together with (4.4), we get $\int_{t_0}^{t_0+\sigma} \varphi_n(s)\, \mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$.

On the other hand, for all $t \in [t_0, t_0 + \sigma]$, $n \in \mathbb{N}$, we have

$$|\varphi_n(t)| = |a(\tau_n, t) M(t)| \leqslant c\, |M(t)| = cM(t).$$

This implies that

$$\kappa(t) \leqslant \varphi_n(t) \leqslant \omega(t), \qquad t \in [t_0, t_0+\sigma],$$

where $\omega(t) := cM(t)$ and $\kappa(t) = -cM(t)$. Also, observe that the integrals $\int_{t_0}^{t_0+\sigma} \kappa(s)\, \mathrm{d}g(s)$ and $\int_{t_0}^{t_0+\sigma} \omega(s)\, \mathrm{d}g(s)$ exist, since M is a Henstock–Kurzweil–Stieltjes integrable function. Since all the hypotheses of Theorem 2.7 are satisfied, we obtain

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} \varphi_n(s)\, \mathrm{d}g(s) = \int_{t_0}^{t_0+\sigma} \varphi(s)\, \mathrm{d}g(s).$$

Hence, the function v_2 is continuous at t_i, for each $i = 1, \dots, m$, proving Assertion 1.

From these facts and by the equality $v(t) = v_1(t) + v_2(t)$, it follow that v is continuous at $t_1, \dots, t_m$.

In the next result, we describe how we can translate the conditions on impulsive Volterra–Stieltjes integral equation to the conditions on Volterra–Stieltjes integral equations. It will be very important in order to prove results for impulsive Volterra–Stieltjes integral equations using known results for Volterra–Stieltjes integral equations.

Lemma 4.2. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \dots < t_m < t_0+\sigma$, $D = \{t_0, \dots, t_m\}$, $I_k : \mathbb{R} \to \mathbb{R}$ for $k \in \{1, \dots, m\}$ and let $a : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$ and $g : [t_0, t_0+\sigma] \to \mathbb{R}$ satisfy conditions (A1)–(A5). Define*

$$\tilde{a}(t,s) = \begin{cases} a(t,s), & t \in [t_0, t_0+\sigma] \text{ and } s \in [t_0, t_0+\sigma]\backslash D, \\ 1, & t \in [t_0, t_0+\sigma] \text{ and } s \in D, \end{cases} \tag{4.5}$$

$$\tilde{f}(x,s) = \begin{cases} f(x,s), & \text{for } x \in \mathbb{R} \text{ and } s \in [t_0, t_0+\sigma]\backslash D, \\ I_k(x), & \text{for } x \in \mathbb{R} \text{ and } s \in D, \end{cases} \tag{4.6}$$

$$\tilde{g}(s) = \begin{cases} g(\tau), & \text{for } s \in [t_0, t_1], \\ g(s) + k, & \text{for } s \in (t_k, t_{k+1}] \text{ and } k \in \{1, \dots, m-1\}, \\ g(s) + m, & \text{for } s \in (t_m, t_0+\sigma]. \end{cases} \tag{4.7}$$

Also, suppose that $I_1, \dots, I_m : \mathbb{R} \to \mathbb{R}$ satisfy the following condition:

(I) There exists constants $M_2, L_2 > 0$ such that

$$|I_k(x)| \leqslant M_2$$

for every $k \in \{1, \dots, m\}$ and $x \in \mathbb{R}$, and

$$|I_k(x) - I_k(y)| \leqslant L_2\, |x-y|$$

for every $k \in \{1, \dots, m\}$ and $x, y \in \mathbb{R}$.

Then the functions $\tilde{a} : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, $\tilde{f} : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$ and $\tilde{g} : [t_0, t_0+\sigma] \to \mathbb{R}$ also satisfy conditions (A1)–(A5) with $\tilde{a}, \tilde{f}, \tilde{g}$ respectively in the place of a, f, g.

Proof. Since g is nondecreasing and left-continuous, $\tilde{g}$ has the same properties by the definition, proving condition (A1). The condition (A2) is an immediate consequence from the definition of $\tilde{a}$.

Notice that (A3) follows by combining the condition (A1) and the hypotheses from $\tilde{f}$ and $\tilde{a}$ together with Lemma 2.10.

To prove the condition (A4), let $x \in G([t_0, t_0+\sigma], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$, $[u_1, u_2] \subset [t_0, t_0+\sigma]$ and $b_{u_2,u_1}(s) := c_1 a(u_2, s) + c_2 a(u_1, s)$. From Lemma 2.10, we obtain

$$\begin{aligned}\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s) &= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x(s),s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b_{u_2,u_1}(t_k)\tilde{f}(x(t_k),t_k)\Delta^+\tilde{g}(t_k)\\ &= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x(s),s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b_{u_2,u_1}(t_k) I_k(x(t_k))\Delta^+\tilde{g}(t_k)\end{aligned}$$

and, therefore,

$$\left|\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s)\right| \leqslant \int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \tag{4.8}$$

On the other hand, notice that $\tilde{g}(v) - \tilde{g}(u) \geqslant g(v) - g(u)$ whenever $t_0 \leqslant u \leqslant v \leqslant t_0 + \sigma$. It implies together with the definition of the Henstock–Kurzweil–Stieltjes integral and Theorem 2.2 the following

$$\int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s) \leqslant \int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s), \tag{4.9}$$

where $\tilde{M}(s) := 1 + M_2 + M_1(s)$ for all $s \in [t_0, t_0+\sigma]$. On the other hand, the function

$$h(t) := \int_{t_0}^{t} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s), \qquad t \in [t_0, t_0+\sigma],$$

is nondecreasing and $\Delta^+ h(t_k) = \tilde{M}(t_k)\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k)$ for $k \in \{1,\dots,m\}$ by Theorem 2.5. Hence

$$\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} \tilde{M}(t_k)\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant h(u_2) - h(u_1).$$

Hence,

$$\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant \int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s) \tag{4.10}$$

Now, by (4.8), (4.9) and (4.10), we get

$$\left|\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s)\right| \leqslant 2\int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s). \tag{4.11}$$

Now, defining $M(t) = 2\tilde{M}(t)$ for all $t \in [t_0, t_0+\sigma]$, we get the statement (A4).

To prove the condition (A5), consider $x, z \in G([t_0, t_0+\sigma], \mathbb{R})$ and $[u_1, u_2] \subset [t_0, t_0+\sigma]$. Using Lemma 2.10 again, we obtain

$$\begin{aligned}&\int_{u_1}^{u_2} a(u_2,s)\left(\tilde{f}(x(s),s) - \tilde{f}(z(s),s)\right)\mathrm{d}\tilde{g}(s)\\ &\quad= \int_{u_1}^{u_2} a(u_2,s)\left(f(x(s),s) - f(z(s),s)\right)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} a(u_2,t_k)(I_k(x(t_k)) - I_k(z(t_k)))\Delta^+\tilde{g}(t_k).\end{aligned}$$

Consequently,

$$\left| \int_{u_1}^{u_2} a(u_2,s) \left(\tilde{f}(x(s),s) - \tilde{f}(z(s),s) \right) \mathrm{d}\tilde{g}(s) \right|$$

$$\leqslant \int_{u_1}^{u_2} L_1(s)\, |a(u_2,s)|\, |x(s) - z(s)| \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\}, \\ u_1 \leqslant t_k < u_2}} L_2\, |a(u_2,t_k)|\, |x(t_k) - z(t_k)| \Delta^+ \tilde{g}(t_k).$$

Therefore,

$$\int_{u_1}^{u_2} L_1(s)\, |a(u_2,s)|\, |x(s) - z(s)| \mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} \tilde{L}(s)\, |a(u_2,s)|\, |x(s) - z(s)| \mathrm{d}\tilde{g}(s),$$

where $\tilde{L}(s) = 1 + L_2 + L_1(s)$ for all $s \in [t_0, t_0 + \sigma]$. Next, we observe that the function

$$\gamma(t) = \int_{t_0}^{t} \tilde{L}(s)\, |a(u_2,s)|\, |x(s) - z(s)|\, \mathrm{d}\tilde{g}(s), \qquad t \in [t_0, t_0 + \sigma],$$

is nondecreasing and

$$\Delta^+ \gamma(t_k) = \tilde{L}(t_k)\, |a(u_2,t_k)|\, |x(t_k) - z(t_k)| \Delta^+ \tilde{g}(t_k),$$

for $k \in \{1, \dots, m\}$. Hence,

$$\sum_{\substack{k \in \{1,\dots,m\}, \\ u_1 \leqslant t_k < u_2}} L_2\, |a(u_2,t_k)|\, |x(t_k) - z(t_k)| \Delta^+ \tilde{g}(t_k) \leqslant \sum_{\substack{k \in \{1,\dots,m\}, \\ u_1 \leqslant t_k < u_2}} \tilde{L}(t_k)\, |a(u_2,t_k)|\, |x(t_k) - z(t_k)| \Delta^+ \tilde{g}(t_k)$$

$$\leqslant \gamma(u_2) - \gamma(u_1).$$

It follows that

$$\left| \int_{u_1}^{u_2} a(u_2,s) \left(\tilde{f}(x(s),s) - \tilde{f}(z(s),s) \right) \mathrm{d}\tilde{g}(s) \right| \leqslant 2 \int_{u_1}^{u_2} \tilde{L}(s)\, |a(u_2,s)|\, |x(s) - z(s)|\, \mathrm{d}\tilde{g}(t).$$

Now, defining $L(t) = 2\tilde{L}(t)$ for all $t \in [t_0, t_0 + \sigma]$, we get the desired result. □

The following theorem describes a strong relation between the solutions of impulsive Volterra–Stieltjes integral equations and the solutions of Volterra–Stieltjes integral equations without impulses. We can omit its proof as it follows by arguments analogous to those used in [12] to prove Theorem 3.1.

Theorem 4.3. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$, $D = \{t_0, \dots, t_m\}$, $I_k : \mathbb{R} \to \mathbb{R}$ for $k \in \{1, \dots, m\}$ and $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$. Assume that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies the condition (A1) and $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ satisfies condition (A2). Furthermore, assume that g and $a(\cdot, s)$, $s \in [t_0, t_0 + \sigma]$, are continuous at each $\tau \in D$. Consider the functions $\tilde{a} : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$, $\tilde{f} : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ and $\tilde{g} : [t_0, t_0 + \sigma] \to \mathbb{R}$ defined in Lemma 4.2, given by (4.5), (4.6) and (4.7) respectively.*

Then $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a solution of

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\}, \\ t_k < t}} I_k(x(t_k)), \tag{4.12}$$

if and only if $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a solution of

$$x(t) = x_0 + \int_{t_0}^{t} \tilde{a}(t,s) \tilde{f}(x(s),s)\, \mathrm{d}\tilde{g}(s). \tag{4.13}$$

As an immediate consequence, we obtain a result about existence and uniqueness of solutions of impulsive Volterra–Stieltjes integral equation. We omit its proof, since it follows directly from the correspondence and the analogue result for Volterra–Stieltjes integral equation.

Theorem 4.4. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$, $D = \{t_0, \ldots, t_m\}$, $I_k : \mathbb{R} \to \mathbb{R}$ for $k \in \{1, \ldots, m\}$ and let $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfy conditions (A1)–(A5). Furthermore, assume that g and $a(\cdot, s)$, $s \in [t_0, t_0 + \sigma]$, are continuous at each $\tau \in D$. Also, suppose that $I_1, \ldots, I_m : \mathbb{R} \to \mathbb{R}$ satisfies condition (I) from Lemma 4.2.*

Then there exists a unique solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ of the impulsive Volterra–Stieltjes integral equation

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < t}} I_k(x(t_k)). \tag{4.14}$$

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