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Sign-changing solutions for a Schrödinger–Kirchhoff–Poisson system with 4-sublinear growth nonlinearity

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Abstract. In this paper we consider the following Schrödinger–Kirchhoff–Poisson-type system

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u+\lambda\phi u=Q(x)|u|^{p-2}u & \text{in } \Omega,\\ -\Delta\phi=u^2 & \text{in } \Omega,\\ u=\phi=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^3$, $a>0, b\geq 0$ are constants and λ is a positive parameter. Under suitable conditions on $Q(x)$ and combining the method of invariant sets of descending flow, we establish the existence and multiplicity of sign-changing solutions to this problem for the case that $2<p<4$ as λ sufficiently small. Furthermore, for $\lambda=1$ and the above assumptions on $Q(x)$, we obtain the same conclusions with $2<p<\frac{12}{5}$.

Keywords: Schrödinger–Kirchhoff–Poisson type system, invariant sets of descending flow, sign-changing solutions.

2020 Mathematics Subject Classification: 35A15, 35J20, 35J50.

1 Introduction

In this paper we are concerned with the existence of sign-changing solutions to the following Schrödinger–Kirchhoff–Poisson-type system

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u+\lambda\phi u=Q(x)|u|^{p-2}u & \text{in } \Omega,\\ -\Delta\phi=u^2 & \text{in } \Omega,\\ u=\phi=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^3$, $a > 0, b \geq 0$ are constants and λ is a positive parameter.

When $a = 1$ and $b = 0$, problem (1.1) reduces to the classical Schrödinger–Poisson system on bounded domain. We rewrite it in the following more general form

$$\begin{cases} -\Delta u + \lambda \phi u = f(x,u) & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is well known that system (1.2) has a great importance in the study of stationary solution $\psi(x,t) = e^{-it}u(x)$ of time-dependent Schrödinger–Poisson equations, which describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. For more details about the physical background of system (1.2), we refer to [3,4,31]. Compared with the researches about system (1.2) on the whole space $\mathbb{R}^3$, there are few works concerning the Schrödinger–Poisson system on bounded domain, see for instance [2,8,12,33,37]. In [33], the authors considered the existence, nonexistence and multiplicity results by using variational methods when $f(x,u) = |u|^{p-1}u$ with $p \in (1,5)$. Siciliano [37] studied the same nonlinearity as in [33], and, by means of Lusternik–Schnirelmann theory, proved that system (1.2) has at least $cat_{\overline{\Omega}}(\overline{\Omega}) + 1$ solutions for p near the critical Sobolev exponent 6, where $cat(\cdot)$ denotes the Lusternik–Schnirelmann category. Alves and Souto [2] studied system (1.2) when f has a subcritical growth and obtained the existence of least energy sign-changing solution by means of variational methods. Using a new sign-changing version of the symmetric mountain pass theorem, Batkam [12] proved the existence of infinitely many sign changing solutions for system (1.2) with critical growth. Bai and He [8] considered system (1.2) with a general 4-superlinear nonlinearity f and proved the existence of ground state solution by the aid of the Nehari manifold; moreover, they also obtained the existence of infinitely many solutions.

On the other hand, if setting $\phi = 0$ and considering the first equation of problem (1.1), we get the Kirchhoff–Dirichlet problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

When $b \neq 0$, problem (1.3) is nonlocal due to the emergence of $b\int_\Omega |\nabla u|^2 dx \Delta u$ and is related to the stationary analogue of the following problem

$$\begin{cases} u_{tt} - \left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which was introduced by Kirchhoff [22] as a generalization of the classical d'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

for free vibration of elastic string, where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material and P_0 is the initial tension. The Kirchhoff's model takes into account the length variation of the string produced by the transverse vibration, which gives rise to the appearance of nonlocal term. For more mathematical and physical relevance on problem (1.3), we refer the reader to [6,16] and the references therein. Recently, different methods and techniques are used to deal with the existence of sign-changing solutions to problem (1.3) or similar Kirchhoff-type equations, and indeed, some interesting results were obtained. For example, the method of invariant sets of descent flow was used in [21,30,44] to obtain the existence of a sign-changing solution for problem (1.3). The authors in [20,36,40] considered problem (1.3) or more general Kirchhoff-type equations respectively, combining constraint variational methods and quantitative deformation lemma. Later, under some more weak assumptions on f (especially, Nehari-type monotonicity condition has been removed), with the aid of some new analytical skills and non-Nehari manifold method, Tang and Cheng [39] improved and generalized some results obtained in [36].

Now we turn our attention to problem (1.1). As far as we know, for the first time Batkam and Santos Júnior [13] introduced this type problem with bi-nonlocal terms and proved that problem (1.1) with $\lambda = 1$ has at least three solutions: one positive, one negative, and one changing its sign by imposing the conditions on the nonlinear term f (more general form than $Q(x)|u|^{p-2}u$) as follows

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exists a constant $c > 0$ such that

$$|f(x,t)| \leq c(1 + |t|^{p-1}), \quad \text{where } 4 < p < 6;$$

(f_2) $f(x,t) = o(|t|)$ uniformly in $x \in \overline{\Omega}$ as $t \to 0$;

(f_3) there exists $\mu > 4$ such that $0 < \mu F(x,t) \leq tf(x,t)$ for all $t \neq 0$, $x \in \overline{\Omega}$, where $F(x,t) = \int_0^t f(x,s)ds$.

Furthermore, in such case, if f is odd with respect to t, the authors obtained an unbounded sequence of sign-changing solutions. After this pioneer work, several interesting results have been obtained about the existence of positive solutions, multiple solutions, ground state solutions and sign-changing solutions, we refer the reader to [5,14,25,27,28,34,35,41,43,45,46,48] and their references. Here, we must point out that, to obtain their results in the above references, various 4-superlinear growth conditions or asymptotical 4-linear assumptions or the Nehari-type monotonicity condition on f are needed, especially for the discussion of sign-changing solutions. So, a natural question is that, for the case that $f(x,u)$ is 4-sublinear, here special form $f(x,u) = Q(x)|u|^{p-2}u$ being considered, does problem (1.1) admit the existence of sign-changing solutions? Meanwhile, due to the oddness of $Q(x)|u|^{p-2}u$ on u, does there exist infinitely many sign-changing solutions as usual?

Motivated by the above discussion, the purpose of this paper is to deal with the existence and multiplicity of sign-changing solutions to problem (1.1) for the case that $2 < p < 4$. For this case, to our best knowledge, during the existing literatures there is no result concerned with sign-changing solutions for problem (1.1). To state our main results, $Q(x)$ is supposed to be satisfied the following condition

(Q) $Q(x) > 0$ and $Q \in L^\infty(\Omega)$.

Now we are in the position to state our first result.

Theorem 1.1. *If $2 < p < 4$ and (Q) holds true, there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$, problem (1.1) admits one sign-changing solution. Moreover, problem (1.1) has infinitely many sign-changing solutions.*

In what follows, we list some difficulties during the process of dealing with sign-changing solutions of nonlocal elliptic problems as usual. Problem (1.1) is a bi-nonlocal problem as the appearance of the two terms $b\int_\Omega |\nabla u|^2 dx \Delta u$ and $\phi_u u$ implies that problem (1.1) is not a pointwise identity, where ϕ_u is defined in Lemma 2.1. This causes some mathematical difficulties in finding sign-changing solutions. In fact, since the nonlocal terms $\frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2$ and $\frac{1}{4}\int_\Omega \phi_u u^2 dx$ in the associated variational functional are homogeneous of order 4, it seems difficult to get the boundedness and compactness for any (PS) sequence or Cerami sequence. Inspired by [21], we overcome this difficulty by adding a reasonable potential $Q(x)$. On the other hand, we observe that

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla u^+|^2 dx + \int_\Omega |\nabla u^-|^2 dx,$$

but the following decomposition relationship

$$\int_\Omega \phi_u u^2 dx = \int_\Omega \phi_{u^+} {u^+}^2 dx + \int_\Omega \phi_{u^-} {u^-}^2 dx$$

does not hold in $H_0^1(\Omega)$. In order to overcome these difficulties, we adopt the idea from [21, 24,26] to introduce an auxiliary operator A, which will be used to construct a pseudo-gradient vector field to ensure existence of the desired invariant sets of the flow. However, since A is merely continuous (see Lemma 3.1 below), it may not be used to define the descending flow. Fortunately, one can construct a suitable locally Lipschitz continuous operator B inheriting the properties of A in a similar way as [11] to define the flow. Finally, by restricting the parameter λ small enough during the minimax arguments in the presence of invariant sets, we complete the proof of Theorem 1.1.

Remark 1.2. As we discussed above, the necessary restriction must be added to the parameter λ to obtain the existence and multiplicity of sign-changing solutions. Indeed, similar requirements have emerged in the literatures to discuss the nonexistence of nontrivial solutions or the existence of positive solutions of Schrödinger–Poisson systems. Explicitly, we observe that, in [31], system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

admits only one trivial solution with $p \in (2,3)$ if $\lambda \geq \frac{1}{4}$. Moreover, the authors in [4] considered the bounded states of system (1.4), and showed that for any $n \in \mathbb{N}$ there exists $\lambda_n \in (0, \frac{1}{4})$ such that for all $\lambda \in (0, \lambda_n)$ system (1.4) with $p \in (2,3)$ has at least n pairs of radially symmetric solutions with positive energies.

However, it will be noted that, different from the whole space $\mathbb{R}^3$ discussed in [4,31], we can also obtain the existence and multiplicity of sign-changing solutions for our problem (1.1) considered on bounded domain without any restriction on the parameter λ. To discuss this case simply, we set $\lambda = 1$. Nevertheless, the range of p will be limited to a small range as follows.

Theorem 1.3. *If $2 < p < \frac{12}{5}$ and (Q) is satisfied, then for $\lambda = 1$, the results of Theorem 1.1 still hold true.*

Remark 1.4. Here, two recent papers [21,38] must be mentioned. In fact, as particular cases, the existence of sign-changing solutions for problems (1.2) and (1.3) are considered, when the nonlinearity is the form of $Q(x)|u|^{p-2}u$ or general form covering the pure power type $Q(x)|u|^{p-2}u$ with $2 < p < 4$ for the case $\Omega = \mathbb{R}^3$. Certainly, different hypotheses on $Q(x)$ are presented to obtain their conclusions. To this point, it should be pointed out that $Q(x)$ can be equal to constants in our Theorems 1.1 and 1.3, which is different from the assumptions on $Q(x)$ in [21,38]. In addition, compared with the situations investigated in [21,38], it is worth pointing out that the technique of constructing nonempty nodal set used in [38] is invalid for our problem. To finish the proof of our Theorems 1.1 and 1.3, as in [21] we apply the method of invariant sets of descending flow. In fact, we make use of an abstract critical point theory developed in [24] that is very useful to deal with elliptic equations, see for instance [9–11,21,26] and the references therein. Meanwhile, it is must be mentioned that, although similar conditions on $Q(x)$ have been given, we could not make the estimation for the energy functional as in [21], and some new difficulties need to be addressed due to the combination of two nonlocal terms $b\int_\Omega |\nabla u|^2 dx \Delta u$ and $\phi_u u$, which is also the reason that we restrict the parameter λ small enough in Theorem 1.1. The difference between the proof of Theorem 1.1 and Theorem 1.3 is just that the functions satisfying particular properties in Properties 3.7 and 3.11 are needed to make necessary changes.

Remark 1.5. Note that, our Theorem 1.3 is valid for the case that $b = 0$, this observation and Remark 1.2 indicate that Schrödinger–Poisson system has differently dynamical behavior on bounded domain Ω and the whole space $\mathbb{R}^3$. Meanwhile, the results of sign-changing solutions in [38] are also based on the fundamental assumption that λ is sufficiently small. Compared this fact with our Theorem 1.3, it is natural to ask whether one can show that Schrödinger–Poisson system defined on the whole space $\mathbb{R}^3$ possesses nontrivial sign-changing solutions such as the case dealt with in [38] without any parameter. In addition, Theorem 1.3 is actually an extension of Theorem 1.1 in the sense that there is not any restriction on the parameter λ in Theorem 1.3. Here, we point out that, using the invariant sets of descending flow, $2 < p < \frac{12}{5}$ is the optimal range (in fact, it is the optimal range to guarantee that (4.2) holds). For $\frac{12}{5} \le p < 4$, it remains an open question about the existence and multiplicity of sign-changing solutions when $\lambda = 1$ in problem (1.1).

This paper is organized as follows. In Section 2, we present some useful preliminary results. Theorems 1.1 and 1.3 are proved in Sections 3 and 4, respectively.

2 Preliminaries

In this section, we first introduce the variational framework associated with problem (1.1). Before that, we define E to be the usual Sobolev space $H_0^1(\Omega)$ with the inner product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

and endowed with the norm $\|u\| = (u, u)^{\frac{1}{2}}$ for $u, v \in E$. From [42, Theorem 1.9], the embedding $E \hookrightarrow L^q(\Omega)$ is compact for any $1 \le q < 6$. The usual norm in the Lebesgue space $L^q(\Omega)$ is denoted by $\|u\|_q$ and C is the positive constant whose precise value can change from line to line. The following result is well known and is a collection of results in [17] and [31].

Lemma 2.1. *For each $u \in H_0^1(\Omega)$, there exists a unique element $\phi_u \in H_0^1(\Omega)$ such that $-\Delta\phi_u = u^2$. Moreover,*

$$\phi_u = \int_\Omega \frac{u^2(y)}{4\pi|x-y|} dy$$

has following properties:

(1) $\phi_u \geq 0$ *and* $\phi_{tu} = t^2\phi_u$, $\forall t > 0$;

(2) *there exists* $C > 0$ *independent of* u *such that* $\|\phi_u\| \leq C\|u\|^2$ *and*

$$\int_\Omega \phi_u u^2 dx \leq C\|u\|^4;$$

(3) *if* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $H_0^1(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} u_n^2 dx = \int_\Omega \phi_u u^2 dx.$$

In view of Lemma 2.1, we can substitute $\phi = \phi_u$ into problem (1.1) and rewrite it as a single equation

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u + \lambda\phi_u u = Q(x)|u|^{p-2}u, \quad u \in E. \tag{2.1}$$

For the equivalent problem (2.1), the corresponding functional $I : E \mapsto \mathbb{R}$

$$I(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \frac{1}{p}\int_\Omega Q(x)|u|^p dx$$

is well defined. In addition, standard discussion shows that $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), \varphi\rangle = \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \cdot \nabla\varphi dx + \lambda\int_\Omega \phi_u u\varphi dx - \int_\Omega Q(x)|u|^{p-2}u\varphi dx \tag{2.2}$$

for any $u, \varphi \in E$. Clearly, critical points of I are weak solutions of problem (2.1).

To estimate the second nonlocal term in (2.2) conveniently, we define

$$D(g,h) = \int_\Omega\int_\Omega \frac{g(x)h(y)}{4\pi|x-y|} dxdy.$$

Obviously, for each $u \in H_0^1(\Omega)$, $D(u^2, u^2) = \int_\Omega \phi_u u^2$. Moreover, the following properties can be reached. For the proof, we refer to [32] and [23, p. 250].

Lemma 2.2.

(1) $D(g,h)^2 \leq D(g,g)D(h,h)$ *for any* $g, h \in L^{\frac{6}{5}}(\Omega)$;

(2) $D(uv, uv)^2 \leq D(u^2, u^2)D(v^2, v^2)$ *for any* $u, v \in L^{\frac{12}{5}}(\Omega)$.

Next we prove a compactness condition for the functional I which will be used later.

Lemma 2.3. *Assume that (Q) holds, then the functional I satisfies the Cerami condition.*

Proof. Let $\{u_n\} \subset E$ be a Cerami sequence of I, that is, $|I(u_n)| \leq C$ and $(1 + \|u_n\|)\|I'(u_n)\|_{E^{-1}} \to 0$ in the dual space E^{-1}. Firstly, we show that $\{u_n\}$ is bounded in E. In fact, for $\mu > 4$, we have

$$\begin{aligned}
C + o_n(1) &\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) a\int_\Omega |\nabla u_n|^2 dx + \left(\frac{1}{4} - \frac{1}{\mu}\right) b\left(\int_\Omega |\nabla u_n|^2 dx\right)^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{\mu}\right)\lambda\int_\Omega \phi_{u_n} u_n^2 dx + \left(\frac{1}{\mu} - \frac{1}{p}\right)\int_\Omega Q(x)|u_n|^p dx.
\end{aligned}$$

Due to the fact that $L^\infty(\Omega) \subset L^{\frac{6}{6-p}}(\Omega)$, by means of Hölder's inequality and the Sobolev embedding theorem, we obtain that

$$\left(\frac{1}{\mu}-\frac{1}{p}\right)\int_\Omega Q(x)|u_n|^p dx \geq \left(\frac{1}{\mu}-\frac{1}{p}\right)\|Q\|_{\frac{6}{6-p}}\|u_n\|_6^p \geq \left(\frac{1}{\mu}-\frac{1}{p}\right) C\left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{p}{2}}, \quad (2.3)$$

which yields that

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{\mu}\right) a\int_\Omega |\nabla u_n|^2 dx + \left(\frac{1}{4}-\frac{1}{\mu}\right) b\left(\int_\Omega |\nabla u_n|^2 dx\right)^2 + \left(\frac{1}{4}-\frac{1}{\mu}\right)\lambda\int_\Omega \phi_{u_n} u_n^2 dx \\ + \left(\frac{1}{\mu}-\frac{1}{p}\right) C\left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{p}{2}} \leq C+1.\end{aligned}$$

Since $2 < p < 4 < \mu$, the above inequality indicates that $\{u_n\}$ is bounded in E. Then, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{in } E; \qquad u_n \to u \quad \text{in } L^q(\Omega), \quad 1 \leq q < 6; \qquad u_n \to u \quad \text{a.e. in } \Omega. \quad (2.4)$$

Since $u_n \to u$ in $L^{\frac{12}{5}}(\Omega)$ ($q = \frac{12}{5}$ in (2.4)), using Hölder's inequality, we have

$$\begin{aligned}&\left|\int_\Omega (\phi_{u_n} u_n - \phi_u u)(u_n - u)\, dx\right| \\ &\quad \leq \|\phi_{u_n}\|_6 \|u_n - u\|_{\frac{12}{5}}^2 + \|\phi_{u_n} - \phi_u\|_6 \|u\|_{\frac{12}{5}} \|u_n - u\|_{\frac{12}{5}} \\ &\quad = o_n(1).\end{aligned}$$

From (Q) and (2.4), using Hölder's inequality again gives that

$$\begin{aligned}&\left|\int_\Omega Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) dx\right| \\ &\quad \leq C(\|u_n\|_p^{p-1}\|u_n - u\|_p + \|u\|_p^{p-1}\|u_n - u\|_p) \\ &\quad = o_n(1).\end{aligned}$$

Hence, the above two facts and the weak convergence of $u_n \rightharpoonup u$ in E bring that

$$\begin{aligned}o_n(1) &= \langle I'(u_n) - I'(u), u_n - u\rangle \\ &= \left(a + b\int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega |\nabla(u_n - u)|^2 dx \\ &\quad + b\left(\int_\Omega |\nabla u_n|^2 dx - \int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \cdot \nabla(u_n - u) dx \\ &\quad + \lambda\int_\Omega (\phi_{u_n} u_n - \phi_u u)(u_n - u)\, dx \\ &\quad - \int_\Omega Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) dx \\ &\geq a\|u_n - u\|^2 + o_n(1),\end{aligned}$$

which implies that $u_n \to u$ in E. □

3 Proof of Theorem 1.1

In this section, we use the method of invariant sets of descending flow to study the existence of sign-changing solutions for problem (1.1). To do this, we introduce an auxiliary operator $A : E \to E$. Explicitly, for any $u \in E$, we define $v = Au$ to be the unique solution for the equation

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta v + \lambda\phi_u v = Q(x)|u|^{p-2}u, \quad v \in E. \tag{3.1}$$

Clearly, u is a fixed point of A if and only if u is a solution of (2.1).

Lemma 3.1. *The operator A is well defined, maps bounded sets to bounded sets and is continuous.*

Proof. For any $u \in E$, define

$$J(v) = \frac{1}{2}\left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla v|^2 dx + \frac{\lambda}{2}\int_\Omega \phi_u v^2 dx - \int_\Omega Q(x)|u|^{p-2}uv dx. \tag{3.2}$$

Then, $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(v), \omega\rangle = \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla v \cdot \nabla\omega dx + \lambda\int_\Omega \phi_u v\omega dx - \int_\Omega Q(x)|u|^{p-2}u\omega dx \tag{3.3}$$

for any $\omega \in E$. From (Q) and the Sobolev embedding theorem, it is easy to verify that J is coercive, bounded below and weakly lower semicontinuous. Thus, J admits a unique minimizer $v = Au \in E$, which is the unique solution to (3.1), that is to say, A is well defined.

Taking $v = \omega = Au$ in (3.3) leads to

$$\left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla Au|^2 dx + \lambda\int_\Omega \phi_u (Au)^2 dx = \int_\Omega Q(x)Au|u|^{p-2}u dx,$$

which implies, using (Q) and the Sobolev embedding theorem, that

$$a\|Au\| \leq C\|u\|^{p-1}.$$

Therefore, Au is bounded whenever u is bounded.

In the following, we prove that A is continuous. Assuming $\{u_n\} \subset E$ with $u_n \to u$ in E and taking $v = Au$, $v_n = Au_n$, we need to prove that $\|v_n - v\| \to 0$ in E. Based on the observation that $\langle J'(v_n) - J'(v), v_n - v\rangle = 0$, that is,

$$\begin{aligned}\left(a + b\int_\Omega |\nabla u_n|^2 dx\right)&\int_\Omega |\nabla(v_n - v)|^2 dx \\ &= b\left(\int_\Omega |\nabla u|^2 dx - \int_\Omega |\nabla u_n|^2 dx\right)\int_\Omega \nabla v \cdot \nabla(v_n - v) dx \\ &\quad + \lambda\int_\Omega (\phi_{u_n} v_n - \phi_u v)(v - v_n) dx \\ &\quad + \int_\Omega Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(v_n - v) dx,\end{aligned} \tag{3.4}$$

and $u_n \to u$ in E, it is sufficient to estimate the second and third terms in the right side of (3.4). Indeed, using Hölder's inequality, one has

$$\begin{aligned}\int_\Omega (\phi_{u_n} v_n - \phi_u v)(v - v_n) dx &\leq \int_\Omega (\phi_{u_n} v - \phi_u v)(v - v_n) dx \\ &\leq \|\phi_{u_n} - \phi_u\|_3 \|v\|_3 \|v - v_n\|_3 \\ &\leq C\|\phi_{u_n} - \phi_u\|_3 \|v\| \|v - v_n\|,\end{aligned} \tag{3.5}$$

where $\|\phi_{u_n} - \phi_u\|_3 \to 0$ due to Lemma 2.1. In addition, according to (Q) and Theorem A.2 in [42], we have

$$Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u) \to 0 \quad \text{in} \quad L^{\frac{p}{p-1}}(\Omega),$$

which, combining with Hölder's inequality, states that

$$\left|\int_\Omega Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(v_n - v)dx\right| \leq C\|Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{\frac{p}{p-1}}\|v_n - v\|. \quad (3.6)$$

Thus, (3.4), (3.5) and (3.6) imply that

$$\begin{aligned} a\|v_n - v\|^2 \leq\ & b\left|\int_\Omega |\nabla u|^2 dx - \int_\Omega |\nabla u_n|^2 dx\right| \|v\|\|v_n - v\| \\ & + C\lambda\|\phi_{u_n} - \phi_u\|_3\|v\|\|v - v_n\| \\ & + C\|Q(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{\frac{p}{p-1}}\|v_n - v\| \\ =\ & o_n(1)\|v_n - v\|, \end{aligned}$$

which means that A is continuous. □

Lemma 3.2.

(1) $\langle I'(u), u - Au\rangle \geq a\|u - Au\|^2$ *for any* $u \in E$;

(2) $\|I'(u)\|_{E^{-1}} \leq [a + (C\lambda + b)\|u\|^2]\|u - Au\|$ *for some* $C > 0$ *and all* $u \in E$.

Proof. Since Au is the solution of (3.1), we see that

$$\begin{aligned} \langle I'(u), u - Au\rangle =\ & \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \cdot \nabla(u - Au)dx \\ & + \lambda\int_\Omega \phi_u u(u - Au)dx - \int_\Omega Q(x)|u|^{p-2}u(u - Au)dx \\ =\ & \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla(u - Au)|^2 dx + \lambda\int_\Omega \phi_u(u - Au)^2 dx, \end{aligned}$$

which implies that $\langle I'(u), u - Au\rangle \geq a\|u - Au\|^2$ for any $u \in E$.

By Hölder's inequality, Lemmas 2.1 and 2.2, for any $\varphi \in E$, we have

$$\begin{aligned} \langle I'(u), \varphi\rangle =\ & \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla(u - Au)\cdot\nabla\varphi dx + \lambda\int_\Omega \phi_u(u - Au)\varphi dx \\ =\ & (a + b\|u\|^2)(u - Au, \varphi) + \lambda D(u^2, (u - Au)\varphi) \\ \leq\ & (a + b\|u\|^2)\|u - Au\|\|\varphi\| + C\lambda\|u\|^2\|u - Au\|\|\varphi\| \\ \leq\ & [a + (C\lambda + b)\|u\|^2]\|u - Au\|\|\varphi\|. \end{aligned}$$

Thus, $\|I'(u)\|_{E^{-1}} \leq [a + (C\lambda + b)\|u\|^2]\|u - Au\|$ for any $u \in E$. □

Lemma 3.3. *Let* $\delta_1 < \delta_2$ *and* $\alpha > 0$, *there exists* $\beta > 0$ *such that* $\|u - Au\| \geq \beta$ *if* $u \in E$, $I(u) \in [\delta_1, \delta_2]$ *and* $\|I'(u)\|_{E^{-1}} \geq \alpha$.

Proof. Let $\mu > 4$, for $u \in E$, using $\langle J'(Au), u\rangle = 0$, we have

$$\begin{aligned} I(u) &- \frac{1}{\mu}\left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u\cdot\nabla(u - Au)dx - \frac{\lambda}{\mu}\int_\Omega \phi_u u(u - Au)dx \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) a\int_\Omega |\nabla u|^2 dx + \left(\frac{1}{4} - \frac{1}{\mu}\right) b\left(\int_\Omega |\nabla u|^2 dx\right)^2 \\ &\quad + \left(\frac{1}{4} - \frac{1}{\mu}\right)\lambda\int_\Omega \phi_u u^2 dx + \left(\frac{1}{\mu} - \frac{1}{p}\right)\int_\Omega Q(x)|u|^p dx. \end{aligned}$$

Using (2.3), Hölder's inequality, Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{\mu}\right) a\|u\|^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right) b\|u\|^4 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\lambda\int_\Omega \phi_u u^2 dx + \left(\frac{1}{\mu} - \frac{1}{p}\right) C\|u\|^p \\ &\le |I(u)| + \frac{1}{\mu}\left(a + b\int_\Omega |\nabla u|^2 dx\right)\left|\int_\Omega \nabla u\cdot\nabla(u - Au)dx\right| + \left|\frac{\lambda}{\mu}\int_\Omega \phi_u u(u - Au)dx\right| \\ &\le |I(u)| + \frac{1}{\mu}\left(a + b\|u\|^2\right)\|u\|\|u - Au\| + \frac{\lambda}{\mu}\left(\int_\Omega \phi_u(u - Au)^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \phi_u u^2 dx\right)^{\frac{1}{2}} \\ &\le |I(u)| + \frac{1}{\mu}\left(a + b\|u\|^2\right)\|u\|\|u - Au\| + C\|u\|\|u - Au\|\left(\int_\Omega \phi_u u^2 dx\right)^{\frac{1}{2}} \\ &\le |I(u)| + \frac{1}{\mu}\left(a + b\|u\|^2\right)\|u\|\|u - Au\| + \frac{C}{2}\left(\|u\|^2 + \int_\Omega \phi_u u^2 dx\right)\|u - Au\|. \end{aligned}$$

If there exists $\{u_n\} \subset E$ with $I(u_n) \in [\delta_1, \delta_2]$ and $\|I'(u_n)\|_{E^{-1}} \ge \alpha$ such that $\|u_n - Au_n\| \to 0$ as $n \to \infty$, since $2 < p < 4$, from the above inequality, we deduce that $\{u_n\}$ is bounded. Then, from Lemma 3.2-(2), we see that $\|I'(u_n)\|_{E^{-1}} \to 0$ as $n \to \infty$, which is a contradiction. Thus, the proof is completed. □

In the sequel, we introduce the positive and negative cones in E defined as follows

$$P^+ := \{u \in E : u \ge 0\} \quad \text{and} \quad P^- := \{u \in E : u \le 0\}.$$

For given $\varepsilon > 0$, two open convex subsets of E are chosen in the following forms

$$P_\varepsilon^+ := \{u \in E : \text{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P_\varepsilon^- := \{u \in E : \text{dist}(u, P^-) < \varepsilon\},$$

where $\text{dist}(u, P^\pm) = \inf_{v\in P^\pm}\|u - v\|$. Obviously, $P_\varepsilon^- = -P_\varepsilon^+$. Let $W = P_\varepsilon^+ \cup P_\varepsilon^-$, then W is an open and symmetric subset of E, and $E \setminus W$ contains only sign-changing functions. To find solutions for problem (1.1) in $E \setminus W$, we establish the following result which provides invariance properties for the convex sets $P_\varepsilon^\pm$.

Lemma 3.4. *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$, there hold*

$$A(\partial P_\varepsilon^+) \subset P_\varepsilon^+ \quad \text{and} \quad A(\partial P_\varepsilon^-) \subset P_\varepsilon^-.$$

Proof. Let $u \in E$ and $v = Au$ satisfying (3.1). Notice that for any $2 \le q \le 6$, there exists $C_q > 0$ such that

$$\|u^+\|_q = \inf_{v\in P^-}\|u - v\|_q \le C_q \inf_{v\in P^-}\|u - v\| = C_q \text{dist}(u, P^-). \tag{3.7}$$

Then, due to the fact that $\operatorname{dist}(v, P^-) \le \|v^+\|$, we have

$$\begin{aligned}
a \operatorname{dist}(v, P^-)\|v^+\| &\le a\|v^+\|^2 \\
&\le \left(a + b\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla v^+|^2 dx + \lambda \int_\Omega \phi_u v v^+ dx \\
&= \int_\Omega Q(x)|u|^{p-2}uv^+ dx \le \int_\Omega Q(x)|u^+|^{p-2}u^+v^+ dx \\
&= \int_\Omega Q(x)|u^+|^{p-1}v^+ dx \le \|Q\|_{\frac{6}{6-p}}\|u^+\|_6^{p-1}\|v^+\|_6 \\
&\le C(\operatorname{dist}(u, P^-))^{p-1}\|v^+\|,
\end{aligned}$$

which means that

$$\operatorname{dist}(v, P^-) \le \frac{C}{a}(\operatorname{dist}(u, P^-))^{p-1}.$$

Therefore, for $\varepsilon_0 \in \left(0, \left(\frac{a}{2C}\right)^{\frac{1}{p-2}}\right)$ with $0 < \varepsilon \le \varepsilon_0$, it holds that

$$\operatorname{dist}(v, P^-) \le \frac{1}{2}\varepsilon < \varepsilon \quad \text{for any} \quad u \in P_\varepsilon^-.$$

That is to say, $A(\partial P_\varepsilon^-) \subset P_\varepsilon^-$. In a similar way, one also has $A(\partial P_\varepsilon^+) \subset P_\varepsilon^+$. □

Let $K = \{u \in E : I'(u) = 0\}$. Since A is merely continuous, it may by itself not be the right operator to construct a descending flow for the functional I, and we need an improved operator $B : E\backslash K \to E$ which is locally Lipschitz continuous and inherits the main properties of A.

Lemma 3.5. *For $0 < \varepsilon \le \varepsilon_0$, there exists a locally Lipschitz continuous odd operator $B : E \setminus K \to E$ such that*

(1) $\frac{1}{2}\|u - Au\| \le \|u - Bu\| \le 2\|u - Au\|$;

(2) $\langle I'(u), u - Bu\rangle \ge \frac{1}{2}a\|u - Au\|^2$;

(3) $B(\partial P_\varepsilon^+) \subset P_\varepsilon^+$, $B(\partial P_\varepsilon^-) \subset P_\varepsilon^-$.

Proof. The proof is similar to that of [9, Lemma 4.1] and [11, Lemma 2.1], so we omit the details. □

By means of the invariant set of descending flow, we are intended to establish the existence of sign-changing solutions for problem (1.1). Here, we use the known abstract critical theorem given by [24, Theorem 2.4], and include its statement for the sake of completeness in the form of a proposition.

Let X be a complete metric space with the metric d, $h \in C^1(X, \mathbb{R})$, $P_1, P_2 \subset X$ be open subsets, $M = P_1 \cap P_2$, $\Sigma = \partial P_1 \cap \partial P_2$ and $W = P_1 \cup P_2$. For $c \in \mathbb{R}$, $h^c = \{x \in X : h(x) \le c\}$ and $K_c = \{x \in X : h(x) = c, h'(x) = 0\}$.

Definition 3.6 ([24]). $\{P_1, P_2\}$ is called an admissible family of invariant sets with respect to h at level c, provided that the following deformation property holds: if $K_c \setminus W = \emptyset$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists a continuous map $\eta : X \to X$ satisfying

(1) $\eta(\overline{P_i}) \subset \overline{P_i}$, $i = 1, 2$;

(2) $\eta|_{h^{c-2\varepsilon}} = Id$;

(3) $\eta(h^{c+\varepsilon} \setminus W) \subset h^{c-\varepsilon}$.

Proposition 3.7 ([24]). *Assume $\{P_1, P_2\}$ is an admissible family of invariant sets with respect to h at level c for $c \geq c_* := \inf_{u \in \Sigma} h(u)$ and there exists a map $\psi_0 : \triangle \to X$ satisfying*

(1) $\psi_0(\partial_i \triangle) \subset P_i,\ i = 1, 2$;

(2) $\psi_0(\partial_0 \triangle) \cap M = \emptyset$;

(3) $c_0 = \sup_{u \in \psi_0(\partial_0 \triangle)} h(u) < c_*$,

where $\triangle = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$, $\partial_0 \triangle = \triangle \cap \{t_1 + t_2 = 1\}$ and $\partial_i \triangle = \triangle \cap \{t_i = 0\}$, $i = 1, 2$. Define

$$c = \inf_{\psi \in \Gamma} \sup_{u \in \psi(\triangle) \setminus W} h(u),$$

where $\Gamma := \{\psi \in C(\triangle, X) : \psi(\partial_i \triangle) \subset P_i, i = 1, 2, \psi|_{\partial_0 \triangle} = \psi_0\}$. Then c is a critical value of h and $K_c \setminus W \neq \emptyset$.

Now we use Proposition 3.7 to obtain the existence of one sign-changing solution for problem (1.1). Here, we choose $X = E$, $h = I$, $P_1 = P_\varepsilon^+$ and $P_2 = P_\varepsilon^-$, then, $M = P_\varepsilon^+ \cap P_\varepsilon^-$, $\Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$ and $W = P_\varepsilon^+ \cup P_\varepsilon^-$. The following lemma implies that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional I at any level $c \in \mathbb{R}$.

Lemma 3.8. *Assume (Q) holds. If $K_c \setminus W = \emptyset$, then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying*

(1) $\sigma(0, u) = u$ for all $u \in E$;

(2) $\sigma(t, u) = u$ for $t \in [0, 1]$, $u \notin I^{-1}([c - \varepsilon', c + \varepsilon'])$;

(3) $\sigma(1, I^{c+\varepsilon} \setminus W) \subset I^{c-\varepsilon}$;

(4) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}, \sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}, t \in [0, 1]$.

Proof. The proof is similar to that of many existing literatures (see [21,26]). We include its proof for the sake of completeness. If $K_c \setminus W = \emptyset$, then $K_c \subset W$. Thus, $2\delta := \mathrm{dist}(K_c, \partial W) > 0$ on account of K_c is compact by Lemma 2.3. For this δ, we have $N_\delta(K_c) := \{u \in E : \mathrm{dist}(u, K_c) < \delta\} \subset W$. Since I satisfies the Cerami condition, there exist $\varepsilon_0, \alpha > 0$ such that

$$\|I'(u)\|_{E^{-1}} \geq \alpha \quad \text{for } u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus N_{\frac{\delta}{2}}(K_c).$$

By Lemmas 3.3 and 3.5-(1), there exists $\beta > 0$ such that

$$\|u - Bu\| \geq \frac{\beta}{2} \quad \text{for } u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus N_{\frac{\delta}{2}}(K_c).$$

Furthermore, owing to Lemma 3.5-(2), we obtain

$$\left\langle I'(u), \frac{u - Bu}{\|u - Bu\|} \right\rangle \geq \frac{1}{8} a \|u - Bu\| \geq \theta := \frac{a\beta}{16}. \tag{3.8}$$

Decreasing ε_0 if necessary, we assume $\varepsilon_0 \leq \frac{\theta\delta}{4}$. Take two even Lipschitz continuous functions $p, q : E \to [0,1]$ such that

$$p(u) = \begin{cases} 0, & u \in N_{\frac{\delta}{4}}(K_c), \\ 1, & u \notin N_{\frac{\delta}{2}}(K_c), \end{cases} \quad \text{and} \quad q(u) = \begin{cases} 0, & u \notin I^{-1}([c-\varepsilon', c+\varepsilon']), \\ 1, & u \in I^{-1}([c-\varepsilon, c+\varepsilon]), \end{cases}$$

and consider the following initial value problem

$$\begin{cases} \frac{d\tau(t,u)}{dt} = -\Phi(\tau(t,u)), \\ \tau(0,u) = u, \end{cases} \tag{3.9}$$

where $\Phi(u) = p(u)q(u)\frac{u-Bu}{\|u-Bu\|}$. Obviously, $\Phi(u)$ is locally Lipschitz continuous, so the existence and uniqueness theory of ODE in Banach space implies that (3.9) admits a unique solution $\tau(\cdot, u) \in C(\mathbb{R}, E)$. Define σ on $[0,1] \times E$ by $\sigma(t,u) := \tau(\frac{2\varepsilon}{\theta}t, u)$, it is sufficient to check (3), because (1)–(2) are obvious, and (4) is a consequence of Lemma 3.5-(3).

To do this, choose $u \in I^{c+\varepsilon} \setminus W$. By (3.9), it easy to see $\frac{dI(\tau(t,u))}{dt} \leq 0$, namely, $I(\tau(t,u))$ is nonincreasing for $t \geq 0$. Then, if there exists $t_0 \in [0, \frac{2\varepsilon}{\theta}]$ such that $I(\tau(t_0,u)) < c - \varepsilon$, we have

$$I(\sigma(1,u)) = I\left(\tau\left(\frac{2\varepsilon}{\theta}, u\right)\right) < c - \varepsilon.$$

Otherwise, for any $t \in [0, \frac{2\varepsilon}{\theta}]$, $I(\tau(t,u)) \geq c - \varepsilon$, then $\tau(t,u) \in I^{-1}([c-\varepsilon, c+\varepsilon])$. We claim that for any $t \in [0, \frac{2\varepsilon}{\theta}]$, $\tau(t,u) \notin N_{\frac{\delta}{2}}(K_c)$. If not, there exists $t_0 \in [0, \frac{2\varepsilon}{\theta}]$ such that $\tau(t_0, u) \in N_{\frac{\delta}{2}}(K_c)$, then, since $N_\delta(K_c) \subset W$, we obtain

$$\frac{\delta}{2} \leq \|\tau(t_0,u) - u\| \leq \int_0^{t_0} \|\tau'(s,u)\| ds \leq t_0 < \frac{2\varepsilon_0}{\theta} \leq \frac{\delta}{2},$$

which is a contradiction. Therefore, $p(\tau(t,u))q(\tau(t,u)) \equiv 1$ for $t \in [0, \frac{2\varepsilon}{\theta}]$. Hence, by (3.8) and (3.9),

$$\begin{aligned} I(\sigma(1,u)) &= I\left(\tau\left(\frac{2\varepsilon}{\theta}, u\right)\right) \\ &= I(u) - \int_0^{\frac{2\varepsilon}{\theta}} \langle I'(\tau(s,u)), \Phi(\tau(s,u))\rangle ds \\ &\leq c + \varepsilon - \frac{2\varepsilon}{\theta}\theta \\ &= c - \varepsilon. \end{aligned}$$

Thus, the proof is completed. □

Lemma 3.9. *If $\varepsilon > 0$ small enough, then $I(u) \geq a\varepsilon^2$ for any $u \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$.*

Proof. For any $u \in \Sigma$, it has $\|u^\pm\| = \|u - u^\mp\| \geq \operatorname{dist}(u, P^\mp) = \varepsilon$. Then, using (Q) and (3.7), we have

$$\begin{aligned} I(u) &= \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \frac{\lambda}{4}\int_\Omega \phi_u u^2 dx - \frac{1}{p}\int_\Omega Q(x)|u|^p dx \\ &\geq \frac{a}{2}\|u\|^2 - \frac{C}{p}\|u\|_p^p \geq 2a\varepsilon^2 - \frac{C}{p}\varepsilon^p \geq a\varepsilon^2 \end{aligned}$$

for $\varepsilon > 0$ small enough. □

Proof of Theorem 1.1 (Existence part). We construct a suitable map ψ_0 satisfying the properties in Proposition 3.7. Choose $v_1 \in P_\varepsilon^-$, $v_2 \in P_\varepsilon^+$ such that $\operatorname{supp}(v_1) \cap \operatorname{supp}(v_2) = \emptyset$ and $\inf_{\operatorname{supp}(v_1)\cup\operatorname{supp}(v_2)} Q(x) > 0$. For

$$\rho = (\rho_1, \rho_2) \in \Delta = \{t \in \mathbb{R}^2 : t = (t_1, t_2), t_1, t_2 \geq 0, t_1 + t_2 \leq 1\},$$

define

$$\psi_0(\rho)(x) = R\left(\rho_1 v_1(R^{-2}x) + \rho_2 v_2(R^{-2}x)\right),$$

where R is a positive constant to be determined later. It is obvious that, for any $\rho = (0, \rho_2) \in \partial_1\Delta$ and $\rho = (\rho_1, 0) \in \partial_2\Delta$, we have

$$\psi_0(\rho)(x) = R\left(\rho_2 v_2(R^{-2}x)\right) \in P_\varepsilon^+ \quad \text{and} \quad \psi_0(\rho)(x) = R\left(\rho_1 v_1(R^{-2}x)\right) \in P_\varepsilon^-,$$

respectively. Thus, $\psi_0(\partial_1\Delta) \subset P_\varepsilon^+$ and $\psi_0(\partial_2\Delta) \subset P_\varepsilon^-$.

From Lemma 3.9, we have $c_\lambda^* := \inf_{u\in\Sigma} I(u) \geq a\varepsilon^2$. Next, we verify that

$$c_0 = \sup_{u\in\psi_0(\partial_0\Delta)} I(u) < c_\lambda^*.$$

Set $u_s = \psi_0(s, 1-s)$ for $s \in [0,1]$, a direct computation shows that

$$\int_\Omega |\nabla u_s|^2 dx = R^4 \int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx,$$
$$\int_\Omega \phi_{u_s} u_s^2 dx = R^{14} \int_\Omega \phi_{\hat{u}_s} \hat{u}_s^2 dx, \quad \text{where } \hat{u}_s = s v_1 + (1-s) v_2,$$
$$\int_\Omega |u_s|^p dx = R^{p+6} \int_\Omega (s^p|v_1|^p + (1-s)^p|v_2|^p) dx.$$

Based on the equalities above, we have

$$\begin{aligned}
I(u_s) &= \frac{a}{2}\int_\Omega |\nabla u_s|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u_s|^2 dx\right)^2 + \frac{\lambda}{4}\int_\Omega \phi_{u_s} u_s^2 dx \\
&\quad - \frac{1}{p}\int_{\operatorname{supp}(v_1)\cup\operatorname{supp}(v_2)} Q(x)|u_s|^p dx \\
&\leq \frac{a}{2}\int_\Omega |\nabla u_s|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u_s|^2 dx\right)^2 + \frac{\lambda}{4}\int_\Omega \phi_{u_s} u_s^2 dx \\
&\quad - \min_{\operatorname{supp}(v_1)\cup\operatorname{supp}(v_2)} Q(x) \int_\Omega |u_s|^p dx \\
&= \frac{a}{2}R^4 \int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx \\
&\quad + \frac{b}{4}R^8 \left(\int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx\right)^2 \\
&\quad + \frac{\lambda}{4}R^{14}\int_\Omega \phi_{\hat{u}_s}\hat{u}_s^2 dx - CR^{p+6}\int_\Omega (s^p|v_1|^p + (1-s)^p|v_2|^p) dx.
\end{aligned} \tag{3.10}$$

Then, taking $0 < \lambda \leq \lambda_R = R^{-6} =: \lambda^*$ in (3.10) (which means that λ is sufficiently small for R large enough), we obtain

$$\begin{aligned}
I(u_s) &\leq \frac{a}{2}R^4 \int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx \\
&\quad + \frac{b}{4}R^8 \left(\int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx\right)^2 \\
&\quad + \frac{1}{4}R^8 \int_\Omega \phi_{\hat{u}_s}\hat{u}_s^2 dx - CR^{p+6}\int_\Omega (s^p|v_1|^p + (1-s)^p|v_2|^p) dx.
\end{aligned} \tag{3.11}$$

Since $2 < p < 4$, by (3.11), it is evident that $I(u_s) \to -\infty$ as $R \to \infty$ uniformly for $s \in [0,1]$. Consequently, choosing R large enough and independent of s, we have

$$c_0 = \sup_{u \in \psi_0(\partial_0 \Delta)} I(u) < c_\lambda^* := \inf_{u \in \Sigma} I(u). \tag{3.12}$$

Moreover, we observe that

$$\int_\Omega |u_s|^2 dx = R^4 \int_\Omega (s^2 |v_1|^2 + (1-s)^2 |v_2|^2) dx \to \infty \quad \text{as } R \to \infty \tag{3.13}$$

uniformly with respect to $s \in [0,1]$, which, combining with (3.7), indicates that $\psi_0(\partial_0 \Delta) \cap M = \emptyset$. Define

$$c = \inf_{\psi \in \Gamma} \sup_{u \in \psi(\Delta) \setminus W} I(u),$$

where $\Gamma := \{\psi \in C(\triangle, E) : \psi(\partial_1 \triangle) \subset P_\varepsilon^+, \psi(\partial_2 \triangle) \subset P_\varepsilon^-, \psi|_{\partial_0 \triangle} = \psi_0\}$, and apply Proposition 3.7, there is a critical point $u \in K_c \setminus W$ which is a sign-changing solution of problem (1.1). $\square$

Next we turn to the existence of infinitely many sign-changing solutions for problem (1.1). To do this, we make use of Theorem 2.5 in [24] recalled below. Explicitly, let X be a complete metric space with the metric d and $h \in C^1(X, \mathbb{R})$, then we say $G : X \to X$ is an isometric involution if G satisfies $G^2 = Id$ and $d(Gx, Gy) = d(x,y)$ for $x, y \in X$. A subset $O \subset X$ is said to be symmetric if $Gx \in O$ for any $x \in O$. The genus of a closed symmetric subset O of $X \setminus \{0\}$ is denoted by $\gamma(O)$.

Definition 3.10 ([24]). Assume G is an isometric involution of X and h is a G-invariant continuous functional on X that is $h(Gx) = h(x)$ for any $x \in X$. We say P is a G-admissible invariant set with respect to h at level c if the following deformation property holds: there exist a symmetric open neighbourhood N of $K_c \setminus (P \cup Q)$ with $\gamma(\overline{N}) < \infty$ and ε_0 such that for $0 < \varepsilon < \varepsilon_0$ there exists a continuous map $\eta : X \to X$ satisfying

(1) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$, here $Q = GP$;

(2) $\eta \circ G = G \circ \eta$;

(3) $\eta|_{h^{c-2\varepsilon}} = Id$;

(4) $\eta(h^{c+\varepsilon} \setminus (N \cup (P \cup Q))) \subset h^{c-\varepsilon}$.

Proposition 3.11 ([24]). *Assume that P is a G-admissible invariant set with respect to h at level c for $c \geq c_* := \inf_{u \in \partial P \cap \partial Q} h(u)$ and for any $n \in \mathbb{N}$ there exists a continuous map $\psi_n : B_n \to X$ satisfying*

(1) $\psi_n(0) \in P \cap Q$;

(2) $\psi_n(\partial B_n) \cap (P \cap Q) = \emptyset$;

(3) $\sup_{u \in \mathrm{Fix}_G \cup \psi_n(\partial B_n)} h(u) < c_*$,

where $B_n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\mathrm{Fix}_G := \{u \in X : Gu = u\}$. Define

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus (P \cup Q)} h(u),$$

where $\Gamma_j := \{B : B = \psi(B_n \setminus Y),\ \psi \in G_n,\ n \geq j,$ and open subset $Y = -Y \subset B_n,\ \gamma(\overline{Y}) \leq n - j\}$ and $G_n := \{\psi : \psi \in C(B_n, X),\ \psi(-t) = G\psi(t),\ t \in B_n,\ \psi(0) \in P \cap Q$ and $\psi|_{\partial B_n} = \psi_n\}$. Then $c_j, j \geq 2$, are critical values of h with $c_j \to \infty$ and $K_{c_j} \setminus (P \cup Q) \neq \emptyset$.

To apply Proposition 3.11, we set $X = E, h = I, G = -Id, P = P_\varepsilon^+$. In addition, thanks to the nonlinearity in problem (1.1) is odd, as a sequence, G is an isometric involution on E, $Q = -P_\varepsilon^+ = P_\varepsilon^-$, and the functional I is G-invariant continuous functional. Since K_c is compact, there exists a symmetric open neighborhood N of $K_c \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ with $\gamma(\overline{N}) < \infty$.

Lemma 3.12. *Assume (Q) holds true, then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there exists a continuous map $\sigma : [0,1] \times E \to E$ satisfying*

(1) $\sigma(0,u) = u$ for $u \in E$;

(2) $\sigma(t,u) = u$ for $t \in [0,1]$, $u \notin I^{-1}([c-\varepsilon', c+\varepsilon'])$;

(3) $\sigma(t,-u) = -\sigma(t,u)$ for $(t,u) \in [0,1] \times E$;

(4) $\sigma(1, I^{c+\varepsilon} \setminus (N \cup (P_\varepsilon^+ \cup P_\varepsilon^-))) \subset I^{c-\varepsilon}$;

(5) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}$, $\sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}$, $t \in [0,1]$.

Proof. The proof is similar to Lemma 3.8. Since I is even, thus σ is odd in u. Here, we omit the details. □

Combining Definition 3.10 with Lemma 3.12, we conclude that P_ε^+ is a G-admissible set for the function I at any level $c \in \mathbb{R}$.

Proof of Theorem 1.1 (Multiplicity part). According to the above discussion, we need to construct an appropriate continuous map $\psi_n : B_n \to E$ to apply Proposition 3.11. In order to achieve this point, for any $n \in \mathbb{N}$, we choose $\{v_i\}_1^n \in E$ with disjoint supports and $\inf_{\mathrm{supp}(v_i)} Q(x) > 0$, and define

$$\psi_n(t)(x) = R_n(t_1 v_1(R_n^{-2}x) + \cdots + t_n v_n(R_n^{-2}x)),$$

where $t = (t_1, t_2, \ldots, t_n) \in B_n$, R_n is a large number such that $\psi_n(\partial B_n) \cap (P_\varepsilon^+ \cap P_\varepsilon^-) = \emptyset$ and

$$\sup_{u \in \psi_n(\partial B_n)} I(u) < 0 < \inf_{u \in \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-} I(u)$$

as in (3.12) and (3.13). Obviously, $\psi_n(0) = 0 \in P_\varepsilon^+ \cap P_\varepsilon^-$ and $\psi_n(-t) = -\psi_n(t)$ for $t \in B_n$. Define

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)} I(u),$$

where Γ_j is given in Proposition 3.11, then it follows that c_j $(j \geq 2)$ are critical values of I with $c_j \to \infty$ as $j \to \infty$, and the corresponding critical points $u_j \in K_{c_j} \setminus (P_\varepsilon^+ \cup P_\varepsilon^-)$ are sign-changing solutions of problem (1.1). □

4 Proof of Theorem 1.3

Under the assumptions of Theorem 1.3, we establish the existence and multiplicity of sign-changing solutions for problem (1.1) in this section. Before proceeding, we point out that the energy functional is still denoted by I, and obviously the conclusions of Lemma 2.3 and Lemmas 3.1–3.9 are effective for $\lambda = 1$. However, due to the fact that $p \in (2, \frac{12}{5})$, we need to construct $\overline{\psi}_0$ different from ψ_0 in Theorem 1.1 to establish the proof of Theorem 1.3.

Proof of Theorem 1.3. Define

$$\overline{\psi}_0 = \overline{\psi}_0(\rho)(x) = R^{-1}\left(\rho_1 v_1(R^{-m}x) + \rho_2 v_2(R^{-m}x)\right),$$

where v_1, v_2, $\rho = (\rho_1, \rho_2)$ are the same as in the proof of Theorem 1.1 and $m \in \left(\frac{p}{3}, \frac{4-p}{2}\right)$ is a constant dependent on p. Next, we check that $\overline{\psi}_0$ satisfies the properties in Proposition 3.11. Similar to the proof of Theorem 1.1, we obtain $\overline{\psi}_0(\partial_1\Delta) \subset P_\varepsilon^+$ and $\overline{\psi}_0(\partial_2\Delta) \subset P_\varepsilon^-$. Therefore, it suffices to verify (2) and (3) of Proposition 3.11. Indeed, set $\bar{u}_s = \overline{\psi}_0(s, 1-s)$ for $s \in [0,1]$, the direct computations show that

$$\begin{aligned}
\int_\Omega |\nabla \bar{u}_s|^2 dx &= R^{-2+m} \int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx, \\
\int_\Omega \phi_{\bar{u}_s} \bar{u}_s^2 dx &= R^{-4+5m} \int_\Omega \phi_{\hat{u}_s} \hat{u}_s^2 dx, \text{ where } \hat{u}_s = s v_1 + (1-s) v_2, \\
\int_\Omega |\bar{u}_s|^p dx &= R^{-p+3m} \int_\Omega (s^p|v_1|^p + (1-s)^p|v_2|^p) dx,
\end{aligned}$$

which signify that

$$\begin{aligned}
I(\bar{u}_s) \le\ & \frac{a}{2} R^{-2+m} \int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx \\
&+ \frac{b}{4} R^{-4+2m} \left(\int_\Omega (s^2|\nabla v_1|^2 + (1-s)^2|\nabla v_2|^2) dx\right)^2 \\
&+ \frac{1}{4} R^{-4+5m} \int_\Omega \phi_{\hat{u}_s} \hat{u}_s^2 dx \\
&- C R^{-p+3m} \int_\Omega (s^p|v_1|^p + (1-s)^2|v_2|^p) dx.
\end{aligned} \tag{4.1}$$

Since $2 < p < \frac{12}{5}$ and $m \in \left(\frac{p}{3}, \frac{4-p}{2}\right)$, we get

$$\max\{-2+m, -4+2m, -4+5m, -p+3m\} = -p+3m > 0. \tag{4.2}$$

Considering the above relationship in (4.1), we are led to $I(\bar{u}_s) \to -\infty$ as $R \to \infty$ uniformly for $s \in [0,1]$. In addition, from Lemma 3.9, we have known that $c_1^* := \inf_{u \in \Sigma} I(u) \ge a\varepsilon^2$. Therefore, choosing R large enough and independent of s can guarantee that

$$\bar{c}_0 = \sup_{u \in \overline{\psi}_0(\partial_0\Delta)} I(u) < c_1^*.$$

Meanwhile, it is obvious that

$$\int_\Omega |\bar{u}_s|^2 dx = R^{-2+3m} \int_\Omega (s^2|v_1|^2 + (1-s)^2|v_2|^2) dx \to \infty \quad \text{as } R \to \infty$$

uniformly for $s \in [0,1]$, which, combining with (3.7), indicates that that $\overline{\psi}_0(\partial_0\Delta) \cap M = \emptyset$. Based on the above facts, define

$$\bar{c} = \inf_{\psi \in \overline{\Gamma}} \sup_{u \in \psi(\Delta) \setminus W} I(u),$$

where $\overline{\Gamma} := \{\psi \in C(\triangle, E) : \psi(\partial_1\triangle) \subset P_\varepsilon^+, \psi(\partial_2\triangle) \subset P_\varepsilon^-, \psi|_{\partial_0\triangle} = \overline{\psi}_0\}$ and apply Proposition 3.7, we obtain the existence of sign-changing solution. The rest of proof with respect to multiplicity is very similar to that of Theorem 1.1. Actually, it is just necessary to use

$$\overline{\psi}_n(t)(x) = R_n^{-1}(t_1 v_1(R_n^{-m}x) + \cdots + t_n v_n(R_n^{-m}x))$$

instead of $\psi_n(t)(x)$ in the process of the proof of Theorem 1.1. Once $\overline{\psi}_n(t)(x)$ is determined as above, the remainder is just to repeat the proof of Theorem 1.1, so we omit the details. □

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