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Multiple positive solutions for a logarithmic Schrödinger–Poisson system with singular nonlinearity

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Abstract. In this article, we devote ourselves to investigate the following logarithmic Schrödinger–Poisson systems with singular nonlinearity

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2} u \log |u| + \frac{\lambda}{u^{\gamma}}, & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $0 < \gamma < 1$, $p \in (4, 6)$ and $\lambda > 0$ is a real parameter. By using the critical point theory for nonsmooth functional and variational method, the existence and multiplicity of positive solutions are established.

Keywords: logarithmic Schrödinger–Poisson system, multiplicity, singularity, positive solutions.

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1 Introduction and main result

In this paper, we consider the following logarithmic Schrödinger–Poisson system with singular term

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2} u \log |u| + \frac{\lambda}{u^{\gamma}}, & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $0 < \gamma < 1$, $p \in (4, 6)$ and $\lambda > 0$ is a real parameter.

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Due to the wide applications in physics and other applied sciences, partial differential equations with logarithmic nonlinearity have attracted much attention in recent years, the logarithmic Schrödinger equation given by

$$-i\frac{\partial\Psi}{\partial t} = -\Delta\Psi + (W(x)+W)\Psi - |\Psi|^{p-1}\log|\Psi|, \quad \Psi : [0,\infty)\times\mathbb{R}^N \to \mathbb{C}, \quad N \geq 1, \tag{1.2}$$

has also received a special attention. This class of equation has some important physics applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, effective quantum gravity and Bose–Einstein condensation, for more details see [28] and the references therein. For the elliptic equations with logarithmic nonlinearity, we can refer to [6,10–12,17,19,23,25] and the references therein. The authors in [10] considered the following logarithmic elliptic equations of the type

$$\begin{cases} -\Delta u + u = u\log u^2, & \text{in } \mathbb{R}^{\mathrm{N}}, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

The authors obtained solutions for this equation by applying the non-smooth critical point theory. In addition, Chao et al. in [11] considered the following Schrödinger equation with logarithmic nonlinearity

$$-\Delta u + V(x)u = u\log u^2, \qquad x \in \mathbb{R}^N,$$

where the potential V is continuous and satisfies the condition $\lim_{|x|\to\infty} V(x) = V_\infty$. When the potential is coercive, the author obtained infinitely many solutions by adapting some arguments of the Fountain theorem, and in the case of bounded potential obtained a ground state solution.

Returning to the singular Schrödinger–Poisson over bounded or unbounded domains, many papers have studied the following problem

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Under various assumptions of nonlocal term f and nonlinear term g, the existence, uniqueness and multiplicity of solutions to system (1.3) has been studied by using the modern variational methods, see [1,8,13–15,20–22,24,26,27].

There are also many references which investigated Schrödinger–Poisson system in bounded domain, see [2,3,9]. It is worth mentioning that the author in [27] considered the following singular Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-\gamma}, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $\eta = \pm 1$, $\gamma \in (0,1)$ is a constant, $\mu > 0$ is a parameter and he proved the existence and uniqueness result for $\eta = 1$ and multiplicity of solutions for $\eta = -1$ and $\mu > 0$ small enough by using Nehari manifold.

In [16] Liu et al. has considered the following singular p-Laplacian equation in $\mathbb{R}^N$

$$\begin{cases} \Delta_p u + f(x)u^{-\alpha} + \lambda g(x)u^{\beta} = 0, \\ u \geq 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $N \geq 3$, $1 < p < N$, $\lambda > 0$, $0 < \alpha < 1$, $\max(p, 2) < \beta + 1 < p^* = \frac{Np}{N-p}$. The existence and multiplicity of positive solutions for this equation are considered under some suitable condition by the critical point theory for non-smooth functional and supper-and sub-solutions method.

On the one hand, we find that most of Schrödinger–Poisson system contain only power terms and not the logarithmic terms $|t|^{p-2}t\log|t|$. This arouses the research interest of the Schrödinger–Poisson systems with logarithmic nonlinear term. On the other hand, it is noted that the logarithmic nonlinear term does not satisfy the monotonicity condition and (AR) condition , which makes system (1.1) more complex and challenging than the case without the logarithmic nonlinear term. Remarkably, the singular term leads to the non-differentiability of the energy functional corresponding to the system (1.1) on $H_0^1(\Omega)$, which make the study of system (1.1) particularly interesting. To our knowledge, the logarithmic Schrödinger–Poisson system with singular term has not been studied. Motivated by the above references, in this paper, we consider logarithmic Schrödinger–Poisson system (1.1) with singular term.

Now our main result is as follows:

Theorem 1.1. *Assume that $0 < \gamma < 1$, $p \in (4, 6)$, then there exists $\Lambda_0 > 0$ such that for any $\lambda \in (0, \Lambda_0)$, system (1.1) has at least two pair of different positive solutions.*

2 Preliminaries

Throughout this paper, we denote the norm of $L^p(\Omega)$ by $|\cdot|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$, where $p \in [1, +\infty)$. Let $H_0^1(\Omega)$ be the usual Sobolev space with the inner product and the norm $(u, v) = \int_\Omega (\nabla u, \nabla v)dx$, $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. We denote by B_r (respectively, ∂B_r) the closed ball (respectively, the sphere) of center zero and radius r. $u_n^+(x) = \max\{u_n(x), 0\}$, $u_n^-(x) = \max\{-u_n(x), 0\}$. $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line. Let S be the best Sobolev constant, namely

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^6 dx\right)^{1/3}}.$$

With the help of the Lax–Milgram theorem, for any given $u \in H_0^1(\Omega)$, the Dirichlet boundary problem $-\Delta\phi = u^2$ in Ω has a unique solution $\phi_u \in H_0^1$. Substituting ϕ_u to the first equation of system (1.1), system (1.1) is transformed into the following equation

$$\begin{cases} -\Delta u + \phi_u u = |u|^{p-2}u\log|u| + \frac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The energy functional corresponding to the equation (2.1) is the following

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega \phi_u u^2 dx + \frac{1}{p^2}\int_\Omega |u|^p dx - \frac{1}{p}\int_\Omega |u|^p \log|u| dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx.$$

From (1.3) and (1.4) in [25], we have

$$\lim_{t\to 0}\frac{t^{p-1}\log|t|}{t}=0 \quad \text{and} \quad \lim_{t\to\infty}\frac{t^{p-1}\log|t|}{t^{q-1}}=0, \tag{2.2}$$

where $q \in (p,6)$, and for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|t|^{p-1}\log|t| \le \epsilon|t| + C_\epsilon|t|^{q-1}, \qquad \forall t \in \mathbb{R}\backslash\{0\}. \tag{2.3}$$

If a function $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (\nabla u, \nabla\varphi)dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx = 0$$

for $\varphi \in H_0^1(\Omega)$, then we say u is a weak solution of (2.1) and (u,ϕ_u) is a pair solution of system (1.1).

Before proving Theorem 1.1, we give the following important lemma.

Lemma 2.1 (See [3,7,18,27]). *For every $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\phi = u^2, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

and

(1) $\|\phi_u\|^2 = \int_\Omega \phi_u u^2 dx$;

(2) $\phi_u \ge 0$. *Moreover,* $\phi_u > 0$ *when* $u \neq 0$;

(3) *For* $t \neq 0, \phi_{tu} = t^2\phi_u$;

(4) *Assume that* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *then* $\phi_{u_n} \to \phi_u$ *in* $H_0^1(\Omega)$ *and*

$$\int_\Omega \phi_{u_n} u_n v dx \to \int_\Omega \phi_u u v dx, \qquad \forall\, v \in H_0^1(\Omega);$$

(5) $\int_\Omega \phi_u u^2 dx = \int_\Omega |\nabla\phi_u|^2 dx \le C\|u\|^4$;

(6) *Set* $\mathcal{F}(u) = \int_\Omega \phi_u u^2 dx$, *then* $\mathcal{F}(u) : H_0^1(\Omega) \to H_0^1(\Omega)$ *is* C^1 *and*

$$\langle \mathcal{F}'(u), v\rangle = 4\int_\Omega \phi_u u v dx, \qquad \forall v \in H_0^1(\Omega);$$

(7) *For* $u, v \in H_0^1(\Omega)$, $\int_\Omega (\phi_u u - \phi_v v)(u-v)dx \ge \frac{1}{2}\|\phi_u - \phi_v\|^2$.

Lemma 2.2 (See [4]). *For all $p, a, s > 0$, we have the following results:*

$$s^p\log(s) \le \frac{1}{ea}s^{p+a}, \tag{2.4}$$

and by simple calculation, we have

$$s^p\log(s) \ge -\frac{1}{ep}.$$

Proof. We can repeat the proof of [4, Lemma 2], so we omit the detailed proof of (2.4). Next, we will prove that another inequality holds.

Let $h(t) = t^p \log t$ for all $t > 0$. Clearly, one can obtain that $t_* = e^{-\frac{1}{p}}$ is the unique minimum point of function h. Thus, $h(t) \geq h(t_*) = -\frac{1}{ep}$ for all $t > 0$. □

In the following, we first recall some concepts and known results of the critical points theory for continuous functional. Let (X, d) be a complete metric space with metric d and $f : X \to R$ be a continuous functional in X. Denote by $|Df|(u)$ the supremum of δ in $[0, \infty)$ such that there exist $r > 0$, and a continuous map $\sigma : U \times [0, r] \to X$ satisfying

$$\begin{cases} f(\sigma(v,t)) \leq f(v) - \delta t, & (v,t) \in U \times [0,r], \\ d(\sigma(v,t), v) \leq t, & (v,t) \in U \times [0,r]. \end{cases} \tag{2.5}$$

The extended real number $|Df|(u)$ is called the weak slope of f at u, we say that $u \in X$ is a critical point of f if $|Df|(u) = 0$, we say that $c \in R$ is a critical value of f if there exists a critical point $u \in X$ of f with $f(u) = c$.

Because of looking for positive solutions of system (1.1), we consider the functional J defined on the closed positive cone P of $H_0^1(\Omega)$, that is,

$$P = \{u \mid u \in H_0^1(\Omega),\ u(x) \geq 0,\ \text{a.e. } x \in \Omega\}.$$

Lemma 2.3. *Assume $|DJ|(u) < +\infty$, then for any $v \in P$ there holds*

$$\begin{aligned} \lambda \int_\Omega \frac{v-u}{u^\gamma} dx \leq & \int_\Omega \nabla u \nabla (v-u) dx + \int_\Omega \phi_u u (v-u) dx - \int_\Omega |u|^{p-1}(v-u) \log |u| dx \\ & + |DJ|(u) \|v-u\|. \end{aligned} \tag{2.6}$$

Proof. We take a similar approach to [16, Lemma 3.1]. Let $|DJ|(u) < c$, $\delta < \frac{1}{2}\|v-u\|$, $v \in P$ and $v \neq u$. Define the mapping $\sigma : U \times [0, \delta] \to P$ by

$$\sigma(z,t) = z + t\frac{v-z}{\|v-z\|},$$

where U is a neighborhood of u. Then $\|\sigma(z,t) - z\| = t$, by (2.5), there exists a pair $(z,t) \in U \times [0,\delta]$ such that$J(\sigma(z,t)) > J(z) - ct$. Consequently, we assume that there exists a sequences $\{u_n\} \subset P$ and $\{t_n\} \subset [0, \infty)$, such that $u_n \to u$, $t_n \to 0^+$, and

$$J(u_n + t_n \frac{v-u_n}{\|v-u_n\|}) \geq J(u_n) - ct_n,$$

that is,

$$J(u_n + s_n(v-u_n)) \geq J(u_n) - cs_n\|v-u_n\|, \tag{2.7}$$

where $s_n = \frac{t_n}{\|v-u_n\|} \to 0^+$ as $n \to \infty$. Let us divide (2.7) by s_n and rewrite it as

$$\begin{aligned} & \frac{\lambda}{1-\gamma} \int_\Omega \frac{[u_n + s_n(v-u_n)]^{1-\gamma} - u_n^{1-\gamma}}{s_n} dx \\ & \quad \leq \frac{1}{2} \int_\Omega \frac{|\nabla(u_n + s_n(v-u_n))|^2 - |\nabla u_n|^2}{s_n} dx + \frac{1}{4} \int_\Omega \frac{\phi_{u_n + s_n(v-u_n)}(u_n + s_n(v-u_n))^2 - \phi_{u_n} u_n^2}{s_n} dx \\ & \qquad + \int_\Omega \frac{H(u_n + s_n(v-u_n)) - H(u_n)}{s_n} + c\|v-u_n\|, \end{aligned}$$

where

$$H(u)=\frac{1}{p^2}\int_\Omega |u|^p dx-\frac{1}{p}\int_\Omega |u|^p\log|u|dx.$$

Letting $n\to\infty$, we claim that we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{H(u_n+s_n(v-u_n))-H(u_n)}{s_n}dx\\
&\quad=\lim_{n\to\infty}\int_\Omega \frac{[u_n+s_n(v-u_n)]^p-u_n^p}{p^2 s_n}dx\\
&\qquad-\lim_{n\to\infty}\int_\Omega \frac{[u_n+s_n(v-u_n)]^p\log|u_n+s_n(v-u_n)|-u_n^p\log|u_n|}{ps_n}dx\\
&\quad=\frac{1}{p}\int_\Omega |u|^{p-1}(v-u)dx-\int_\Omega |u|^{p-1}(v-u)\log|u|dx-\frac{1}{p}\int_\Omega |u|^{p-1}(v-u)dx\\
&\quad=-\int_\Omega |u|^{p-1}(v-u)\log|u|dx.
\end{aligned}\tag{2.8}$$

Indeed, we have only to justify the limit

$$\int_\Omega |u_n|^p\log|u_n|dx\to\int_\Omega |u|^p\log|u|dx.\tag{2.9}$$

Since $u_n(x)\to u(x)$ a.e. in Ω and $u\to u^p\log(u)$ is continuous, then we get

$$u_n^p\log u_n\to u^p\log u\quad \text{a.e. in } \Omega.$$

Furthermore,

$$u^p\log u\le \frac{1}{ea}u^{p+a},$$

where a is a positive number small enough to ensure the compact embedding $H_0^1(\Omega)\hookrightarrow L^{p+a}(\Omega)$. By Lemma 2.2, for n large enough, we have

$$-\frac{1}{ep}\le u_n^p\log u_n\le \frac{1}{ea}u^{p+a}+1\in L^1(\Omega).$$

By using dominating convergence theorem, we justify (2.9). Thus, (2.8) holds.

Notice that

$$\begin{aligned}
\int_\Omega \frac{[u_n+s_n(v-u_n)]^{1-\gamma}-u_n^{1-\gamma}}{s_n(1-\gamma)}dx&=\int_\Omega \frac{[u_n+s_n(v-u_n)]^{1-\gamma}-[(1-s_n)u_n]^{1-\gamma}}{s_n(1-\gamma)}dx\\
&\quad+\int_\Omega \frac{[(1-s_n)u_n]^{1-\gamma}-u_n^{1-\gamma}}{s_n(1-\gamma)}dx\\
&=\int_\Omega \frac{[u_n+s_n(v-u_n)]^{1-\gamma}-[(1-s_n)u_n]^{1-\gamma}}{s_n(1-\gamma)}dx\\
&\quad+\frac{(1-s_n)^{1-\gamma}-1}{s_n(1-\gamma)}\int_\Omega u_n^{1-\gamma}dx\\
&=J_{1,n}+J_{2,n}.
\end{aligned}$$

Clearly, $J_{1,n}=\int_\Omega \frac{\xi_n^{-r}s_n v}{s_n}dx=\int_\Omega \frac{v}{\xi_n^\gamma}dx$, where $\xi_n\in(u_n-u_ns_n, u_n+s_n(v-u_n))$, which implies that $\xi_n\to u(u_n\to u)$ as $s_n\to 0^+$. Since $J_{1,n}\ge 0$ for all n, applying Fatou's Lemma to $J_{1,n}$, we obtain

$$\liminf_{n\to\infty} J_{1,n}\ge\int_\Omega \frac{v}{u^\gamma}dx,$$

for any $v \in P$. For $J_{2,n}$, by the dominated convergence theorem, we get

$$\lim_{n\to\infty} J_{2,n} = -\int_\Omega u^{1-\gamma} dx.$$

From the above information, for every $v \in P$, it follows

$$\lambda \int_\Omega \frac{v-u}{u^\gamma} dx \le \int_\Omega \nabla u \nabla (v-u) dx + \int_\Omega \phi_u u (v-u) dx - \int_\Omega |u|^{p-1}(v-u)\log|u| dx + c\|v-u\|.$$

Since $|DJ|(u) < c$ is arbitrary, this leads us to the proof of Lemma 2.3. □

Lemma 2.4. *J satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset P$ be (PS) sequence of J, that is

$$|DJ|(u_n) \to 0, \qquad J(u_n) \to c \quad \text{as } n \to \infty.$$

By Lemma 2.3, for any $v \in P$, we have

$$\begin{aligned}\lambda \int_\Omega \frac{v-u_n}{u_n^\gamma} dx &\le \int_\Omega \nabla u_n \nabla (v-u_n) dx + \int_\Omega \phi_{u_n} u_n (v-u_n) dx \\ &\quad - \int_\Omega u_n^{p-1}(v-u_n)\log|u_n| dx + o(1)\|v-u_n\|,\end{aligned} \tag{2.10}$$

taking $v = 2u_n \in P$ in (2.10), we get

$$\lambda \int_\Omega u_n^{1-\gamma} dx \le \int_\Omega |\nabla u_n|^2 dx + \int_\Omega \phi_{u_n} u_n^2 dx - \int_\Omega u_n^p \log|u_n| dx + o(1)\|u_n\|. \tag{2.11}$$

Since $J(u_n) \to c$,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{4}\int_\Omega \phi_{u_n} u_n^2 dx + \frac{1}{p^2}\int_\Omega |u_n|^p dx - \frac{1}{p}\int_\Omega |u_n|^p \log|u_n| dx \\ &\qquad - \frac{\lambda}{1-\gamma}\int_\Omega |u_n|^{1-\gamma} dx = c + o(1).\end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\begin{aligned}&\frac{p-2}{2p}\int_\Omega |\nabla u_n|^2 + \frac{p-4}{4p}\int_\Omega \phi_{u_n} u_n^2 dx + \frac{1}{p^2}\int_\Omega |u_n|^p dx \\ &\quad \le \lambda\left(\frac{1}{1-\gamma} - \frac{1}{p}\right)\int_\Omega u_n^{1-\gamma} dx + c + o(1) + o(1)\|u_n\| \\ &\quad \le C\lambda \|u_n\|^{1-\gamma} + C + o(1)\|u_n\|.\end{aligned}$$

Which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exists a subsequence, still denoted by itself, and a function $u \in H_0^1(\Omega)$, such that $u_n \rightharpoonup u$ in $H_0^1(\Omega), u_n(x) \to u(x)$ *a.e.* in Ω as $n \to \infty$. Choosing $v = u_m$ as the test function in (2.10), we have

$$\begin{aligned}\lambda \int_\Omega \frac{u_m-u_n}{u_n^\gamma} dx &\le \int_\Omega \nabla u_n \nabla (u_m-u_n) dx + \int_\Omega \phi_{u_n} u_n (u_m-u_n) dx \\ &\quad - \int_\Omega u_n^{p-1}(u_m-u_n)\log|u_n| dx + o(1)\|u_m-u_n\|.\end{aligned}$$

By changing the role of u_m and u_n, we have a similar inequality, by adding the two inequalities, there holds

$$\begin{aligned}
\|u_n-u_m\|^2 \le{}& \lambda \int_\Omega (u_n-u_m)\left(\frac{1}{u_n^\gamma}-\frac{1}{u_m^\gamma}\right)dx + \int_\Omega (\phi_{u_m}u_m-\phi_{u_n}u_n)(u_n-u_m)dx \\
&+\int_\Omega \left(u_n^{p-1}\log|u_n| - u_m^{p-1}\log|u_m|\right)(u_n-u_m)dx + o(1)\|u_m-u_n\| \\
\le{}& \int_\Omega (\phi_{u_m}u_m-\phi_{u_n}u_n)(u_n-u_m)dx \\
&+\int_\Omega \left(u_n^{p-1}\log|u_n| - u_m^{p-1}\log|u_m|\right)(u_n-u_m)dx + o(1)\|u_m-u_n\| \\
\le{}& -\frac{1}{2}\|\phi_{u_m}-\phi_{u_n}\|^2 + \int_\Omega u_n^p(u_n-u_m)dx + \int_\Omega u_m^p(u_n-u_m)dx + o(1)\|u_m-u_n\|.
\end{aligned}$$

Note that

$$\|\phi_{u_m}-\phi_{u_n}\| \to 0, \qquad \int_\Omega u_n^p(u_n-u_m)dx \to 0, \qquad \int_\Omega u_m^p(u_n-u_m)dx \quad \text{as } n\to\infty.$$

We have $\lim_{n\to\infty}\|u_n-u_m\| = 0$. Therefore, $u_n \to u$ in $H_0^1(\Omega)$ as $n\to\infty$. The proof is complete. □

Lemma 2.5. *Assume that $|DJ|(u)=0$, then u is a weak solution of problem (2.1). Namely $u^{-\gamma}\varphi \in L^1(\Omega)$ for all $\varphi \in H_0^1(\Omega)$, it holds that*

$$\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx = \int_\Omega |u|^{p-1}\varphi\log|u|dx + \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx. \tag{2.13}$$

Proof. By Lemma 2.3, we have

$$\lambda\int_\Omega \frac{v-u}{u^\gamma}dx \le \int_\Omega \nabla u\nabla(v-u)dx + \int_\Omega \phi_u u(v-u)dx - \int_\Omega |u|^{p-1}(v-u)\log|u|dx,$$

for every $v\in P$. Letting $s\in\mathbb{R}, \varphi\in H_0^1(\Omega)$, taking $(u+s\varphi)^+ \in P$ as a test function in (2.6), one has

$$\begin{aligned}
0 \le{}& \int_\Omega \nabla u\nabla((u+s\varphi)^+-u)dx + \int_\Omega \phi_u u\left((u+s\varphi)^+-u\right)dx \\
&-\int_\Omega |u|^{p-1}\left((u+s\varphi)^+-u\right)\log|u|dx - \lambda\int_\Omega \frac{(u+s\varphi)^+-u}{u^\gamma}dx \\
={}& s\left[\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx\right] \\
&-\int_{u+s\varphi<0}\nabla u\nabla(u+s\varphi)dx - \int_{u+s\varphi<0}\phi_u u(u+s\varphi)dx + \int_{u+s\varphi<0}|u|^{p-1}(u+s\varphi)\log|u|dx \\
&+\lambda\int_{u+s\varphi<0}\frac{u+s\varphi}{u^\gamma}dx \\
\le{}& s\left[\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx\right] \\
&-s\int_{u+s\varphi<0}[\nabla u\nabla\varphi + \phi_u u\varphi]dx + \int_{u+s\varphi<0}|u|^{p-1}(u+s\varphi)\log|u|dx.
\end{aligned}$$

Since $\nabla u(x) = 0$ for a.e. $x\in\Omega$ with $u(x)=0$ and meas$\{x\in\Omega \mid u(x)+s\varphi(x)<0, u(x)>0\} \to 0$ as $s\to 0$, we have

$$\int_{u+s\varphi<0}[\nabla u\nabla\varphi + \phi_u u\varphi]dx = \int_{\substack{u+s\varphi<0,\\ u>0}}[\nabla u\nabla\varphi + \phi_u u\varphi]dx \to 0,$$

and

$$\int_{u+s\varphi<0} |u|^{p-1}(u+s\varphi)\log|u|dx = \int_{\substack{u+s\varphi<0,\\ u>0}} |u|^{p-1}(u+s\varphi)\log|u|dx \to 0 \quad \text{as } s\to 0.$$

Therefore

$$0 \le s\left(\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx\right) + o(s),$$

as $s\to 0$. we obtain that

$$\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx \ge 0.$$

By the arbitrariness of φ, this inequality also holds for $-\varphi$,

$$\int_\Omega \nabla u\nabla\varphi dx + \int_\Omega \phi_u u\varphi dx - \int_\Omega |u|^{p-1}\varphi\log|u|dx - \lambda\int_\Omega \frac{\varphi}{u^\gamma}dx = 0.$$

Hence, we can deduce that (2.13) holds. The proof of Lemma 2.5 is complete. □

3 Proof of Theorem 1.1

In this section, we firstly prove that the problem (2.1) has a negative energy solution.

Lemma 3.1. *Given $0<\gamma<1$, there exist constants $r,\rho,\Lambda_0>0$ such that the functional J satisfies the following conditions for $0<\lambda<\Lambda_0$:*

(i) *$J(u)|_{u\in S_\rho}\ge r$, $\inf_{u\in B_\rho} J(u)<0$;*

(ii) *There exists $e\in H_0^1(\Omega)$ with $\|e\|>\rho$ such that $J(e)<0$.*

Proof. (i) By (2.12) in [25], we have

$$\int_\Omega |u|^p\log|u|dx \le \frac{1}{2}\|u\|^2 + C_1\|u\|^q. \tag{3.1}$$

Therefore, one has

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2dx + \frac{1}{4}\int_\Omega \phi_u u^2dx + \frac{1}{p^2}\int_\Omega |u|^pdx - \frac{1}{p}\int_\Omega |u|^p\log|u|dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &\ge \frac{p-1}{2p}\|u\|^2 + \frac{1}{4}\int_\Omega \phi_u u^2dx - C_1\|u\|^q - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &\ge \frac{p-1}{2p}\|u\|^2 - C_1\|u\|^q - C_2\lambda\|u\|^{1-\gamma}. \end{aligned}$$

Where $q\in(p,6)$. Which implies that there exist constants $r,\rho,\Lambda_0>0$, such that $J(u)|_{u\in S_\rho}\ge r$ for every $\lambda\in(0,\Lambda_0)$. Moreover, for $u\in H_0^1(\Omega)\setminus\{0\}$, it holds that

$$\lim_{t\to 0^+}\frac{J(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx < 0.$$

So we obtain that $J(tu)<0$ for all $u\ne 0$ and t small enough. Therefore, for $\|u\|$ small enough, one has

$$m_1 = \inf_{u\in B_\rho} J(u) < 0. \tag{3.2}$$

(ii) For every $u^+ \in H_0^1(\Omega)$, $u^+ \neq 0$ and $t > 0$, we have

$$\begin{aligned} J(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_\Omega \phi_u u^2 dx + \frac{t^p}{p^2}\int_\Omega |u|^p dx \\ &\quad - \frac{t^p}{p}\int_\Omega u^p \log|tu| dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore we can certainly find $e \in H_0^1(\Omega)$ such that $\|e\| > \rho$ and $J(e) < 0$. The proof is complete. □

Theorem 3.2. *Suppose* $0 < \lambda < \Lambda_0$, *then system* (1.1) *has a positive function pair solution* $(u_*, \phi_{u_*}) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *satisfying* $J(u_*) < 0$.

Proof. First, we claim that there exists $u_* \in B_\rho$, such that $J(u_*) = m_1 < 0$.

By the definition of m_1, we know that there exists a minimizing sequence $\{u_n\} \subset B_\rho \subset P$ such that $\lim_{n\to\infty} J(u_n) = m_1 < 0$. Since $J(|u_n|) = J(u_n)$, we may assume that $u_n(x) > 0$ for almost every x in Ω. Clearly, this minimizing sequence is of course bounded in B_ρ, up to a subsequence, there exists $u_* > 0$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_*, & \text{strongly in } L^q(\Omega),\ 1 \le q < 2^*, \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega, \end{cases} \tag{3.3}$$

as $n \to \infty$. Set $\omega_n = u_n - u_*$, by the Brézis–Lieb Lemma, one has

$$\|u_n\|^2 = \|\omega_n\|^2 + \|u_*\|^2 + o(1). \tag{3.4}$$

Hence, by Lemma 2.4, we have that

$$m_1 = \lim_{n\to\infty} J(u_n) = J(u_*) + \frac{1}{2}\lim_{n\to\infty}\|\omega_n\|^2 \ge J(u_*),$$

from $u_* \in B_\rho$ and by definition of m_1 equality holds. Hence, we obtain $J(u_*) = m_1 < 0$ and $u_* \not\equiv 0$. From the above arguments we know that u_* is a local minimizer of J.

Now, we prove that u_* is a critical point of J. Note that $u_* \ge 0$ and $u_* \not\equiv 0$. Then for any $\psi \in P \subset H_0^1(\Omega)$, let $t > 0$ such that $u_* + t\psi \in H_0^1(\Omega)$ and one has

$$\begin{aligned} 0 &\le J(u_* + t\psi) - J(u_*) \\ &= \frac{1}{2}\|u_* + t\psi\|^2 + \frac{1}{4}\int_\Omega \phi_{u_*+t\psi}(u_* + t\psi)^2 dx + \frac{1}{p^2}\int_\Omega |u_* + t\psi|^p dx \\ &\quad - \frac{1}{p}\int_\Omega |u_* + t\psi|^p \log|u_* + t\psi| dx - \frac{\lambda}{1-\gamma}\int_\Omega |u_* + t\psi|^{1-\gamma} dx \\ &\quad - \frac{1}{2}\|u_*|^2 - \frac{1}{4}\int_\Omega \phi_{u_*} u_*^2 dx - \frac{1}{p^2}\int_\Omega |u_*|^p dx \\ &\quad + \frac{1}{p}\int_\Omega |u_*|^p \log|u_*| dx + \frac{\lambda}{1-\gamma}\int_\Omega |u_*|^{1-\gamma} dx. \end{aligned} \tag{3.5}$$

Actually, from (3.5), we also get

$$\begin{aligned}&\frac{\lambda}{1-\gamma}\int_\Omega\left[(u_*+t\psi)^{1-\gamma}-(u_*)^{1-\gamma}\right]dx\\&\quad\le\frac{1}{2}\left(\|u_*+t\psi\|^2-\|u_*\|^2\right)dx+\frac{1}{4}\int_\Omega\left[\phi_{u_*+t\psi}(u_*+t\psi)^2-\phi_{u_*}{u_*}^2\right]dx\\&\qquad+\frac{1}{p^2}\int_\Omega\left[(u_*+t\psi)^p-{u_*}^p\right]dx-\frac{1}{p}\int_\Omega\left[(u_*+t\psi)^p\log|u_*+t\psi|-{u_*}^p\log|u_*|\right]dx.\end{aligned}$$

Dividing by $t>0$ and passing to the limit as $t\to 0^+$, it gives

$$\begin{aligned}\frac{\lambda}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega\frac{(u_*+t\psi)^{1-\gamma}-(u_*)^{1-\gamma}}{t}dx\le\int_\Omega\nabla u_*\nabla\psi dx+\int_\Omega\phi_{u_*}u_*\psi dx\\-\int_\Omega|u_*|^{p-1}\psi\log|u_*|dx.\end{aligned}\tag{3.6}$$

Notice that

$$\frac{\lambda}{1-\gamma}\int_\Omega\frac{(u_*+t\psi)^{1-\gamma}-(u_*)^{1-\gamma}}{t}dx=\lambda\int_\Omega(u_*+\xi t\psi)^{-\gamma}\psi dx.$$

Where $\xi\to 0^+$ and $(u_*+\xi t\psi)^{-\gamma}\psi\to(u_*)^{-\gamma}\psi$ a.e. $x\in\Omega$ as $t\to 0^+$, since $(u_*+\xi t\psi)^{-\gamma}\psi\ge 0$. Thus by using Fatou's Lemma, we have

$$\lambda\int_\Omega(u_*)^{-\gamma}\psi dx\le\frac{\lambda}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega\frac{(u_*+t\psi)^{1-\gamma}-(u_*)^{1-\gamma}}{t}dx.$$

Therefore, we deduce from (3.6) and the above estimate that

$$\int_\Omega(\nabla u_*,\nabla\psi)dx+\int_\Omega\phi_{u_*}u^*\psi dx-\int_\Omega|u_*|^{p-1}\psi\log|u_*|dx-\lambda\int_\Omega(u_*)^{-\gamma}\psi dx\ge 0,\quad\psi\ge 0.\tag{3.7}$$

Since $J(u_*)<0$, this together with Lemma 3.1, imply that $u_*\notin S_\rho$, therefore we obtain $\|u_*\|<\rho$. For u_* there is $\delta_1\in(0,1)$ such that $(1+t)u_*\in B_\rho$ for $|t|\le\delta_1$. Define $k:[-\delta_1,\delta_1]$ by $k(t)=J((1+t)u_*)$. Clearly, $k(t)$ achieves its minimum at $t=0$, namely

$$k'(t)|_{t=0}=\|u_*\|^2+\int_\Omega\phi_{u_*}(u_*)^2dx-\int_\Omega|u_*|^p\log|u_*|dx-\lambda\int_\Omega(u_*)^{1-\gamma}dx=0.\tag{3.8}$$

Suppose for any $v\in H_0^1(\Omega)$, and $\varepsilon>0$. Define $\Psi\in P$ by

$$\Psi=(u_*+\varepsilon v)^+.$$

By (3.7) and (3.8), we have

$$
\begin{aligned}
0 \le & \int_\Omega \left[(\nabla u_*, \nabla \Psi) + \phi_{u_*} u_* \Psi - |u_*|^{p-1} \Psi \log |u_*| - \lambda (u_*)^{-\gamma} \Psi \right] dx \\
= & \int_{\{u_* + \varepsilon v > 0\}} (\nabla u_*, \nabla (u_* + \varepsilon v)) dx \\
& + \int_{\{u_* + \varepsilon v > 0\}} \left[\phi_{u_*} u_* (u_* + \varepsilon v) - |u_*|^{p-1} (u_* + \varepsilon v) \log |u_*| - \lambda (u_*)^{-\gamma} (u_* + \varepsilon v) \right] dx \\
= & \left(\int_\Omega - \int_{\{u_* + \varepsilon v \le 0\}} \right) [(\nabla u_*, \nabla (u_* + \varepsilon v)) \\
& + \phi_{u_*} u_* (u_* + \varepsilon v) - |u_*|^{p-1} (u_* + \varepsilon v) \log |u_*| - \lambda (u_*)^{-\gamma} (u_* + \varepsilon v)] dx \\
\le & \|u_*\|^2 + \int_\Omega \phi_{u_*} u_*^2 dx - \int_\Omega |u_*|^p \log |u_*| dx - \lambda \int_\Omega (u_*)^{1-\gamma} dx \\
& + \varepsilon \int_\Omega \left[(\nabla u_*, \nabla v) + \phi_{u_*} u_* v - |u_*|^{p-1} v \log |u_*| - \lambda (u_*)^{-\gamma} v \right] dx \\
& - \int_{\{u_* + \varepsilon v \le 0\}} [(\nabla u_*, \nabla (u_* + \varepsilon v)) + \phi_{u_*} u_* (u_* + \varepsilon v)] dx \\
& + \int_{\{u_* + \varepsilon v \le 0\}} \left[|u_*|^{p-1} (u_* + \varepsilon v) \log |u_*| + \lambda (u_*)^{-\gamma} (u_* + \varepsilon v) \right] dx \\
\le & \varepsilon \int_\Omega \left[(\nabla u_*, \nabla v) + \phi_{u_*} u_* v - |u_*|^{p-1} v \log |u_*| - \lambda (u_*)^{-\gamma} v \right] dx \\
& - \varepsilon \int_{\{u_* + \varepsilon v \le 0\}} (\nabla u_* \nabla v + \phi_{u_*} u_* v) dx.
\end{aligned} \tag{3.9}
$$

Since the measure of the domain of integration $\{u_* + \varepsilon v \le 0\} \to 0$ as $\varepsilon \to 0$, it follows that

$$
\lim_{\varepsilon \to 0} \int_{\{u_* + \varepsilon v \le 0\}} (\nabla u_* \nabla v + \phi_{u_*} u_* v) dx = 0.
$$

Therefore, dividing by ε and setting $\varepsilon \to 0$ in (3.9), one has

$$
\int_\Omega (\nabla u_*, \nabla v) dx + \int_\Omega \phi_{u_*} u_* v dx - \int_\Omega |u_*|^{p-1} v \log |u_*| dx - \lambda \int_\Omega (u_*)^{-\gamma} v dx \ge 0. \tag{3.10}
$$

By the arbitrariness of v, the inequality also holds for $-v$,

$$
\int_\Omega (\nabla u_*, \nabla v) dx + \int_\Omega \phi_{u_*} u_* v dx - \int_\Omega |u_*|^{p-1} v \log |u_*| dx - \lambda \int_\Omega (u_*)^{-\gamma} v dx = 0. \tag{3.11}
$$

Since $u_* \not\equiv 0$. From (3.10), there holds

$$
-\Delta u_* + \phi_{u_*} u_* \ge 0.
$$

Note that $\phi_{u_*} > 0$, then, by the strong maximum principle, it suggests that $u_* > 0$ in Ω. From the above arguments, we obtain that (u_*, ϕ_{u_*}) is a positive solution of system (1.1) with $J(u_*) = m_1 < 0$. This proof is complete. □

Now, we only need prove that system (1.1) has another positive solution.

Theorem 3.3. *Suppose* $0 < \lambda < \Lambda_0$, *then system* (1.1) *has a positive function pair solution* $(v_*, \phi_{v_*}) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *such that* $J(v_*) > 0$.

Proof. By Lemma 3.1, J satisfies the geometric structure of mountain pass Lemma. Applying the Mountain pass Lemma [5] and Lemma 2.4, there exists a sequence $\{v_n\}$ such that

$$|DJ|(v_n) \to 0, \qquad J(v_n) \to c \quad \text{as } n \to \infty.$$

According to Lemma 2.4, we know that $\{v_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{v_n\}$, we may assume that $v_n \to v_*$ in $H_0^1(\Omega)$, and

$$J(v_*) = \lim_{n\to\infty} J(v_n) = c, \qquad |DJ|(v_n) \to 0.$$

Similar to Theorem 3.2, v_* satisfies equation (2.1) with $J(v_*) = c > 0$. Thus (v_*, ϕ_{v_*}) is a positive solution of system (1.1). Thereby, we obtain that the function pairs (u_*, ϕ_{u_*}) and (v_*, ϕ_{v_*}) are different positive solutions. This completes the proof of Theorem 1.1. □

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