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# Existence results for a clamped beam equation with integral boundary conditions

*Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday*

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**Abstract.** In this paper we investigate the existence of positive solutions of fourth-order non autonomous differential equations with integral boundary conditions, the nonlinearity is a continuous function that depends on the spatial variable and its the second-order derivative. The approach relies an extension of Krasnoselskii's fixed point theorem in a cone. Some examples are given to illustrate our results.

**Keywords:** Green's functions, Fourth-order boundary value problem, integral boundary conditions, positive solutions, extension of Krasnoselskii's fixed point theorem in a cone.

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## 1 Introduction

Fourth-order boundary value problems with integral boundary conditions arises in the mathematical modeling of viscoelastic and inelastic flows, thermos-elasticity, deformation of beams and plate deflection theory [12, 14, 22].

In [2], Cabada and Enguiça characterized the inverse positive character of operator $u^{(4)} + M\,u$ coupled with the, so called, clamped beam boundary conditions

$$u^{(4)}(t) + Mu(t) = \sigma(t), \quad t \in I := [0, 1] \tag{1.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0. \tag{1.2}$$

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Using oscillation theory [23], on [2] are obtained the exact values on the real parameter $M \in (-m_1^4, m_0^4)$, for which the related Green's function $g_M$ is strictly positive in $(0,1) \times (0,1)$. To be concise, $m_1 \cong 4.73004$ is the first positive root of equation

$$\cos m \, \cosh m = 1,$$

and $-m_1^4$ coincides with the first negative eigenvalue of operator $u^{(4)}$ coupled to boundary conditions (1.2).

Moreover, $m_0 \approx 5.553$ is the smaller positive solution of equation

$$\tanh \frac{m}{\sqrt{2}} = \tan \frac{m}{\sqrt{2}}, \tag{1.3}$$

and, as it is showed at [4], $m_0^4$ is the first positive eigenvalue of operator $u^{(4)}$ coupled to boundary conditions $u(0) = u'(0) = u''(0) = u(1) = 0$.

These results have been extended in [7] (and further in [8]) for any $n$-th order linear differential operator.

The existence of positive solutions for nonlinear problems are deduced by using the upper and lower solutions method and fixed point theorems in cones. In those cases, the nonlinearity depends only on the function $u$. For these problems the dependence on the second derivative of their nonlinearity has taken less attention.

In this work we will study the existence of positive solution of a more general fourth order problem related to clamped beam:

$$u^{(4)}(t) + Mu(t) = f(t, u(t), u''(t)), \quad t \in I, \tag{1.4}$$

subject to the perturbed functional boundary conditions:

$$u(1) = u'(0) = u'(1) = 0, \; u(0) = \lambda \int_0^1 u(s)\nu(s)\, ds. \tag{1.5}$$

Where $M \in (-m_1^4, 4\pi^4)$, $\nu \in L^1(I)$ is a positive weight function a.e. on $(0,1)$ and $\lambda$ is a positive parameter bounded from above by a constant that will be introduced later. We suppose that the function $f$ satisfy the following regularity assumption

$(H_0)$ $f : I \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is a continuous function.

Equation (1.4) models the stationary states of the deflection of an elastic beam. The boundary conditions (1.5) can be thought of as having the end at 1 clamped, and having some mechanism at end 0 that controls the displacement according to feedback from devices measuring the displacements along parts of the beam.

This paper is a continuation of the work done in [5] for problem

$$u^{(4)}(t) + Mu(t) + f(t, u(t)) = 0, \quad t \in I,$$

subject to the perturbed functional boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \; u(1) = \lambda \int_0^1 u(s)\, ds.$$

A standard approach to study positive solutions of a boundary value problem such as (1.4)–(1.5) consists of finding the corresponding Green's function $G_M$ and seek solutions as fixed points of the Hammerstein integral equations with kernel $G_M$. The majority of methods are based on classical fixed point index theory and Krasnoselskii's fixed point theorem in a cone. The majority of authors work in a suitable cone $K$ in a Banach space which is made using the property of Green's function. Sometimes the Green's function associated to this integral equation can change its sign. In theses cases, the authors should work in a cone smaller than $K$ (see [17–19, 21]). The construction of a such cone requires more concise properties of the Green's function (see [3, 6, 13]).

We note that in our problem, the nonlinearity $f$ depends on the second order derivatives. Using the classical Krasnoselskii's expansion/contraction theorem, we need to study the sign of the second order derivative of the Green's function and look for a nonnegative function $\phi$ such that

$$(C_1) \qquad \left|\frac{\partial^2 G_M}{\partial t^2}(t,s)\right| \leq \phi(s), \quad (t,s) \in I \times I,$$

and

$$(C_2) \qquad \frac{\partial^2 G_M}{\partial t^2}(t,s) \geq c\,\phi(s), \quad (t,s) \in [a,b] \times I,$$

for some $[a,b] \subset I$ and $c \in (0,1)$.

In our case, the explicit form of second derivative of Green's function $\frac{\partial^2 G_M}{\partial t^2}$ is very complicated and the previous inequalities ($(C_1)$ and $(C_2)$) become hard to be checked. So, we apply an extension of Krasnoselskii's fixed point theorem that was used in [15, 16, 20, 24]. With this result, we do not need to prove the inequalities $(C_1)$ and $(C_2)$. Here we need only the conditions $(C_1)$ and $(C_2)$ for the Green's function $G_M$. As far as we know, Problem (1.4)–(1.5) have not been previously studied. At the end of this paper, some examples are given to show that the theoretical results can be computed.

This paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas to prove our main results and through this section we prove that the Green's function associated to (1.1), (1.5) satisfies some suitable properties. In Section 3, we show the existence of at least one positive solution. In section 4, some examples are presented to illustrate our main results.

## 2 Preliminaries and Green's function properties

In this section we introduce some preliminary results which will be used along the paper. First, we provide some background definitions cited from cone theory in Banach spaces. After that, we introduce some definitions and properties of the Green's function $G_M$ related to problem (1.1), (1.5).

**Definition 2.1.** Let $E$ be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

$(i)$ $\alpha u \in P$ for all $u \in P$ and all $\alpha \geq 0$;

$(ii)$ $u, -u \in P$ implies $u = 0$.

In the sequel, we enunciate the a fixed point theorem due to Guo and Ge [16].

**Lemma 2.2** ([16, Theorem 2.1])**.** *Let $E$ be a Banach space and $P \subset E$ a cone. Suppose $\alpha, \beta : E \to [0, \infty)$ are two continuous convex functionals satisfying*

$$\alpha(\mu u) = |\mu|\alpha(u), \quad \beta(\mu u) = |\mu|\beta(u), \quad u \in E, \ \mu \in \mathbb{R}$$

*and* $\|u\| \leq N \max\{\alpha(u), \beta(u)\}$, *for* $u \in E$ *and* $\alpha(u_1) \leq \alpha(u_2)$ *for* $u_1, u_2 \in P$, $u_1 \leq u_2$, *where* $N > 0$ *is a constant.*

*Let* $r_2 > r_1 > 0$, $L > 0$ *be constants and* $\Omega_i = \{u \in E : \alpha(u) < r_i, \beta(u) < L\}$, $i = 1, 2$. *be two bounded open sets in E. Set* $D_i = \{u \in E : \alpha(u) = r_i\}$. *Assume that* $T : P \to P$ *is a completely continuous operator satisfying*

($C_1$) $\alpha(Tu) < r_1$, $u \in D_1 \cap P$; $\alpha(Tu) > r_2$, $u \in D_2 \cap P$,

($C_2$) $\beta(Tu) < L$, $u \in P$,

($C_3$) *there is a* $p \in (\Omega_2 \cap P) \setminus \{0\}$ *such that* $\alpha(p) \neq 0$ *and* $\alpha(u + \mu p) \geq \alpha(u)$ *for all* $u \in P$ *and* $\mu \geq 0$.

*Then T has at least one fixed point in* $(\Omega_2 \setminus \overline{\Omega_1}) \cap P$.

Moreover, we enunciate the following result concerning the expression of the Green's function $g_M$, related to the linear Problem (1.1), (1.5). The proof can be found in [1, 2]. To this end, we introduce the following condition:

$$M < 0 \quad \text{and} \quad \cos\left(\sqrt[4]{-M}\right) \cosh\left(\sqrt[4]{-M}\right) = 1. \tag{2.1}$$

**Lemma 2.3.** *Let* $\sigma \in C(I)$ *and* $M \in \mathbb{R}$. *Then problem* (1.1)–(1.2) *has a unique solution if and only if* (2.1) *does not hold.*

*In such a case, it is given by the following expression:*

$$u(t) = \int_0^1 g_M(t,s)\sigma(s)\, ds.$$

*Here, for* $M = -m^4 < 0$, *we have*

$$g_M(t,s) = \begin{cases} g_1(t,s,m) & \text{if } 0 \leq s \leq t \leq 1 \\ g_1(s,t,m) & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

*with*

$$\begin{aligned} g_1(t,s,m) = {} & \frac{1}{8m^3\left((1+e^{2m})\cos(m) - 2e^m\right)} \\ & \times \Big\{ e^{-m(4s+t)}\left(-2e^{mt}\cos(mt) + e^{2mt} + 1\right)\Big(\left(e^{5ms} - e^{m(3s+2)}\right)\cos(m) \\ & + e^{3sm+m} - e^{5sm+m} + e^{4ms}\left(-1 + e^{2m}\right)\cos(m - ms) + e^{5ms}\sin(m) \\ & + e^{m(3s+2)}\sin(m) - 2e^{4sm+m}\sin(ms) - e^{4ms}\sin(m - ms) - e^{4sm+2m}\sin(m - ms)\Big) \\ & - 2e^{m(s-t)} + 2e^{m(t-s)}\left((1+e^{2m})\cos(m) - 2e^m\right) \\ & + e^{-m(4s+t)}\Big(\left(e^{5ms} + e^{m(3s+2)}\right)\cos(m) - e^{3sm+m} - e^{5sm+m} - 2e^{4sm+m}\cos(ms) \\ & + e^{4ms}\cos(m - ms) + e^{4sm+2m}\cos(m - ms) - e^{5ms}\sin(m) + e^{m(3s+2)}\sin(m) \\ & + e^{4ms}\sin(m - ms) - e^{4sm+2m}\sin(m - ms)\Big)\left(2e^{mt}\sin(mt) - e^{2mt} + 1\right) \\ & - 4\sin(m(t-s))\left((1+e^{2m})\cos(m) - 2e^m\right)\Big\}. \end{aligned}$$

*If $M=0$, it is given by*

$$g_0(t,s)=-\frac{1}{6}\begin{cases} s^2(t-1)^2(2ts+s-3t) & \text{if } 0\le s\le t\le 1,\\ t^2(s-1)^2(2ts+t-3s) & \text{if } 0<t\le s\le 1.\end{cases}$$

*Moreover, when $M=m^4>0$ it follows the expression*

$$g_M(t,s)=\begin{cases} g_2(t,s,m) & \text{if } 0\le s\le t\le 1\\ g_2(s,t,m) & \text{if } 0\le t\le s\le 1,\end{cases}$$

$$\begin{aligned} g_2(t,s,m)={}&\frac{e^{-\frac{m(-6+3s+t)}{\sqrt{2}}}}{2\sqrt{2}m^3\left(1+e^{2\sqrt{2}m}+2e^{\sqrt{2}m}\left(-2+\cos\left(\sqrt{2}m\right)\right)\right)}\Bigg\{-2\left(-1+e^{\sqrt{2}mt}\right)\\ &\times\left(\left(e^{\sqrt{2}m(-2+s)}-e^{2\sqrt{2}m(-1+s)}\right)\cos\left(\frac{m(-2+s)}{\sqrt{2}}\right)+\left(-e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\right.\\ &\times\cos\left(\frac{ms}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m(-2+s)}-e^{\sqrt{2}m(-1+s)}+e^{2\sqrt{2}m(-1+s)}-e^{\sqrt{2}m(-3+2s)}\right)\\ &\left.\times\sin\left(\frac{ms}{\sqrt{2}}\right)\right)\sin\left(\frac{mt}{\sqrt{2}}\right)+\left(\left(e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\cos\left(\frac{m(-2+s)}{\sqrt{2}}\right)\right.\\ &+\left(-2e^{\sqrt{2}m(-2+s)}+e^{\sqrt{2}m(-1+s)}-2e^{2\sqrt{2}m(-1+s)}+e^{\sqrt{2}m(-3+2s)}\right)\cos\left(\frac{ms}{\sqrt{2}}\right)\\ &+\left(-e^{\sqrt{2}m(-2+s)}+e^{2\sqrt{2}m(-1+s)}\right)\sin\left(\frac{m(-2+s)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m(-1+s)}-e^{\sqrt{2}m(-3+2s)}\right)\\ &\left.\times\sin\left(\frac{ms}{\sqrt{2}}\right)\right)\left(\left(-1+e^{\sqrt{2}mt}\right)\cos\left(\frac{mt}{\sqrt{2}}\right)-\left(1+e^{\sqrt{2}mt}\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\right)\Bigg\}.\end{aligned}$$

Using the expressions given in Lemma 2.3, coupled to the definition of a Green's function [8] and, as a particular case of [8, Theorem 2.14 and Theorem 5.1], we deduce the following properties for function $g_M$:

**Corollary 2.4.** *Assuming that condition (2.1) does not hold. Then, function $g_M$, defined in Lemma 2.3, satisfies the following properties:*

1. *$g_M$ is symmetric, that is $g_M(t,s)=g_M(s,t)$, for all $t,s\in I$.*
2. *$g_M(0,s)=\frac{\partial g_M}{\partial t}(0,s)=g_M(1,s)=\frac{\partial g_M}{\partial t}(1,s)=0$, for all $s\in I$.*
3. *$g_M(t,1)=\frac{\partial g_M}{\partial s}(t,1)=g_M(t,0)=\frac{\partial g_M}{\partial s}(t,0)=0$, for all $t\in I$.*

   *Moreover, if $M\in(-m_1^4,m_0^4)$ the following inequalities are fulfilled:*

4. *$g_M(t,s)>0$ for all $t,s\in(0,1)$.*
5. *$\frac{\partial^2 g_M}{\partial t^2}(0,s)>0$ and $\frac{\partial^2 g_M}{\partial t^2}(1,s)>0$, for all $s\in(0,1)$.*
6. *$\frac{\partial^2 g_M}{\partial s^2}(t,1)>0$ and $\frac{\partial^2 g_M}{\partial s^2}(t,0)>0$, for all $t\in(0,1)$.*

To obtain the expression of the solution of Problem (1.1),(1.5), we must study the solution of a suitable non-homogeneous boundary value problem as follows

**Lemma 2.5** ([2, Theorem 3.12]). *The following problem:*

$$\begin{cases} u^{(4)}(t) + Mu(t) = 0, & t \in I, \\ u(1) = u'(0) = u'(1) = 0, \\ u(0) = 1, \end{cases} \tag{2.2}$$

*has no solution if and only if* (2.1) *holds.*

*In any other case, it has a unique solution, denoted by $w_M$, which is given by the following expression:*

$$w_M(t) = \begin{cases} \dfrac{\cos\left(\frac{mt}{\sqrt{2}}\right)\cosh\left(\frac{m(t-2)}{\sqrt{2}}\right) - \sin\left(\frac{mt}{\sqrt{2}}\right)\sinh\left(\frac{m(t-2)}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} \\ \quad + \dfrac{\left(\cos(\frac{m(t-2)}{\sqrt{2}}) - 2\cos(\frac{mt}{\sqrt{2}})\right)\cosh\left(\frac{mt}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} \\ \quad + \dfrac{\sin\left(\frac{m(t-2)}{\sqrt{2}}\right)\sinh\left(\frac{mt}{\sqrt{2}}\right)}{\cos\left(\sqrt{2}m\right) + \cosh\left(\sqrt{2}m\right) - 2} & \text{if } m > 0 \text{ and } M = m^4, \\ (t-1)^2(1+2t) & \text{if } M = 0, \\ \dfrac{-\cos(m-mt) + \cosh(m)(\cos(mt) - \cosh(mt))}{2\cos(m)\cosh(m) - 2} \\ \quad + \dfrac{\cos(m)\cosh(mt) - \sin(mt)\sinh(m)}{2\cos(m)\cosh(m) - 2} \\ \quad + \dfrac{(\sin(m) + \sinh(m))\sinh(mt))}{2\cos(m)\cosh(m) - 2} & \text{if } m > 0 \text{ and } M = -m^4. \end{cases} \tag{2.3}$$

In [2, Theorem 3.12] it is proved that if $M > 0$, then $w_M(t) > 0$ for all $t \in [0,1)$ if and only $M \in (0, 4\pi^4]$. It is obvious that $w_0(t) > 0$ for all $t \in [0,1)$.

To study the sign in the negative case, $M = -m^4$, we must introduce the concept of disconjugate equation given in [10].

**Definition 2.6.** Let $a_k \in C^{n-k}(I)$ for $k = 1, \dots, n$. The general $n$-th order linear differential equation $u^{(n)}(t) + a_1(t)\, u^{(n-1)}(t) + \cdots + a_{n-1}(t)\, u'(t) + a_n(t)\, u(t) = 0$ defined on any arbitrary interval $[a,b]$ is said to be disconjugate on an interval $J \subset [a,b]$ if every non trivial solution has, at most, $n-1$ zeros on $J$, multiple zeros being counted according to their multiplicity.

Moreover, we use the characterization for an equation to be disconjugate given in [9, Theorem 2.1] for a general $n$th order linear equation. Next, we enunciate the particular case for operator $u^{(4)} + M\, u$.

**Lemma 2.7.** *The linear equation $u^{(4)}(t) + M\, u(t) = 0$ is disconjugate on the interval $I$ if and only if $M \in (-m_1^4, m_0^4)$.*

As a consequence, due to the continuity of the expression of $w_M$ with respect to $M$, since $w_0''(1) = 6$, we have that if there is some $\bar{M} \in (-m_1^4, 0)$ for which $w_{\bar{M}}$ takes some negative values on $(0,1)$, then it must exists $M^* \in (\bar{M}, 0)$ such that one of the two following situations holds:

There is $t_0 \in (0,1)$ such that $w_{M^*}(t_0) = w'_{M^*}(t_0) = w_{M^*}(1) = w'_{M^*}(1) = 0$,

which contradicts Lemma 2.7, or

$$w_{M^*}(1) = w'_{M^*}(1) = w''_{M^*}(1) = 0.$$

But, in this last case, we have that

$$w''_{-m^4}(1) = \frac{m^2(\cos(m) - \cosh(m))}{\cos(m)\cosh(m) - 1},$$

which never takes the value zero for $m > 0$.

Therefore, if $M \in (-m_1^4, 0)$ then $w_M(t) > 0$ for all $t \in [0,1)$.

From the expression of $w''_M(1)$ we have that $w_M < 0$ in a neighborhood of $t = 1$ for $M$ smaller and close enough to $-m_1^4$.

Now, suppose that there is some $M_1 < -m_1^4$ for wich $w_{M_1} > 0$ on $[0,1)$. Let $-m_1^4 < M_2 < 0$, we have that for all $t \in [0,1)$, the following property is fulfilled:

$$w^{(4)}_{M_2}(t) - w^{(4)}_{M_1}(t) = -M_2\,(w_{M_2} - w_{M_1})(t) - (M_2 - M_1)w_{M_1}(t) < -M_2\,(w_{M_2} - w_{M_1})(t).$$

Now, since $w_{M_2} - w_{M_1}$ satisfies the boundary conditions (1.2), from Corollary 2.4, we deduce that $0 < w_{M_2} < w_{M_1}$ on $(0,1)$. But this contradicts the fact that

$$\lim_{M \to -m_1^{4+}} \{w_M(t)\} = +\infty, \quad \text{for all } t \in (0,1).$$

So, we have proved the following result:

**Lemma 2.8.** *$w_M > 0$ on $[0,1)$ if and only if $M \in (-m_1^4, 4\pi^4)$.*

Now, by denoting

$$C_M = \int_0^1 w_M(\tau)\nu(\tau)\,d\tau, \tag{2.4}$$

we are in a position to obtain the explicit expression of the Green's function related to the equation (1.1) coupled to boundary conditions (1.5). The result is the following.

**Lemma 2.9.** *Let $\sigma \in L^1(I)$, $\lambda > 0$ and $M \in \mathbb{R}$ be such that (2.1) does not hold. Then problem*

$$\begin{cases} u^{(4)}(t) + Mu(t) = \sigma(t), & t \in I, \\ u'(0) = u(1) = u'(1) = 0, \\ u(0) = \lambda \displaystyle\int_0^1 u(s)\nu(s)\,ds \end{cases} \tag{2.5}$$

*has a unique solution if and only if*

$$\lambda\, C_M \neq 1.$$

*In such a case, it is given by the following expression*

$$u_M(t) = \int_0^1 G_M(t,s)\sigma(s)\,ds$$

*where*

$$G_M(t,s) = g_M(t,s) + \frac{\lambda w_M(t)}{1 - \lambda C_M} \int_0^1 g_M(\tau,s)\nu(\tau)\,d\tau, \tag{2.6}$$

*$w_M$ and $C_M$ are defined in (2.3) and (2.4) respectively and $g_M$ is showed in Lemma 2.3.*

*Proof.* Since (2.1) does not hold, we have that Problems (1.1)–(1.2) and (2.2) are uniquely solvable. Let $v_M$ and $w_M$ be the unique solutions of each problem respectively. Then, it is clear that

$$u_M(t) = v_M(t) + \lambda\, w_M(t) \int_0^1 u_M(s) \nu(s)\, ds$$

is the unique solution of problem (2.5).

As a consequence, for all $t \in I$, the following equalities are fulfilled:

$$u_M(t) = \int_0^1 g_M(t,s)\sigma(s)\, ds + \lambda w_M(t) \int_0^1 u_M(s)\nu(s)\, ds. \tag{2.7}$$

Let $A_M = \int_0^1 u_M(\tau)\nu(\tau)\, d\tau$, then, from the previous equality, we deduce that

$$A_M = \int_0^1 \int_0^1 g_M(\tau,s)\nu(\tau)\sigma(s)\, ds\, d\tau + \lambda A_M \int_0^1 w_M(\tau)\nu(\tau)\, d\tau$$

or, which is the same,

$$A_M = \frac{\displaystyle\int_0^1 \sigma(s) \int_0^1 g_M(\tau,s)\nu(\tau)\, d\tau\, ds}{1 - \lambda \int_0^1 w_M(\tau)\nu(\tau)\, d\tau}.$$

Replacing this value in (2.7), we arrive at the following expression for function $u_M$:

$$u_M(t) = \int_0^1 g_M(t,s)\sigma(s)\, ds + \lambda w_M(t) \frac{\displaystyle\int_0^1 \sigma(s) \int_0^1 g_M(\tau,s)\nu(\tau)\, d\tau\, ds}{1 - \lambda \int_0^1 w_M(\tau)\nu(\tau)\, d\tau},$$

and the proof is concluded. □

Assuming that (2.1) does not hold, let $z_M$ be the unique solution of the following boundary value problem:

$$z^{(4)}(t) + Mz(t) = \nu(t) \quad t \in I, \quad z(0) = z(1) = z'(0) = z'(1) = 0, \tag{2.8}$$

which is given by the following expression

$$z_M(t) = \int_0^1 g_M(t,s)\nu(s)\, ds.$$

Moreover, if $M \in (-m_1^4, m_0^4)$, since $\nu(t) > 0$ a.e. $t \in (0,1)$, from Corollary 2.4, we have that $z_M(t) > 0$ for all $t \in (0,1)$, $z_M''(0) > 0$ and $z_M''(1) > 0$.

We point out that, by direct computations, it is possible to obtain the explicit expression of function $z_M$ for any particular choice of function $\nu$.

A careful analysis of the Green's function $G_M$ allows us to deduce the following result:

**Theorem 2.10.** *Let $G_M(t,s)$ be the Green's function related to problem (1.1), (1.5) given by expression (2.6). Then if $M \in (-m_1^4, 4\pi^4)$ and $\lambda \in (0, 1/C_M)$ we have that $G_M(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$. Moreover there exist $R > 0$ and $h \in C(I)$, such that $h(1) = 0$ and $h > 0$ on $[0,1)$, for which the following inequalities are fulfilled:*

$$h(t)\, \frac{\lambda}{1 - \lambda\, C_M} z_M(s) \le G_M(t,s) \le R\, \frac{\lambda}{1 - \lambda\, C_M} z_M(s), \text{ for all } (t,s) \in I \times I. \tag{2.9}$$

*Proof.* First, notice that $4\pi^4 < m_0^4$. So, since $M \in (-m_1^4, 4\pi^4)$ we have, from Corollary 2.4, that $g_M > 0$ on $(0,1)\times(0,1)$ and, as a direct consequence of $\lambda \in (0, 1/C_M)$ and the fact that $w_M > 0$ on $[0,1)$ for all $M \in (-m_0^4, 4\pi^4)$ (Lemma 2.8), we conclude, from (2.6), that $G_M(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$.

Now, we denote by

$$\varphi(t,s) = \frac{G_M(t,s)}{G_M(0,s)} = \frac{1-\lambda C_M}{\lambda} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} + w_M(t). \tag{2.10}$$

It is clear that function $\varphi$ is continuous on $[0,1]\times(0,1)$, $\varphi(0,s) = 1$ and $\varphi(1,s) = 0$ for all $s \in I$.

Using the properties of $g_M$ showed in Lemma 2.3 and those of $z_M$ previously explained, by means of L'Hôpital's rule, we deduce, for all $t \in (0,1)$:

$$\lim_{s\to 0^+} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} = \lim_{s\to 0^+} \frac{g_M(t,s)}{z_M(s)} = \lim_{s\to 0^+} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z_M''(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,0)}{z_M''(0)} > 0.$$

Thus,

$$\lim_{s\to 0^+} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda} \left( \frac{\frac{\partial^2 g_M}{\partial s^2}(t,0)}{z_M''(0)} \right) + w_M(t) := l_1(t) > 0 \quad \text{for all } t \in [0,1).$$

Analogously, if $t \in (0,1)$, we have

$$\lim_{s\to 1^-} \frac{g_M(t,s)}{\displaystyle\int_0^1 g_M(s,r)v(r)\,dr} = \lim_{s\to 1^-} \frac{g_M(t,s)}{z_M(s)} = \lim_{s\to 1^-} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z_M''(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,1)}{z_M''(1)} > 0$$

and

$$\lim_{s\to 1^-} \varphi(t,s) = \frac{1-\lambda C_M}{\lambda} \left( \frac{\frac{\partial^2 g_M}{\partial s^2}(t,1)}{z_M''(1)} \right) + w_M(t) := l_2(t) > 0 \quad \text{for all } t \in [0,1).$$

The limits $l_1(t)$ and $l_2(t)$ exist and are finite, so $\varphi$ has removable discontinuities at $s = 0, 1$, and we can extend it to a function $\tilde{\varphi} \in C(I \times I)$.

Therefore $h(t) = \min_{s\in[0,1]} \tilde{\varphi}(t,s)$ is a continuous function such that

$$h(1) = 0 \quad \text{and} \quad 0 < h(t) \le \tilde{\varphi}(t,s) \le R \text{ for all } (t,s) \in [0,1)\times[0,1],$$

where $R = \max_{(t,s)\in I\times I} \tilde{\varphi}(t,s)$. □

**Corollary 2.11.** *Let $G_M(t,s)$ be Green's function related to problem (1.1), (1.5) given by expression (2.6). Then if $M \in (-m_1^4, 4\pi^4)$ and $\lambda \in (0, 1/C_M)$ we have that for all positive constant $\delta \in (0,1)$ there exists $\gamma(\delta) \in (0,1)$ for which the following inequality is fulfilled:*

$$\gamma(\delta) \frac{\lambda}{1-\lambda\, C_M} z_M(s) \le G_M(t,s), \quad \textit{for all } (t,s) \in [0,\delta]\times I. \tag{2.11}$$

*Proof.* The result follows from the fact that function $h$ is continuous on $I$ and strictly positive on $[0,1)$. □

## 3 Existence of positive solutions

In this section, we are concerned with the existence of positive solutions of the boundary value problem (1.4)–(1.5). Firstly, we shall give a result of completely continuous operator. Then, we shall derive the existence results. Consider the vectorial space

$$E = \{u \in C^2(I); \quad u'(0) = u'(1) = 0\}$$

with the weighted norm $\|u\| = \|u\|_\infty + \|u''\|_\infty$.

Since, for any $u \in E$, and all $t \in I$, it is satisfied that

$$u'(t) = \int_0^t u''(s)\,ds.$$

We deduce that

$$\|u\| \le \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \le \|u\|_\infty + 2\,\|u''\|_\infty \le 2\,\|u\|,$$

we have that $\|\cdot\|$ is an equivalent norm to the usual one in $E$. As consequence, $E$ is a Banach Space with the weighted norm $\|\cdot\|$.

The following result is a direct consequence of the results showed in previous sections.

Let $T$ the operator from $E$ to $E$ defined by

$$(T\,u)(t) = \int_0^1 G_M(t,s) f(s, u(s), u''(s))\,ds. \tag{3.1}$$

**Lemma 3.1.** *Assume that $f$ satisfies condition $(H_0)$, then, $u \in C^2(I)$ is a solution of (1.4)–(1.5) if and only if $u$ is a fixed point of operator $T$ defined on (3.1).*

Now, by considering function $h$ and constant $R$, obtained in Theorem 2.10, we look for the fixed points of operator $T$ at the following cone,

$$K = \left\{ u \in C^2(I) \text{ and } u(t) \ge \frac{h(t)}{R}\,\|u\|_\infty \text{ for all } t \in I \right\}. \tag{3.2}$$

**Lemma 3.2.** *If condition $(H_0)$ is fulfilled, then operator $T : K \to K$, defined in (3.1), is completely continuous.*

*Proof.* From the non-negativeness of functions $f$ and $G_M$ we deduce that $(T\,u)(t) \ge 0$ for all $t \in I$ and $u \in K$. Using that $G_M \in C^2(I \times I)$, from the continuity of function $f$ we deduce the completely continuous character of operator $T$ as a direct application of Arzelà–Ascoli Theorem [11].

Let $u \in K$, by (2.9), we have that the following inequalities are fulfilled for all $t \in I$

$$\begin{aligned}
(T\,u)(t) &= \int_0^1 G_M(t,s) f(s,u(s),u''(s))\,ds \\
&\ge h(t)\,\frac{\lambda}{1-\lambda\,C_M}\int_0^1 z_M(s)\, f(s,u(s),u''(s))\,ds \\
&\ge \frac{h(t)}{R}\int_0^1 \max_{t\in I}\{G_M(t,s)\} f(s,u(s),u''(s))\,ds \\
&\ge \frac{h(t)}{R}\max_{t\in I}\int_0^1 G_M(t,s) f(s,u(s),u''(s))\,ds \\
&= \frac{h(t)}{R}\|T\,u\|_\infty.
\end{aligned}$$

Moreover, from Corollary 2.4, (2), we have that

$$(T\,u)'(0) = (T\,u)'(1) = 0,$$

and, as a consequence, $T\,u \in K$ for all $u \in K$ and the proof is complete. □

In the sequel, for any pair $\delta, \gamma$ satisfying (2.11) we introduce the following cone as follows:

$$K_\gamma^\delta = \left\{ u \in K \text{ and } \min_{t\in[0,\delta]} u(t) \geq \frac{\gamma}{R} \|u\|_\infty \right\}. \tag{3.3}$$

As in the proof of Lemma 3.2, one can verify the following result.

**Lemma 3.3.** *Assuming condition* $(H_0)$, *we have that* $T(K_\gamma^\delta) \subset K_\gamma^\delta$.

Define the convex functionals $\alpha(u) = \|u\|_\infty$, $\beta(u) = \|u''\|_\infty$. Then, we have that

$$\|u\| \leq 2\max\{\alpha(u), \beta(u)\},$$

$$\alpha(\mu u) = |\mu|\alpha(u),\ \beta(\mu u) = |\mu|\beta(u), \quad u \in E,\ \mu \in \mathbb{R},$$

and since for all $u \in K$, it is satisfied that $u \geq 0$ on $I$, we have that if $u_1, u_2 \in K$ are such that $u_1 \leq u_2$ on $I$, then $\alpha(u_1) \leq \alpha(u_2)$.

In the following, we introduce the positive constants:

$$m = \max_{t\in I} \int_0^\delta G_M(t,s)\,ds, \tag{3.4}$$

$$M_1 = \max_{t\in I} \int_0^1 G_M(t,s)\,ds \tag{3.5}$$

and

$$M_2 = \max_{t\in I} \int_0^1 \left| \frac{\partial^2 G_M}{\partial t^2}(t,s) \right| ds. \tag{3.6}$$

We suppose that there are $L > b > \frac{\gamma}{R} b > c > 0$ such that $f$ satisfies the following growth conditions:

$(H_1)$ $f(t,u,v) < \frac{c}{M_1}$, for $(t,u,v) \in I \times [0,c] \times [-L,L]$,

$(H_2)$ $f(t,u,v) \geq \frac{b}{m}$, for $(t,u,v) \in I \times [\frac{\gamma}{R} b, b] \times [-L,L]$,

$(H_3)$ $f(t,u,v) < \frac{L}{M_2}$, for $(t,u,v) \in I \times [0,b] \times [-L,L]$.

**Theorem 3.4.** *Assume that conditions* $(H_0)$–$(H_3)$ *are fulfilled. Then the boundary value problem* (1.4)–(1.5) *has at least one positive solution u satisfying*

$$c < \|u\|_\infty < b, \quad \|u''\|_\infty < L.$$

*Proof.* Take

$$\Omega_1 = \{u \in E : \|u\|_\infty < c, \|u''\|_\infty < L\}, \quad \Omega_2 = \{u \in E : \|u\|_\infty < b, \|u''\|_\infty < L\}$$

two boundary open sets in $E$, and

$$D_1 = \{u \in E : \|u\|_\infty = c\}, \quad D_2 = \{u \in E : \|u\|_\infty = b\}.$$

As in [16], we define the following double truncated continuous function as follows:

$$f^*(t,u,v)=\begin{cases} f(t,u,v) & \text{if } (t,u,v)\in I\times[0,b]\times\mathbb{R},\\ f(t,b,v) & \text{if } (t,u,v)\in I\times[b,\infty)\times\mathbb{R},\end{cases}$$

and

$$f_1(t,u,v)=\begin{cases} f^*(t,u,-L) & \text{if } (t,u,v)\in I\times[0,\infty)\times(-\infty,-L],\\ f^*(t,u,v) & \text{if } (t,u,v)\in I\times[0,\infty)\times[-L,L],\\ f^*(t,u,L) & \text{if } (t,u,v)\in I\times[0,\infty)\times[L,\infty).\end{cases}$$

As a direct consequence, we have that $f_1$ satisfies the following properties:

$(H_1^1)$ $f_1(t,u,v)<\frac{c}{M_1}$, for $(t,u,v)\in I\times[0,c]\times\mathbb{R}$,

$(H_2^1)$ $f_1(t,u,v)\geq\frac{b}{m}$, for $(t,u,v)\in I\times[\frac{\gamma}{R}b,\infty)\times\mathbb{R}$,

$(H_3^1)$ $f_1(t,u,v)<\frac{L}{M_2}$, for $(t,u,v)\in I\times[0,\infty)\times\mathbb{R}$.

Now, we define the operator

$$(T_1\,u)(t)=\int_0^1 G_M(t,s)f_1(s,u(s),u''(s))\,ds,$$

whose fixed points coincide with the solutions of problem

$$u^{(4)}(t)+Mu(t)=f_1(t,u(t),u''(t)),\quad t\in I, \tag{3.7}$$

coupled to boundary conditions (1.5).

As in Lemmas 3.2 and 3.3 it is not difficult to verify that $T_1:K_\gamma^\delta\to K_\gamma^\delta$ is a completely continuous operator.

Let $p=\frac{1}{2}b\in(\Omega_2\cap K_\gamma^\delta)\setminus\{0\}$. It is easy to see that $\alpha(u+\mu p)\geq\alpha(u)$ for all $u\in K_\gamma^\delta$ and $\mu\geq 0$.

In view of $(H_1)$ and $\alpha(u)=c$, $u\in D_1\cap K_\gamma^\delta$, we have that

$$\alpha(T_1u)=\max_{t\in I}\left|\int_0^1 G_M(t,s)f_1(s,u(s),u''(s))\,ds\right|<\max_{t\in I}\int_0^1 G_M(t,s)\frac{c}{M_1}\,ds\leq c.$$

Hence, $\alpha(T_1u)<c$.

Therefore, using $(H_2)$ and the fact that $u(s)\geq\frac{\gamma}{R}\alpha(u)$ for all $s\in[0,\delta]$, we have for all $u\in D_2\cap K_\gamma^\delta$ the following inequality is fulfilled

$$\alpha(T_1u)=\max_{t\in I}\left|\int_0^1 G_M(t,s)f_1(s,u(s),u''(s))\,ds\right|>\max_{t\in I}\int_0^\delta G_M(t,s)\frac{b}{m}\,ds\geq b.$$

Hence, $\alpha(T_1u)>b$.

$$\beta(T_1u)=\max_{t\in I}\left|\int_0^1\frac{\partial^2 G_M}{\partial t^2}(t,s)f_1(s,u(s),u''(s))\,ds\right|<\max_{t\in I}\int_0^1\left|\frac{\partial^2 G_M}{\partial t^2}(t,s)\right|\frac{L}{M_2}\,ds\leq L.$$

Hence, $\beta(T_1u)<L$.

Therefore, $u$ is a positive solution for the boundary value problem (3.7), (1.5) satisfying

$$c<\|u\|_\infty<b,\quad \|u''\|_\infty<L.$$

From the definition of function $f_1$, we conclude that the obtained solutions are also solutions of (1.4)–(1.5) and the proof is complete. □

## 4 Examples

In the sequel, we will obtain the different bounds and results for the particular case when $M = 0$ and $v(t) = 1$ for all $t \in I$. That is, we want to prove the existence of positive solutions of the problem:

$$L_0 u(t) = u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0,1] \tag{4.1}$$

subject to the boundary conditions:

$$u(1) = u'(0) = u'(1) = 0, \quad u(0) = \lambda \int_0^1 u(s)\, ds. \tag{4.2}$$

It is immediate to verify that

$$C_0 := \int_0^1 (t-1)^2(1+2t)dt = \frac{1}{2}.$$

As a consequence: $0 < \lambda < 2$.

Now, let us obtain the correspondent $\delta$, $\gamma$ and $R$. The expression of the related Green's function is given in Lemma 2.3.

Using the notation in Theorem 2.10, we have

$$\widetilde{\varphi}(t,s) = \begin{cases} \phi_1(t,s) & \text{if } 0 < s < t < 1, \\ \psi_1(t) & \text{if } s = 0, \\ \psi_2(t) & \text{if } s = 1, \\ \phi_2(t,s) & \text{if } 0 < t \le s < 1. \end{cases}$$

So we have

$$\phi_1(t,s) = \frac{(-1+t)^2\left(-4(s+2st) - 4t(-3+\lambda) + \lambda + s^2(1+2t)\lambda\right)}{(-1+s)^2\lambda},$$

$$\phi_2(t,s) = \frac{2t^3(-2+\lambda) + 2st^2(-3+2t)(-2+\lambda) + s^2(-1+t)^2(1+2t)\lambda}{s^2\lambda},$$

$$\psi_1(t) = \frac{(-1+t)^2(-4t(-3+\lambda)+\lambda)}{\lambda}$$

and

$$\psi_2(t) = 1 + \frac{t^2(12 - 9\lambda + 4t(-3+2\lambda))}{\lambda}.$$

It is clear that

$$\frac{\partial \widetilde{\varphi}}{\partial s}(t,s) = \begin{cases} \frac{\partial \phi_1(t,s)}{\partial s} & \text{if } 0 < s < t < 1, \\ 0 & \text{if } s = 0 \text{ or } s = 1, \\ \frac{\partial \phi_2(t,s)}{\partial s} & \text{if } 0 < t \le s < 1, \end{cases}$$

where

$$\frac{\partial \phi_1(t,s)}{\partial s} = -\frac{2(-1+t)^2(1+s+2(-2+s)t)(-2+\lambda)}{(-1+s)^3\lambda}$$

and

$$\frac{\partial \phi_2(t,s)}{\partial s} = -\frac{2t^2(-3s+2(1+s)t)(-2+\lambda)}{s^3\lambda}.$$

Let $\alpha_1(t)=\frac{4t-1}{2t+1}$ and $\alpha_2(t)=\frac{2t}{3-2t}$, it is obvious that

$$\frac{\partial \phi_1(t,s)}{\partial s}=0 \text{ if and only if } s=\alpha_1(t)$$

and

$$\frac{\partial \phi_2(t,s)}{\partial s}=0 \text{ if and only if } s=\alpha_2(t).$$

- If $t\in[0,\frac{1}{4}]$, in this case $\phi_1(t,\cdot)$ is decreasing on $[0,t]$ and $\phi_2(t,\cdot.)$ is decreasing on $[t,1]$ . In this case for all $t\in[0,\frac{1}{4}]$, $\max_{s\in I}\widetilde{\varphi}(t,s)=\psi_1(t)$ and $h(t)=\min_{s\in I}\widetilde{\varphi}(t,s)=\psi_2(t)$.
- If $t\in[\frac{1}{4},\frac{1}{2}]$, $\alpha_1(t)\in[0,t]$ in this case $\phi_1(t,\cdot)$ is increasing on $[0,\alpha_1(t)]$ and it is decreasing on $[\alpha_1(t),t]$ and $\phi_2(t,\cdot.)$ is decreasing on $[t,1]$. Then for all $t\in[\frac{1}{4},\frac{1}{2}]$ we have

$$\max_{s\in I}\widetilde{\varphi}(t,s)=\phi_1(t,\alpha_1(t))=\frac{(-1+t)(1+2t)(-2+4t(-1+\lambda)-\lambda)}{2\lambda} \tag{4.3}$$

  and

$$h(t)=\min_{s\in I}\widetilde{\varphi}(t,s)=\min\{\psi_1(t),\psi_2(t)\}=\psi_2(t). \tag{4.4}$$

- If $t\in[\frac{1}{2},\frac{3}{4}]$, $\alpha_2(t)\in[t,1]$ in this case $\phi_2(t,\cdot.)$ is increasing on $[t,\alpha_2(t)]$ and is decreasing on $[\alpha_2(t),1]$ and $\phi_1(t,\cdot)$ is increasing on $[0,t]$. Then for all $t\in[\frac{1}{2},\frac{3}{4}]$ we have

$$\max_{s\in I}\widetilde{\varphi}(t,s)=\phi_2(t,\alpha_2(t))=\frac{2\lambda+t(-3+2t)(-6+4t+3\lambda)}{2\lambda} \tag{4.5}$$

  and

$$h(t)=\min_{s\in I}\widetilde{\varphi}(t,s)=\min\{\psi_1(t),\psi_2(t)\}=\psi_1(t). \tag{4.6}$$

- If $t\in[\frac{3}{4},1]$, in this case $\phi_1(t,\cdot)$ is increasing on $[0,t]$ and $\phi_2(t,\cdot.)$ is increasing on $[t,1]$. In this case for all $t\in[\frac{3}{4},1]$, $\max_{s\in I}\widetilde{\varphi}(t,s)=\psi_2(t)$ and $h(t)=\psi_1(t)$.

In conclusion we obtain

$$\max_{s\in I}\widetilde{\varphi}(t,s)=\begin{cases}\psi_1(t) & \text{if } t\in\left[0,\frac{1}{4}\right],\\ \phi_1(t,\alpha_1(t)) & \text{if } t\in\left[\frac{1}{4},\frac{1}{2}\right],\\ \phi_2(t,\alpha_2(t)) & \text{if } t\in\left[\frac{1}{2},\frac{3}{4}\right],\\ \psi_2(t) & \text{if } t\in\left[\frac{3}{4},1\right],\end{cases}$$

and

$$\min_{s\in I}\widetilde{\varphi}(t,s)=\begin{cases}\psi_2(t) & \text{if } t\in\left[0,\frac{1}{2}\right],\\ \psi_1(t) & \text{if } t\in\left[\frac{1}{2},1\right].\end{cases}$$

Let $R_1=\max_{t\in\left[\frac{1}{4},\frac{1}{2}\right]}\phi_1(t,\alpha_1(t))$, $R_2=\max_{t\in\left[\frac{1}{2},\frac{3}{4}\right]}\phi_2(t,\alpha_2(t))$, $R_3=\max_{t\in\left[0,\frac{1}{4}\right]}\psi_1(t)$ and $R_4=\max_{t\in\left[\frac{3}{4},1\right]}\psi_2(t)$. We deduce that $R=\max_{(t,s)\in I\times I}\widetilde{\varphi}(t,s)=\max\{R_1,R_2,R_3,R_4\}$ and

$$\gamma=\min_{t\in[0,\delta]}h(t)=\begin{cases}\min\{1,\psi_2(\delta)\} & \text{if } \delta\in\left(0,\frac{1}{2}\right],\\ \min\{1,\psi_1(\delta)\} & \text{if } \delta\in\left[\frac{1}{2},1\right).\end{cases}$$

Choosing $\delta=0.9$ and $\lambda=1$. By computation we obtain

$R = \max\{1.6875, 1, 1.6875, 1\} = 1.6875$, $\gamma = \min_{t\in[0,\delta]} h(t) = 0.082$, $\frac{\gamma}{R} = 0.0485926$.

By simple calculation, we have that $M_2 = \max_{t\in I} \int_0^1 \left|\frac{\partial^2 G_0}{\partial t^2}(t,s)\right| ds \approx 0.1$, $m = \max_{t\in I} \int_0^\delta G_0(t,s)\, ds \approx 0.00417006$ and $M_1 = \max_{t\in I} \int_0^1 G_0(t,s)\, ds \approx 0.0042$.

**Example 4.1.** Let

$$f(t,u,v) = \frac{t}{100} + 4.71241u + 0.000416894\, u^3 + (0.00521618 + 0.000125066\, u^2)\frac{|v|}{90}.$$

Choosing $b = 60$, $\frac{\gamma}{R}b = 2.91556$, $\frac{\gamma}{Rm} = 11.6527$, $c = 0.5$ and $L = 400$. By simple calculation, $f$ satisfy $(H_0)$ and we have that:

$$f(t,u,v) \le 2.38958 < \frac{c}{M_1} = 119.048 \quad \text{for all } (t,u,v) \in I \times [0,c] \times [-L,L],$$
$$f(t,u,v) \ge 13.7496 > \frac{\gamma}{Rm} = 11.6527 \quad \text{for all } (t,u,v) \in I \times \left[\frac{\gamma}{R}b, b\right] \times [-L,L]$$

and

$$f(t,u,v) \le 374.828 < \frac{L}{M_2} = 4000 \quad \text{for all } (t,u,v) \in I \times [0,b] \times [-L,L].$$

With the use of Theorem 3.4, the boundary value problem (1.4)–(1.5) has at least one positive solution $u$ satisfying

$$0.5 < \|u\|_\infty < 60, \quad \|u''\|_\infty < 400.$$

**Example 4.2.** Let

$$f(t,u,v) = a(t)u + b(t)u^3 + c(t)|v|^\alpha,\ \alpha \in (0,1)$$

where

$$a(t) = \begin{cases} t+10 & \text{if } t \in [0, \frac{1}{2}], \\ -t+11 & \text{if } t \in [\frac{1}{2}, \frac{3}{4}], \\ 3t+8 & \text{if } t \in [\frac{3}{4}, 1], \end{cases} \qquad b(t) = \begin{cases} e^t & \text{if } t \in [0, \frac{1}{2}], \\ 2e^{\frac{1}{2}}t & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$c(t) = \begin{cases} \left(\frac{10^{-3}}{9}\right)^\alpha \frac{1}{2^\alpha} & \text{if } t \in [0, \frac{1}{2}], \\ \left(\frac{10^{-3}}{9}\right)^\alpha t^\alpha & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

For this we have, for all $t \in [0,1]$, $10 \le a(t) \le 11$, $1 \le b(t) \le 2e^{\frac{1}{2}}$ and $\left(\frac{10^{-3}}{9}\right)^\alpha \frac{1}{2^\alpha} \le c(t) \le \left(\frac{10^{-3}}{9}\right)^\alpha$. Choosing $c = 1$, $b = 30$, $\frac{\gamma}{R}b = 1.457778$ and $L = 9 \times 10^3$. By simple calculation, we have that:

$$f(t,u,v) \le 15.2974 < \frac{c}{M_1} = 238.09 \quad \text{for all } (t,u,v) \in I \times [0,c] \times [-L,L],$$
$$f(t,u,v) \ge 17.6757 > \frac{\gamma}{Rm} = 11.6527 \quad \text{for all } (t,u,v) \in I \times \left[\frac{\gamma}{R}b, b\right] \times [-L,L]$$

and

$$f(t,u,v) \le 89361.9486 < \frac{L}{M_2} = 90000 \quad \text{for all } (t,u,v) \in I \times [0,b] \times [-L,L].$$

With the use of Theorem 3.4, the boundary value problem (1.4)–(1.5) has at least one positive solution $u$ satisfying

$$1 < \|u\|_\infty < 30, \quad \|u''\|_\infty < 9 \times 10^3.$$

## References

[1] A. Cabada, J. A. Cid, B. Máquez-Villamarín, Computation of Green's functions for boundary value problems with Mathematica, *Appl. Math. Comput.* **219**(2012), 1919–1936. https://doi.org/10.1016/j.amc.2012.08.035; MR2983897

[2] A. Cabada, R. R. Enguiça, Positive solutions of fourth order problems with clamped beam boundary conditions, *Nonlinear Anal.* **74**(2011), 3112–3122. https://doi.org/10.1016/j.na.2011.01.027; MR2793550

[3] A. Cabada, R. Enguiça, L. López-Somoza, Positive solutions for second-order boundary-value problems with sign changing Green's functions, *Electronic J. Differential Equations* **2017**, No. 245, 1–17. MR3711198

[4] A. Cabada, C. Fernández-Gómez, Constant sign solutions of two-point fourth order problems; *Appl. Math. Comput.* **263**(2015), 122–133. https://doi.org/10.1016/j.amc.2015.03.112; MR3348530

[5] A. Cabada, R. Jebari, Multiplicity results for fourth order problems related to the theory of deformations beams, *Discrete Contin. Dyn. Syst. Ser. B* **25**(2020), No. 2, 489–505. https://doi.org/10.3934/dcdsb.2019250; MR4043575

[6] A. Cabada, L. López-Somoza, F. Minhós, Existence, non-existence and multiplicity results for a third order eigenvalue three-point boundary value problem, *J. Nonlinear Sci. Appl.* **10**(2017), 5445–5463. https://doi.org/10.22436/jnsa.010.10.28; MR3725419

[7] A. Cabada, L. Saavedra, The eigenvalue characterization for the constant sign Green's functions of $(k, n-k)$ problems, *Bound. Value Probl.* **2016**, No. 44, 35 pp. https://doi.org/10.1186/s13661-016-0547-1; MR3459347

[8] A. Cabada, L. Saavedra, Characterization of constant sign Green's function for a two-point boundary-value problem by means of spectral theory, *Electron. J. Differential Equations* **2017**, No. 146, 1–96. MR3690173

[9] A. Cabada, L. Saavedra, Characterization of non-disconjugacy for a one parameter family of $n$th-order linear differential equations, *Appl. Math. Lett.* **65**(2017), 98–105. https://doi.org/10.1016/j.aml.2016.10.01; MR3575178

[10] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin-New York, 1971. https://doi.org/10.1007/BFb0058618; MR0460785

[11] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7; MR787404

[12] E. Feireisl, Nonzero time periodic solutions to an equation of Petrovsky type with nonlinear boundary conditions: slow oscillations of beams on elastic bearings, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **20**(1993), 133–146. MR1216001

[13] C. Gao, F. Zhang, R. Ma, Existence of positive solutions of second-order periodic boundary value problems with sign-changing Green's function, *Acta Math. Appl. Sin. Engl. Ser.* **33**(2017), 263–268. https://doi.org/10.1007/s10255-017-0657-2; MR3646983

[14] M. R. Grossinho, T. F. Ma, Symmetric equilibria for a beam with a nonlinear elastic foundation, *Portugal. Math.* **51**(1994), 375–393. MR1295208

[15] Y. Ji, Y. Guo, Chang-long Yu, Positive solutions to $(n-1,1)$ $m$-point boundary value problems with dependence on the first order derivative, *Appl. Math. Mech. (English Ed.)* **30**(2009), No. 4, 527–536. https://doi.org/10.1007/s10483-009-0413-x; MR2513432

[16] Y. Guo, W. Ge, Positive solutions for three-point boundary value problems with dependence on the first order derivative, *J. Math. Anal. Appl.* **290**(2004), 291–301. https://doi.org/10.1016/j.jmaa.2003.09.061; MR2032241

[17] G. Infante, Positive solutions of nonlocal boundary value problems with singularities, *Discrete Contin. Dyn. Syst.*, Dynamical systems, differential equations and applications. 7th AIMS Conference, suppl. (2009), 377–384. https://doi.org/10.3934/proc.2009.2009.377; MR2641414

[18] G. Infante, F. Minhós, Nontrivial solutions of systems of Hammerstein integral equations with first derivative dependence, *Mediterr. J. Math.* **14**(2017), Paper No. 242, 18 pp. https://doi.org/10.1007/s00009-017-1044-1; MR3735472

[19] G. Infante, J. R. L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, *J. Math. Anal. Appl.* **272**(2002), 30–42. https://doi.org/10.1016/S0022-247X(02)00125-7; MR1930701

[20] Y. Liu, Z. Weiguo, L. Xiping, A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann–Liouville derivative, *Appl. Math. Lett.* **25**(2012), 1986–1992. https://doi.org/10.1016/j.aml.2012.03.018; MR2957792

[21] F. Minhós, R. de Sousa, On the solvability of third-order three point systems of differential equations with dependence on the first derivative, *Bull. Braz. Math. Soc. (N.S.)* **48**(2017), 485–503. https://doi.org/10.1007/s00574-016-0025-5; MR3712344

[22] G. Perla Menzala, E. Zuazua, Explicit exponential decay rates for solutions of von Kármán's system of thermoelastic plates, *C. R. Acad. Sci. Paris Sér. I Math.* **324**(1997), 49–54. https://doi.org/10.1016/S0764-4442(97)80102-4; MR1435586

[23] J. Schröder, On linear differential inequalities, *J. Math. Anal. Appl.* **22**(1968), 188–216. https://doi.org/10.1016/0022-247X(68)90169-8; MR223935

[24] X. Yang, Z. Wei, W. Dong, Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.* **17**(2012), 85–92. https://doi.org/10.1016/j.cnsns.2011.05.007; MR2825989