Continuity, Compactness, Fixed Points, and Integral Equations

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ABSTRACT. An integral equation, $x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds$ with $a(t)$ bounded, is studied by means of a Liapunov functional. There results an *a priori* bound on solutions. This gives rise to an interplay between continuity and compactness and leads us to a fixed point theorem of Schaefer type. It is a very flexible fixed point theorem which enables us to show that the solution inherits properties of $a(t)$, including periodic or almost periodic solutions in a Banach space.

Key words: Continuity, compactness, fixed points, integral equations, Liapunov functionals.

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1. Introduction

Consider the equation

(1) $$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds$$

with a, D, D_{st}, and g continuous. Suppose also that there exist $A > 0$, $B > 0$, and $G > 0$ so that

(2) $$|a(t)| \leq A, xg(x) > 0 \text{ if } x \neq 0, G = \max_{0 \leq |x| \leq 1} |g(x)|,$$

(3) $$D(t,s) \geq 0, D_s(t,s) \geq 0, D_{st}(t,s) \leq 0, D(t,t) \leq B,$$

(4) $$\lim_{s \to -\infty} (t-s)D(t,s) = 0 \text{ for each fixed } t,$$

and

$$\int_{-\infty}^{t}[D(t,s)+\Big(D_s(t,s)-D_{st}(t,s)\Big)(t-s)^2]ds \text{ is continuous .} \tag{5}$$

This is a direct generalization of the case $g(x) = x$ and $D(t,s) = e^{-(t-s)}$ which is reducible to a trivial ordinary differential equation.

Equation (1) has received a fair amount of attention in the literature as may be seen in [8]. In this context the reader may also consult [2-4], for example. The result which motivates the paper is that there is a constant L such that if ϕ is a solution and if for $t < 0$ it is bounded, then $|\phi(t)| < L$ for $-\infty < t < \infty$.

In the following we will study solutions of (1) on several different intervals: $(-\infty, 0]$, $[-\infty, \infty)$, $[t_0, \infty)$. We always mean that the solution satisfies Equation (1) on these intervals. In particular:

a) In the case of $[t_0, \infty)$, we begin with an initial function on $(-\infty, t_0)$; from that initial function we construct a solution satisfying (1) on $[t_0, \infty)$, while agreeing with the initial function on $(-\infty, t_0)$. There may be a discontinuity at t_0 and the initial function need not satisfy (1).

b) In the other two cases, the solution is its own initial function on any interval $(-\infty, t_0]$.

We prove our results in several steps. First we show that if there is a solution on $(-\infty, 0]$ then it can be extended to $(-\infty, \infty)$ and it is bounded by a constant L. Then we find a solution of (1) on $(-\infty, 0]$ using a fixed point theorem. Finally we obtain a solution on $(-\infty, \infty)$ in one step.

Thus, if $(X, \|\cdot\|)$ is the space of bounded continuous functions on $(-\infty, 0]$ with the supremum norm and if

$$M = \{\phi \in X | \|\phi\| \leq L\}, \tag{6}$$

then to find such a solution is to find a fixed point of the mapping $P : M \to X$ defined by $\phi \in M$ implies

$$(P\phi)(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(\phi(s))ds, \qquad -\infty < t \leq 0. \tag{7}$$

When we consider known fixed point theorems, we are led to the hypothesis that PM be contained in a compact subset of X. And that seems impossible, even though PM may be equicontinuous. There then arises a very interesting interplay between continuity of P and compactness of PM, as well as several other consequences which we now enumerate.

(i) Under general conditions including (5), it is possible to find a continuous function $h : (-\infty, 0] \to [1, \infty)$ with $h(t)$ tending monotonically to ∞ as $t \to -\infty$ so that P is continuous on M in the norm defined by

$$|\phi|_h := \sup_{t\leq 0} |\phi(t)|/h(t). \tag{8}$$

If Y is the set of continuous functions ψ on $(-\infty, 0]$ for which $|\psi|_h < \infty$, then PM may be contained in a compact subset of $(Y, |\cdot|_h)$.

(ii) If we consider the classical fixed point theorem of Schaefer [10] we see that it can be extended in a simple way to cover the present situation. Immediately, we find that P has a fixed point in M so that (1) has a solution ϕ which is bounded on $(-\infty, 0]$ and so $|\phi(t)| < L$ on $(-\infty, \infty)$.

(iii) But there is another useful conclusion. Given any property S of bounded continuous functions, if a convex and complete (in the topology of $(Y, |\cdot|_h)$) subset M^* of M is required to satisfy property S and if $\phi \in M^*$ implies $P\phi$ also has property S, then the fixed point also has property S. Under the conditions of our basic theorem, two interesting examples are:

(a) If there is a $T > 0$ with $a(t + T) = a(t)$ and $D(t + T, s + T) = D(t, s)$ then the fixed point is also $T-$periodic.

(b) If $a(t)$ is almost periodic in the space $(Y, |\cdot|_h)$ and if $D(t, s) = D(t - s)$, then the fixed point is almost periodic.

2. A fixed point theorem

Let $h : (-\infty, 0] \to [1, \infty)$ be a continuous decreasing function with $h(0) = 1$ and $h(r) \to \infty$ as $r \to -\infty$. Let

$$(Y, |\cdot|_h)$$

be the Banach space of continuous $\phi : (-\infty, 0] \to R^n$ for which

$$|\phi|_h := \sup_{t\leq 0} |\phi(t)|/h(t) < \infty,$$

and let

$$(X, \|\cdot\|)$$

be the Banach space of bounded continuous $\phi : (-\infty, 0] \to R^n$ with the supremum norm.

We will need a fixed point theorem patterned after the following result of Schaefer [10]. Our formulation here is that of Smart [9; p. 29].

THEOREM (Schaefer). Let B be a normed space, P a continuous mapping of B into B which is compact on each bounded subset K of B. Then either

(i) the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or

(ii) the set of solutions $\{x|$ there exists $\lambda \in (0, 1)$, for which $x = \lambda Px\}$ is unbounded.

That theorem will fail us here owing to problems with continuity, compactness, norms, and global considerations. The following modification fits our circumstances.

THEOREM 1. Let $L > 0$ and $J > 0$ be constants,

$$M = \{\phi \in X | \|\phi\| \leq L\},$$

and let $P : M \to X$ satisfy:

(i) P is continuous in $|\cdot|_h$.

(ii) For each $\phi \in M$, $|(P\phi)(t_1) - (P\phi)(t_2)| \leq J|t_1 - t_2|$ for $-\infty < t_2 < t_1 \leq 0$.

(iii) If $0 \leq \lambda \leq 1$ and if $\phi = \lambda P\phi$ for $\phi \in M$, then $\|\phi\| < L$.

Then P has a fixed point in M.

Proof. Define $H : M \to M$ by

a) $H\phi = P\phi$ if $\|P\phi\| \leq L$

and

b) $H\phi = \lambda P\phi$, $0 < \lambda < 1$, $\|\lambda P\phi\| = L$ if $\|P\phi\| > L$.

Since P is continuous in $|\cdot|_h$, so is H. Since $h(r) \to \infty$ as $r \to -\infty$ and (ii) holds, HM is contained in a compact subset of M in the space $(Y, |\cdot|_h)$ (cf. [1; p.169.]). By Schauder's second theorem (Smart [9; p. 25]) H has a fixed point: $H\phi = \phi$ or $\lambda P\phi = \phi$. If $\|P\phi\| \leq L$ then $\lambda = 1$ and P has the fixed point as claimed. If $\|P\phi\| > L$ then $\|\lambda P\phi\| = L = \|\phi\|$, and this contradicts (iii).

COROLLARY. Let the conditions of Theorem 1 hold and let M^* be a convex subset of M. Suppose that for H defined in the proof of Theorem 1 we have $H : M^* \to M^*$ and that if f_n is a sequence in M^* and if there is an $f \in Y$ such that

$$\lim_{n\to\infty} |f_n - f|_h = 0$$

then $f \in M^*$. Under these conditions P has a fixed point in M^*.

The function h plays a central role in what can be said about solutions and the first thing we want to do is show what its properties must be.

LEMMA 1. Suppose there are positive constants C and L so that $|x_i| \le L$ for $i = 1, 2$ imply that $|g(x_1) - g(x_2)| \le C|x_1 - x_2|$. Suppose also that there is an $E > 0$ and a continuous $h : (-\infty, 0] \to [1, \infty)$ such that $h(0) = 1, h(t) \to \infty$ as $t \to -\infty$ so that

$$\sup_{-\infty<t\le 0} \int_{-\infty}^{t} [D(t,s)h(s)/h(t)]ds \le E. \tag{9}$$

holds. Let P be defined in (7) and $M = \{\phi \in X | \|\phi\| \le L\}$. Then P is continuous on M in $|\cdot|_h$.

Proof. For a given $\varepsilon > 0$ we must find $\delta > 0$ such that $[\phi, \psi \in M, |\phi - \psi|_h < \delta]$ imply that $|P\phi - P\psi|_h < \varepsilon$. As $\phi, \psi \in M$ we have

$$|g(\phi(s)) - g(\psi(s))| \le C|\phi(s) - \psi(s)|$$

for $-\infty < s \le 0$. Hence, $-\infty < t \le 0$ implies that

$$\begin{aligned} |(P\phi)(t) - (P\psi(t))|/h(t) &\le \int_{-\infty}^{t} D(t,s)C|\phi(s) - \psi(s)|/h(t)ds \\ &\le [C/h(t)]|\phi - \psi|_h \int_{-\infty}^{t} D(t,s)h(s)ds \\ &\le C|\phi - \psi|_h E. \end{aligned}$$

For the given ε we take $\delta \le \varepsilon/CE$. This completes the proof.

Given (1), we must find an appropriate h and constant E so that (9) holds. The reader should experience little difficulty in constructing a suitable h when D is an elementary function satisfying (5). In the Appendix we show that it can always be done in the convolution case, $0 \le D(t,s) = D(t-s)$.

The fixed point is a bounded continuous function. A main goal is to refine the solution and those refinements take place in the space $(Y, |\cdot|_h)$ as indicated in the corollary above.

We are interested in periodc and almost periodic solutions which must be defined in that norm. This will be treated in Example 2 of Section 4.

3. A Uniform Bound

Equation (1) can have an initial function $\phi : (-\infty, t_0] \to R$; in that case, in order for the solution to be continuous at t_0 it is required that

$$\phi(t_0) = a(t_0) - \int_{-\infty}^{t_0} D(t_0, s)g(\phi(s))ds.$$

If ϕ is bounded and continuous, then (5) implies that this integral exists. On the other hand, (1) may have a solution ψ on all of R; in that case, ψ is its own initial function on any interval $(-\infty, t_0]$. Suitable existence theory is found in [5].

The Liapunov functional used below was introduced in [2] and has been further studied in [3] and [4].

THEOREM 2. Let (2) - (5) hold. Then there exists a positive constant J such that if $x(t)$ is a continuous solution of (1) on an interval $[t_0, \infty)$ with bounded continuous initial function ϕ, then for

$$V(t, x(\cdot)) = \int_{-\infty}^{t} D_s(t, s)\left(\int_s^t g(x(v))dv\right)^2 ds \tag{10}$$

we have

$$V'(t, x(\cdot)) \le -x(t)g(x(t)) + J|a(t)| \tag{11}$$

and

$$(a(t) - x(t))^2 \le D(t, t)V(t, x(\cdot)). \tag{12}$$

Proof. A computation yields

$$\begin{aligned} V'(t, x(\cdot)) =& \int_{-\infty}^{t} D_{st}(t, s)\left(\int_s^t g(x(v))dv\right)^2 ds + 2g(x)\int_{-\infty}^{t} D_s(t, s)\int_s^t g(x(v))dvds \\ \le& 2g(x)\left[D(t, s)\int_s^t g(x(v))dv\,\Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} D(t, s)g(x(s))ds\right] \\ =& 2g(x)[a(t) - x(t)] = -xg(x) + g(x)[2a(t) - x(t)] \end{aligned}$$

where we have used the boundedness of the initial function and (4) to conclude that the first term in the second line of the above inequality is zero. Now if $|x| \leq 2A$ then

$$g(x)[2a(t) - x(t)] \leq 2g(x)a(t) \leq 2|a(t)| \sup_{|x|\leq 2A} |g(x)| =: J|a(t)|$$

while for $|x| > 2A$ we find that $2a(t) - x(t)$ has the same sign as $-x(t)$ and $g(x)$ has the opposite sign, therefore $g(x)[2a(t) - x(t)] < 0$. The existence of the first integrals shown in V' follows from (5).

For the lower bound on V we set

$$\begin{aligned} Q(t) :=& \left(\int_{-\infty}^{t} D_s(t,s)ds \int_s^t g(x(v))dvds \right)^2 \\ \leq& \int_{-\infty}^{t} D_s(t,s)ds \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t g(x(v))dv\right)^2 ds \\ =& D(t,t)V(t,x(\cdot)). \end{aligned}$$

where we have taken $\lim_{s\to-\infty} D(t,s) = 0$ using (4).

On the other hand,

$$\begin{aligned} Q(t) =& \left(D(t,s) \int_s^t g(x(v))dv \Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} D(t,s)g(x(s))ds \right)^2 \\ =& (a(t) - x(t))^2. \end{aligned}$$

This completes the proof.

THEOREM 3. Let (2) - (5) hold and let $x(t)$ be a bounded, continuous solution of (1) on an interval $(-\infty, t_1]$. If, in addition, there is a $K > 0$ with

$$\int_{-\infty}^{t} D_s(t,s)[(t-s)^2 + 1]ds \leq K \qquad (t \in (-\infty,\infty)) \tag{13}$$

then $x(t)$ can be defined on all of R and

$$(a(t) - x(t))^2 \leq B[1 + 2((G + JA)^2 + 1)K]. \tag{14}$$

Proof. If we obtain the bound (14) on the solution so long as it can be defined, it will follow from existence theory [1; p. 193] that it can be defined for all t.

Since x is bounded and (13) holds, $V(t) = V(t, x(\cdot))$ is bounded on the interval $(-\infty, t_1]$. Then we have two cases.

1. If $V(t)$ is bounded as long as $x(t)$ can be defined, then there exists a t^* such that $V(s) \le V(t^*) + 1$ for all s where $x(s)$ is defined.
2. If $V(t)$ is not bounded, then we have $\limsup_{s\to\infty} V(s) = \infty$, and hence for all t we can find a $t^* \ge t$ such that $V(s) \le V(t^*)$ for all $s \le t^*$.

In both cases we obtained, that for all t we can find a t^* such that $V(t) \le V(t^*) + 1$ and $V(s) \le V(t^*) + 1$ for all $s \le t^*$. Then we have

$$-1 \le V(t^*) - V(s) \le -\int_s^{t^*} |x(u)g(x(u))|du + JA(t^* - s),$$

and hence

$$\begin{aligned}
(a(t) - x(t))^2 \le& D(t,t)V(t) \\
\le& D(t,t)[V(t^*) + 1] \\
\le& B\left[1 + \int_{-\infty}^{t^*} D_s(t^*, s)\left(\int_s^{t^*} [G + |x(u)g(x(u))|]du\right)^2 ds\right] \\
\le& B\left[1 + \int_{-\infty}^{t^*} D_s(t^*, s)(G(t^* - s) + JA(t^* - s) + 1)^2 ds\right] \\
=& B\left[1 + 2(G + JA)^2 \int_{-\infty}^{t^*} D_s(t^*, s)(t^* - s)^2 ds + 2\int_{-\infty}^{t^*} D_s(t^*, s)ds\right] \\
\le& B[1 + 2[(G + JA)^2 + 1]K].
\end{aligned}$$

4. A summary

It is easy to verify that if in (1) we replace D by λD and $a(t)$ by $\lambda a(t)$, then (10) - (12) and (14) contain only those changes. Thus, the conclusion of Theorem 3 is valid for

$$(1_\lambda) \qquad x(t) = \lambda[a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds]$$

for $0 \le \lambda \le 1$.

I. Let the conditions of Theorem 3 hold: (2) - (5). Then from (14) we conclude that there is an L such that if (1_λ) has a solution ϕ bounded on $(-\infty, 0]$, then $|\phi(t)| \le L$ on all of R. Condition (iii) of Theorem 1 is satisfied.

II. Let the conditions of the Lemma hold. That is, (9) is valid and there is a constant $C > 0$ so that $|x_i| \le L$ imply that $|g(x_1) - g(x_2)| \le C|x_1 - x_2|$. Then P defined by (7) is continuous on

$$M = \{\phi \in X | \|\phi\| \le L\} \tag{16}$$

in the norm $|\cdot|_h$. Condition (i) of Theorem 1 is satisfied.

III. Let (2) - (5) hold and suppose that a satisfies a Lipschitz condition. In addition, let

$$\int_{-\infty}^{t} |D_t(t,s)| ds \tag{17}$$

be bounded. Then we can obtain a bound on the derivative of the integral in (7) for $\phi \in M$ and conclude that (ii) of Theorem 1 holds: $P\phi$ is Lipschitz on M. Condition (ii) of Theorem 1 is satisfied.

Thus, if I, II, III hold, then (1) has a solution in M; to this point, we then say that (1) has a solution bounded on R.

We now focus on the corollary to Theorem 1. The idea is to refine the result that we have a bounded solution. If we add a condition S to M and if $\phi \in M$ implies $P\phi$ satisfies S then there may be a solution of (1) also satisfying S. When we do this we frequently want a condition to hold on all of R instead of just on $(-\infty, 0]$. We can revise the above work in a simple way. In Section 2 leave h as stated, except that it be an even function. Then

$$(Y, |\cdot|_h) \tag{18}$$

is the Banach space of continuous $\phi : R \to R^n$ for which

$$|\phi|_h := \sup_{t \in R} |\phi(t)|/h(t) < \infty \tag{19}$$

and $(X, \|\cdot\|)$ is the Banach space of bounded continuous $\phi : R \to R^n$ with the supremum norm. Then M and P are defined with these changes and we get a fixed point in M.

EXAMPLE 1. Let I, II, III hold. If there is a $T > 0$ so that $a(t + T) = a(t)$ and if $D(t + T, s + T) = D(t,s)$, then (1) has a T-periodic solution.

With the corollary to Theorem 1 in mind, we take M^* as the subset of M consisting of continuous $T-$periodic functions. A sequence in M^* with limit in the norm $|\cdot|_h$ is certainly periodic. It is easy to verify that $\phi \in M$ implies $P\phi$ is periodic.

The existence of a bounded solution was new, but the existence of a periodic solution had been obtained in [4] by different methods. The next result seems to be entirely new.

EXAMPLE 2. Let I, II, III hold. If $a(t)$ is almost periodic (AP) in a sense to be defined and if $D(t,s) = D(t-s)$ then we obtain conditions to ensure that (1) has an AP solution.

We will define almost periodic functions parallel to that of Corduneanu [7; p. 153].

DEFINITION 2. A continuous function $f : R \to R$ is said to be almost periodic if for any $\varepsilon > 0$ there is an $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least one number τ with $|f(t+\tau) - f(t)|/h(t) < \varepsilon$ for all $t \in R$.

We now need a result which will allow us to use the corollary of Theorem 1 in Section 2 for almost periodic functions in a Banach space.

DEFINITION 3. A family F of AP functions in $(Y, |\cdot|_h)$ is said to be equi-AP if for each $\varepsilon > 0$ there is an $l > 0$ such that any interval on R of length l contains at least one τ which is an ε-translation number for every member of F: if $f \in F$ then $|f(t+\tau) - f(t)| < \varepsilon h(t)$ for all $t \in R$.

For this discussion, refer back to I and II where L, M, and C are fixed. In particular, we can assume without loss of generality that

$$A \leq L$$

so that we know that $a \in M$.

Let $a(t)$ in (1) be AP in Y, let P be defined by (7) for all t, and let C be the Lipschitz constant for g on M. We now specify the set M^* of the corollary to Theorem 1. Since $a(t) \in AP$, for each $\varepsilon > 0$ there is an $l(\varepsilon)$ such than any interval of length l contains at least one τ which is an $\varepsilon-$translation number of $a(t)$. More precisely: for each $\varepsilon > 0$ there is an $l(\varepsilon)$ such that for all $c \in R$ there exists $\tau_{\varepsilon,c} \in [c, c+l]$ such that $|a(t+\tau_{\varepsilon,c}) - a(t)| < \varepsilon h(t)$. Now, for each ε and c let us fix one $\tau_{\varepsilon,c}$. Then let $K \geq 1$ and define

$$M^{**} := \{x \in Y : |x(t+\tau_{\varepsilon,c}) - x(t)| \leq \varepsilon K h(t) \text{ for all } \varepsilon > 0, c \in R\}.$$

Certainly, the set is not empty since it contains $a(t)$. Moreover, it is convex: for $\phi, \psi \in M^{**}$, $0 \le k \le 1$ and $\xi = k\phi + (1-k)\psi$ we obtain the relation

$$\begin{aligned}|\xi(t+\tau_{\varepsilon,c}) - \xi(t)| &= |k\phi(t+\tau_{\varepsilon,c}) + (1-k)\psi(t+\tau_{\varepsilon,c}) - k\phi(t) - (1-k)\psi(t)| \\ &= |k(\phi(t+\tau_{\varepsilon,c}) - \phi(t)) + (1-k)(\psi(t+\tau_{\varepsilon,c}) - \psi(t))| \\ &\le k\varepsilon Kh(t) + (1-k)\varepsilon Kh(t) = \varepsilon Kh(t).\end{aligned}$$

Define

$$M^* = M^{**} \cap M.$$

THEOREM 4. Let $\{f_n\}$ be a sequence in M^*, let $f \in Y$, and let $|f_n - f|_h \to 0$ as $n \to \infty$. Then $f \in AP$ and $f \in M^*$.

Proof. Let $\varepsilon > 0$ and $c \in R$ be given. Then we have a fixed $\tau = \tau_{\varepsilon,c}$. Now, there is a sequence $\varepsilon_n \to 0$ with $|f(t) - f_n(t)| \le \varepsilon_n h(t)$. This yields

$$\begin{aligned}|f(t+\tau) - f(t)| &\le |f(t+\tau) - f_n(t+\tau)| + |f_n(t+\tau) - f_n(t)| + |f_n(t) - f(t)| \\ &\le \varepsilon_n h(t+\tau) + \varepsilon Kh(t) + \varepsilon_n h(t).\end{aligned}$$

Let $n \to \infty$ and obtain $|f(t+\tau) - f(t)| \le \varepsilon Kh(t)$, which proves the statement.

To prepare our mapping P defined in (7), we first extend that definition to all $t \in R$ and ask that

(20) $$a \in AP, \qquad D(t,s) = D(t-s)$$

and that there is an E with

(21) $$E = \sup_{t \in R} \int_0^\infty |D(u)|(h(t-u)/h(t))du.$$

REMARK. We are going to introduce a condition here, $EC < 1$ (C is the Lipschitz-contant for g defined in Lemma 1), which bears much study. One can verify that it will make P into a local contraction. Furthermore, one can argue that it may map a certain bounded set into itself so that the boundedness theorem for the integral equation might not even be needed. Such nice relations depend on very special relationships between L and C, so we avoid trying for a contraction mapping. But whether we use a contraction or our corollary to Theorem 1, the real problem here is one of completeness. Our condition offers a way to achieve that property.

LEMMA 2. Suppose that $EC < 1$ and let $K = 1/(1 - EC)$. If H is the function defined in Theorem 1 then $H : M^* \to M^*$.

Proof: For $\tau = \tau_{\varepsilon,c}$ and $\phi \in M^*$ we have

$$
\begin{aligned}
&|(P\phi)(t+\tau) - (P\phi)(t)|/h(t)\\
&\quad \le |a(t+\tau) - a(t)|/h(t) + \int_{-\infty}^{t} [|D(t-s)(g(\phi(s+\tau)) - g(\phi(s))|/h(t)]ds\\
&\quad \le \varepsilon + \int_{-\infty}^{t} |D(t-s)C\varepsilon Kh(s)/h(t)|ds\\
&\quad \le \varepsilon + \varepsilon CKE = (1 + CKE)\varepsilon = K\varepsilon.
\end{aligned}
$$

This proves that P maps M^* into M^{**}. It then follows that $H : M^* \to M^*$.

Using the corollary to Theorem 1 we can say:

THEOREM 5. Suppose that I, II, III hold, as well as (19) - (21) and the conditions of Lemma 1. Then (1) has an AP solution.

There are many other special properties of $a(t)$ which the solution will inherit in exactly the same way. A basic idea throughout the study of integral equations is that the solution should follow $a(t)$ under very general conditions. Our result shows how to verify such properties.

4. Appendix

We now show the details for constructing h when $0 \le D(t,s) = D(t-s)$ with $\int_0^\infty D(u)du < \infty$ so that (9) will be satisfied. Under these conditions on D, in Burton-Grimmer [6; pp. 207-8] there is constructed a continuous increasing function $p_2(t)$ tending to infinity with t so that $\int_0^\infty D(u)p_2(u)du < \infty$. Referring to that argument, we can construct a sequence $\{t_n\} \to \infty$ (with $t_0 = 0$) and a function h so that $t_{n+1} - t_n \to \infty$ with n, $h(t_n) = 2^n$, h is linear on the intervals $[t_n, t_{n+1}]$ and $\int_0^\infty D(u)h(u)du < \infty$. First of all we see, that

$$t_{k+l} = t_k + (t_{k+1} - t_k) + ... + (t_{k+l} - t_{k+l-1}) \ge t_k + (t_1 - t_0) + ... + (t_l - t_{l-1}) = t_k + t_l$$

Let $a \in (t_{k-1}, t_k]$ and $b \in (t_{l-1}, t_l]$ be arbitrary numbers, then

$$h(a+b) \le h(t_k + t_l) \le h(t_{k+l}) = 2^{k+l} \le 4 \cdot 2^{k-1} \cdot 2^{l-1} = 4h(t_{k-1})h(t_{l-1}) \le 4h(a)h(b)$$

Now, let h be an even function and have

$$\int_{-\infty}^{t} [D(t-s)h(s)/h(t)]ds = \int_{0}^{\infty} [D(u)h(t-u)/h(t)]du$$
$$\leq \int_{0}^{\infty} [D(u)4h(-u)h(t)/h(t)]du < \infty.$$

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