On the Existence of Almost Periodic Solutions of Neutral Functional Differential Equations

Jianhua Xu[1),2)], Zhicheng Wang[1)] and Zuxiu Zheng [2)]

Abstract. This paper discuss the existence of almost periodic solutions of neutral functional differential equations. Using a Liapunov function and the Razumikhin's technique, we obtain the existence, uniqueness and stability of almost periodic solutions.

Key words: Almost periodic solution; Neutral functional differential equation; Liapunov function.

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In the theory of functional differential equations, the existence, uniqueness and stability of almost periodic solutions is an important subject. Hale[1], Yoshizawa[2] and Yuan[3,4] et al, have provided some existence results for certain kind of retarded functional differential equations by means of Liapunov functions. The focus of our present work is to establish the existence of almost periodic solutions of neutral functional differential equations by using the Razumikhin-type argument. The problem of uniqueness and stability of the solution is also addressed. As a corollary to our results, the corresponding theorem of Yuan[4] is included and the proof in [4] is also simplified.

Consider the following almost periodic neutral functional differential equation

$$\frac{d}{dt}Dx_t = f(t, x_t) \tag{1}$$

and its product systems

$$\begin{cases} \dfrac{d}{dt}Dx_t = f(t, x_t) \\ \dfrac{d}{dt}Dy_t = f(t, y_t) \end{cases} \tag{1*}$$

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1). Dept. of Appl. Math., Hunan University, Changsha 410082, P. R. China

2). Dept. of Math., Anhui University, Hefei 230039, P. R. China

where $D : C \to R^n$ is linear, autononous and atomic at zero(see Hale [9]), $C := C([-\tau, 0], R^n)$, $f : R \times C \to R^n$ is continous and local Lipschitzian with respect to $\phi \in C$. Namely, for any $H > 0$, there is $K_0 = K_0(H) > 0$ such that for $\phi, \psi \in C_H$,

$$|f(t, \phi) - f(t, \psi)| \leq K_0|\phi - \psi|,$$

where $C_H := \{\phi \in C : |\phi| \leq H\}$.

Under the above hypotheses, there is a unique solution $x(t) = x(\sigma, \phi)(t)$ of Eq. (1) through a given intial value $(\sigma, \phi) \in R \times C_{H^*}$ (see [9]).

In addition, we always suppose that $f : R \times C_{H^*} \to R^n$ is almost periodic in t uniformly for $\phi \in C_{H^*}$ (see [8]).

Definition. *Let $C_D = \{\phi \in C : D\phi = 0\}$. D is said to be stable if the zero solution of the homogeneous difference equation $Dy_t = 0$, $t \geq 0$, $y_0 = \psi \in C_D$ is uniformly asymptotically stable.*

It is shown (see [9]) that when D is linear autonomous and atomic at zero, D is stable if and only if D is uniformly stable. Namely, there are two constant $a, b > 0$ such that for any $h \in C(R^+, R^n)$, the solutions of the equation

$$Dy_t = h(t), \quad t \geq \sigma$$

satisfies

$$|y_t| \leq be^{-a(t-\sigma)}|y_\sigma| + b \sup_{\sigma \leq u \leq t} |h(u)|, \quad t \geq \sigma. \tag{2}$$

Suppose that $V : R^+ \times R^n \times R^n \to R^+$ is continuous. For any $\phi, \psi \in C$, we define the derivative of V along the solution of (1^*) by

$$\dot{V}_{(1^*)}(t, \phi, \psi) = \limsup_{h \to 0^+}[V(t + h, Dx_{t+h}(t, \phi), Dy_{t+h}(t, \psi)) - V(t, D\phi, D\psi)].$$

Similar to the proof in [8, p.207], we can obtain

Lemma 1. *Suppose $p : R \to R$ is the unique almost periodic solution of (1) with $p_t \in C_H$ for $t \in R$. Then $\operatorname{mod}(p) \subset \operatorname{mod}(f)$.*

Lemma 2[3]**.** *Suppose D is stable, and Eq.(1) has a solution $\xi : R \to R$ with $|\xi_t| \leq H < H^*$ for $t \geq 0$. If ξ is an asymptotically almost periodic function, then Eq.(1) has an almost periodic solution.*

In what follows, we assume D is a stable operator, $||D|| = K$. Let $0 \leq u(s) \leq v(s)$, $s \geq 0$, be continuous and nondecreasing functions, $u(s) \to \infty$ as $s \to \infty$, $v(0) = 0$, and suppose that there is a continuous function $\alpha : R^+ \to R$ satisfying $v(K\eta) \leq u(\alpha(\eta))$. Let $\beta(\eta)$ be an arbitrary function of $\eta > 0$ such that $\beta(\eta) > b\alpha(\eta)$ for $\eta > 0$ (where $b > 0$ is defined in inequality (2)). Also assume $\alpha(0) = \beta(0) = 0$. The main result of this work is as follows:

Theorem. *Suppose $f(t,\phi)$ is almost periodic in $t \in R$ uniformly for $\phi \in C_{H^*}$. If there exists a Liapunov function $V : R^+ \times R^n \times R^n \to R^+$ such that*

(i). $u(|x-y|) \le V(t,x,y) \le v(|x-y|)$ for $(t,x,y) \in R^+ \times R^n \times R^n$;

(ii). $|V(t,x_1,y_1)-V(t,x_2,y_2)| \le L(|x_1-x_2|+|y_1-y_2|)$, where $L > 0$, $(t,x_i,y_i) \in R^+ \times \Omega \times \Omega$, $i = 1,2$, $\Omega = \{x \in R^n : |x| < H^*\}$;

(iii). For $t \in R$, $\phi,\psi \in C_{H^*}$ with $F(V(t,D\phi,D\psi)) \ge V(t+\theta,\phi(\theta),\psi(\theta))$ for $\theta \in [-\tau,0]$, we have

$$\dot{V}_{(1^*)}(t,\phi,\psi) \le -\omega(|D\phi - D\psi|),$$

where $F : [0,\infty) \to R^+$ is continuous and nondecreasing such that $F(v(K\eta)) > v(\beta(\eta))$, $\eta > 0$.

Moreover, Assume that Eq. (1) has a bounded solution $\xi : R \to R$ with $|\xi_t| \le H < H^*$ for $t \ge 0$. Then Eq. (1) has a unique almost periodic solution $p : R \to R$ with $|p(t)| \le H$ for $t \in R$, $\mod (p) \subset \mod (f)$, and p is uniformly asymptotically stable.

We first prove the following two lemmas.

Lemma 3. *Assume all conditions of the Theorem are satisfied. If a sequence $\{\alpha_n\}$ is given so that $f(t+\alpha_n,\phi)$ coverges uniformly on $R^+ \times C_H$, then for any $\varepsilon > 0$, there is a positive integer $k_0(\varepsilon)$, such that for $m \ge k \ge k_0$,*

$$\begin{aligned} A_{m,k}(t) = \limsup_{h\to 0^+} \frac{1}{h} |V(t+h, D\xi_{t+h}, D\xi_{t+\alpha_m-\alpha_k+h}) \\ - V(t+h, Dx_{t+h}(t,\xi_t), Dy_{t+h}(t,\xi_t+\alpha_m-\alpha_k))| \le \varepsilon. \end{aligned}$$

Proof. For any $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that for $m \ge k \ge k_0$, we have

$$|f(t+\alpha_k,\phi) - f(t+\alpha_m,\phi)| \le \frac{\varepsilon}{2L}, \quad (t,\phi) \in R^+ \times C_H.$$

By condition (ii),

$$\begin{aligned} A_{m,k}(t) &\le \limsup_{h\to 0^+} \frac{L}{h}(|D\xi_{t+h} - Dx_{t+h}(t,\xi_t)| \\ &\quad + |D\xi_{t+\alpha_m-\alpha_k+h} - Dy_{t+h}(t,\xi_{t+\alpha_m-\alpha_k})|) \\ &= \limsup_{h\to 0^+} \frac{L}{h} |D(\xi_{t+\alpha_m-\alpha_k+h} - y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k}))|. \end{aligned} \tag{3}$$

Note that $\eta_s := \xi_{s+\alpha_m-\alpha_k}$ satisfies

$$\begin{cases} \dfrac{d}{ds} D\eta_s = f(s+\alpha_m-\alpha_k, \eta_s), \\ \eta_t = \xi_{t+\alpha_m-\alpha_k}, \quad s \ge t. \end{cases}$$

Let $B_{m,k}(t) = \max_{t\le s\le t+h} |D(\eta_s - y_s)|$, where $y_s := y_s(t, \xi_{t+\alpha_m-\alpha_k})$. Thus,

$$
\begin{aligned}
B_{m,k} &\le \max_{t\le s\le t+h} \int_t^s |f(s+\alpha_m-\alpha_k,\eta_s) - f(s,y_s)|ds \\
&= \int_t^{t+h} |f(s+\alpha_m-\alpha_k,\eta_s) - f(s,y_s)|ds \\
&\le \int_t^{t+h} |f(s+\alpha_m-\alpha_k,\eta_s) - f(s,\eta_s)|ds \\
&\quad + \int_t^{t+h} |f(s,\eta_s) - f(s,y_s)|ds \\
&\le h\frac{\varepsilon}{2L} + K_0 \int_t^{t+h} |\eta_s - y_s|ds.
\end{aligned}
$$

By (2) we have

$$
\begin{aligned}
|\eta_s - y_s| &\le b \sup_{t\le u\le s} |D(\eta_u - y_u)| \\
&\le bB_{m,k}(t), \quad t \le s \le t+h.
\end{aligned}
$$

Hence

$$
B_{m,k}(t) \le h\frac{\varepsilon}{2L} + K_0 bhB_{m,k}(t). \tag{4}
$$

Let $h > 0$ be sufficiently small such that $K_0bh < 1/2$. By (4) we get

$$
B_{m,k} \le \frac{\varepsilon h}{2L(1-K_0bh)} \le \frac{\varepsilon}{L}h.
$$

Then

$$
|D(\xi_{t+\alpha_m-\alpha_k+h} - y_{t+h}(t,\xi_{t+\alpha_m-\alpha_k}))| \le \frac{\varepsilon}{L}h.
$$

Form (3), it follows that

$$
A_{m,k}(t) \le \limsup_{h\to o^+} \frac{L}{h}\frac{\varepsilon h}{L} = \varepsilon.
$$

Lemma 4. *Assume $t_0 \in R$ and $|y_t| \le 2H$ and $|Dy_t| \le \alpha(\delta)(\delta > 0)$ for $t \ge t_0$. Then there exists $t_1 > t_0, t_1 = t_1(\delta, t_0)$, such that $|y_t| \le \beta(\delta)$ for $t \ge t_1$.*

Proof. From inequality (2),

$$
|y_t| \le be^{-a(t-t_0)}|y_{t_0}| + b\sup_{t_o\le u\le t} |Dy_t| \le 2Hbe^{-a(t-t_0)} + b\alpha(\delta).
$$

Choose
$$t_1 > t_0 + \frac{1}{a} ln \frac{2Hb}{\beta(\delta) - b\alpha(\delta)},$$
then
$$|y_t| \leq 2Hb\frac{\beta(\delta) - b\alpha(\delta)}{2Hb} + b\alpha(\delta) = \beta(\delta) \quad for \quad t \geq t_1.$$
This complete the proof of Lemma 4.

Proof of the Theorem. Let $S = Cl\{\xi_t : t \geq 0\}$. It is easy to see that S is a compact set in C(see, for example, [4]). Let $\alpha' = \{\alpha'_n\}, a'_n \to \infty$ as $n \to \infty$, be a given sequence. Since $f(t, \phi)$ is almost periodic in t uniformly for $\phi \in C_{H^*}$, there exists a subsequence $\{\alpha_n\} \subset \alpha'$ such that $\lim_{n\to\infty} f(t+\alpha_n, \phi)$ exists uniformly on $R \times S$. Also we can suppose that $\{\alpha_n\}$ is increasing.

From the condition $F(v(K\eta)) > v(\beta(\eta)), \eta > 0$, we know that there exists a sequence $\{z_n\}_{n=1,2,\cdots}$, $z_0 = 2H$ such that
$$F(v(Kz_n)) = v(\beta(z_{n-1})), \quad n = 1, 2, \cdots.$$
Obviously, z_n is decreasing and tends to zero as $n \to \infty$. For any given $\varepsilon > 0$, we may assume $\varepsilon < \beta(2H)$, and select a N such that $\beta(z_N) < \varepsilon$. In the following, we prove that there exists $l_0 = l_0(\varepsilon)$ such that

(5) $$|\xi(t + \alpha_k) - \xi(t + \alpha_m)| < \varepsilon,$$

for $m \geq k \geq l_0$ and $t \in R^+$. Let
$$\gamma = \frac{1}{2} \inf_{kz_N \leq s \leq 2HK} \omega(s) > 0.$$

First, we prove that there is a $T_1 > 0$ such that

(6) $$V(t) := V(t, D\xi_t, D\xi_{t+\alpha_m-\alpha_k}) \leq v(kz_1),$$

for $t \geq T_1 + v(2HK)/\gamma$ and $m \geq k \geq k_0(\gamma)$. From
$$\begin{aligned} u(|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})|) \leq V(t) &\leq v(|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})|) \\ &\leq v(2HK) \leq u(\alpha(2H)), \end{aligned}$$
we deduce
$$|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})| \leq \alpha(2H), \quad t \geq 0.$$
Applying Lemma 4, there is a $T_1 \geq 0$ such that

(7) $$|\xi_t - \xi_{t+\alpha_m-\alpha_k}| \leq \beta(2H), t \geq T_1.$$

We now consider the following two cases:

Case 1. $V(t) > v(Kz_1)$ for $T_1 \le t \le T_1 + v(2HK)/\gamma$. In this case we have

$$\begin{aligned}F(V(t)) &\ge F(v(Kz_1)) = v(\beta(2H))\\ &\ge v(|\xi_t - \xi_{t+\alpha_m-\alpha_k}|)\\ &\ge V(t+\theta, \xi(t+\theta), \xi(t+\alpha_m-\alpha_k+\theta)), \quad -\tau \le \theta \le 0,\end{aligned}$$

which yields

$$\dot{V}_{(1^*)}(t) \le -\omega(|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})|).$$

Since

$$v(|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})|) \ge V(t) > v(Kz_1),$$

we obtain

$$|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})| \ge Kz_1 \ge Kz_N.$$

Moreover,

$$|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})| \le 2KH.$$

Then

$$\dot{V}_{(1^*)}(t) \le -2\gamma.$$

Applying Lemma 3 with $m \ge k \ge k_0(\gamma_0)$, we obtain that

$$\begin{aligned}V'(t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h) - V(t)]\\ &\le \limsup_{h\to 0^+} \frac{1}{h}[V(t+h) - V(t+h, Dx_{t+h}(t,\xi_t), Dy_{t+h}(t,\xi_{t+\alpha_m-\alpha_k})]\\ &\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, Dx_{t+h}(t,\xi_t), Dy_{t+h}(t,\xi_{t+\alpha_m-\alpha_k})) - V(t)]\\ &\le \gamma - 2\gamma = -\gamma \quad \mathit{for} \quad T_1 \le t \le T_1 + \frac{v(2HK)}{\gamma}.\end{aligned}$$

Thus,

$$V(t) \le V(T_1) - \gamma(t - T_1) \quad \mathit{for} \quad T_1 \le t \le T_1 + \frac{v(2HK)}{\gamma},$$

which yields

$$V(T_1 + \frac{v(2HK)}{\gamma}) \le v(2HK) - \gamma(T_1 + \frac{v(2HK)}{\gamma} - T_1) = 0.$$

This contradicts $V(t) > v(Kz_1)$.

Case 2. There is a $t_1 \in [T_1, T_1 + v(2HK)/\gamma]$ such that $V(t_1) \le v(Kz_1)$. In this case, we can suppose that there is $t_2 \ge t_1$ such that $V(t_2) = v(Kz_1)$. Then,

$$\begin{aligned} F(V(t_2)) &= F(v(Kz_1)) = v(\beta(2H)) \\ &\ge v(|\xi_{t_2} - \xi_{t_2+\alpha_m-\alpha_k}|) \\ &\ge V(t_2+\theta, \xi(t_2+\theta), \xi(t_2+\alpha_m-\alpha_k+\theta)), \end{aligned}$$

where $-\tau \le \theta \le 0$. Thus, condition(iii) implies

$$\dot{V}_{(1^*)}(t_2) \le -\omega(|D\xi_{t_2} - \xi_{t_2+\alpha_m-\alpha_k})|).$$

An argument similar to Case 1 leads to

$$V'(t_2) \le -\gamma < 0 \quad \textit{for} \quad m \ge k \ge k_0(\gamma).$$

Consequently, in both Case 1 and Case 2, (6) turns to be true.

By the same reasoning as above, we obtain that if

$$V(t) \le v(Kz_j) \;\; (j = 1, 2, ..., N-1) \;\; \textit{for all} \quad t \ge T_j + \frac{v(2HK)}{\gamma},$$

then there exists $T_{j+1} > T_j + v(2HK)/\gamma$ such that

$$V(t) \le v(Kz_{j+1}) \quad \textit{for all} \quad t \ge T_{j+1} + \frac{v(2HK)}{\gamma}.$$

Finally,

$$V(t) \le v(Kz_N) \quad \textit{for all} \quad t \ge T_{N+1}.$$

Thus, we have

$$u(|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})|) \le V(t) \le v(Kz_N) \le u(\alpha(z_N)).$$

Therefore,

$$|D(\xi_t - \xi_{t+\alpha_m-\alpha_k})| \le \alpha(z_N).$$

Applying Lemma 4, there is a $T^* > T_{N+1}$ such that

$$|\xi_t - \xi_{t+\alpha_m-\alpha-k}| \le \beta(z_N) < \varepsilon \quad \text{for} \quad t \ge T^*.$$

Then,

$$|\xi(t) - \xi(t+\alpha_m-\alpha_k)| \le \varepsilon, \tag{8}$$

for all $t \geq T^*, m \geq k \geq k_0$. We can select $l_0 \geq k_0$ such that $a_{l_0} \geq T^*$. Therefore, (8) implies

$$|\xi(t+\alpha_k) - \xi(t+\alpha_m)| \leq \varepsilon, \quad t \in R^+ m \geq k \geq l_0.$$

Thus, $\xi(t)$ is an asymptotially almost periodic solution of Eq.(1). Applying Lemma 2, Eq.(1) has an almost periodic solution p with $p(t) \in C_H$ for $t \in R$.

Similarly to the proof above, we can obtain that p is quasi- uniformly asymptotically stable. At last, we prove that p is uniformly stable. For any $\varepsilon \geq 0$ and $t_0 \in R$, let $\delta_1 > 0$ so that $\beta(\delta_1) < \varepsilon$. Denote

$$\delta := \frac{1}{b}(\beta(\delta_1) - b\alpha(\delta_1)) > 0.$$

We will prove that when $|\phi - p_{t_0}| < \delta$, we have

$$V(t_1, Dx_t, Dp_t) \leq v(K\delta_1), \quad t \geq t_0,$$

where $x(t) := x(t_0, \phi)(t)$. Suppose that there is a $t_1 > t_0$, such that

$$V(t, Dx_{t_1}, Dp_{t_1}) = v(K\delta_1)$$

and

$$V(t, Dx_t, Dp_t) \leq v(K\delta_1) \quad \textit{for} \quad t_0 \leq t \leq t_1.$$

Then,

$$u(|D(x_t - p_t)|) \leq V(t, Dx_t, Dp_t) \leq v(K\delta_1) \leq u(\alpha(\delta_1)), t_0 \leq t \leq t_1.$$

Therefore, $|D(x_t - p_t)| \leq \alpha(\delta_1)$. From inequality (2), we have

$$\begin{aligned} |x_{t_1} - p_{t_1}| &\leq be^{-a(t_1 - t_0)}|\phi - p_{t_0}| + b \sup_{t_0 \leq u \leq t_1} |D(x_u - p_u)| \\ &\leq b\delta + b\alpha(\delta_1) = \beta(\delta_1). \end{aligned}$$

Consequently,

$$\begin{aligned} F(V(t_1, Dx_{t_1}, Dp_{t_1})) &= F(v(K\delta_1)) > v(\beta(\delta_1)) \geq v(|x_{t_1} - p_{t_1}|) \\ &\geq V(t_1 + \theta, x(t_1 + \theta), p(t_1 + \theta)), \quad -\tau \leq \theta \leq 0. \end{aligned}$$

Then, from condition (iii), we have

$$V'(t_1, x_{t_1}, p_{t_1}) \leq -\omega(|D(x_{t_1} - p_{t_1})|) \leq 0.$$

Thus,

$$V(t, Dx_t, Dp_t)|) \leq v(K\delta_1), \quad t \geq t_0,$$

which yields

$$u(|D(x_t - p_t)|) \leq v(K\delta_1) \leq u(\alpha(\delta_1)).$$

That is,

$$|D(x_t - p_t)| \leq \alpha(\delta_1), \quad t \geq t_0.$$

It follows from (2) that

$$|x_t - p_t| \leq \beta(\delta_1) < \varepsilon,$$

and this implies that p is uniformly stable. Since p is asymptotically stable, it follows that for any almost periodic solution $\bar{p}(t)$ of Eq. (1), $|\bar{p}(t)| < H$ for $t \in R$, we have

$$|p(t) - \bar{p}(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Using the almost periodicity, we obtain $p(t) = \bar{p}(t)$ for all $t \in R$. This implies that Eq. (1) has only one almost periodic solution in C_H. And, from Lemma 2, we have $mod(p) \subset mod(f)$, completing the proof.

We conclude the paper with an example to illustrate the theorem.

Example. Consider the following equation

$$\frac{d}{dt}[x(t) - e^{-1}x(t-r)] = -x(t) + (p(t) - e^{-1}p(t-r))' + p(t), \tag{9}$$

where $r = \frac{1}{2}(1 - ln2)$, $p : R \to R$ is an almost periodic function such that p' is uniformly continous on R.

Let $V(x, y) = (x - y)^2, u(s) = v(s) = s^2, F(s) = A^2 s$, where $A > \frac{1}{(1-e^{-1})}$, $\alpha(\eta) = (1 + e^{-1})\eta, \beta(\eta) = \frac{e+1}{e-1}\eta$, and $\psi(t) = e^{-2t} + p(t)$ is a bounded solution of Eq.(9). Then it is easy to see that the conditions of the Theorem are satisfied, thus, Eq.(9) has a unique almost periodic solution $x(t) = p(t)$, which is uniformly asymptotically stable.

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