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Existence results for a class of *p*–*q* Laplacian semipositone boundary value problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^N$; $N > 1$ with a smooth boundary or $\Omega = (0,1)$. We study positive solutions to the boundary value problem of the form:

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $q \in [2,p)$, λ is a positive parameter, and $f : [0,\infty) \mapsto \mathbb{R}$ is a class of C^1, non-decreasing and p-sublinear functions at infinity (i.e. $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0$) that are negative at the origin (semipositone). We discuss the existence of positive solutions for $\lambda \gg 1$. Further, when $p = 4, q = 2$, $\Omega = (0,1)$ and $f(s) = (s+1)^\gamma - 2$; $\gamma \in (0,3)$, we provide the exact bifurcation diagram for positive solutions. In particular, we observe two positive solutions for a finite range of λ and a unique positive solution for $\lambda \gg 1$.

Keywords: *p*–*q* Laplacian, semipositone problems, positive solutions.

2020 Mathematics Subject Classification: 35G30, 35J62, 35J92.

1 Introduction

In [3], authors discussed results which imply the existence of positive solutions for $\lambda \gg 1$ for the boundary value problem:

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$; $N > 1$ with a smooth boundary, λ is a positive parameter, and $\Delta_s u = \operatorname{div} |\nabla u|^{s-2}\nabla u)$; $s > 1$, and $f : [0,\infty) \to \mathbb{R}$ satisfies:

(H1) f is C^1, non-decreasing, p-sublinear at infinity $\big($i.e. $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0\big)$,

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(**H2**) $f(0) < 0$,

(**H3**) $\lim_{t\to\infty} f(t) = \infty$.

In the literature, such problems where $f(0) < 0$, are referred as semipositone problems. It is well known that establishing the existence of a positive solution for semipositone problems are challenging, see [1,4,9,10] and references therein.

In recent years, there has been considerable interest to study boundary value problems involving the p–q Laplacian operator ($-\Delta_p - \Delta_q$, $q \in (1, p)$), for examples, see [2,5,8,11] and the references therein. Such operators often occur in the mathematical modelling of chemical reactions and plasma physics. In this article, we extend this study of p–q Laplacian boundary value problem for a class of semipositone reaction terms. Namely, we study the boundary value problem

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\;\; \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

for $q \in [2, p)$. We establish the following result.

Theorem 1.1. *Assume* (**H1**), (**H2**) *hold and there exists* $A > 0, \sigma > 0$ *such that*

$$f(s) \geq As^{\sigma}, \quad \textit{for } s \gg 1.$$

Then (1.2) *has a positive solution for* $\lambda \gg 1$.

Remark 1.2. It is easy to see that (1.2) does not admit any positive solution for $\lambda \approx 0$. This follows due to the p-sublinear condition at infinity which implies there exists a $M > 0$ such that $f(s) \leq Ms^{p-1}, \forall s > 0$. Hence, if u is a positive solution, multiplying (1.2) by u and integrating we obtain

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^q dx \leq \lambda M \int_\Omega |u|^p dx$$

which implies

$$\lambda \geq \left(\frac{1}{M}\right) \left(\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}\right) \geq \frac{\lambda_{1,p}}{M},$$

where $\lambda_{1,p} > 0$ is the principal eigenvalue of $-\Delta_p$ on Ω with Dirichlet boundary condition.

We will use the method of sub-super solutions to establish Theorem 1.1. We will adapt and extend the ideas used in [3] to construct a crucial positive sub-solution.

Finally, for the case when $\Omega = (0, 1)$, $p = 4$ and $q = 2$, namely to the two-point boundary value problem:

$$\begin{aligned} -[(u')^3]' - [(u')]' &= \lambda f(u) \qquad \text{in } (0,1), \\ u(0) = 0 &= u(1) \end{aligned} \tag{1.3}$$

with $f(s) = (s+1)^r - 2$; $r \in (0, 3)$, we will provide exact bifurcation diagrams for positive solutions in Section 4. Bifurcation diagrams we obtained are of the form given in Figure 1.1. Note that this bifurcation diagram implies the existence of two positive solutions for certain finite range of λ and a unique positive solution for $\lambda \gg 1$.

The rest of the paper is organized as follows. In Section 2, we will recall some important results that are required for the development of this article. Section 3 is dedicated to the proof of Theorem 1.1, and Section 4 is devoted to obtaining the bifurcation diagram of positive solutions to (1.3).

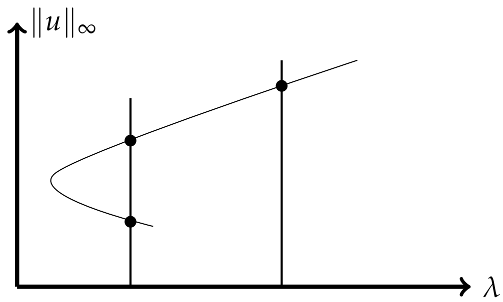

Figure 1.1: Bifurcation diagram for positive solutions to (1.3)

2 Preliminaries

In this section, we recall some results concerning a sub-super solution method for p–q Laplacian boundary value problem. First, by a weak solution of (1.2) we mean a function $u \in W_0^{1,p}(\Omega)$ which satisfies:

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\phi + \int_\Omega |\nabla u|^{q-2}\nabla u.\nabla\phi = \lambda \int_\Omega f(u)\phi, \qquad \forall\phi \in C_0^\infty(\Omega).$$

However, in this paper, we in fact study $C^1(\overline{\Omega})$ solution. Next, by a sub-solution (super solution) of (1.2) we mean a function $v \in W^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ such that $v \leq (\geq)\, 0$ on $\partial\Omega$ and satisfies:

$$\int_\Omega |\nabla v|^{p-2}\nabla v.\nabla\phi + \int_\Omega |\nabla v|^{q-2}\nabla v.\nabla\phi \leq (\geq)\ \lambda \int_\Omega f(v)\phi, \qquad \forall\phi \in C_0^\infty(\Omega),\ \phi \geq 0 \quad \text{in } \Omega.$$

Then the following sub-super solution result holds.

Lemma 2.1. *Let ψ, z be sub and super solutions of (1.2) respectively such that $\psi \leq z$ in Ω. Then (1.2) has a solution $u \in C^1(\overline{\Omega})$ such that $\psi \leq u \leq z$.*

Proof. We refer to Corollary 1 of [6] for the proof. □

3 Proof of Theorem 1.1

In this section, we use sub-super solution method to prove Theorem 1.1. We adapt and extend the ideas used in [3] to construct a crucial positive sub-solution.

Construction of a sub-solution : Let λ_1 be the principal eigenvalue and $\phi_1 \in C^\infty(\overline{\Omega})$ be the corresponding eigenfunction of

$$\begin{aligned} -\Delta\phi_1 &= \lambda_1\phi_1 \quad && \text{in } \Omega, \\ \phi_1 &= 0 && \text{on } \partial\Omega \end{aligned}$$

such that $\phi_1 > 0$ in Ω and $\|\phi_1\|_\infty = 1$. Then $\Delta_p\phi_1$, $\Delta_q\phi_1$ are in $L^\infty(\Omega)$, since $2 \leq q < p$. Further, by Hopf's lemma $|\nabla\phi_1| > 0$ on $\partial\Omega$. Now we consider

$$\psi = \lambda^r\phi_1^\beta, \quad \text{where } \beta = \frac{p}{p-1} \text{ and } r \in \left(\frac{1}{p-1}, \frac{1}{p-1-\sigma}\right).$$

Note that without loss of generality we can assume $\sigma < q-1$. Then, for $s = p, q$,

$$-\Delta_s \psi = \lambda^{r(s-1)} \beta^{s-1} \phi_1^{(\beta-1)(s-1)} [-\Delta_s \phi_1] - \lambda^{r(s-1)} \beta^{s-1} (\beta-1)(s-1) \frac{|\nabla \phi_1|^s}{\phi_1^{s-\beta(s-1)}}.$$

Note that $s - \beta(s-1) = 0$ when $s = p$ and $s - \beta(s-1) > 0$ when $s = q$. Also, $|\nabla \phi_1| > 0$ on $\partial\Omega$, $\phi_1 = 0$ on $\partial\Omega$ and $\phi_1 \in C^\infty(\overline{\Omega})$. Therefore, by continuity, there exists a δ neighborhood of $\partial\Omega$, say $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) \leq \delta\}$ such that

$$-\Delta_s \psi < 0 \quad \text{in } \Omega_\delta \tag{3.1}$$

for $s = p, q$. Further, since $\Delta_p \phi_1 \in L^\infty(\Omega)$ we see that $\exists\, \epsilon_p > 0$ (independent of λ) such that

$$-\Delta_p \psi \leq -\lambda^{r(p-1)} \epsilon_p \quad \text{in } \Omega_\delta.$$

As $r(p-1) > 1$, for $\lambda \gg 1$ it follows that

$$-\Delta_p \psi \leq -\lambda^{r(p-1)} \epsilon_p \leq \lambda f(0) \leq \lambda f(\psi) \quad \text{in } \Omega_\delta.$$

Hence, by (3.1) for $\lambda \gg 1$ we have

$$-\Delta_p \psi - \Delta_q \psi \leq \lambda f(\psi) \quad \text{in } \Omega_\delta. \tag{3.2}$$

Next let $\mu > 0$ be such that $\phi_1^\beta \geq \mu$ in $\Omega \setminus \Omega_\delta$ and $M_s > 0$ ($s = p, q$) be such that $-\Delta_s \psi \leq M_s \lambda^{r(s-1)}$ in Ω. Since $r < \frac{1}{s-1-\sigma}$ ($s = p, q$), it follows that for $\lambda \gg 1$ we have

$$\begin{aligned} -\Delta_s \psi \leq M_s \lambda^{r(s-1)} &\leq \left(\frac{\lambda A}{2}\right) (\lambda^r \mu)^\sigma \\ &\leq \left(\frac{\lambda}{2}\right) f(\psi) \quad \text{in } \Omega \setminus \Omega_\delta. \end{aligned}$$

Thus, for $\lambda \gg 1$, we obtain

$$-\Delta_p \psi - \Delta_q \psi \leq \lambda f(\psi) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{3.3}$$

Combining (3.2) and (3.3), for $\lambda \gg 1$ we see that

$$-\Delta_p \psi - \Delta_q \psi \leq \lambda f(\psi) \;\; \text{in } \Omega. \tag{3.4}$$

Therefore, ψ is a sub-solution of (1.2) when $\lambda \gg 1$.

Construction of a super solution: Let $R > 0$ be such that $\overline{\Omega} \subseteq B_R(0)$, where $B_R(0)$ is the open ball of radius R centered at origin. Now consider

$$\eta(r) = \frac{1 - (\frac{r}{R})^{p'}}{p'} \quad \text{on } B_R,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Notice that $0 \leq \eta \leq 1$. Also for $0 \leq r \leq R$,

$$\eta'(r) = -\frac{r^{p'-1}}{R^{p'}},$$

$$-\Delta_s \eta = -\left(|\eta'(r)|^{s-2} \eta'(r)\right)' = \left(\frac{r^{(p'-1)(s-1)}}{R^{p'(s-1)}}\right)' \geq 0 \quad \text{in } B_R, \tag{3.5}$$

for $s = p, q$. In particular,

$$- \Delta_p \eta = \frac{1}{R^p}. \tag{3.6}$$

Now let $Z = M(\lambda)\eta$, where $M(\lambda) \gg 1$ so that $\frac{[M(\lambda)]^{p-1}}{f(M(\lambda))} \geq \lambda R^p$. Note that this is possible by (**H1**). Then, using that f is non-decreasing, (3.5) and (3.6) we have

$$- \Delta_p Z - \Delta_q Z \geq -\Delta_p Z = \frac{M(\lambda)^{p-1}}{R^p} \geq \lambda f(M(\lambda)) \geq \lambda f(Z) \text{ in } B_R. \tag{3.7}$$

Clearly $Z \geq 0$ on $\partial\Omega$ and hence it is a super solution of (1.2).

Proof of Theorem 1.1. Let ψ be a sub-solution of (1.2) for $\lambda \gg 1$ (as constructed in (3.4)). Then, we can construct a super solution Z of (1.2) (as constructed in (3.7)). Further, since $Z > 0$ in $\overline{\Omega}$, we can choose $M(\lambda) \gg 1$ such that $Z \geq \psi$ in $\overline{\Omega}$. Hence by Lemma 2.1, (1.2) has a positive solution $u_\lambda \in [\psi, Z]$ for $\lambda \gg 1$ and Theorem 1.1 is proven. □

4 Bifurcation diagram for positive solutions to (1.3)

Here we adapt and extend the method used by Laetsch in [7] where he studied the boundary value problem: $-u'' = \lambda f(u); (0,1)$, $u(0) = 0 = u(1)$. First we note that since (1.3) is autonomous, any positive solution u must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$. Let $u(\frac{1}{2}) = \rho$ (say).

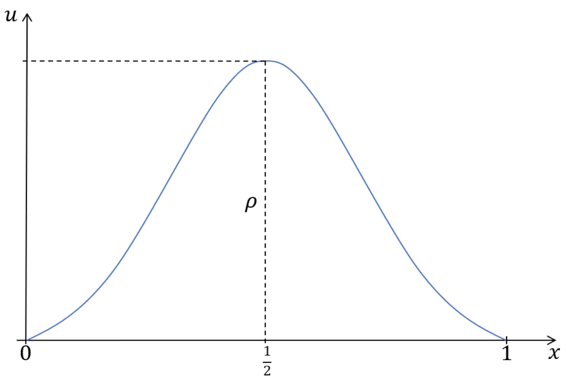

Figure 4.1: Shape of a positive solution to (1.3)

Now multiplying (1.3) by u' and integrating we obtain

$$-\frac{3}{4}[(u')^4]' - \frac{1}{2}[(u')^2]' = \lambda(F(u))' \quad \text{in } (0,1)$$

where $F(s) = \int_0^s f(z)dz$. Further integrating we obtain

$$3[u'(x)]^4 + 2[u'(x)]^2 = 4\lambda[F(\rho) - F(u(x))] \quad \text{in } [0, \tfrac{1}{2}]$$

and hence

$$u'(x) = \frac{\sqrt{[1 + 12\lambda(F(\rho) - F(u(x)))]^{\frac{1}{2}} - 1}}{\sqrt{3}} \quad \text{in } [0, \tfrac{1}{2}]. \tag{4.1}$$

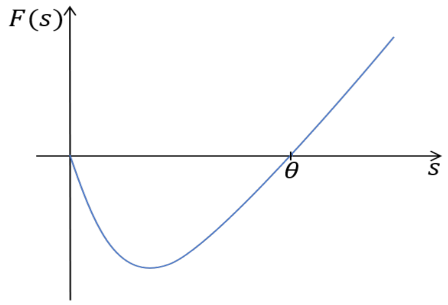

Figure 4.2: Shape of a function F

Noting that $u'(0) = \frac{\sqrt{[1+12\lambda F(\rho)]^{\frac{1}{2}}-1}}{\sqrt{3}}$, it is easy to see that ρ must be greater or equal to θ where θ is the position zero of F. Integrating (4.1) we get

$$\int_0^{u(x)} \frac{ds}{\sqrt{\left[1+12\lambda(F(\rho)-F(s))\right]^{\frac{1}{2}}-1}} = \frac{x}{\sqrt{3}} \quad \text{in } [0, \tfrac{1}{2}), \tag{4.2}$$

and setting $x \to (\frac{1}{2})^-$ we obtain

$$G(\lambda,\rho) = \int_0^{\rho} \frac{ds}{\sqrt{\left[1+12\lambda(F(\rho)-F(s))\right]^{\frac{1}{2}}-1}} = \frac{1}{2\sqrt{3}}. \tag{4.3}$$

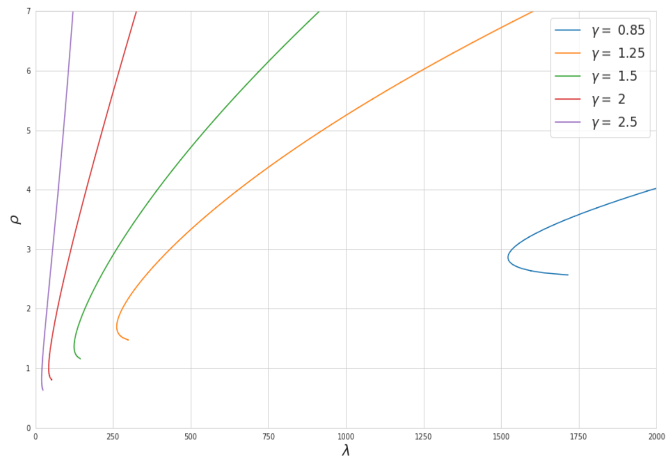

Figure 4.3: Bifurcation diagrams for (1.3) when $f(s) = (s+1)^{\gamma} - 2;\ \gamma = 0.85, 1.25, 1.5, 2.0, 2.5$.

It can be shown that that for $\lambda > 0$ and $\rho \geq \theta$, $G(\lambda,\rho)$ is well defined. Further, if $\lambda > 0$, $\rho \geq \theta$ satisfies (4.3), then (4.2) yields a C^2 function $u : [0, \frac{1}{2}) \to [0, \rho)$ such that $u(x) \to \rho$ as $x \to (\frac{1}{2})^-$. Extending this function on $[0,1]$ so that $u(\frac{1}{2}) = \rho$, and it is symmetric about $x = \frac{1}{2}$, it can be shown that it will be a positive solution of (1.3). Hence the bifurcation diagram for positive solutions to (1.3) is given by:

$$S = \left\{(\lambda,\rho) \mid \lambda > 0, \rho \geq \theta \ \&\ G(\lambda,\rho) = \tfrac{1}{2\sqrt{3}}\right\}. \tag{4.4}$$

Now, when $f(s) = (s+1)^{\gamma} - 2;\ \gamma \in (0,3)$, we compute S using *Mathematica*. In particular, here are the bifurcation diagrams we obtained for $\gamma = 0.85, 1.25, 1.5, 2.0$ and 2.5 (see Figure 4.3).

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