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Multiplicity of positive solutions for a class of nonlocal problem involving critical exponent

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Abstract. In this paper, we study the following critical nonlocal problem

$$\begin{cases} -\left(a-\lambda b\int_\Omega |\nabla u|^2dx\right)\Delta u=\lambda|u|^{p-2}u+Q(x)|u|^2u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases}$$

where $a>0$, $b\geq 0$, $2<p<4$, $\lambda>0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^4$ and $Q(x)\in C(\overline{\Omega})$ is a nonnegative function. By virtue of variational methods and delicate estimates, we prove that problem admits k positive solutions for $\lambda>0$ sufficiently small, provided that the maximum of $Q(x)$ is achieved at k interior points in Ω.

Keywords: nonlocal problem, variational methods, critical nonlinearity, multiple positive solutions.

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1 Introduction

In this paper, we concern with the multiplicity of positive solutions to the nonlocal problem

$$\begin{cases} -\left(a-\lambda b\int_\Omega |\nabla u|^2dx\right)\Delta u=\lambda|u|^{p-2}u+Q(x)|u|^2u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $a>0$, $b\geq 0$, $2<p<4$, $\lambda>0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^4$ ($2^*=4$ is the critical exponent in dimension four) and $Q(x)\in C(\overline{\Omega})$ is a nonnegative function satisfying:

(Q_1) There exist k different points $x^1,x^2,\ldots,x^k\in\Omega$ such that $Q(x^j)$ are strict local maximums and satisfy

$$Q(x^j)=Q_M=\max\{Q(x):x\in\overline{\Omega}\}>0,\quad j=1,2,\ldots,k;$$

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(Q_2) $Q_M - Q(x) = O(|x - x^j|^2)$ for x near x^j, $j = 1, 2, \ldots, k$.

In the past decade, the following Kirchhoff type problem involving critical growth on a bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x,u) + K(x)|u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

has attracted considerable attention, where $a, b > 0$ are constants, $2^* = 2N/(N-2)$ with $N \geq 3$ and $K(x)$ is a nonnegative continuous function. Kirchhoff type problem is often viewed as nonlocal due to the presence of the term $b\int_\Omega |\nabla u|^2 dx$ which implies that such problem is no longer pointwise identity. It is commonly known that Kirchhoff type problem has a mechanical and biological motivation, see [1,8]. Under different hypotheses on $g(x,u)$ and $K(x)$, there are many interesting results of positive solutions to (1.2) by using variational methods, see e.g. [6,7,15]. In particular, Fan [6] showed how the topology of the maximum set of $K(x)$ affects the number of positive solutions to (1.2) via Ljusternik–Schnirelmann category theory when $N = 3$ and $f(x,u) = f(x)u^q$ with $f(x) \in L^{\frac{6}{6-q}}(\Omega)$ and $3 < q < 5$. There are also several existence results for (1.2) in the whole space $\mathbb{R}^N$, see [5,11,12] and the references therein.

In (1.2), if we replace $a + b\int_\Omega |\nabla u|^2 dx$ by $a - b\int_\Omega |\nabla u|^2 dx$, it turns to be a new nonlocal one. This kind of nonlocal problem presents some interesting difficulties different from Kirchhoff type problem. Such nonlocal problem with subcritical growth

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f_\lambda(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

has been studied by some researchers, where $f_\lambda(x) \in L^{\frac{2^*}{2^*-p}}(\Omega)$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. If $f_\lambda(x) \equiv 1$ and $2 < p < 2^*$, Yin and Liu [23] obtained two nontrivial solutions to (1.3); Qian [18] proved the existence and asymptotic behavior of ground state sign-changing solutions for (1.3); Wang et al. [22] proved that (1.3) has infinitely many sign-changing solutions. For $1 \leq p < 2^*$, Duan et al. [4] established the existence of multiple positive solutions to (1.3). In [10], the multiplicity result of positive solutions to (1.3) was obtained for $0 < p < 1$. When $f_\lambda(x)$ has indefinite sign, Lei et al. [9] and Qian and Chao [16] proved the existence of positive solution to (1.3) for $1 < p < 2$ and $3 < p < 6$, respectively. For more results about (1.3) with general nonlinearities and its variants on unbounded domain, we refer the interested readers to [19,20,24]. To the best of our knowledge, there is little result for (1.3) when $f(x,u)$ exhibits a critical exponent. Only Wang et al. [21] investigated the existence of two positive solutions for the following problem involving critical exponent

$$\begin{cases} -\left(a - b\int_{\mathbb{R}^4} |\nabla u|^2 dx\right)\Delta u = \lambda g(x) + |u|^2 u, x \in \mathbb{R}^4, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^4), \end{cases}$$

under the assumptions $\lambda > 0$ is sufficiently small and $g(x) \in L^{4/3}(\mathbb{R}^4)$ is a nonnegative function.

When $a = 1$, $b = 0$, $\mathbb{R}^4$ and $Q(x)|u|^2u$ are replaced by $\mathbb{R}^N$ and $Q(x)|u|^{2^*-2}u$, respectively, (1.1) is reduced to the following local one

$$\begin{cases} -\Delta u = \lambda|u|^{p-2}u + Q(x)|u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

which does not depend on the nonlocal term $\int_\Omega |\nabla u|^2 dx$ any more. The study by Cao and Noussair [3] is the first to investigate the effect of the shape of the graph of $Q(x)$ on the number of positive solutions to (1.4) with $p = 2$. More precisely, they proved that for small enough $\lambda > 0$, (1.4) has k positive solutions if the maximum of $Q(x)$ is achieved at exactly k different points of Ω, by applying Nehari manifold method. Liao et al. [13] extended the result of [3] in the sense that a more wider range of p is covered. In [17], Qian and Chen got a similar but more complicated result for (1.4) with an additional fast increasing weight.

Motivated by the idea of [3, 6, 21], it is natural and interesting to ask: can we apply the shape of the graph of $Q(x)$ to prove the multiplicity of positive solutions for the critical nonlocal problem (1.1) as in Kirchhoff problem (1.2)? In the present paper, we will give a positive answer to this question.

Our main results can be stated as follows.

Theorem 1.1. *Assume that $a > 0$, $b \geq 0$, $2 < p < 4$ and Ω is a smooth bounded domain in $\mathbb{R}^4$. If the conditions (Q_1) and (Q_2) hold, then there exists $\Lambda_0 > 0$, such that for each $\lambda \in (0, \Lambda_0)$, (1.1) has at least k positive solutions.*

Since the result of Theorem 1.1 still holds for $b = 0$, then we obtain the following corollary related to the multiplicity result of positive solutions for a semilinear problem with critical exponent.

Corollary 1.2. *Assume that $a > 0$, $2 < p < 4$ and Ω is a smooth bounded domain in $\mathbb{R}^4$. If the conditions (Q_1) and (Q_2) hold, then there exists $\Lambda_1 > 0$, such that for each $\lambda \in (0, \Lambda_1)$, the problem*

$$\begin{cases} -a\Delta u = \lambda|u|^{p-2}u + Q(x)|u|^2u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

has at least k positive solutions.

Associated with (1.1), we define the functional I_λ on $H_0^1(\Omega)$ by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 - \frac{\lambda b}{4}\|u\|^4 - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{4}\int_\Omega Q(x)|u|^4 dx,$$

where $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. Then $I_\lambda \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$. Moreover, there exists a one to one correspondence between the critical points of I_λ on $H_0^1(\Omega)$ and the weak solutions of (1.1). Here, we say that u is a weak solution of (1.1), if $u \in H_0^1(\Omega)$ and for all $v \in H_0^1(\Omega)$, there holds

$$(a - \lambda b\|u\|^2)\int_\Omega \nabla u \nabla v dx - \lambda\int_\Omega |u|^{p-2}uv dx - \int_\Omega Q(x)|u|^2 uv dx = 0.$$

The proof of Theorem 1.1 is based on variational methods. Since (1.1) has a negative nonlocal term, the approaches used in [6] to deal with Kirchhoff problem do not work here. Indeed, we shall apply the ideas introduced by Cao and Noussair [3]. However, in the present paper, there are some new difficulties caused by the competing effect of the nonlocal term

with the nonlinear terms and the non-compactness due to the critical exponent. To overcome these difficulties, we need to add the factor λ of $|u|^{p-2}u$ to the nonlocal term $-b\int_\Omega |\nabla u|^2 dx$ in problem (1.1). This modification will play an important role in our arguments (see Lemma 2.2 below). Moreover, inspired by [21], we consider our problem in dimension 4 and make some delicate estimates in order to get the compactness condition. We also point out that it is not clear whether the multiplicity result in Theorem 1.1 still holds for critical problem (1.1) in other dimension, from which it follows that the critical exponent 2^* is no longer equal to 4.

In Section 2, we present some lemmas which will be used to prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1.

2 Notations and preliminaries

Throughout the paper, for simplicity we write $\int u$ instead of $\int_\Omega u(x)dx$. $H_0^1(\Omega)$ and $L^r(\Omega)$ are the usual Sobolev spaces equipped with the standard norms $\|u\|$ and $|u|_r$, respectively. $\mathcal{D}^{1,2}(\mathbb{R}^4) = \{u \in L^4(\mathbb{R}^4) : \nabla u \in L^2(\mathbb{R}^4)\}$. Denote by $B_r(x)$ the ball centered at x with radius $r > 0$. Let $\overline{B}_r(x)$ and $\partial B_r(x)$ denote the closure and the boundary of $B_r(x)$, respectively. We use $\to$ ($\rightharpoonup$) to denote the strong (weak) convergence. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C$ as $\varepsilon \to 0$, and $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$. C and C_i denote various positive constants whose exact values are not essential. Let S be the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, that is,

$$S = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int |\nabla u|^2}{\left(\int |u|^4\right)^{1/2}}.$$

The Nehari manifold corresponding to I_λ is defined by

$$\mathcal{M}_\lambda = \{u \in H_0^1(\Omega) \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

By the condition (Q_1), we can take $\eta > 0$ sufficiently small such that $B_{2\eta}(x^j) \subset \Omega$ are disjoint and $Q(x) < Q(x^j)$ for $x \in \overline{B}_{2\eta}(x^j) \setminus \{x^j\}$, $j = 1, 2, \dots, k$. Following the argument of [3], we define a barycenter map $\beta : H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^4$ by setting

$$\beta(u) = \frac{\int x|u|^4}{\int |u|^4}.$$

With the help of the map above, we will first separate the Nehari manifold $\mathcal{M}_\lambda$, then study minimization problems of I_λ on its proper subset. We point out that, a key role of β is to insure that the minimizers of the considered minimization problems are distinct.

For $j = 1, 2, \dots, k$, we consider the following subsets of $\mathcal{M}_\lambda$,

$$\mathcal{M}_\lambda^j = \{u \in \mathcal{M}_\lambda : \beta(u) \in B_\eta(x^j)\} \quad \text{and} \quad \mathcal{O}_\lambda^j = \{u \in \mathcal{M}_\lambda : \beta(u) \in \partial B_\eta(x^j)\}.$$

Correspondingly, study the following minimization problems

$$m_\lambda^j = \inf_{u\in\mathcal{M}_\lambda^j} I_\lambda(u) \quad \text{and} \quad \widetilde{m}_\lambda^j = \inf_{u\in\mathcal{O}_\lambda^j} I_\lambda(u).$$

For all $\varepsilon > 0$ and $x_0 \in \mathbb{R}^4$, we define

$$U_{\varepsilon,x_0} = \frac{(8)^{1/2}\varepsilon}{(\varepsilon^2 + |x - x_0|^2)},$$

which solves $-\Delta u = |u|^2 u$ in $\mathbb{R}^4$. For $j = 1, 2, \ldots, k$ fixed, define a cut off function $\varphi_j \in C_0^\infty(\mathbb{R}^4)$ such that $0 \le \varphi_j \le 1$, $\varphi_j(x) = 1$ for $|x - x^j| < \rho$ and $\varphi_j(x) = 0$ for $|x - x^j| \ge 2\rho$ with $0 < \rho < \eta/2$. Let $u_{\varepsilon,j} = \varphi_j(x - x^j)U_{\varepsilon,x^j}(x)$. By [2], we have for $2 < p < 4$,

$$\|u_{\varepsilon,j}\|^2 = S^2 + O(\varepsilon^2),$$

$$|u_{\varepsilon,j}|_4^4 = S^2 + O(\varepsilon^4),$$

$$|u_{\varepsilon,j}|_p^p = O(\varepsilon^{4-p}).$$

Lemma 2.1. *For $j = 1, 2, \ldots, k$ and $\lambda > 0$, we have*

$$m_\lambda^j < \frac{a^2 S^2}{4(\lambda b S^2 + Q_M)}. \tag{2.1}$$

Proof. It is easy to see that there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon u_{\varepsilon,j} \in \mathcal{M}_\lambda$ and $I_\lambda(t_\varepsilon u_{\varepsilon,j}) = \sup_{t>0} I_\lambda(t u_{\varepsilon,j})$. By the symmetry of $u_{\varepsilon,j}$ about x^j, we further obtain $t_\varepsilon u_{\varepsilon,j} \in \mathcal{M}_\lambda^j$. Thus, to complete the proof of lemma, it suffices to prove that

$$\sup_{t>0} I_\lambda(t u_{\varepsilon,j}) < \frac{a^2 S^2}{4(\lambda b S^2 + Q_M)}. \tag{2.2}$$

At this point, we can suppose that $t_\varepsilon \ge C_1 > 0$ for any $\varepsilon > 0$ small. Otherwise, there is a sequence $\varepsilon_n \to 0^+$ such that $t_{\varepsilon_n} \to 0$. By the continuity of I_λ and the boundedness of $\{u_{\varepsilon_n,j}\}$,

$$\sup_{t>0} I_\lambda(t u_{\varepsilon_n,j}) = I_\lambda(t_{\varepsilon_n} u_{\varepsilon_n,j}) \to 0 < \frac{a^2 S^2}{4(\lambda b S^2 + Q_M)},$$

that is, the proof is complete. Similarly, we also suppose that $t_\varepsilon \le C_2$ for some positive constant C_2 and any $\varepsilon > 0$ small.

To proceed, set

$$h(t) = \frac{at^2}{2}\|u_{\varepsilon,j}\|^2 - \frac{\lambda b t^4}{4}\|u_{\varepsilon,j}\|^4 - \frac{t^4}{4}\int Q_M |u_{\varepsilon,j}|^4.$$

We easily see that $h(t)$ achieves its maximum at

$$\begin{aligned} t_{max} &= \left(\frac{a\|u_{\varepsilon,j}\|^2}{\lambda b\|u_{\varepsilon,j}\|^4 + Q_M|u_{\varepsilon,j}|_4^4}\right)^{1/2} \\ &= \left(\frac{aS^2 + O(\varepsilon^2)}{\lambda b S^4 + Q_M S^2 + O(\varepsilon^2)}\right)^{1/2} \\ &= \left(\frac{aS^2}{\lambda b S^4 + Q_M S^2}\right)^{1/2} + O(\varepsilon^2), \end{aligned}$$

with

$$h(t_{max}) = \frac{a^2 S^2}{4(\lambda b S^2 + Q_M)} + O(\varepsilon^2). \tag{2.3}$$

Using condition (Q_2), we also have

$$\int (Q_M - Q(x)) |u_{\varepsilon,j}|^4 = O(\varepsilon^2). \tag{2.4}$$

By (2.3) and (2.4),

$$\begin{aligned}
\sup_{t>0} I_\lambda(tu_{\varepsilon,j}) &= I_\lambda(t_\varepsilon u_{\varepsilon,j}) \\
&= h(t_\varepsilon) + \frac{t_\varepsilon^4}{4} \int (Q_M - Q(x)) |u_{\varepsilon,j}|^4 - \frac{\lambda}{p} t_\varepsilon^p \int |u_{\varepsilon,j}|^p \\
&\le h(t_{max}) + \frac{C_2^4}{4} \int (Q_M - Q(x)) |u_{\varepsilon,j}|^4 - \frac{\lambda}{p} C_1^p \int |u_{\varepsilon,j}|^p \\
&= \frac{a^2 S^2}{4(\lambda b S^2 + Q_M)} + O(\varepsilon^2) - O(\varepsilon^{4-p}).
\end{aligned}$$

Since $2 < p < 4$, (2.2) holds for $\varepsilon > 0$ small enough. This ends the proof. □

Lemma 2.2. *Assume that condition (Q_1) holds. Then there exists $\Lambda_0 > 0$ such that*

$$\widetilde{m}_\lambda^j > \frac{a^2 S^2}{4Q_M}$$

for $j = 1, 2, \ldots, k$, and $\lambda \in (0, \Lambda_0)$.

Proof. Let us argue by contradiction and suppose that there exist sequences $\lambda_n \to 0$, and $\{u_n\} \subset \mathcal{O}_{\lambda_n}^j$ satisfying

$$I_{\lambda_n}(u_n) \to c \le \frac{a^2 S^2}{4Q_M},$$

and

$$a \int |\nabla u_n|^2 - \lambda_n b \left(\int |\nabla u_n|^2 \right)^2 = \lambda_n \int |u_n|^p + \int Q(x) |u_n|^4. \tag{2.5}$$

By $\{u_n\} \subset \mathcal{O}_{\lambda_n}^j$, one has for n large,

$$\begin{aligned}
c + 1 &\ge I_{\lambda_n}(u_n) - \frac{1}{p} \langle I'_{\lambda_n}(u_n), u_n \rangle \\
&= a \left(\frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 + \lambda_n b \left(\frac{1}{p} - \frac{1}{4} \right) \|u_n\|^4 + \left(\frac{1}{p} - \frac{1}{4} \right) \int Q(x) |u_n|^4 \\
&\ge a \left(\frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Using (2.5) and Sobolev embedding, we also have

$$a\|u_n\|^2 = \lambda_n b \|u_n\|^4 + \lambda_n |u_n|_p^p + \int Q(x) |u_n|^4 \le \lambda_n b \|u_n\|^4 + \lambda_n C \|u_n\|^p + Q_M S^{-2} \|u_n\|^4$$

from which we infer that

$$\|u_n\| \ge C_3 > 0.$$

Noting that $\lambda_n \to 0$, we then deduce from (2.5) that there is a constant $C_4 > 0$ such that

$$\int Q(x)|u_n|^4 \geq C_4 > 0,$$

for all $n \in \mathbb{N}$. Thus, we are able to choose $t_n > 0$ such that $v_n = t_n u_n$ satisfies

$$a\int |\nabla v_n|^2 = \int Q_M |v_n|^4. \tag{2.6}$$

This and Sobolev inequality give that $\frac{aS^2}{Q_M} \leq \|v_n\|^2$. Moreover,

$$t_n = \left(\frac{\int Q(x)|u_n|^4 + \lambda_n b\left(\int |\nabla u_n|^2\right)^2 + \lambda_n \int |u_n|^p}{\int Q_M |u_n|^4} \right)^{1/2}.$$

It follows that $\{t_n\}$ is uniformly bounded. Then, we can assume $\lim_{n\to\infty} t_n = t_0$. By $Q(x) \leq Q_M$, $\lambda_n \to 0$ and the boundedness of $\{u_n\}$, we see that $t_0 \leq 1$. We show next that the case $t_0 \leq 1$ leads to a contradiction. Since for $t_0 \leq 1$, we have

$$\begin{aligned}
\frac{a^2S^2}{4Q_M} &\leq \lim_{n\to\infty} \frac{1}{4} a\int |\nabla v_n|^2 = \lim_{n\to\infty} \frac{1}{4} a t_n^2 \int |\nabla u_n|^2 \\
&= \lim_{n\to\infty} t_n^2 \left[\left(\frac{1}{2} - \frac{1}{4}\right) \left(a\int |\nabla u_n|^2 - \lambda_n b \left(\int |\nabla u_n|^2\right)^2 - \lambda_n \int |u_n|^p \right) \right. \\
&\qquad \left. + \lambda_n b \left(\frac{1}{2} - \frac{1}{4}\right) \left(\int |\nabla u_n|^2\right)^2 + \lambda_n \left(\frac{1}{2} - \frac{1}{p}\right) \int |u_n|^p \right] \\
&= \lim_{n\to\infty} t_n^2 I_{\lambda_n}(u_n) = t_0^2 c \leq c \leq \frac{a^2S^2}{4Q_M},
\end{aligned}$$

then it follows that

$$c = \frac{a^2S^2}{4Q_M} \quad \text{and} \quad \lim_{n\to\infty} \int |\nabla v_n|^2 = \frac{aS^2}{Q_M}. \tag{2.7}$$

Let $w_n = v_n/|v_n|_4$, then $|w_n|_4 = 1$. Moreover, by (2.6) and (2.7),

$$\lim_{n\to\infty} \int |\nabla w_n|^2 = \lim_{n\to\infty} \frac{\|v_n\|^2}{|v_n|_4^2} = \lim_{n\to\infty} \frac{\|v_n\|^2}{(a\|v_n\|^2/Q_M)^{1/2}} = S,$$

namely, $\{w_n\}$ is a minimizing sequence for S. According to [14], we can find a point $y_0 \in \overline{\Omega}$ such that

$$|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{y_0} \quad \text{and} \quad |w_n|_4 \rightharpoonup d\nu = \delta_{y_0} \tag{2.8}$$

with the above convergence holding weakly in the sense of measure, where δ_{y_0} is a Dirac mass at y_0. Then

$$\beta(u_n) = \frac{\int x|u_n|^4}{\int |u_n|^4} = \frac{\int x|v_n|^4}{\int |v_n|^4} = \frac{\int x|w_n|^4}{\int |w_n|^4} \to y_0, \quad \text{as } n \to \infty.$$

This together with $\beta(u_n) \in \partial B_\eta(x^j)$ imply that $y_0 \in \partial B_\eta(x^j)$. Thus, from (2.6) and (2.8), we conclude that

$$\begin{aligned}\lim_{n\to\infty} I_{\lambda_n}(u_n) &= \lim_{n\to\infty} t_n^2\left[\left(\frac{1}{2}-\frac{1}{4}\right)\left(a\int|\nabla u_n|^2-\lambda_n b\left(\int|\nabla u_n|^2\right)^2-\lambda_n\int|u_n|^p\right)\right.\\ &\qquad\left.+\lambda_n b\left(\frac{1}{2}-\frac{1}{4}\right)\left(\int|\nabla u_n|^2\right)^2+\lambda_n\left(\frac{1}{2}-\frac{1}{p}\right)\int|u_n|^p\right]\\ &\le \lim_{n\to\infty}\frac{1}{4}\int Q(x)|u_n|^4\\ &= \lim_{n\to\infty}\frac{1}{4}\int Q(x)|v_n|^4\\ &= \frac{Q(y_0)}{4Q_M}\lim_{n\to\infty}\int Q_M|v_n|^4\\ &= \frac{Q(y_0)}{4Q_M}\lim_{n\to\infty} a\int|\nabla v_n|^2\\ &= \frac{Q(y_0)}{4Q_M}\frac{a^2S^2}{Q_M} < \frac{a^2S^2}{4Q_M},\end{aligned}$$

which contradicts with (2.7). This completes the proof. □

Lemma 2.3. *For any $u \in \mathcal{M}_\lambda^j$, there exist $\rho > 0$ and a differential function $g = g(w)$ defined for $w \in H_0^1(\Omega)$, $w \in B_\rho(0)$ satisfying that*

$$g(0) = 1, \quad g(w)(u-w) \in \mathcal{M}_\lambda^j$$

and

$$\langle g'(0), \phi\rangle = \frac{(2a-4\lambda b\|u\|^2)\int\nabla u\nabla\phi-\lambda p\int|u|^{p-2}u\phi-4\int Q(x)|u|^2u\phi}{a\|u\|^2-3\lambda b\|u\|^4-\lambda(p-1)\int|u|^p-3\int Q(x)|u|^4}.$$

Proof. Define $F : \mathbb{R}^+ \times H_0^1(\Omega) \to \mathbb{R}$ by

$$F(t,w) = at\|u-w\|^2-\lambda bt^3\|u-w\|^4-\lambda t^{p-1}\int|u-w|^p-t^3\int Q(x)|u-w|^4.$$

By $u \in \mathcal{M}_\lambda^j$, we get $F(1,0) = 0$ and

$$\begin{aligned}F_t(1,0) &= a\|u\|^2-3\lambda b\|u\|^4-\lambda(p-1)\int|u|^p-3\int Q(x)|u|^4\\ &= a(2-p)\|u\|^2-\lambda b(4-p)\|u\|^4-(4-p)\int Q(x)|u|^4\\ &< 0.\end{aligned}$$

Thus, we can use the implicit function theorem for F at the point $(1,0)$ and obtain $\overline{\rho} > 0$ and a functional $g = g(w) > 0$ defined for $w \in H_0^1(\Omega)$, $\|w\| < \overline{\rho}$ satisfying that

$$g(0) = 1, \quad g(w)(u-w) \in \mathcal{M}_\lambda, \quad \forall\, w \in H_0^1(\Omega),\ \|w\| < \overline{\rho}.$$

By the continuity of the maps g and β, we can further take $\rho > 0$ possibly smaller ($\rho < \overline{\rho}$) such that

$$\beta\left(g(w)(u-w)\right) \in B_\eta(x^j), \quad \forall\, w \in H_0^1(\Omega),\ \|w\| < \rho,$$

which means that $g(w)(u-w) \in \mathcal{M}_\lambda^j$.

Moreover, we also have for all $\phi \in H_0^1(\Omega),\ r > 0$,

$$\begin{aligned}
&F(1,0+r\phi) - F(1,0)\\
&\quad = a\|u-r\phi\|^2 - \lambda b\|u-r\phi\|^4 - \lambda\int |u-r\phi|^p - \int Q(x)|u-r\phi|^4\\
&\qquad - a\|u\|^2 + \lambda b\|u\|^4 + \lambda\int |u|^p + \int Q(x)|u|^4\\
&\quad = -a\int \left(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\right)\\
&\qquad + \lambda b\left[2\int |\nabla u|^2\int \left(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\right) - \left(\int \left(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\right)\right)^2\right]\\
&\qquad - \lambda\int \Big(|u-r\phi|^p - |u|^p\Big) - \int Q(x)\Big(|u-r\phi|^4 - |u|^4\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle F_w, \phi\rangle|_{t=1,w=0} &= \lim_{r\to 0}\frac{F(1,0+r\phi)-F(1,0)}{r}\\
&= -\left(2a - 4\lambda b\|u\|^2\right)\int \nabla u\nabla\phi + p\lambda\int |u|^{p-2}u\phi + 4\int Q(x)|u|^2u\phi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle g'(0),\phi\rangle &= -\frac{\langle F_w,\phi\rangle}{F_t}\Big|_{t=1,w=0}\\
&= \frac{\left(2a-4\lambda b\|u\|^2\right)\int \nabla u\nabla\phi - \lambda p\int |u|^{p-2}u\phi - 4\int Q(x)|u|^2u\phi}{a\|u\|^2 - 3\lambda b\|u\|^4 - \lambda(p-1)\int |u|^p - 3\int Q(x)|u|^4}.
\end{aligned}$$

The proof is completed. □

Lemma 2.4. *There exist $\Lambda_0 > 0$ and a sequence $\{u_n\} \subset \mathcal{M}_\lambda^j$ such that*

$$u_n \geq 0, \quad I_\lambda(u_n) \to m_\lambda^j, \quad I_\lambda'(u_n) \to 0,$$

for $j = 1, 2, \ldots, k$, and $\lambda \in (0, \Lambda_0)$.

Proof. Note that $\overline{\mathcal{M}}_\lambda^j = \mathcal{M}_\lambda^j \cup \mathcal{O}_\lambda^j$ and $\mathcal{O}_\lambda^j$ is the boundary of $\overline{\mathcal{M}}_\lambda^j$. In view of Lemmas 2.1 and 2.2, we know that there exists $\Lambda_0 > 0$ such that

$$m_\lambda^j < \widetilde{m}_\lambda^j$$

for $\lambda \in (0, \Lambda_0)$, $j = 1, 2, \ldots, k,$. This implies that

$$m_\lambda^j = \inf\{I_\lambda(u) : u \in \overline{\mathcal{M}}_\lambda^j\}.$$

Then, for each $j = 1, 2, \ldots, k$, we can apply Ekeland's variational principle to construct a minimizing sequence $\{u_n\} \subset \mathcal{M}_\lambda^j$ satisfying the following properties :

(i) $\lim_{n\to\infty} I_\lambda(u_n) = m_\lambda^j$,

(ii) $I_\lambda(u_n) \le I_\lambda(w) + \frac{1}{n}\|w - u_n\|$, for each $w \in \overline{\mathcal{M}}_\lambda^j$.

Since $I_\lambda(|u|) = I_\lambda(u)$, we may assume $u_n \ge 0$. Using Lemma 2.3 with $u = u_n$, we get $\rho_n > 0$, a differential function $g_n(w)$ defined for $w \in H_0^1(\Omega)$, $w \in B_{\rho_n}(0)$ such that $g_n(w)(u_n - w) \in \mathcal{M}_\lambda^j$. Let $0 < \delta < \rho_n$ and let $w_\delta = \delta u$ with $\|u\| = 1$. Fix n and set $z_\delta = g_n(w_\delta)(u_n - w_\delta)$. By $z_\delta \in \mathcal{M}_\lambda^j$ and the property (ii), one has

$$I_\lambda(z_\delta) - I_\lambda(u_n) \ge -\frac{1}{n}\|z_\delta - u_n\|.$$

Then, by mean value theorem

$$\langle I'_\lambda(u_n), z_\delta - u_n\rangle + o(\|z_\delta - u_n\|) \ge -\frac{1}{n}\|z_\delta - u_n\|.$$

Thus,

$$\langle I'_\lambda(u_n), (u_n - w_\delta) + \big(g_n(w_\delta) - 1\big)(u_n - w_\delta) - u_n\rangle \ge -\frac{1}{n}\|z_\delta - u_n\| + o(\|z_\delta - u_n\|)$$

which yields that

$$-\delta\langle I'_\lambda(u_n), u\rangle + \big(g_n(w_\delta) - 1\big)\langle I'_\lambda(u_n), u_n - w_\delta\rangle \ge -\frac{1}{n}\|z_\delta - u_n\| + o(\|z_\delta - u_n\|).$$

Combining this with $\langle I'_\lambda(z_\delta), g_n(w_\delta)(u_n - w_\delta)\rangle = 0$, we obtain

$$\langle I'_\lambda(u_n), u\rangle \le \frac{1}{n}\frac{\|z_\delta - u_n\|}{\delta} + \frac{o(\|z_\delta - u_n\|)}{\delta} + \frac{g_n(w_\delta) - 1}{\delta}\langle I'_\lambda(u_n), u_n - w_\delta\rangle. \tag{2.9}$$

By Lemma 2.3 and the boundedness of $\{u_n\}$, we easily see that

$$\|z_\delta - u_n^j\| = \|\,(g_n(w_\delta) - 1)\,(u_n^j - w_\delta) - w_\delta\| \le |g_n(w_\delta) - 1|C_5 + \delta$$

and

$$\lim_{\delta\to 0}\frac{|g_n(w_\delta) - 1|}{\delta} = \langle g'_n(0), u\rangle \le \|g'_n(0)\| \le C_6.$$

Therefore, for fixed n, we can conclude by passing $\delta \to 0$ in (2.9) that

$$\langle I'_\lambda(u_n), u\rangle \le \frac{C}{n},$$

which implies that $I'_\lambda(u_n) \to 0$ as $n \to \infty$, and Lemma 2.4 is proved. □

Lemma 2.5. *For all $\lambda > 0$, if $\{u_n\} \subset \mathcal{M}_\lambda$ is a sequence satisfying*

$$I_\lambda(u_n) \to c < \frac{a^2S^2}{4(\lambda bS^2 + Q_M)} \quad \text{and} \quad I'_\lambda(u_n) \to 0,$$

as $n \to \infty$, then $\{u_n\}$ has a convergent subsequence.

Proof. As in the proof of Lemma 2.2, it is easy to verify that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, we may assume that for some $u_* \in H_0^1(\Omega)$,

$$\begin{aligned} &u_n \rightharpoonup u_* \quad \text{in } H_0^1(\Omega),\\ &u_n \to u_* \quad \text{in } L^r(\Omega), \quad 1 \le r < 4,\\ &u_n \to u_* \quad \text{a.e. in } \Omega. \end{aligned}$$

Denote $v_n = u_n - u_*$ and we claim that $\|v_n\| \to 0$. If not, there is a subsequence (still denoted by $\{v_n\}$) such that $\|v_n\| \to L$ with $L > 0$. By $\langle I'_\lambda(u_n), u_*\rangle = o(1)$ and the weak convergence of u_n, we see that

$$0 = a\|u_*\|^2 - \lambda b(L^2 + \|u_*\|^2)\|u_*\|^2 - \lambda \int |u_*|^p - \int Q(x)|u_*|^4. \tag{2.10}$$

Moreover, by $\langle I'_\lambda(u_n), u_n\rangle = 0$, we can apply the Brézis–Lieb Lemma to get

$$\begin{aligned} 0 = {}& a(\|v_n\|^2 + \|u_*\|^2) - \lambda b(\|v_n\|^4 + 2\|v_n\|^2\|u_*\|^2 + \|u_*\|^4)\\ &- \lambda \int |u_*|^p - \int Q(x)|v_n|^4 - \int Q(x)|u_*|^4 + o(1). \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$o(1) = a\|v_n\|^2 - \lambda b\|v_n\|^4 - \lambda b\|v_n\|^2\|u_*\|^2 - \int Q(x)|v_n|^4 \tag{2.12}$$

and consequently,

$$a\|v_n\|^2 - \lambda b\|v_n\|^4 - \lambda b\|v_n\|^2\|u_*\|^2 = \int Q(x)|v_n|^4 + o(1) \le Q_M S^{-2}\|v_n\|^4 + o(1).$$

Passing the limit as $n \to \infty$, we obtain that

$$L^2 \ge \frac{S^2(a - \lambda b\|u_*\|^2)}{\lambda b S^2 + Q_M} \ge 0. \tag{2.13}$$

By (2.10) and (2.13), we have

$$\begin{aligned} I_\lambda(u_*) &= \frac{a}{2}\|u_*\|^2 - \frac{\lambda b}{4}\|u_*\|^4 - \frac{\lambda}{p}\int |u_*|^p - \frac{1}{4}\int Q(x)|u_*|^4\\ &= \frac{\lambda b}{4}\|u_*\|^4 + \frac{\lambda b}{2}L^2\|u_*\|^2 + \lambda\left(\frac{1}{2} - \frac{1}{p}\right)\int |u_*|^p + \frac{1}{4}\int Q(x)|u_*|^4\\ &\ge \frac{\lambda b}{4}\|u_*\|^4 + \frac{\lambda b}{2}\frac{S^2(a - \lambda b\|u_*\|^2)}{\lambda b S^2 + Q_M}\|u_*\|^2\\ &= \frac{\lambda b(\lambda b S^2 + Q_M)\|u_*\|^4}{4(\lambda b S^2 + Q_M)} + \frac{\lambda a b S^2\|u_*\|^2}{2(\lambda b S^2 + Q_M)} - \frac{\lambda^2 b^2 S^2\|u_*\|^4}{2(\lambda b S^2 + Q_M)}\\ &\ge \frac{\lambda a b S^2\|u_*\|^2}{2(\lambda b S^2 + Q_M)} - \frac{\lambda^2 b^2 S^2\|u_*\|^4}{4(\lambda b S^2 + Q_M)}. \end{aligned} \tag{2.14}$$

Furthermore, using (2.12)–(2.14), we deduce that

$$\begin{aligned}
c + o(1) &= I_\lambda(u_n) \\
&= \frac{a}{2}\|u_n\|^2 - \frac{\lambda b}{4}\|u_n\|^4 - \frac{\lambda}{p}\int |u_n|^p - \frac{1}{4}\int Q(x)|u_n|^4 \\
&= \frac{a}{2}\|u_*\|^2 - \frac{\lambda b}{4}\|u_*\|^4 - \frac{\lambda}{p}\int |u_*|^p - \frac{1}{4}\int Q(x)|u_*|^4 \\
&\quad + \frac{a}{2}\|v_n\|^2 - \frac{\lambda b}{4}\|v_n\|^4 - \frac{\lambda b}{2}\|v_n\|^2\|u_*\|^2 - \frac{1}{4}\int Q(x)|v_n|^4 + o(1) \\
&= I(u_*) + \frac{a}{4}\|v_n\|^2 - \frac{\lambda b}{4}\|v_n\|^2\|u_*\|^2 + o(1) \\
&= I(u_*) + \frac{a - \lambda b\|u_*\|^2}{4}L^2 + o(1) \\
&\geq I(u_*) + \frac{a^2S^2}{4(\lambda bS^2 + Q_M)} - \frac{\lambda abS^2\|u_*\|^2}{2(\lambda bS^2 + Q_M)} + \frac{\lambda^2 b^2 S^2\|u_*\|^4}{4(\lambda bS^2 + Q_M)} + o(1) \\
&\geq \frac{a^2S^2}{4(\lambda bS^2 + Q_M)} + o(1)
\end{aligned}$$

a contradiction to the assumption $c < \frac{a^2S^2}{4(\lambda bS^2+Q_M)}$. Therefore, the claim holds, namely, $u_n \to u_*$ in $H_0^1(\Omega)$. This completes the proof of Lemma 2.5. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 2.4, we know that there exists Λ_0 such that for each $\lambda \in (0, \Lambda_0)$ and $j = 1, 2, \ldots, k$, there is a minimizing sequence $\{u_n^j\} \subset \mathcal{M}_\lambda^j$ satisfying $u_n^j \geq 0$, $I_\lambda(u_n^j) \to m_\lambda^j$ and $I'_\lambda(u_n^j) \to 0$. From Lemmas 2.1 and 2.5, it follows that $u_n^j \to u^j$ and $u^j \geq 0$ is a weak solution of (1.1). Furthermore, standard elliptic regularity argument and strong maximum principle imply that u^j is a positive solution. Finally, u^j, $j = 1, 2, \ldots, k$, are different positive solutions since $\beta(u^j) \in B_\eta(x^j)$ and $B_\eta(x^j)$ are disjoint. The proof is completed. □

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References

[1] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.* **49**(2005), No. 1, 85–93. https://doi.org/10.1016/j.camwa.2005.01.008; MR2123187; Zbl 1130.35045

[2] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic problems involving critical Sobolev exponent, *Comm. Pure Appl. Math.* **36**(1983), No. 4, 437–477. https://doi.org/10.1002/cpa.3160360405; MR0709644; Zbl 0541.35029

[3] D. M. Cao, E. S. Noussair, Multiple positive and nodal solutions for semilinear elliptic problems with critical exponents, *Indiana Univ. Math. J.* **44**(1995), No. 4, 1249–1271. https://doi.org/10.1512/iumj.1995.44.2027; MR1386768; Zbl 0849.35030

[4] Y. Duan, X. Sun, H. Y. Li, Existence and multiplicity of positive solutions for a nonlocal problem, *J. Nonlinear Sci. Appl.* **10**(2017), No. 11, 6056–6061. https://doi.org/10.22436/jnsa.010.11.40; MR3738822; Zbl 1412.35021

[5] Y. Duan, X. Sun, J.F. Liao, Multiplicity of positive solutions for a class of critical Sobolev exponent problems involving Kirchhoff-type nonlocal term, *Comput. Math. Appl.* **75**(2018), No. 12, 4427–4437. https://doi.org/10.1016/j.camwa.2018.03.041; MR3800181; Zbl 1421.35106

[6] H. N. Fan, Multiple positive solutions for a class of Kirchhoff type problems involving critical Sobolev exponents, *J. Math. Anal. Appl.* **431**(2015), No. 1, 150–168. https://doi.org/10.1016/j.jmaa.2015.05.053; MR3357580; Zbl 1319.35050

[7] G. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, *J. Math. Anal. Appl.* **401**(2013), No. 2, 706–713. https://doi.org/10.1016/j.jmaa.2012.12.053; MR3018020; Zbl 1307.35110

[8] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[9] C. Y. Lei, J. F. Liao, H. M. Suo, Multiple positive solutions for nonlocal problems involving a sign-changing potential, *Electron. J. Differential Equations* **2017**, No. 9, 1–8. MR3609137; Zbl 1357.35107

[10] C. Y. Lei, C. M. Chu, H. M. Suo, Positive solutions for a nonlocal problem with singularity, *Electron. J. Differential Equations* **2017**, No. 85, 1–9. MR3651882; Zbl 1370.35086

[11] G. B. Li, H.Y. Ye, Existence of positive solutions for nonlinear Kirchhoff type problems in $\mathbb{R}^3$ with critical Sobolev exponent, *Math. Method. Appl. Sci.* **37**(2014), No. 16, 2570–2584. https://doi.org/10.1002/mma.3000; MR3271105; Zbl 1303.35009

[12] S. H. Liang, J. H. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.* **17**(2014), 126–136. https://doi.org/10.1016/j.nonrwa.2013.10.011; MR3158465; Zbl 1302.35031

[13] J. F. Liao, J. Liu, P. Zhang, C. L. Tang, Existence and multiplicity of positive solutions for a class of elliptic equations involving critical Sobolev exponents, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **110**(2016), No. 2, 483–501. https://doi.org/10.1007/s13398-015-0244-4; MR3534502; Zbl 1372.35104

[14] P. L. Lions, The concentration-compactness principle in the calculus of variations: the limit case II, *Rev. Mat. Iberoamericana* **1**(1985), No. 2, 45–121. https://doi.org/10.4171/RMI/12; MR0850686; Zbl 0704.49006

[15] D. Naimen, The critical problem of Kirchhoff type elliptic equations in dimension four, *J. Differential Equations* **257**(2014), No. 4, 1168–1193. https://doi.org/10.1016/j.jde.2014.05.002; MR3210026; Zbl 1301.35022

[16] X. T. Qian, W. Chao, Existence of positive solutions for nonlocal problems with indefinite nonlinearity, *Bound. Value Probl.* **2020**, Paper No. 40, 13 pp. https://doi.org/10.1186/s13661-020-01343-2; MR4069873

[17] X. T. Qian, J. Q. Chen, Multiple positive and sign-changing solutions of an elliptic equation with fast increasing weight and critical growth, *J. Math. Anal. Appl.* **465**(2018), No. 2, 1186–1208. https://doi.org/10.1016/j.jmaa.2018.05.058; MR3809352; Zbl 1455.35069

[18] X.T. Qian, Ground state sign-changing solutions for a class of nonlocal problem, *J. Math. Anal. Appl.* **495**(2021), No. 2, 1–15. https://doi.org/10.1016/j.jmaa.2020.124753; MR4182964; Zbl 1459.35165

[19] X. T. Qian, W. Chao, Existence and concentration of ground state solutions for a class of nonlocal problem in $\mathbb{R}^N$, *Nonlinear Anal.* **203**(2021), 112170, 15 pp. https://doi.org/10.1016/j.na.2020.112170; MR4164555; Zbl 1459.35166

[20] Z. Y. Tang, Z. Q. Ou, Infinitely many solutions for a nonlocal problem, *J. Appl. Anal. Comput.* **10**(2020), No. 5, 1912–1917. https://doi.org/10.11948/20190286; MR4147814; Zbl 07331967

[21] Y. Wang, H. M. Suo, C. Y. Lei, Multiple positive solutions for a nonlocal problem involving critical exponent, *Electron. J. Differential Equations* **2017**, No. 275, 1–11. MR3723548; Zbl 1386.35007

[22] Y. Wang, X. Yang, Infinitely many solutions for a new Kirchhoff-type equation with subcritical exponent, *Appl. Anal.*, published online: 25 May 2020. https://doi.org/10.1080/00036811.2020.1767288

[23] G. S. Yin, J. S. Liu, Existence and multiplicity of nontrivial solutions for a nonlocal problem, *Bound. Value Probl.* **2015**, 2015:26, 1–7. https://doi.org/10.1186/s13661-015-0284-x; MR3311505; Zbl 1332.35099

[24] J. Zhang, Z. Y. Zhang, Existence of nontrivial solution for a nonlocal problem with subcritical nonlinearity, *Adv. Difference Equations* **2018**, Paper No. 359, 1–8. https://doi.org/10.1186/s13662-018-1823-4; MR3863087; Zbl 1448.35203