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Existence and multiplicity of positive solutions for singular ϕ-Laplacian superlinear problems with nonlinear boundary conditions

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Abstract. We prove the existence and multiplicity of positive solutions to the singular ϕ-Laplacian BVP

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)\left(f(u) - \frac{a}{u^\alpha}\right), & t \in (0,1), \\ u(0) = 0,\ u'(1) + H(u(1)) = 0 \end{cases}$$

for a certain range of the parameter $\lambda > 0$, where $a > 0$, $\alpha \in (0,1)$, ϕ is an odd, increasing and convex homeomorphism on $\mathbb{R}$, and f is ϕ-superlinear at ∞.

Keywords: ϕ-Laplacian, infinite semipositone, positive solutions.

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1 Introduction

Consider the one-dimensional ϕ-Laplacian problem

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)\left(f(u) - \frac{a}{u^\alpha}\right),\ t \in (0,1), \\ u(0) = 0,\ u'(1) + H(u(1)) = 0, \end{cases} \tag{1.1}$$

where $a > 0$, $\alpha \in (0,1)$, λ is a positive parameter, and the following conditions are assumed:

(A1) $r : [0,1] \to (0,\infty)$ is continuous and nondecreasing.

(A2) $H : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $H(0) = 0$.

(A3) $g : (0,1) \to (0,\infty)$ is continuous with $g/p^\alpha \in L^1(0,1)$, where $p(t) = \min(t, 1-t)$.

(A4) $\phi : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism such that ϕ is convex on $[0,\infty)$ and

$$\limsup_{x\to\infty} \frac{\phi(\sigma x)}{\phi(x)} < \infty$$

for all $\sigma > 0$.

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(A5) $f : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, and

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = \infty.$$

(A6) There exist constants $m > 0$ and $\beta \in (0,1)$ such that

$$f(x) \geq \left(\frac{x}{m}\right)^{\beta} f(m)$$

for $x \in [0,m]$.

By a positive solution of (1.1), we mean a function $u \in C^1[0,1]$ with $u > 0$ on $(0,1]$ and $r(t)\phi(u')$ absolutely continuous on $[0,1]$ that satisfies (1.1).

Our main result is

Theorem 1.1.

(i) Let (A1)–(A6) hold. Then there exist a positive number K and an interval $I \subset (0,\infty)$ such that if $f(m) \geq K$ then problem (1.1) has at least two positive solutions for $\lambda \in I$.

(ii) Let (A1)–(A5) hold. Then there exists a positive number $\lambda_0 > 0$ such that for $\lambda < \lambda_0$, problem (1.1) has a positive solution u_λ with $u_\lambda(t) \to \infty$ as $\lambda \to 0^+$ uniformly on compact subsets of $(0,1]$.

Example 1.2. Let r satisfy (A1), $a, \alpha, \gamma, \delta > 0$ with $\alpha + \gamma < 1$, and $\phi(x) = \sum_{i=1}^{n} a_i |x|^{p_i-2}x$, where $a_i > 0, p_i \geq 2$ for $i = 1, \ldots, n$, and $H(z) = z^\delta$. Consider the BVP

$$\begin{cases} -(r(t)\phi(u'))' = \frac{\lambda}{t^\gamma}\left(f(u) - \frac{a}{u^\alpha}\right), \ t \in (0,1), \\ u(0) = 0, \ u'(1) + (u(1))^\delta = 0. \end{cases} \tag{1.2}$$

Let $q > \max_{1\leq i\leq n}(p_i - 1)$Then

(i) By Theorem 1.1 (i) with $m = 1$, problem (1.2) with

$$f(u) = \begin{cases} Ku^\beta, \ 0 \leq u \leq 1, \\ Ku^q, \ u > 1, \end{cases}$$

where $0 < \beta < 1$, has two positive solutions for λ in a certain range of $(0,\infty)$, provided that K is large enough.

(ii) By Theorem 1.1 (ii), problem (1.2) with $f(u) = u^q e^{-\frac{b}{1+u}}$, where $b \geq 0$, has a large positive solution for $\lambda > 0$ small.

A problem of the form (1.1) occurs in the study of positive radial solutions to the p-Laplacian problems in an exterior domain

$$\begin{cases} -\Delta_p u = \lambda K(|x|)f(u) \text{ in } \Omega = \{x \in \mathbb{R}^N : |x| > r_0 > 0\}, \\ \frac{\partial u}{\partial n} + \tilde{c}(u)u = 0 \text{ on } |x| = r_0, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $N > p$, $\tilde{c} : [0,\infty) \to [0,\infty)$, n denotes the outer unit normal vector on $\partial\Omega$, as it reduced (see [10]) via the Kevin transformation $r = |x|$, $t = (r/r_0)^{\frac{p-N}{p-1}}$ to the ODE problem

$$\begin{cases} -(\phi(u'))' = \lambda h(t) f(u), \ t \in (0,1), \\ u(0) = 0, \ u'(1) + c(u(1))u(1) = 0, \end{cases}$$

where $\phi(z) = |z|^{p-2}z$, $h(t) = \left(\frac{p-1}{N-p} r_0\right)^p t^{\frac{p(N-1)}{p-N}} K(r_0 t^{\frac{p-1}{p-N}})$, and $c(s) = \frac{p-1}{N-p} r_0 \tilde{c}(s)$. It also arises in the study of radial solutions for the (p,q)-Laplacian problem in an annulus i.e.

$$\begin{cases} -\Delta_p u - \Delta_q u = f(|x|, u), \ \ a < |x| < b, \\ u = 0 \ \text{ on } |x| = a, \\ \frac{\partial u}{\partial n} - H(u) = 0 \text{ on } |x| = b, \end{cases}$$

where $p > q > 1$, which stems from a variety of applied areas (see e.g. [2,3]). We are motivated by related results on the existence and multiplicity of positive radial solutions for the system

$$\begin{cases} -\Delta u_i = \lambda K_i(|x|) f_i(u_j) \text{ in } \Omega = \{x \in \mathbb{R}^N : r_0 < |x| < r_1\}, \\ d_i \frac{\partial u_i}{\partial n} + \tilde{c}_i(u_i) u_i = 0 \ \text{ on } |x| = r_0, \\ u_i = 0 \text{ on } |x| = r_1, \end{cases}$$

in [6,7], where $d_i = 0$, $\tilde{c}_i \equiv 1, r_1 < \infty$ in [6] and $d_i \geq 0, \tilde{c}_i > 0, r_1 = \infty$ in [7], as well as its extension to p-Laplacian systems in [12]. The results in [6,7,12] have been obtained under the assumptions that the reaction terms satisfy a combined superlinear at ∞ and are allowed to have semipositone structures at 0 i.e. $f_i(0^+) \in [-\infty, 0)$. Searching for positive solutions in the semipositone case is known to be challenging due to the lack of the maximum principle. Our main result here on the one hand allows the p-Laplacian operator to be replaced by a general homeomorphism on $\mathbb{R}$, and on the other hand permits nonlinear boundary conditions that can not be linearized e.g. $u'(1) + \sqrt[3]{u(1)} = 0$, which are not allowed in [6,7,12] . We obtain the existence of a large positive solution to (1.1) for $\lambda > 0$ small when f is merely ϕ-superlinear at ∞, and the existence of two positive solutions for λ in a certain interval in $(0,\infty)$ if in addition f satisfies a concavity condition on $[0,m]$ for some $m > 0$ and $f(m)$ is large enough. It is worth noting that problem (1.1) is of infinite semipositone nature as $\lim_{u\to 0^+} \left(f(u) - \frac{a}{u^\alpha}\right) = -\infty$. Our approach is based on a Krasnoselskii's fixed point theorem in a Banach space.

We refer to [4,5,8,9,11] for results in the PDE case related to (1.1), where [9,11] are of particular relevance to this study. In [9], an overview of recent developments on elliptic variational problems with functional satisfying nonstandard growth of (p,q)-type is provided. Related existence results for positive solutions of the Brézis–Nirenberg type critical semipositone problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

can be found in [11], where Ω is a bounded domain in $\mathbb{R}^N$, $p^* = \frac{Np}{N-p}$, and λ, μ are positive parameters.

2 Preliminary results

For the rest of the paper, we define $r_0 = \min_{t\in[0,1]} r(t)$, $r_1 = \max_{t\in[0,1]} r(t)$, and $H(z) = H(0)$ for $z < 0$. The norm in $L^p(0,1)$ will be denoted by $\|\cdot\|_p$.

We first recall the following fixed point result of Krasnoselskii type in a Banach space (see e.g. [1, Theorem 12.3]).

Lemma A. *Let E be a Banach space and $T : E \to E$ be a completely continuous operator. Suppose there exist $h \in E, h \neq 0$ and positive constants r, R with $r \neq R$ such that*

(a) If $y \in E$ satisfies $y = \theta Ty$, $\theta \in (0,1]$ then $\|y\| \neq r$,

(b) If $y \in E$ satisfies $y = Ty + \xi h$, $\xi \geq 0$ then $\|y\| \neq R$.

Then T has a fixed point $y \in E$ with $\min(r,R) < \|y\| < \max(r,R)$.

Lemma 2.1.

(i) $|\phi^{-1}(x) - \phi^{-1}(y)| \leq 2\phi^{-1}(|x-y|)$ for all $x, y \in \mathbb{R}$.

(ii) $\phi^{-1}(x-y) \geq \phi^{-1}(x) - 2\phi^{-1}(y)$ for all $x, y \in \mathbb{R}$ with $y \geq 0$.

Proof. *(i)* Without loss of generality, we need only to consider two cases.

Case 1. $x \geq y \geq 0$.

Since ϕ^{-1} is concave on $[0,\infty)$,

$$\phi^{-1}(x-y) + \phi^{-1}(y) \geq \phi^{-1}(x),$$

which implies

$$\phi^{-1}(x) - \phi^{-1}(y) \leq \phi^{-1}(x-y) \leq 2\phi^{-1}(x-y)$$

i.e. (i) holds.

Case 2. $x \geq 0 \geq y$.

Then $\phi^{-1}(x) \leq \phi^{-1}(x-y)$ and $-\phi^{-1}(y) = \phi^{-1}(-y) \leq \phi^{-1}(x-y)$, from which (i) follows.

(ii) Since $y \geq 0$, it follows from (i) that

$$\phi^{-1}(x-y) - \phi^{-1}(x) \geq -2\phi^{-1}(|y|) = -2\phi^{-1}(y)$$

i.e. (ii) holds. □

Lemma 2.2. *Let $h \in L^1(0,1)$ and $u \in C^1[0,1]$ satisfy*

$$\begin{cases} (r(t)\phi(u'))' \leq h \quad \text{on } (0,1) \\ u(0) \geq 0,\ u'(1) + H(u(1)) \geq 0. \end{cases} \tag{2.1}$$

Suppose $\|u\|_\infty > \phi^{-1}(\|h\|_1/r_0)$. Then $u(1) \geq 0$.

Proof. Suppose on the contrary that $u(1) < 0$. Then the boundary condition at 1 implies that $u'(1) \geq 0$. Let $\tau \in [0,1]$ be such that $\|u\|_\infty = |u(\tau)|$. Integrating the inequality in (2.1) on $[t,1]$ we get

$$r(t)\phi(u'(t)) = r(1)\phi(u'(1)) - \int_t^1 (r(s)\phi(u'))'ds \geq -\|h\|_1,$$

whence

$$u'(t) \geq -\phi^{-1}\left(\frac{\|h\|_1}{r(t)}\right) \geq -\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right) \tag{2.2}$$

for $t \in [0,1]$. Next, integrating (2.2) on $[0,\tau]$ and $[\tau,1]$ give

$$u(\tau) \geq u(0) - \phi^{-1}\left(\frac{\|h\|_1}{r_0}\right) \geq -\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right) \tag{2.3}$$

and

$$-u(\tau) \geq u(1) - u(\tau) \geq -\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right) \tag{2.4}$$

respectively. Combining (2.3) and (2.4), we deduce that

$$\|u\|_\infty \leq \phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)$$

a contradiction. Thus $u(1) \geq 0$. □

Lemma 2.3. *Let $h \in L^1(0,1)$ with $h \geq 0$ and $u \in C^1[0,1]$ satisfy*

$$\begin{cases} (r(t)\phi(u'))' \leq h \quad \text{on } (0,1), \\ u(0) \geq 0, \quad u(1) \geq 0. \end{cases}$$

Then

(i) $u(t) \geq \big(u(1) - 2\phi^{-1}(\frac{\|h\|_1}{r_0})\big)t$ *for* $t \in [0,1]$.

(ii) $u(t) \geq \big(\|u\|_\infty - 4\phi^{-1}(\frac{\|h\|_1}{r_0})\big)p(t)$ *for* $t \in [0,1]$. *In particular,*

$$u(t) \geq \frac{1}{5}\|u\|_\infty p(t)$$

for $t \in [0,1]$, *provided that* $\|u\|_\infty \geq 5\phi^{-1}(\frac{\|h\|_1}{r_0})$.

Proof. Define

$$w(t) = \phi^{-1}\left(\phi(u'(t)) + \frac{1}{r(t)}\int_t^1 h\right) - u'(t), \qquad z(t) = \int_0^t w$$

for $t \in [0,1]$. Then $w, z \geq 0$ on $[0,1]$ and in view of Lemma 2.1 (i),

$$w(t) \leq 2\phi^{-1}\left(\frac{1}{r(t)}\int_t^1 h\right) \leq 2\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)$$

which implies

$$z(t) \leq 2\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)t$$

for $t \in [0,1]$. Since

$$(r(t)\phi(u'+z'))' = (r(t)\phi(u'+w))' = (r(t)\phi(u'))' - h \leq 0 \quad \text{on } (0,1),$$

$r(t)\phi(u'+z')$ is nonincreasing on $[0,1]$. This, together with (A1), gives the concavity of $u+z$ on $[0,1]$. Since $(u+z)(0) \geq 0$, we obtain

$$(u+z)(t) \geq t(u+z)(1) \geq tu(1),$$

whence

$$u(t) \geq tu(1) - z(t) \geq t\left(u(1) - 2\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right)$$

i.e. (i) holds. Since $(u+z)(1) \geq 0$, we deduce from the concavity of $u+z$ on $[0,1]$ that

$$u(t) + z(t) \geq \|u+z\|_\infty p(t)$$

for $t \in [0,1]$. Consequently,

$$u(t) \geq (\|u\|_\infty - \|z\|_\infty)p(t) - z(t)$$

for $t \in [0,1]$, which implies

$$u(t) \geq \left(\|u\|_\infty - 4\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right)t \tag{2.5}$$

for $t \in [0,1/2]$. Similarly, by defining

$$w_0(t) = \phi^{-1}\left(\phi(u'(t)) - \frac{1}{r(t)}\int_0^t h\right) - u'(t), \qquad z_0(t) = \int_t^1 w_0$$

for $t \in [0,1]$, and using $w_0, z_0 \leq 0$ on $[0,1]$,

$$|z_0(t)| \leq 2\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)(1-t)$$

for $t \in [0,1]$, together with

$$\begin{cases} (r(t)\phi(u' - z_0'))' = (r(t)\phi(u' + w_0))' = (r(t)\phi(u'))' - h \leq 0 \text{ on } (0,1), \\ (u - z_0)(0) \geq 0, \quad (u - z_0)(1) \geq 0, \end{cases}$$

we obtain as above that

$$u(t) \geq \left(\|u\|_\infty - 4\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right)(1-t) \tag{2.6}$$

for $t \in [1/2,1]$. Combining (2.5) and (2.6), we obtain

$$u(t) \geq \left(\|u\|_\infty - 4\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right)p(t)$$

for $t \in [0,1]$. In particular, $u(t) \geq \frac{1}{5}\|u\|_\infty p(t)$ if $\|u\|_\infty \geq 5\phi^{-1}(\|h\|_1/r_0)$, which completes the proof. □

3 Proof of the main result

Let $E = C[0,1]$ be equipped with $\|\cdot\|_\infty$.

Proof of Theorem 1.1. (i) Since ϕ is convex on $[0,\infty)$ and $\phi(0)=0$, it follows that

$$\lim_{x\to 0^+}\frac{\phi(x)}{x^\beta}=0.$$

Hence there exists $\gamma\in(0,m)$ such that

$$\frac{\phi(8\gamma)}{(\gamma/5)^\beta}<\frac{r_0 g_0\phi(m)}{64r_1(4m)^\beta}\left(\int_0^1\frac{g}{p^\alpha}\right)^{-1}, \tag{3.1}$$

where $g_0=\inf_{[1/4,3/4]} g$. Suppose $f(m)>K$, where

$$K=\max\left\{\frac{a}{(\gamma/5)^\alpha},\frac{a\phi(m)}{4(\gamma/5)^\alpha\phi(\gamma/5)},\frac{16ar_1(4m)^\beta}{g_0r_0(\gamma/5)^{\alpha+\beta}}\left(\int_0^1\frac{g}{p^\alpha}\right)\right\}.$$

Let

$$I=\left(\frac{16r_1\phi(8\gamma)}{g_0f(\gamma/20)},\frac{r_0\phi(m)}{4f(m)}\left(\int_0^1\frac{g}{p^\alpha}\right)^{-1}\right).$$

Then $I\neq\varnothing$. Indeed, it follows from (A6) that

$$f\left(\frac{\gamma}{20}\right)\geq\left(\frac{\gamma}{20m}\right)^\beta f(m)=\left(\frac{\gamma}{5}\right)^\beta\frac{f(m)}{(4m)^\beta},$$

which, together with (3.1), implies

$$\frac{16r_1\phi(8\gamma)}{g_0f(\gamma/20)}\leq\frac{16r_1(4m)^\beta}{g_0f(m)}\left(\frac{\phi(8\gamma)}{(\gamma/5)^\beta}\right)<\frac{r_0\phi(m)}{4f(m)}\left(\int_0^1\frac{g}{p^\alpha}\right)^{-1}.$$

We shall verify that (1.1) has at least two positive solutions for $\lambda\in I$. For $\lambda\in I$ and $v\in E$, define $T_\lambda v=u$, where u is the solution of

$$\begin{cases}-(r(t)\phi(u'))'=\lambda g(t)\left(f(\tilde v)-\frac{a}{\tilde v^\alpha}\right), & 0<t<1,\\ u(0)=0,\ u'(1)+H(u(1))=0,\end{cases}$$

where $\tilde v(t)=\max\left(v(t),\gamma p(t)/5\right)$. Note that u is given by

$$u(t)=\int_0^t\phi^{-1}\left(\frac{C-\lambda\int_0^s g(z)\left(f(\tilde v-\frac{a}{\tilde v^\alpha}\right)dz}{r(s)}\right)ds \tag{3.2}$$

for $t\in(0,1)$, where C is the unique number such that $u'(1)+H(u(1))=0$ i.e.

$$\phi^{-1}\left(\frac{C-\lambda\int_0^1 g(z)\left(f(\tilde v)-\frac{a}{\tilde v^\alpha}\right)dz}{r(1)}\right)+H\left(\int_0^1\phi^{-1}\left(\frac{C-\lambda\int_0^s g(z)\left(f(\tilde v)-\frac{a}{\tilde v^\alpha}\right)dz}{r(s)}\right)ds\right)=0.$$

Note that

$$|C|\leq\lambda\int_0^1 g(t)\left|f(\tilde v)-\frac{a}{\tilde v^\alpha}\right|dt,$$

from which (3.2) gives

$$|u|_{C^1}\leq\phi^{-1}\left(\frac{2\lambda\int_0^1 g(t)\left|f(\tilde v)-\frac{a}{\tilde v^\alpha}\right|dt}{r_0}\right)\leq\phi^{-1}\left(\frac{2\lambda\int_0^1 g(t)\left(f(\tilde v)+\frac{a}{(\gamma/5)^\alpha p^\alpha(t)}dt\right)}{r_0}\right), \tag{3.3}$$

where $|u|_{C^1} = \max(\|u\|_\infty, \|u'\|_\infty)$. From this and standard arguments, it follows that $T_\lambda : E \to E$ is a completely continuous operator. Next, we verify the two conditions of Lemma A.

(a) Let $u \in E$ satisfy $u = \theta T_\lambda u$ for some $\theta \in (0,1]$. Then $\|u\|_\infty \neq m$.

Suppose on the contrary that $\|u\|_\infty = m$. Then $\|\tilde{u}\|_\infty \leq m$. Since $u/\theta = T_\lambda u$, we deduce from (3.3) and the assumption $f(m) > \frac{a}{(\gamma/5)^\alpha}$ that

$$\left\|\frac{u}{\theta}\right\|_\infty \leq \phi^{-1}\left(\frac{2\lambda}{r_0}\int_0^1 g(t)\left(f(m) + \frac{a}{(\gamma/5)^\alpha p^\alpha(t)}\right)dt\right)$$
$$\leq \phi^{-1}\left(\frac{2\lambda}{r_0}\int_0^1 g(t)\left(f(m) + \frac{f(m)}{p^\alpha(t)}\right)dt\right) \leq \phi^{-1}\left(\frac{4\lambda f(m)}{r_0}\int_0^1 \frac{g}{p^\alpha}\right).$$

Hence

$$m \leq \phi^{-1}\left(\frac{4\lambda f(m)}{r_0}\int_0^1 \frac{g}{p^\alpha}\right),$$

which implies $\lambda \geq \frac{r_0\phi(m)}{4f(m)}\left(\int_0^1 \frac{g}{p^\alpha}\right)^{-1}$, a contradiction with $\lambda \in I$. Thus $\|u\|_\infty \neq m$.

(b) *Let $u \in E$ satisfy $u = T_\lambda u + \xi$ for some $\xi \geq 0$. Then $\|u\|_\infty \notin \{\gamma, R\}$ for $R >> 1$.*

Since $u - \xi = T_\lambda u$, u satisfies

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)\left(f(\tilde{u}) - \frac{a}{\tilde{u}^\alpha}\right),\ 0 < t < 1, \\ u(0) = \xi \geq 0,\ u'(1) + H(u(1)) = H(u(1)) - H(u(1) - \xi) \geq 0. \end{cases}$$

Note that

$$\lambda g(t)\left(f(\tilde{u} - \frac{a}{\tilde{u}^\alpha}\right) \geq -\frac{\lambda a g(t)}{(\gamma/5)^\alpha p^\alpha(t)} = -h_\lambda(t),$$

where $h_\lambda(t) = \frac{\lambda a}{(\gamma/5)^\alpha}\left(\frac{g(t)}{p^\alpha(t)}\right)$. Since $f(m) > \frac{a\phi(m)}{4(\gamma/5)^\alpha\phi(\gamma/5)}$ and $\lambda \in I$, we get

$$\lambda < \frac{r_0\phi(m)}{4f(m)}\left(\int_0^1 \frac{g}{p^\alpha}\right)^{-1} \leq \frac{r_0(\gamma/5)^\alpha\phi(\gamma/5)}{a}\left(\int_0^1 \frac{g}{p^\alpha}\right)^{-1}.$$

This implies $\phi(\gamma/5) > \frac{\lambda a}{r_0(\gamma/5)^\alpha}\left(\int_0^1 \frac{g}{p^\alpha}\right)$, i.e.

$$\gamma > 5\phi^{-1}\left(\frac{\lambda a}{r_0(\gamma/5)^\alpha}\left(\int_0^1 \frac{g}{p^\alpha}\right)\right) = 5\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right). \tag{3.4}$$

Suppose $\|u\|_\infty \in \{\gamma, R\}$ with $R > \gamma$. Since $\|u\|_\infty \geq \gamma > 5\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right)$ in view of (3.4), it follows from Lemma 2.3 (ii) that

$$u(t) \geq \frac{1}{5}\|u\|_\infty p(t) \quad \text{for } t \in [0,1]. \tag{3.5}$$

In particular, $u(t) \geq (\gamma/5)p(t)$ i.e. $\tilde{u} \equiv u$, and $u(t) \geq \|u\|_\infty/20$ for $t \in [1/4, 3/4]$. Consequently, u satisfies

$$\begin{cases} -(r(t)\phi(u'))' \geq \lambda g(t)\left(f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha p^\alpha(t)}\right),\ \frac{1}{4} < t < \frac{3}{4}, \\ u(1/4) \geq 0,\ u(3/4) \geq 0. \end{cases}$$

By the comparison principle, $u \geq v$ on $[1/4, 3/4]$, where v is the solution of

$$\begin{cases} -(r(t)\phi(v'))' = \lambda g(t)\left(f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha p^\alpha(t)}\right),\ \frac{1}{4} < t < \frac{3}{4}, \\ v(1/4) = 0,\ v(3/4) = 0. \end{cases}$$

Let $t_0 \in (1/4, 3/4)$ be such that $v'(t_0) = 0$. Then upon integrating, we obtain

$$v'(t) = \phi^{-1}\left(\frac{\lambda}{r(t)}\int_t^{t_0} g(s)\left(f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha p^\alpha(s)}\right)ds\right) \tag{3.6}$$

for $t \in [1/4, 3/4]$. We shall distinguish two cases.

Case 1. $t_0 > 1/2$.

Integrating (3.6) on $[1/4, 3/8]$ gives

$$\begin{aligned} v(3/8) &= \int_{1/4}^{3/8} \phi^{-1}\left(\frac{\lambda}{r(t)}\int_t^{t_0} g(s)\left(f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha p^\alpha(s)}\right)ds\right)dt \\ &\geq \int_{1/4}^{3/8} \phi^{-1}\left(\frac{\lambda}{r(t)}\left(\int_{3/8}^{1/2} g(s) f\left(\frac{\|u\|_\infty}{20}\right)ds - \frac{a}{(\gamma/5)^\alpha}\int_0^1 \frac{g}{p^\alpha}\right)dt\right) \\ &\geq \frac{1}{8}\phi^{-1}\left(\lambda\left(\frac{g_0}{8r_1} f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha r_0}\int_0^1 \frac{g}{p^\alpha}\right)\right). \end{aligned} \tag{3.7}$$

Since

$$f(m) > \frac{16ar_1(4m)^\beta}{g_0 r_0 (\gamma/5)^{\alpha+\beta}}\left(\int_0^1 \frac{g}{p^\alpha}\right),$$

it follows from (A6) that

$$\frac{g_0}{8r_1} f\left(\frac{\|u\|_\infty}{20}\right) \geq \frac{g_0}{8r_1} f\left(\frac{\gamma}{20}\right) \geq \frac{g_0}{8r_1}\left(\frac{\gamma}{20m}\right)^\beta f(m) \geq \frac{2a}{(\gamma/5)^\alpha r_0}\int_0^1 \frac{g}{p^\alpha}.$$

Hence (3.7) gives

$$v(3/8) \geq \frac{1}{8}\phi^{-1}\left(\frac{\lambda g_0}{16r_1} f\left(\frac{\|u\|_\infty}{20}\right)\right),$$

which implies

$$\|u\|_\infty \geq \frac{1}{8}\phi^{-1}\left(\frac{\lambda g_0}{16r_1} f\left(\frac{\|u\|_\infty}{20}\right)\right). \tag{3.8}$$

Case 2. $t_0 \leq 1/2$.

Integrating (3.6) on $[5/8, 3/4]$ gives

$$\begin{aligned} v(5/8) &= \int_{5/8}^{3/4} \phi^{-1}\left(\frac{\lambda}{r(t)}\int_{t_0}^{t} g(s)\left(f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha p^\alpha(s)}\right)ds\right)dt \\ &\geq \int_{5/8}^{3/4} \phi^{-1}\left(\frac{\lambda}{r(t)}\left(\int_{1/2}^{5/8} g(s) f\left(\frac{\|u\|_\infty}{20}\right)ds - \frac{a}{(\gamma/5)^\alpha}\int_0^1 \frac{g}{p^\alpha}\right)dt\right) \\ &\geq \frac{1}{8}\phi^{-1}\left(\lambda\left(\frac{g_0}{8r_1} f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{(\gamma/5)^\alpha r_0}\int_0^1 \frac{g}{p^\alpha}\right)\right) \\ &\geq \frac{1}{8}\phi^{-1}\left(\frac{\lambda g_0}{16r_1} f\left(\frac{\|u\|_\infty}{20}\right)\right), \end{aligned} \tag{3.9}$$

i.e. (3.8) holds. Thus (3.8) holds in either case. If $\|u\|_\infty = \gamma$ then (3.8) gives

$$\gamma \geq \frac{1}{8}\phi^{-1}\left(\frac{\lambda g_0}{16r_1} f\left(\frac{\gamma}{20}\right)\right),$$

which implies

$$\lambda \leq \frac{16r_1\phi(8\gamma)}{g_0 f(\gamma/20)},$$

a contradiction with $\lambda \in I$. Thus $\|u\|_\infty \neq \gamma$.

Since
$$\frac{f\left(\frac{\|u\|_\infty}{20}\right)}{\phi(8\|u\|_\infty)} \leq \frac{16r_1}{\lambda g_0}$$
in view of (3.8) and $\lim_{z\to\infty} \frac{f(z/20)}{\phi(8z)} = \infty$ in view of (A4) and (A6), it follows that $\|u\|_\infty \neq R$ for $R >> 1$.

By Lemma A, T_λ has two fixed points $u_{i,\lambda}, i = 1,2$, such that $\gamma < \|u_{1,\lambda}\|_\infty < m$, $m < \|u_{2,\lambda}\|_\infty < R$. Since $\|u_{i,\lambda}\|_\infty \geq \gamma$, it follows from (3.5) with $\xi = 0$ that $u_{i,\lambda}(t) \geq \frac{\gamma}{5}p(t)$ for $t \in [0,1]$ i.e. $\tilde{u}_{i,\lambda} = u_{i,\lambda}$ on $[0,1]$ for $i = 1,2$. Hence $u_{i,\lambda}$, $i = 1,2$, are positive solutions of (1.1).

(ii) We shall modify the above proof. Let $\lambda > 0$ satisfy
$$\phi^{-1}\left(\frac{2\lambda}{r_0}\int_0^1 g(t)\left(f(5) + \frac{a}{p^\alpha(t)}\right)dt\right) < 5.$$
For $v \in E$, define $S_\lambda v = u$, where u is the solution of
$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)\left(f(\tilde{v}) - \frac{a}{\tilde{v}^\alpha}\right), \ 0 < t < 1, \\ u(0) = 0, \ u'(1) + H(u(1)) = 0, \end{cases}$$
where $\tilde{v}(t) = \max(v(t), p(t))$. ($S_\lambda$ is T_λ in part (i) with $\gamma = 5$.) Then $S_\lambda : E \to E$ is completely continuous. We verify that

(c) *Let $u \in E$ satisfy $u = \theta T_\lambda u$ for some $\theta \in (0,1]$. Then $\|u\|_\infty \neq 5$.*

Suppose $\|u\|_\infty = 5$. Then, as in part (a) above, we get
$$5 = \|u\|_\infty \leq \phi^{-1}\left(\frac{2\lambda}{r_0}\int_0^1 g(t)\left(f(5) + \frac{a}{p^\alpha(t)}\right)dt\right) < 5,$$
a contradiction with the choice of λ. Thus $\|u\|_\infty \neq 5$.

(d) *Let $u \in E$ satisfy $u = T_\lambda u + \xi$ for some $\xi \geq 0$. Then $\|u\|_\infty \neq R$ for $R >> 1$.*

Suppose $\|u\|_\infty = R$. Using the same arguments as in part (b) above with $\gamma = 5$ and note that for R large
$$R = \|u\|_\infty > 5\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right),$$
where $h_\lambda(t) = \lambda a\left(\frac{g(t)}{p^\alpha(t)}\right)$. Hence
$$u(t) \geq \frac{\|u\|_\infty}{5}p(t) \geq p(t) \quad \text{for } t \in [0,1], \tag{3.10}$$
i.e. $\tilde{u} = u$ on $[0,1]$. As in (3.7) and (3.9) above, we obtain
$$\|u\|_\infty \geq \frac{1}{8}\phi^{-1}\left(\lambda\left(\frac{g_0}{8r_1}f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)\right)$$
i.e.
$$\frac{\frac{g_0}{8r_1}f\left(\frac{\|u\|_\infty}{20}\right) - \frac{a}{r_0}\int_0^1 \frac{g}{p^\alpha}}{\phi(8\|u\|_\infty)} \leq \frac{1}{\lambda}. \tag{3.11}$$

Since the left side of (3.11) tends to ∞ as $\|u\|_\infty$ goes to ∞, we deduce that $\|u\|_\infty < R$ for $R >> 1$, which proves (d). By Lemma A, S_λ has a fixed point u_λ with $\|u_\lambda\|_\infty > 5$, which together with (3.10) with $\xi = 0$ imply that $\tilde{u}_\lambda = u_\lambda$ on $[0,1]$, i.e. u_λ is a positive solution of (1.1). We show next that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. Since $H(u_\lambda(1)) \geq H(0) = 0$, it follows from the boundary condition of u_λ at 1 that $u'_\lambda(1) \leq 0$. Hence u_λ satisfies

$$\begin{cases} -(r(t)\phi(u'_\lambda))' \leq \lambda g(t) f(\|u_\lambda\|_\infty), & 0 < t < 1, \\ u_\lambda(0) = 0, \; u'_\lambda(1) \leq 0. \end{cases}$$

By the comparison principle, $u_\lambda \leq v_\lambda$ on $[0,1]$, where v_λ is the solution of

$$\begin{cases} -(r(t)\phi(v'_\lambda))' = \lambda g(t) f(\|u_\lambda\|_\infty), & 0 < t < 1, \\ v_\lambda(0) = 0, \; v'_\lambda(1) = 0. \end{cases} \tag{3.12}$$

Note that

$$v_\lambda(t) = \int_0^t \phi^{-1}\left(\frac{\lambda f(\|u_\lambda\|_\infty)}{r(s)} \int_s^1 g\right) ds \leq \phi^{-1}\left(\frac{\lambda \|g\|_1 f(\|u_\lambda\|_\infty)}{r_0}\right)$$

for $t \in [0,1]$. Hence

$$\|u_\lambda\|_\infty \leq \phi^{-1}\left(\frac{\lambda \|g\|_1 f(\|u_\lambda\|_\infty)}{r_0}\right),$$

i.e.

$$\frac{f(\|u_\lambda\|_\infty)}{\phi(\|u_\lambda\|_\infty)} \geq \frac{r_0}{\lambda \|g\|_1}.$$

Consequently,

$$\lim_{\lambda \to 0^+} \frac{f(\|u_\lambda\|_\infty)}{\phi(\|u_\lambda\|_\infty)} = \infty$$

and since $\|u_\lambda\|_\infty > 5$, it follows from (A5) that $\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty$.

Next, we show that $u_\lambda(1) \to \infty$ as $\lambda \to 0^+$. Since $u'_\lambda(1) \leq 0$, it follows upon integrating the equation in (1.1) and using Lemma 2.1 (ii) that

$$\begin{aligned} u_\lambda(1) &= \int_0^1 \phi^{-1}\left(\frac{\lambda \int_t^1 g(s)\left(f(u_\lambda) - \frac{a}{u_\lambda^\alpha}\right) ds - r(1)\phi(|u'_\lambda(1)|)}{r(t)}\right) dt \\ &\geq \int_0^1 \phi^{-1}\left(\frac{\lambda \int_t^1 g(s)\left(f(u_\lambda) - \frac{a}{u_\lambda^\alpha}\right) ds}{r(t)}\right) dt - 2\int_0^1 \phi^{-1}\left(\frac{r(1)\phi(|u'_\lambda(1)|)}{r(t)}\right) dt. \end{aligned}$$

Thus

$$u_\lambda(1) + 2\int_0^1 \phi^{-1}\left(\frac{r(1)\phi(H(u_\lambda(1)))}{r(t)}\right) dt \geq \int_0^1 \phi^{-1}\left(\frac{\lambda \int_t^1 g(s)\left(f(u_\lambda) - \frac{a}{u_\lambda^\alpha}\right) ds}{r(t)}\right) dt. \tag{3.13}$$

Since

$$\int_0^1 \phi^{-1}\left(\frac{\lambda\int_t^1 g(s)\left(f(u_\lambda)-\frac{a}{u_\lambda^\alpha}\right)ds}{r(t)}\right)dt \geq \int_0^1 \phi^{-1}\left(\frac{\lambda\int_t^{3/4} g(s)f(u_\lambda)ds}{r_1}-\frac{\lambda a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)dt$$

$$\geq \int_0^{1/2} \phi^{-1}\left(\frac{\lambda\int_t^{3/4} g(s)f(u_\lambda)ds}{r_1}-\frac{\lambda a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)dt-\frac{1}{2}\phi^{-1}\left(\frac{\lambda a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)$$

$$\geq \frac{1}{2}\phi^{-1}\left(\frac{\lambda\left(\int_{1/2}^{3/4} g\right)f\left(\frac{\|u_\lambda\|_\infty}{4}\right)}{r_1}-\frac{\lambda a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)-\frac{1}{2}\phi^{-1}\left(\frac{\lambda a}{r_0}\int_0^1 \frac{g}{p^\alpha}\right)\to\infty \quad \text{as } \lambda\to\infty,$$

we deduce from (3.13) that $u_\lambda(1)\to\infty$ as $\lambda\to\infty$. Since u_λ satisfies

$$(r(t)\phi(u_\lambda'))' \leq h_\lambda \quad \text{on } (0,1),$$

where $h_\lambda(t)=\lambda a(g(t)/p^\alpha(t))$, Lemma 2.3 (i) gives

$$u_\lambda(t)\geq\left(u_\lambda(1)-2\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right)\right)t$$

for $t\in[0,1]$. Consequently, $u_\lambda(t)\to\infty$ as $\lambda\to 0^+$ uniformly on compact subsets of $(0,1]$, which completes the proof of Theorem 1.1. □

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