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New class of practically solvable systems of difference equations of hyperbolic-cotangent-type

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. The systems of difference equations

$$x_{n+1} = \frac{u_n v_{n-2} + a}{u_n + v_{n-2}}, \quad y_{n+1} = \frac{w_n s_{n-2} + a}{w_n + s_{n-2}}, \quad n \in \mathbb{N}_0,$$

where a, u_0, w_0, v_j, s_j $j = -2, -1, 0$, are complex numbers, and the sequences u_n, v_n, w_n, s_n are x_n or y_n, are studied. It is shown that each of these sixteen systems is practically solvable, complementing some recent results on solvability of related systems of difference equations.

Keywords: system of difference equations, general solution, solvability of difference equations, hyperbolic-cotangent-type system of difference equations.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, be the sets of natural, whole, real and complex numbers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $p, q \in \mathbb{Z}$ and $p \leq q$, then $j = \overline{p,q}$ is a notation for $j = p, p+1, \dots, q$.

First important results on solvability of difference equations and systems belong to de Moivre [5–7], D. Bernoulli [3], Euler [9], Lagrange [15] and Laplace [16]. They found a few methods for solving linear difference equations with constant coefficients, as well as methods for solving some linear difference equations with nonconstant coefficients and some nonlinear difference equations. Many books containing basic methods for solving difference equations and systems have appeared since (see, e.g., [4, 11–13, 18, 19, 21, 22]). It is interesting to note

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that many difference equations and systems have naturally appeared as some mathematical models for problems in combinatorics, population dynamics and other branches of sciences (see, e.g., [5–7, 11–13, 15–17, 20, 21, 31, 49]). The fact that it is difficult to find new methods for solving difference equations and systems has influenced on a lack of considerable interest in the topic for a long time. Use of computers seems renewed some interest in the topic in the last two decades.

During the '90s has started some interest in concrete difference equations and systems. Papaschinopoulos and Schinas have influenced on the study of such systems (see, e.g., [23–28, 32, 33]). Work [29] is on solvability, whereas [24–26, 28, 32, 33] can be regarded as ones on solvability in a wider sense, since they are devoted to finding invariants of the systems studied therein. Beside their study, have appeared several papers by some other authors which essentially rediscovered some known results. These facts motivated us to study the solvability of difference equations and systems (see, e.g., [1, 34–48] and many other related references therein).

Let $k, l \in \mathbb{N}_0$, $a \in \mathbb{R}$ (or $\mathbb{C}$), and

$$z_{n+1} = \frac{z_{n-k}z_{n-l} + a}{z_{n-k} + z_{n-l}}, \quad n \in \mathbb{N}_0. \tag{1.1}$$

Equation (1.1) have been studied by several authors. Convergence of positive solutions to the equation follows from a result in [14] (see [2]). For some generalizations of the result in [14], see [8] and [27]. The fact that equation (1.1) resembles the hyperbolic-cotangent sum formula has been a good hint for solvability of the equation. Some special cases of the equation were studied in [30]. In [43] was presented a natural way for showing solvability of the equation.

The following systems

$$x_{n+1} = \frac{u_{n-k}v_{n-l} + a}{u_{n-k} + v_{n-l}}, \quad y_{n+1} = \frac{w_{n-k}s_{n-l} + a}{w_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k, l \in \mathbb{N}_0$, $a, u_{-j}, w_{-j}, v_{-j'}, s_{-j'} \in \mathbb{C}$, $j = \overline{0,k}$, $j' = \overline{0,l}$, and u_n, v_n, w_n, s_n are x_n or y_n, are natural extensions of equation (1.1) (for studying the systems in the form, we have been also motivated by [34]).

The case $k = 0$, $l = 1$, was studied in [47] and [48], and also in [41] where we presented another method. We have also shown therein the theoretical solvability of the systems in (1.2). The case $k = 1$, $l = 2$, has been recently studied in [40]. Here we study practical solvability of the systems in (1.2) in the case $k = 0$ and $l = 2$, continuing our research in [40, 41, 43, 47, 48]. We use and combine some methods from these, as well as the following works: [35–39, 42, 46]. The investigation of the case has been announced in [41].

2 Main results

First we mention two lemmas. The first one belongs to Lagrange (see, e.g., [10, 13, 46]), while the second one should be folklore (for a proof see [40]), and have been applied for several times recently (see, e.g., [38, 39, 46]).

Lemma 2.1. *Let t_l, $l = \overline{1,m}$, be the roots of $p_m(t) = \alpha_m t^m + \cdots + \alpha_1 t + \alpha_0$, $\alpha_m \neq 0$, and assume that $t_l \neq t_j$, when $l \neq j$. Then*

$$\sum_{l=1}^{m} \frac{t_l^j}{p'_m(t_l)} = 0, \quad j = \overline{0, m-2}, \quad \textit{and} \quad \sum_{l=1}^{m} \frac{t_l^{m-1}}{p'_m(t_l)} = \frac{1}{\alpha_m}.$$

Lemma 2.2. *Consider the equation*

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_m x_{n-m}, \quad n \geq l, \tag{2.1}$$

where $l \in \mathbb{Z}$, $a_j \in \mathbb{C}$, $j = \overline{1,m}$, $a_m \neq 0$. Let t_k, $k = \overline{1,m}$, be the roots of $q_m(t) = t^m - a_1 t^{m-1} - a_2 t^{m-2} - \cdots - a_m$, and assume that $t_k \neq t_s$, when $k \neq s$.

Then, the solution to equation (2.1) satisfying the initial conditions

$$x_{j-m} = 0, \quad j = \overline{l, l+m-2}, \quad \text{and} \quad x_{l-1} = 1, \tag{2.2}$$

is

$$x_n = \sum_{k=1}^{m} \frac{t_k^{n+m-l}}{q'_m(t_k)}, \tag{2.3}$$

for $n \geq l - m$.

We transform the systems in (1.2) with $k = 0$ and $l = 2$ to some more suitable ones. We have

$$x_{n+1} \pm \sqrt{a} = \frac{(u_n \pm \sqrt{a})(v_{n-2} \pm \sqrt{a})}{u_n + v_{n-2}} \quad \text{and} \quad y_{n+1} \pm \sqrt{a} = \frac{(w_n \pm \sqrt{a})(s_{n-2} \pm \sqrt{a})}{w_n + s_{n-2}},$$

for $n \in \mathbb{N}_0$, and consequently

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{u_n+\sqrt{a}}{u_n-\sqrt{a}} \cdot \frac{v_{n-2}+\sqrt{a}}{v_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{w_n+\sqrt{a}}{w_n-\sqrt{a}} \cdot \frac{s_{n-2}+\sqrt{a}}{s_{n-2}-\sqrt{a}}, \tag{2.4}$$

for $n \in \mathbb{N}_0$.

Hence, the following systems are studied

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.5}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.6}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.7}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.8}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.9}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.10}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.11}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{2.12}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{2.13}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}},\tag{2.14}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\tag{2.15}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\tag{2.16}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}},\tag{2.17}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}},\tag{2.18}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\tag{2.19}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\tag{2.20}$$

for $n\in\mathbb{N}_0$.

Let

$$\zeta_n=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\quad\text{and}\quad \eta_n=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}},$$

then

$$x_n=\sqrt{a}\,\frac{\zeta_n+1}{\zeta_n-1}\quad\text{and}\quad y_n=\sqrt{a}\,\frac{\eta_n+1}{\eta_n-1},\tag{2.21}$$

and the systems (2.5)–(2.20) respectively become

$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2},\tag{2.22}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2},\tag{2.23}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2},\tag{2.24}$$
$$\zeta_{n+1}=\zeta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2},\tag{2.25}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2},\tag{2.26}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2},\tag{2.27}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2},\tag{2.28}$$
$$\zeta_{n+1}=\eta_n\zeta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2},\tag{2.29}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2},\tag{2.30}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2},\tag{2.31}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2},\tag{2.32}$$
$$\zeta_{n+1}=\zeta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2},\tag{2.33}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\zeta_{n-2},\tag{2.34}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\zeta_{n-2},\tag{2.35}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\zeta_n\eta_{n-2},\tag{2.36}$$
$$\zeta_{n+1}=\eta_n\eta_{n-2},\quad \eta_{n+1}=\eta_n\eta_{n-2},\tag{2.37}$$

for $n \in \mathbb{N}_0$.

To study the systems we use some ideas in [35–40, 42, 46]. The case $a = 0$ is simple (see [41]). Hence, it is omitted.

2.1 System (2.22)

First, note that

$$\zeta_n = \eta_n, \quad n \in \mathbb{N}. \tag{2.38}$$

Let

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 1, \tag{2.39}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1}, \quad n \in \mathbb{N}. \tag{2.40}$$

Use of (2.40) implies

$$\zeta_n = (\zeta_{n-2}\zeta_{n-4})^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} = \zeta_{n-2}^{a_1+b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{a_1} = \zeta_{n-2}^{a_2} \zeta_{n-3}^{b_2} \zeta_{n-4}^{c_2},$$

for $n \geq 2$, where

$$a_2 := a_1 + b_1, \quad b_2 := c_1, \quad c_2 := a_1.$$

Assume

$$\zeta_n = \zeta_{n-k}^{a_k} \zeta_{n-k-1}^{b_k} \zeta_{n-k-2}^{c_k}, \tag{2.41}$$

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = a_{k-1}, \tag{2.42}$$

for a $k \geq 2$ and $n \geq k$.

If we use (2.40) in (2.41), we obtain

$$\begin{aligned} \zeta_n &= (\zeta_{n-k-1}\zeta_{n-k-3})^{a_k} \zeta_{n-k-1}^{b_k} \zeta_{n-k-2}^{c_k} \\ &= \zeta_{n-k-1}^{a_k+b_k} \zeta_{n-k-2}^{c_k} \zeta_{n-k-3}^{a_k} \\ &= \zeta_{n-k-1}^{a_{k+1}} \zeta_{n-k-2}^{b_{k+1}} \zeta_{n-k-3}^{c_{k+1}}, \end{aligned}$$

where

$$a_{k+1} := a_k + b_k, \quad b_{k+1} := c_k, \quad c_{k+1} := a_k.$$

In this way, by using induction, we proved that (2.41) and (2.42) hold for every $2 \leq k \leq n$.

From (2.39) and (2.42) we have

$$a_n = a_{n-1} + a_{n-3}, \tag{2.43}$$

not only for $n \geq 4$, but even for all $n \in \mathbb{Z}$, and

$$a_0 = 1, \quad a_{-1} = a_{-2} = 0, \quad a_{-3} = 1, \quad a_{-4} = 0. \tag{2.44}$$

By taking $k = n$ in (2.41), and employing (2.42) and (2.43), it follows that

$$\zeta_n = \zeta_0^{a_n} \zeta_{-1}^{b_n} \zeta_{-2}^{c_n} = \zeta_0^{a_n} \zeta_{-1}^{a_{n-2}} \zeta_{-2}^{a_{n-1}}, \tag{2.45}$$

not only for $n \in \mathbb{N}$, but even for $n \geq -2$.

Combining (2.38) and (2.45), we have

$$\eta_n = \zeta_0^{a_n} \zeta_{-1}^{a_{n-2}} \zeta_{-2}^{a_{n-1}}, \tag{2.46}$$

for $n \in \mathbb{N}$.

Now note that the characteristic polynomial

$$P_3(\lambda) = \lambda^3 - \lambda^2 - 1 = 0 \tag{2.47}$$

is associated with (2.43), and it has three different roots, say λ_j, $j = \overline{1,3}$. They are routinely found [10].

By using Lemma 2.2, we see that

$$a_n = \sum_{j=1}^{3} \frac{\lambda_j^{n+2}}{P_3'(\lambda_j)}, \quad n \in \mathbb{Z}, \tag{2.48}$$

is the solution to (2.43) satisfying the initial conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$.

From (2.21), (2.45) and (2.46), the following corollary follows.

Corollary 2.3. *If $a \neq 0$, then the general solution to (2.5) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N},$$

where a_n is given by (2.48).

2.2 System (2.23)

First note that (2.45) holds, and that

$$\eta_n = \eta_{n-1}\zeta_{n-3}, \quad n \in \mathbb{N}. \tag{2.49}$$

By using (2.45) in (2.49), we obtain

$$\begin{aligned}\eta_n &= \eta_0 \prod_{j=1}^{n} \zeta_{j-3}\\ &= \eta_0 \prod_{j=1}^{n} \zeta_0^{a_{j-3}} \zeta_{-1}^{a_{j-5}} \zeta_{-2}^{a_{j-4}}\\ &= \eta_0 \zeta_0^{\sum_{j=1}^{n} a_{j-3}} \zeta_{-1}^{\sum_{j=1}^{n} a_{j-5}} \zeta_{-2}^{\sum_{j=1}^{n} a_{j-4}},\end{aligned} \tag{2.50}$$

for $n \in \mathbb{N}_0$.

Employing (2.43) and (2.44), it follows that

$$\sum_{j=1}^{n} a_{j-3} = \sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - 1, \tag{2.51}$$

$$\sum_{j=1}^{n} a_{j-5} = \sum_{j=1}^{n}(a_{j-2} - a_{j-3}) = a_{n-2}, \tag{2.52}$$

$$\sum_{j=1}^{n} a_{j-4} = \sum_{j=1}^{n}(a_{j-1} - a_{j-2}) = a_{n-1}, \tag{2.53}$$

for $n \in \mathbb{N}_0$.

From (2.50)–(2.53), it follows that

$$\eta_n = \eta_0 \zeta_0^{a_n - 1} \zeta_{-1}^{a_{n-2}} \zeta_{-2}^{a_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.54}$$

From (2.21), (2.45) and (2.54), the following corollary follows.

Corollary 2.4. *If $a \neq 0$, then the general solution to (2.6) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N}_0,$$

where a_n is given by (2.48).

2.3 System (2.24)

First note that (2.45) holds, and that

$$\eta_n = \zeta_{n-1}\eta_{n-3},$$

for $n \in \mathbb{N}$, that is,

$$\eta_{3n+i} = \zeta_{3n-1+i}\eta_{3(n-1)+i}, \tag{2.55}$$

for $n \in \mathbb{N}$, $i = -2, -1, 0$.

From (2.45) and (2.55), we have

$$\begin{aligned}\eta_{3n} &= \eta_0 \prod_{j=1}^{n} \zeta_{3j-1}\\ &= \eta_0 \prod_{j=1}^{n} \zeta_0^{a_{3j-1}} \zeta_{-1}^{a_{3j-3}} \zeta_{-2}^{a_{3j-2}}\\ &= \eta_0 \zeta_0^{\sum_{j=1}^{n} a_{3j-1}} \zeta_{-1}^{\sum_{j=1}^{n} a_{3j-3}} \zeta_{-2}^{\sum_{j=1}^{n} a_{3j-2}},\end{aligned} \tag{2.56}$$

for $n \in \mathbb{N}_0$,

$$\begin{aligned}\eta_{3n+1} &= \eta_{-2} \prod_{j=0}^{n} \zeta_{3j}\\ &= \eta_{-2} \prod_{j=0}^{n} \zeta_0^{a_{3j}} \zeta_{-1}^{a_{3j-2}} \zeta_{-2}^{a_{3j-1}}\\ &= \eta_{-2} \zeta_0^{\sum_{j=0}^{n} a_{3j}} \zeta_{-1}^{\sum_{j=0}^{n} a_{3j-2}} \zeta_{-2}^{\sum_{j=0}^{n} a_{3j-1}},\end{aligned} \tag{2.57}$$

for $n \geq -1$, and

$$\begin{aligned}\eta_{3n+2} &= \eta_{-1} \prod_{j=0}^{n} \zeta_{3j+1}\\ &= \eta_{-1} \prod_{j=0}^{n} \zeta_0^{a_{3j+1}} \zeta_{-1}^{a_{3j-1}} \zeta_{-2}^{a_{3j}}\\ &= \eta_{-1} \zeta_0^{\sum_{j=0}^{n} a_{3j+1}} \zeta_{-1}^{\sum_{j=0}^{n} a_{3j-1}} \zeta_{-2}^{\sum_{j=0}^{n} a_{3j}},\end{aligned} \tag{2.58}$$

for $n \ge -1$.

Employing (2.43) and (2.44), it follows that

$$\sum_{j=1}^{n} a_{3j-3} = \sum_{j=1}^{n}(a_{3j-2} - a_{3j-5}) = a_{3n-2}, \tag{2.59}$$

$$\sum_{j=1}^{n} a_{3j-2} = \sum_{j=1}^{n}(a_{3j-1} - a_{3j-4}) = a_{3n-1}, \tag{2.60}$$

$$\sum_{j=1}^{n} a_{3j-1} = \sum_{j=1}^{n}(a_{3j} - a_{3j-3}) = a_{3n} - 1, \tag{2.61}$$

$$\sum_{j=0}^{n} a_{3j-2} = \sum_{j=0}^{n}(a_{3j-1} - a_{3j-4}) = a_{3n-1}, \tag{2.62}$$

$$\sum_{j=0}^{n} a_{3j-1} = \sum_{j=0}^{n}(a_{3j} - a_{3j-3}) = a_{3n} - 1, \tag{2.63}$$

$$\sum_{j=0}^{n} a_{3j} = \sum_{j=0}^{n}(a_{3j+1} - a_{3j-2}) = a_{3n+1}, \tag{2.64}$$

$$\sum_{j=0}^{n} a_{3j+1} = \sum_{j=0}^{n}(a_{3j+2} - a_{3j-1}) = a_{3n+2}, \tag{2.65}$$

Use of (2.59)–(2.65) in (2.56)–(2.58), yield

$$\eta_{3n} = \eta_0 \zeta_0^{a_{3n}-1} \zeta_{-1}^{a_{3n-2}} \zeta_{-2}^{a_{3n-1}}, \tag{2.66}$$

for $n \in \mathbb{N}_0$,

$$\eta_{3n+1} = \eta_{-2} \zeta_0^{a_{3n+1}} \zeta_{-1}^{a_{3n-1}} \zeta_{-2}^{a_{3n}-1}, \tag{2.67}$$

for $n \ge -1$, and

$$\eta_{3n+2} = \eta_{-1} \zeta_0^{a_{3n+2}} \zeta_{-1}^{a_{3n}-1} \zeta_{-2}^{a_{3n+1}}, \tag{2.68}$$

for $n \ge -1$.

From (2.21), (2.45), (2.66)–(2.68), the following corollary follows.

Corollary 2.5. *If $a \neq 0$, then the general solution to (2.7) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \ge -2,$$

$$y_{3n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n-1}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{3n+1} = \sqrt{a}\,\frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n}-1}+1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n}-1}-1}, \quad n \ge -1,$$

$$y_{3n+2} = \sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n+1}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{3n+1}}-1}, \quad n \ge -1,$$

where sequence a_n is given by (2.48).

2.4 System (2.25)

Note that (2.45) holds, and that

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},$$

for $n \geq -2$.

From this and (2.21) the following corollary follows.

Corollary 2.6. *If $a \neq 0$, then the general solution to (2.8) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$
$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \geq -2$, where a_n is given by (2.48).

2.5 System (2.26)

The relations in (2.26) yield

$$\zeta_n = \zeta_{n-2}\zeta_{n-3}\zeta_{n-4}, \quad n \geq 2. \tag{2.69}$$

Let

$$b_1 = c_1 = d_1 = 1, \quad e_1 = 0, \tag{2.70}$$

then (2.69) can be written as

$$\zeta_n = \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}, \quad n \geq 2. \tag{2.71}$$

Use of (2.69) in (2.71) yield

$$\begin{aligned}\zeta_n &= \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\\ &= (\zeta_{n-4}\zeta_{n-5}\zeta_{n-6})^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\\ &= \zeta_{n-3}^{c_1}\zeta_{n-4}^{b_1+d_1}\zeta_{n-5}^{b_1+e_1}\zeta_{n-6}^{b_1}\\ &= \zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\end{aligned}$$

for $n \geq 4$, where

$$b_2 := c_1, \quad c_2 := b_1 + d_1, \quad d_2 := b_1 + e_1, \quad e_2 := b_1.$$

Suppose that

$$\zeta_n = \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k} \tag{2.72}$$

and

$$b_k = c_{k-1}, \quad c_k = b_{k-1} + d_{k-1}, \quad d_k = b_{k-1} + e_{k-1}, \quad e_k = b_{k-1}, \tag{2.73}$$

for a $k \geq 2$ and $n \geq k + 2$.

Employing (2.69) in (2.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\\ &= (\zeta_{n-k-3}\zeta_{n-k-4}\zeta_{n-k-5})^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\\ &= \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{b_k+d_k}\zeta_{n-k-4}^{b_k+e_k}\zeta_{n-k-5}^{b_k}\\ &= \zeta_{n-k-2}^{b_{k+1}}\zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}},\end{aligned}$$

where

$$b_{k+1} := c_k, \quad c_{k+1} := b_k + d_k, \quad d_{k+1} := b_k + e_k, \quad e_{k+1} := b_k,$$

for a $k \geq 2$ and every $n \geq k+3$. Hence, (2.72) and (2.73) really hold for $2 \leq k \leq n-2$.

From (2.70) and (2.73) we have

$$b_n = b_{n-2} + b_{n-3} + b_{n-4}, \tag{2.74}$$

not only for $n \geq 5$, but also for every $n \in \mathbb{Z}$, and that

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-2} = b_{-3} = b_{-4} = 0, \quad b_{-5} = 1.$$

Letting $k = n-2$ in (2.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_1^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\\ &= (\eta_0\zeta_{-2})^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\\ &= \eta_0^{b_{n-2}}\zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{b_{n-2}+e_{n-2}}\\ &= \eta_0^{b_{n-2}}\zeta_0^{b_{n-1}}\zeta_{-1}^{b_{n-3}+b_{n-4}}\zeta_{-2}^{b_{n-2}+b_{n-3}},\end{aligned} \tag{2.75}$$

for $n \geq -2$.

By using (2.75) in the second equation in (2.26), it follows that

$$\begin{aligned}\eta_n &= \zeta_{n-1}\zeta_{n-3}\\ &= \eta_0^{b_{n-3}+b_{n-5}}\zeta_0^{b_{n-2}+b_{n-4}}\zeta_{-1}^{b_{n-3}+b_{n-4}}\zeta_{-2}^{b_{n-2}+b_{n-3}},\end{aligned} \tag{2.76}$$

for $n \in \mathbb{N}_0$.

The characteristic polynomial associated with equation (2.74) is

$$P_4(\lambda) = \lambda^4 - \lambda^2 - \lambda - 1.$$

Since

$$P_4(\lambda) = (\lambda+1)(\lambda^3 - \lambda^2 - 1),$$

we have that three roots of P_4, coincide with the roots, λ_j, $j = \overline{1,3}$, of polynomial (2.47), whereas $\lambda_4 = -1$.

Lemma 2.2 shows that the solution to (2.74) satisfying the initial conditions $b_{-4} = b_{-3} = b_{-2} = 0$ and $b_{-1} = 1$, is

$$b_n = \sum_{j=1}^{4} \frac{\lambda_j^{n+4}}{P_4'(\lambda_j)}, \quad n \in \mathbb{Z}. \tag{2.77}$$

From (2.21), (2.75) and (2.76), the following corollary follows.

Corollary 2.7. *If $a \neq 0$, then the general solution to (2.9) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n \geq -2$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}}-1},$$

for $n \in \mathbb{N}_0$, where b_n is given by (2.77).

2.6 System (2.27)

Clearly, we have

$$\zeta_n = \eta_n, \quad n \in \mathbb{N},$$

from which along with (2.27), it follows that

$$\zeta_{n+1} = \zeta_n\zeta_{n-2}, \quad n \in \mathbb{N}.$$

Hence, by using (2.45) it follows that

$$\begin{aligned}\zeta_n &= \zeta_1^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\\ &= (\eta_0\zeta_{-2})^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\\ &= \eta_0^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}},\end{aligned} \tag{2.78}$$

for $n \geq -1$, where a_n is the solution to (2.43) satisfying the initial conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$. Hence

$$\eta_n = \eta_0^{a_{n-1}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}}, \tag{2.79}$$

for $n \in \mathbb{N}$.

From (2.21), (2.78) and (2.79), the following corollary follows.

Corollary 2.8. *If $a \neq 0$, then the general solution to (2.10) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \geq -1$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}$, where a_n is given by (2.48).

2.7 System (2.28)

From (2.28) we easily get

$$\zeta_n = \zeta_{n-2}\zeta_{n-3}^2\zeta_{n-6}^{-1}, \quad n \ge 4. \tag{2.80}$$

Let

$$b_1 = 1, \quad c_1 = 2, \quad d_1 = e_1 = 0, \quad f_1 = -1, \quad g_1 = 0. \tag{2.81}$$

From this and (2.80), we have

$$\begin{aligned}
\zeta_n &= \zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1}\\
&= (\zeta_{n-4}\zeta_{n-5}^2\zeta_{n-8}^{-1})^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1}\\
&= \zeta_{n-3}^{c_1}\zeta_{n-4}^{b_1+d_1}\zeta_{n-5}^{2b_1+e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{g_1}\zeta_{n-8}^{-b_1}\\
&= \zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2}\zeta_{n-8}^{g_2},
\end{aligned}$$

for $n \ge 6$, where

$$b_2 := c_1, \ \ c_2 := b_1 + d_1, \ \ d_2 := 2b_1 + e_1, \ \ e_2 := f_1, \ \ f_2 := g_1, \ \ g_2 := -b_1.$$

Assume

$$\zeta_n = \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}, \tag{2.82}$$

for a $k \ge 2$ and all $n \ge k+4$, and

$$\begin{aligned}
&b_k = c_{k-1}, \quad c_k = b_{k-1} + d_{k-1}, \quad d_k = 2b_{k-1} + e_{k-1},\\
&e_k = f_{k-1}, \quad f_k = g_{k-1}, \quad g_k = -b_{k-1}.
\end{aligned} \tag{2.83}$$

We have

$$\begin{aligned}
\zeta_n &= \zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}\\
&= (\zeta_{n-k-3}\zeta_{n-k-4}^2\zeta_{n-k-7}^{-1})^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}\\
&= \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{b_k+d_k}\zeta_{n-k-4}^{2b_k+e_k}\zeta_{n-k-5}^{f_k}\zeta_{n-k-6}^{g_k}\zeta_{n-k-7}^{-b_k}\\
&= \zeta_{n-k-2}^{b_{k+1}}\zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}}\zeta_{n-k-6}^{f_{k+1}}\zeta_{n-k-7}^{g_{k+1}},
\end{aligned}$$

for $n \ge k+5$, where

$$\begin{aligned}
&b_{k+1} = c_k, \quad c_{k+1} = b_k + d_k, \quad d_{k+1} = 2b_k + e_k,\\
&e_{k+1} = f_k, \quad f_{k+1} = g_k, \quad\quad\ \ g_{k+1} = -b_k.
\end{aligned}$$

Hence (2.82) and (2.83) really hold when $2 \le k \le n-4$.

From (2.81) and (2.83) we have

$$b_n = b_{n-2} + 2b_{n-3} - b_{n-6}, \tag{2.84}$$

not only for $n \ge 7$, but for all $n \in \mathbb{Z}$, and

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-j} = 0, \ j = \overline{2,6}, \quad b_{-7} = -1, \quad b_{-8} = 0.$$

By taking $k = n - 4$ in (2.82), it follows that

$$\begin{aligned}\zeta_n &= \zeta_3^{b_{n-4}}\zeta_2^{c_{n-4}}\zeta_1^{d_{n-4}}\zeta_0^{e_{n-4}}\zeta_{-1}^{f_{n-4}}\zeta_{-2}^{g_{n-4}}\\ &= (\eta_0\eta_{-1}\zeta_0\zeta_{-2})^{b_{n-4}}(\eta_{-2}\zeta_0\zeta_{-1})^{c_{n-4}}(\eta_0\zeta_{-2})^{d_{n-4}}\zeta_0^{e_{n-4}}\zeta_{-1}^{f_{n-4}}\zeta_{-2}^{g_{n-4}}\\ &= \zeta_0^{b_{n-4}+c_{n-4}+e_{n-4}}\zeta_{-1}^{c_{n-4}+f_{n-4}}\zeta_{-2}^{b_{n-4}+d_{n-4}+g_{n-4}}\eta_0^{b_{n-4}+d_{n-4}}\eta_{-1}^{b_{n-4}}\eta_{-2}^{c_{n-4}}\\ &= \zeta_0^{b_{n-1}-b_{n-4}}\zeta_{-1}^{b_{n-3}-b_{n-6}}\zeta_{-2}^{b_{n-2}-b_{n-5}}\eta_0^{b_{n-2}}\eta_{-1}^{b_{n-4}}\eta_{-2}^{b_{n-3}}, \end{aligned} \tag{2.85}$$

for $n \geq -2$.

Using (2.85) in the first equation in (2.28), we obtain

$$\begin{aligned}\eta_n &= \zeta_{n+1}/\zeta_{n-2}\\ &= \zeta_0^{b_n-2b_{n-3}+b_{n-6}}\zeta_{-1}^{b_{n-2}-2b_{n-5}+b_{n-8}}\zeta_{-2}^{b_{n-1}-2b_{n-4}+b_{n-7}}\eta_0^{b_{n-1}-b_{n-4}}\eta_{-1}^{b_{n-3}-b_{n-6}}\eta_{-2}^{b_{n-2}-b_{n-5}}\\ &= \zeta_0^{b_{n-2}}\zeta_{-1}^{b_{n-4}}\zeta_{-2}^{b_{n-3}}\eta_0^{b_{n-1}-b_{n-4}}\eta_{-1}^{b_{n-3}-b_{n-6}}\eta_{-2}^{b_{n-2}-b_{n-5}}, \end{aligned} \tag{2.86}$$

for $n \geq -2$.

The characteristic polynomial associated with equation (2.84) is

$$P_6(t) = t^6 - t^4 - 2t^3 + 1 = (t^3 - t^2 - 1)(t^3 + t^2 - 1).$$

Let t_j, $j = \overline{1,6}$, be its roots. Clearly $t_j = \lambda_j$, $j = \overline{1,3}$, (the roots of polynomial (2.47)), whereas the other three roots of P_6 are the roots of the polynomial $t^3 + t^2 - 1$.

Thus, the solution to (2.84) satisfying the initial conditions $b_{-j} = 0$, $k = \overline{2,6}$, and $b_{-1} = 1$, is

$$b_n = \sum_{j=1}^{6} \frac{t_j^{n+6}}{P_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.87}$$

From (2.21), (2.85) and (2.86), the following corollary follows.

Corollary 2.9. *If $a \neq 0$, then the general solution to (2.11) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\beta_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\beta_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\beta_{n-2}}-1},$$

for $n \geq -2$, where the sequence b_n is given by (2.87) and $\beta_n := b_n - b_{n-3}$.

2.8 System (2.29)

This system is obtained from (2.24) by interchanging letters ζ and η.

Hence, we have

$$\zeta_{3n} = \zeta_0\eta_0^{a_{3n}-1}\eta_{-1}^{a_{3n-2}}\eta_{-2}^{a_{3n-1}}, \quad n \in \mathbb{N}_0, \tag{2.88}$$

$$\zeta_{3n+1} = \zeta_{-2}\eta_0^{a_{3n+1}}\eta_{-1}^{a_{3n-1}}\eta_{-2}^{a_{3n}-1}, \quad n \geq -1, \tag{2.89}$$

$$\zeta_{3n+2} = \zeta_{-1}\eta_0^{a_{3n+2}}\eta_{-1}^{a_{3n}-1}\eta_{-2}^{a_{3n+1}}, \quad n \geq -1, \tag{2.90}$$

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \quad n \geq -2. \tag{2.91}$$

From (2.21), (2.88)–(2.91), the following corollary follows.

Corollary 2.10. *If $a \neq 0$, then the general solution to (2.12) is*

$$
\begin{aligned}
y_n &= \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,\\
x_{3n} &= \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n-1}}-1}, \quad n \in \mathbb{N}_0,\\
x_{3n+1} &= \sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n}-1}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n}-1}-1}, \quad n \geq -1,\\
x_{3n+2} &= \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n+1}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{3n+2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{3n}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{3n+1}}-1}, \quad n \geq -1,
\end{aligned}
$$

where sequence a_n is given by (2.48).

2.9 System (2.30)

From (2.30), we have

$$\zeta_n = \zeta_{n-1}\zeta_{n-4}\zeta_{n-6}, \tag{2.92}$$

for $n \geq 4$.

Let

$$a_1 = 1, \quad b_1 = c_1 = 0, \quad d_1 = 1, \quad e_1 = 0, \quad f_1 = 1, \tag{2.93}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}, \quad n \geq 4. \tag{2.94}$$

From (2.92) and (2.94), it follows that

$$
\begin{aligned}
\zeta_n &= \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\\
&= (\zeta_{n-2}\zeta_{n-5}\zeta_{n-7})^{a_1}\zeta_{n-2}^{b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}\\
&= \zeta_{n-2}^{a_1+b_1}\zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{a_1+e_1}\zeta_{n-6}^{f_1}\zeta_{n-7}^{a_1}\\
&= \zeta_{n-2}^{a_2}\zeta_{n-3}^{b_2}\zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2},
\end{aligned}
$$

for $n \geq 5$, where

$$a_2 := a_1 + b_1, \quad b_2 := c_1, \quad c_2 := d_1, \quad d_2 := a_1 + e_1, \quad e_2 := f_1, \quad f_2 := a_1.$$

Similar to the case of equation (2.80) it is shown

$$\zeta_n = \zeta_{n-k}^{a_k}\zeta_{n-k-1}^{b_k}\zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}, \tag{2.95}$$

for a $k \geq 2$ and $n \geq k+3$, and that

$$
\begin{aligned}
a_k &= a_{k-1} + b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1},\\
d_k &= a_{k-1} + e_{k-1}, \quad e_k = f_{k-1}, \quad f_k = a_{k-1}.
\end{aligned} \tag{2.96}
$$

From (2.93) and (2.96), we have

$$a_n = a_{n-1} + a_{n-4} + a_{n-6}, \tag{2.97}$$

not only for $n \geq 7$, but for all $n \in \mathbb{Z}$, and that

$$a_0 = 1, \quad a_{-j} = 0, \ j = \overline{1,5}, \quad a_{-6} = 1, \quad a_{-7} = 0, \quad a_{-8} = -1.$$

Letting $k = n-3$ in (2.95), it follows that

$$\begin{aligned}
\zeta_n &= \zeta_3^{a_{n-3}} \zeta_2^{b_{n-3}} \zeta_1^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= (\zeta_0\eta_0\eta_{-1}\eta_{-2})^{a_{n-3}} (\zeta_0\eta_{-1}\eta_{-2})^{b_{n-3}} (\zeta_0\eta_{-2})^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= \zeta_0^{a_{n-3}+b_{n-3}+c_{n-3}+d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-3}+b_{n-3}} \eta_{-2}^{a_{n-3}+b_{n-3}+c_{n-3}} \\
&= \zeta_0^{a_n} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-2}} \eta_{-2}^{a_{n-1}},
\end{aligned} \tag{2.98}$$

for $n \geq -2$.

If in the second equation in (2.30) is used (2.98), we get

$$\begin{aligned}
\eta_n &= \zeta_{n-1}\zeta_{n-3} \\
&= \zeta_0^{a_{n-1}+a_{n-3}} \zeta_{-1}^{a_{n-6}+a_{n-8}} \zeta_{-2}^{a_{n-5}+a_{n-7}} \eta_0^{a_{n-4}+a_{n-6}} \eta_{-1}^{a_{n-3}+a_{n-5}} \eta_{-2}^{a_{n-2}+a_{n-4}},
\end{aligned} \tag{2.99}$$

for $n \geq -2$.

The characteristic polynomial

$$\widetilde{P}_6(t) = t^6 - t^5 - t^2 - 1 = (t^3 - t^2 - 1)(t^3 + 1)$$

associated with equation (2.97) has the roots

$$t_1 = \lambda_1, \quad t_2 = \lambda_2, \quad t_3 = \lambda_3, \quad t_4 = -1, \quad t_{5,6} = e^{\pm i\frac{\pi}{3}},$$

where λ_j, $j = \overline{1,3}$, are the roots of polynomial (2.47).

Hence, the solution to equation (2.97) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{1,5}$, and $a_0 = 1$ is

$$a_n = \sum_{j=1}^{6} \frac{t_j^{n+5}}{\widetilde{P}_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.100}$$

From (2.21), (2.98) and (2.99), the following corollary follows.

Corollary 2.11. *If $a \neq 0$, then the general solution to (2.13) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}-1},$$

for $n \geq -2$, where the sequence a_n is given by (2.100) and $b_n = a_n + a_{n-2}$.

2.10 System (2.31)

From (2.31), we have

$$\zeta_n = \zeta_{n-1}^2 \zeta_{n-2}^{-1} \zeta_{n-6}, \tag{2.101}$$

for $n \geq 4$.

Let

$$a_1 = 2, \quad b_1 = -1, \quad c_1 = d_1 = e_1 = 0, \quad f_1 = 1, \tag{2.102}$$

then

$$\zeta_n = \zeta_{n-1}^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1}, \tag{2.103}$$

for $n \geq 4$.

Employing (2.101) in (2.103), it follows that

$$\begin{aligned}
\zeta_n &= \zeta_{n-1}^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \\
&= (\zeta_{n-2}^2 \zeta_{n-3}^{-1} \zeta_{n-7})^{a_1} \zeta_{n-2}^{b_1} \zeta_{n-3}^{c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \\
&= \zeta_{n-2}^{2a_1+b_1} \zeta_{n-3}^{-a_1+c_1} \zeta_{n-4}^{d_1} \zeta_{n-5}^{e_1} \zeta_{n-6}^{f_1} \zeta_{n-7}^{a_1} \\
&= \zeta_{n-2}^{a_2} \zeta_{n-3}^{b_2} \zeta_{n-4}^{c_2} \zeta_{n-5}^{d_2} \zeta_{n-6}^{e_2} \zeta_{n-7}^{f_2},
\end{aligned}$$

for $n \geq 5$, where

$$a_2 := 2a_1 + b_1, \quad b_2 := -a_1 + c_1, \quad c_2 := d_1, \quad d_2 := e_1, \quad e_2 := f_1, \quad f_2 := a_1.$$

Similar to equation (2.80), it is shown that

$$\zeta_n = \zeta_{n-k}^{a_k} \zeta_{n-k-1}^{b_k} \zeta_{n-k-2}^{c_k} \zeta_{n-k-3}^{d_k} \zeta_{n-k-4}^{e_k} \zeta_{n-k-5}^{f_k}, \tag{2.104}$$

and

$$\begin{aligned}
&a_k = 2a_{k-1} + b_{k-1}, \quad b_k = -a_{k-1} + c_{k-1}, \quad c_k = d_{k-1}, \\
&d_k = e_{k-1}, \quad e_k = f_{k-1}, \quad f_k = a_{k-1},
\end{aligned} \tag{2.105}$$

for a $2 \leq k \leq n-3$.

From (2.102) and (2.105) we have

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-6}, \tag{2.106}$$

not only for $n \geq 7$, but for all $n \in \mathbb{Z}$, and that

$$a_0 = 1, \quad a_{-j} = 0, \ j = \overline{1,5}, \quad a_{-6} = 1, \quad a_{-7} = 0.$$

For $k = n-3$, from (2.104), we have

$$\begin{aligned}
\zeta_n &= \zeta_3^{a_{n-3}} \zeta_2^{b_{n-3}} \zeta_1^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= (\zeta_0 \eta_0 \eta_{-1} \eta_{-2})^{a_{n-3}} (\zeta_0 \eta_{-1} \eta_{-2})^{b_{n-3}} (\zeta_0 \eta_{-2})^{c_{n-3}} \zeta_0^{d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \\
&= \zeta_0^{a_{n-3}+b_{n-3}+c_{n-3}+d_{n-3}} \zeta_{-1}^{e_{n-3}} \zeta_{-2}^{f_{n-3}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-3}+b_{n-3}} \eta_{-2}^{a_{n-3}+b_{n-3}+c_{n-3}} \\
&= \zeta_0^{a_n - a_{n-1}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \eta_0^{a_{n-3}} \eta_{-1}^{a_{n-2}-a_{n-3}} \eta_{-2}^{a_{n-1}-a_{n-2}},
\end{aligned} \tag{2.107}$$

for $n \geq -2$.

From (2.31) and (2.107), it follows that

$$\begin{aligned}\eta_n &= \zeta_{n+3}/\zeta_{n+2}\\ &= \zeta_0^{a_{n+3}-2a_{n+2}+a_{n+1}}\zeta_{-1}^{a_{n-2}-a_{n-3}}\zeta_{-2}^{a_{n-1}-a_{n-2}}\eta_0^{a_n-a_{n-1}}\eta_{-1}^{a_{n+1}-2a_n+a_{n-1}}\eta_{-2}^{a_{n+2}-2a_{n+1}+a_n}\\ &= \zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}-a_{n-3}}\zeta_{-2}^{a_{n-1}-a_{n-2}}\eta_0^{a_n-a_{n-1}}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}},\end{aligned} \tag{2.108}$$

for $n \ge -2$ ((2.108) is also obtained from (2.107) due to the symmetry of system (2.31)).

The characteristic polynomial associated with (2.106) is

$$\widehat{P}_6(t) = t^6 - 2t^5 + t^4 - 1 = (t^3 - t^2 - 1)(t^3 - t^2 + 1).$$

Let $\widetilde{t}_j$, $j = \overline{1,6}$, be the roots of polynomial $\widehat{P}_6$. Then, the solution to (2.106) such that $a_{-j} = 0$, $j = \overline{1,5}$, and $a_0 = 1$, is

$$a_n = \sum_{j=1}^{6} \frac{\widetilde{t}_j^{\,n+5}}{\widehat{P}_6'(\widetilde{t}_j)}, \quad n \in \mathbb{Z}. \tag{2.109}$$

From this and (2.21) the following corollary follows.

Corollary 2.12. *If $a \neq 0$, then the general solution to (2.14) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{\Delta a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{\Delta a_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\Delta a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}-1},$$

for $n \ge -2$, where the sequence a_n is given by (2.109) and $\Delta a_n = a_{n+1} - a_n$.

2.11 System (2.32)

From (2.32) we see that

$$\zeta_n = \eta_n, \quad n \in \mathbb{N},$$

implying that

$$\zeta_n = \zeta_{n-1}\zeta_{n-3},$$

for $n \ge 4$.

Employing (2.45) it follows that

$$\begin{aligned}\zeta_n &= \zeta_3^{a_{n-3}}\zeta_2^{a_{n-5}}\zeta_1^{a_{n-4}}\\ &= (\zeta_0\eta_0\eta_{-1}\eta_{-2})^{a_{n-3}}(\zeta_0\eta_{-1}\eta_{-2})^{a_{n-5}}(\zeta_0\eta_{-2})^{a_{n-4}}\\ &= \zeta_0^{a_{n-1}}\eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},\end{aligned} \tag{2.110}$$

for $n \in \mathbb{N}$, where a_n is the solution to equation (2.43) such that $a_{-1} = a_{-2} = 0$ and $a_0 = 1$, and consequently

$$\eta_n = \zeta_0^{a_{n-1}}\eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}, \tag{2.111}$$

for $n \in \mathbb{N}_0$.

From (2.21), (2.110) and (2.111), the following corollary follows.

Corollary 2.13. *If $a \neq 0$, then the general solution to (2.15) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \in \mathbb{N}_0$, where a_n is given by (2.48).

2.12 System (2.33)

This system is get from (2.23) by interchanging letters ζ and η. Hence, we have

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}$$

for $n \geq -2$, and

$$\zeta_n = \zeta_0\eta_0^{a_n-1}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}},$$

for $n \in \mathbb{N}_0$.

From this and (2.21) the following corollary follows.

Corollary 2.14. *If $a \neq 0$, then the general solution to (2.16) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \geq -2,$$

where a_n is given by (2.48).

2.13 System (2.34)

From (2.34) we have

$$\zeta_n = \zeta_{n-2}\zeta_{n-4}^2\zeta_{n-6}, \tag{2.112}$$

for $n \geq 4$.

Let

$$a_1 := 1, \quad b_1 := 2, \quad c_1 := 1. \tag{2.113}$$

Then, from (2.112) and (2.113), it follows that

$$\zeta_{2n} = \zeta_{2n-2}^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}, \tag{2.114}$$

for $n \geq 2$.

Employing (2.112) in (2.114), it follows that

$$\begin{aligned}\zeta_{2n} &= \zeta_{2n-2}^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}\\ &= (\zeta_{2n-4}\zeta_{2n-6}^2\zeta_{2n-8})^{a_1}\zeta_{2n-4}^{b_1}\zeta_{2n-6}^{c_1}\\ &= \zeta_{2n-4}^{a_1+b_1}\zeta_{2n-6}^{2a_1+c_1}\zeta_{2n-8}^{a_1}\\ &= \zeta_{2n-4}^{a_2}\zeta_{2n-6}^{b_2}\zeta_{2n-8}^{c_2},\end{aligned}$$

for $n \geq 3$, where

$$a_2 := a_1 + b_1, \quad b_2 := 2a_1 + c_1, \quad c_2 := a_1.$$

Similar to equation (2.40), it follows that

$$\zeta_{2n} = \zeta_{2(n-k)}^{a_k}\zeta_{2(n-k-1)}^{b_k}\zeta_{2(n-k-2)}^{c_k}, \tag{2.115}$$

and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = 2a_{k-1} + c_{k-1}, \quad c_k = a_{k-1}, \tag{2.116}$$

for a $k \geq 2$ and all $n \geq k+1$.

From (2.113) and (2.116) we have

$$a_n = a_{n-1} + 2a_{n-2} + a_{n-3}, \tag{2.117}$$

and

$$a_0 = 1, \quad a_{-1} = 0, \quad a_{-2} = 0, \quad a_{-3} = 1.$$

Letting $k = n-1$ in (2.115), it follows that

$$\begin{aligned}\zeta_{2n} &= \zeta_2^{a_{n-1}}\zeta_0^{b_{n-1}}\zeta_{-2}^{c_{n-1}}\\ &= (\zeta_0\zeta_{-2}\eta_{-1})^{a_{n-1}}\zeta_0^{a_n-a_{n-1}}\zeta_{-2}^{a_{n-2}}\\ &= \zeta_0^{a_n}\zeta_{-2}^{a_{n-1}+a_{n-2}}\eta_{-1}^{a_{n-1}},\end{aligned} \tag{2.118}$$

for $n \geq -1$.

Similarly is get

$$\begin{aligned}\zeta_{2n-1} &= \zeta_3^{a_{n-2}}\zeta_1^{b_{n-2}}\zeta_{-1}^{c_{n-2}}\\ &= (\zeta_{-1}\eta_0^2\eta_{-2})^{a_{n-2}}(\eta_0\eta_{-2})^{a_{n-1}-a_{n-2}}\zeta_{-1}^{a_{n-3}}\\ &= \zeta_{-1}^{a_{n-2}+a_{n-3}}\eta_0^{a_{n-1}+a_{n-2}}\eta_{-2}^{a_{n-1}},\end{aligned} \tag{2.119}$$

for $n \in \mathbb{N}_0$.

Combining (2.34) and (2.118), it follows that

$$\begin{aligned}\eta_{2n-1} &= \zeta_{2n-2}\zeta_{2n-4}\\ &= \zeta_0^{a_{n-1}+a_{n-2}}\zeta_{-2}^{a_{n-2}+2a_{n-3}+a_{n-4}}\eta_{-1}^{a_{n-2}+a_{n-3}}\\ &= \zeta_0^{a_{n-1}+a_{n-2}}\zeta_{-2}^{a_{n-1}}\eta_{-1}^{a_{n-2}+a_{n-3}},\end{aligned}$$

for $n \in \mathbb{N}_0$, whereas by combining (2.34) and (2.119), it follows that

$$\begin{aligned}\eta_{2n} &= \zeta_{2n-1}\zeta_{2n-3}\\ &= \zeta_{-1}^{a_{n-2}+2a_{n-3}+a_{n-4}}\eta_0^{a_{n-1}+2a_{n-2}+a_{n-3}}\eta_{-1}^{a_{n-1}+a_{n-2}}\\ &= \zeta_{-1}^{a_{n-1}}\eta_0^{a_n}\eta_{-2}^{a_{n-1}+a_{n-2}},\end{aligned}$$

for $n \ge -1$.

The characteristic polynomial associated with (2.117) is

$$\widehat{P}_3(t) = t^3 - t^2 - 2t - 1.$$

Let t_j, $j = \overline{1,3}$, be the roots of the polynomial. Then, the solution to (2.117) such that $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, is

$$a_n = \sum_{j=1}^{3} \frac{t_j^{n+2}}{\widehat{P}_3'(t_j)}, \quad n \in \mathbb{Z}. \tag{2.120}$$

From this and (2.21) the following corollary follows.

Corollary 2.15. *If $a \neq 0$, then the general solution to (2.17) is*

$$x_{2n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_n}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-1}}-1}, \qquad n \ge -1$$

$$x_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1}, \quad n \in \mathbb{N}_0$$

$$y_{2n} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}-1}, \qquad n \ge -1$$

$$y_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-1}+a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}+a_{n-3}}-1}, \quad n \in \mathbb{N}_0,$$

where the sequence a_n is given by (2.120).

2.14 System (2.35)

This system is get from (2.30) by interchanging letters ζ and η. Hence, we have

$$\zeta_n = \eta_0^{a_{n-1}+a_{n-3}}\eta_{-1}^{a_{n-6}+a_{n-8}}\eta_{-2}^{a_{n-5}+a_{n-7}}\zeta_0^{a_{n-4}+a_{n-6}}\zeta_{-1}^{a_{n-3}+a_{n-5}}\zeta_{-2}^{a_{n-2}+a_{n-4}},$$

for $n \ge -2$, and

$$\eta_n = \eta_0^{a_n}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}}\zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}},$$

for $n \ge -2$.

From this and (2.21) the following corollary follows.

Corollary 2.16. *If $a \neq 0$, then the general solution to (2.18) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-6}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-5}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-2}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-1}}-1},$$

for $n \ge -2$, where the sequence a_n is given by (2.100) and $b_n = a_n + a_{n-2}$.

2.15 System (2.36)

This system is obtained from (2.26) by interchanging letters ζ and η. Hence, we have

$$\zeta_n = \zeta_0^{b_{n-3}+b_{n-5}} \eta_0^{b_{n-2}+b_{n-4}} \eta_{-1}^{b_{n-3}+b_{n-4}} \eta_{-2}^{b_{n-2}+b_{n-3}}, \quad n \in \mathbb{N}_0,$$
$$\eta_n = \zeta_0^{b_{n-2}} \eta_0^{b_{n-1}} \eta_{-1}^{b_{n-3}+b_{n-4}} \eta_{-2}^{b_{n-2}+b_{n-3}}, \quad n \geq -2.$$

From this and (2.21) the following corollary follows.

Corollary 2.17. *If $a \neq 0$, then the general solution to (2.19) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}+b_{n-5}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}+b_{n-4}} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}} - 1},$$

for $n \in \mathbb{N}_0$, and

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-3}+b_{n-4}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}+b_{n-3}} - 1},$$

for $n \geq -2$, where b_n is given by (2.77).

2.16 System (2.37)

System (2.37) is get from system (2.22) by interchanging letters ζ and η only. Hence, we have

$$\zeta_n = \eta_0^{a_n} \eta_{-1}^{a_{n-2}} \eta_{-2}^{a_{n-1}}, \quad n \in \mathbb{N},$$
$$\eta_n = \eta_0^{a_n} \eta_{-1}^{a_{n-2}} \eta_{-2}^{a_{n-1}}, \quad n \geq -2.$$

From this and (2.21) the following corollary follows.

Corollary 2.18. *If $a \neq 0$, then the general solution to (2.20) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}} - 1}, \quad n \in \mathbb{N},$$
$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_n} \left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-2}} \left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-1}} - 1}, \quad n \geq -2,$$

where a_n is given by (2.48).

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