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New oscillation criteria for third-order differential equations with bounded and unbounded neutral coefficients

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Abstract. This paper examines the oscillatory behavior of solutions to a class of third-order differential equations with bounded and unbounded neutral coefficients. Sufficient conditions for all solutions to be oscillatory are given. Some examples are considered to illustrate the main results and suggestions for future research are also included.

Keywords: oscillation, third-order, neutral differential equation.

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1 Introduction

In this paper, we wish to obtain some new criteria for the oscillation of all solutions of the third-order differential equations with bounded and unbounded neutral coefficients of the form

$$(x(t) + p(t)x(\tau(t)))''' + q(t)x^{\beta}(\sigma(t)) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, and β is the ratio of odd positive integers with $0 < \beta \leq 1$. Throughout the paper, we will always assume that:

(C1) $p, q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ not identically zero for large t;

(C2) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, τ is strictly increasing, σ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

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(C3) there exist a constant $\theta \in (0,1)$ and $t_\theta \geq t_0$ such that

$$\left(\frac{t}{\tau(t)}\right)^{2/\theta} \frac{1}{p(t)} \leq 1, \quad t \geq t_\theta. \tag{1.2}$$

By a *solution* of equation (1.1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \geq t_0$ such that $x(t) + p(t)x(\tau(t)) \in C^3([t_x,\infty),\mathbb{R})$ and x satisfies (1.1) on $[t_x,\infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x,\infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq t_x;$$

we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of equation (1.1) is said to be *oscillatory* if it has arbitrarily large zeros, and it is called *nonoscillatory* otherwise. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

Neutral differential equations are differential equations in which the highest order derivative of the unknown function appears both with and without deviating arguments. As stated in many sources, besides their theoretical interest, equations of this type have numerous applications in the natural sciences and technology. For example, they appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and as the Euler equation in some variational problems; we refer the reader to the monograph by Hale [14] for these and other applications.

Oscillatory and asymptotic behavior of solutions to various classes of third and higher odd-order neutral differential equations have been attracting attention of researchers during the last few decades, and we mention the papers [1, 3–13, 15, 18–26] and the references cited therein for examples of some recent contributions in this area. However, except for the papers [3,4,12,23,26], all the above cited papers were concerned with the case where $p(t)$ is bounded, i.e., the cases where $0 \leq p(t) \leq p_0 < 1$, $-1 < p_0 \leq p(t) \leq 0$, and $0 \leq p(t) \leq p_0 < \infty$ were considered, and so the results established in these papers cannot be applied to the case $p(t) \to \infty$ as $t \to \infty$. Based on this observation, the aim of this paper is to establish some new oscillation criteria that can be applied not only to the case where $p(t) \to \infty$ as $t \to \infty$ but also to the case where $p(t)$ is a bounded function. We would like to point out that the results established here are motivated by oscillation results of Koplatadze et all. [17], where a nth order linear differential equation with a deviating argument was considered. Since our equation considered here is fairly simple, it would be possible to extend our results to the more general equations studied in the papers cited above and to the others types that include equation (1.1) as a special case. For these reasons, it is our hope that the present paper will stimulate additional interest in research on third and higher odd-order neutral differential equations in general, and those with unbounded neutral coefficients in particular.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1); since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

2 Main results

For the reader's convenience, we define:

$$z(t) := x(t) + p(t)x(\tau(t)),$$

$$h(t) := \tau^{-1}(\sigma(t)), \quad g(t) := \tau^{-1}(\eta(t)), \quad \eta \in C^1([t_0, \infty)),$$

$$\pi_1(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{2/\theta} \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right]$$

and

$$\pi_2(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right],$$

where τ^{-1} is the inverse function of τ (if τ is invertible) and $\theta \in (0, 1)$. It is also important to notice that condition (1.2) in (C3) ensures the nonnegativity of the functions $\pi_1(t)$.

Lemma 2.1 (See [2, Lemma 1]). *Suppose that the function h satisfies $h^{(i)}(t) > 0$, $i = 0, 1, 2, \dots, m$, and $h^{(m+1)}(t) \le 0$ on $[T, \infty)$ and $h^{(m+1)}(t)$ is not identically zero on any interval of the form $[T', \infty)$, $T' \ge T$. Then for every $\theta \in (0, 1)$,*

$$\frac{h(t)}{h'(t)} \ge \theta \frac{t}{m},$$

eventually.

Lemma 2.2. *Assume that x is an eventually positive solution of (1.1), say for $t_1 \ge t_0$. Then there exists a $t_2 \ge t_1$ such that the corresponding function z satisfies one of the following two cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \le 0$,

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) \le 0$

for $t \ge t_2$.

Proof. This result follows immediately from Kiguradze's lemma [16], so we omit its proof. □

Lemma 2.3. *Let $x(t)$ be an eventually positive solution of (1.1) with $z(t)$ satisfying case (I) of Lemma 2.2 for $t \ge t_2$ for some $t_2 \ge t_1$. Then for every $\theta \in (0, 1)$ there exists a $t_\theta \ge t_2$ such that*

$$\left(\frac{z(t)}{t^{2/\theta}}\right)' \le 0 \quad \text{for } t \ge t_\theta. \tag{2.1}$$

Proof. Since z satisfies case (I) of Lemma 2.2 for $t \ge t_2$ for some $t_2 \ge t_1$, by Lemma 2.1, there exists a $t_\theta \ge t_2$ for every $\theta \in (0, 1)$ such that

$$z(t) \ge \frac{\theta}{2} t z'(t) \quad \text{for } t \ge t_\theta. \tag{2.2}$$

It follows from (2.2) that

$$\left(\frac{z(t)}{t^{2/\theta}}\right)' = \frac{\theta t z'(t) - 2z(t)}{\theta t^{2/\theta + 1}} \le 0 \quad \text{for } t \ge t_\theta.$$

This completes the proof of the lemma. □

Lemma 2.4. *Let $x(t)$ be an eventually positive solution of (1.1) with z(t) satisfying case (I) of Lemma 2.2. Assume that*

$$\int_{t_0}^{\infty} \int_u^{\infty} q(s) \pi_1^{\beta}(\sigma(s)) h^{\beta}(s) ds du = \infty. \tag{2.3}$$

Then:

(i) *z satisfies the inequality*

$$z'''(t) + q(t)\pi_1^{\beta}(\sigma(t))z^{\beta}(h(t)) \le 0 \tag{2.4}$$

for large t;

(ii) $z'(t) \to \infty$ *as* $t \to \infty$;

(iii) $z(t)/t$ *is increasing.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. From the definition of z, we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{2.5}$$

Now $\tau(t) \le t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \le \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t)). \tag{2.6}$$

Since $z(t)$ satisfies case (I) for $t \ge t_2$, by Lemma 2.3, there exists a $t_\theta \ge t_2$ such that (2.1) holds for $t \ge t_\theta$. From (2.1) and (2.6), we observe that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \le \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{2/\theta} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{2/\theta}}. \tag{2.7}$$

Using (2.7) in (2.5) yields

$$x(t) \ge \pi_1(t)z(\tau^{-1}(t)) \quad \text{for } t \ge t_\theta. \tag{2.8}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_3 \ge t_\theta$ such that $\sigma(t) \ge t_\theta$ for all $t \ge t_3$. Thus, it follows from (2.8) that

$$x(\sigma(t)) \ge \pi_1(\sigma(t))z(\tau^{-1}(\sigma(t))) \quad \text{for } t \ge t_3. \tag{2.9}$$

Using (2.9) in (1.1) gives

$$z'''(t) + q(t)\pi_1^{\beta}(\sigma(t))z^{\beta}(h(t)) \le 0 \ \text{ for } t \ge t_3, \tag{2.10}$$

i.e., (2.4) holds.

Next, we claim that condition (2.3) implies $z'(t) \to \infty$ as $t \to \infty$. If this is not the case, then there exists a constant $k > 0$ such that $\lim_{t\to\infty} z'(t) = k$, and so $z'(t) \le k$. Since $z'(t)$ is positive and increasing on $[t_2, \infty)$, there exist a $t_3 \ge t_2$ and a constant $c > 0$ such that

$$z'(t) \ge c \quad \text{for } t \ge t_3,$$

which implies

$$z(t) \ge dt$$

for $t \ge t_4$, for some $t_4 \ge t_3$ and some $d > 0$. Since $\lim_{t\to\infty} h(t) = \infty$, we can choose $t_5 \ge t_4$ such that $h(t) \ge t_4$ for all $t \ge t_5$, so

$$z(h(t)) \ge dh(t).$$

Using this in (2.10) gives

$$z'''(t) + d^\beta q(t)\pi_1^\beta(\sigma(t))h^\beta(t) \le 0 \ \text{ for } t \ge t_5.$$

Integrating this inequality from t to ∞, we obtain

$$z''(t) \ge d^\beta \int_t^\infty q(s)\pi_1^\beta(\sigma(s))h^\beta(s)ds.$$

Now integrating from t_5 to t yields

$$k \ge z'(t) \ge d^\beta \int_{t_5}^t \int_u^\infty q(s)\pi_1^\beta(\sigma(s))h^\beta(s)dsdu,$$

which contradicts (2.3) and proves the claim.

Finally, from the fact that $z'(t) \to \infty$ as $t \to \infty$, we see that

$$z(t) = z(t_2) + \int_{t_2}^t z'(s)ds \le z(t_2) + (t - t_2)z'(t) \le tz'(t),$$

which implies

$$\left(\frac{z(t)}{t}\right)' = \frac{tz'(t) - z(t)}{t^2} \ge 0,$$

i.e., (iii) holds. The proof of the lemma is now complete. □

Lemma 2.5. *Let $x(t)$ be an eventually positive solution of (1.1) with z(t) satisfying case (I) of Lemma 2.2. If*

$$\int_{t_0}^\infty q(s)\pi_1^\beta(\sigma(s))h^{2\beta/\theta}(s)ds = \infty, \tag{2.11}$$

then

$$\lim_{t\to\infty} \frac{z(t)}{t^{2/\theta}} = 0. \tag{2.12}$$

Proof. Since $z(t)$ satisfies case (I) for $t \ge t_2$ for some $t_2 \ge t_1$, by Lemma 2.3, there exists a $t_\theta \ge t_2$ such that (2.1) holds for $t \ge t_\theta$, i.e., $z(t)/t^{2/\theta}$ is decreasing for $t \ge t_\theta$. We now claim that (2.11) implies

$$\lim_{t\to\infty} \frac{z(t)}{t^{2/\theta}} = 0.$$

If this is not the case, then there exist a constant $b > 0$ and a $t_3 \ge t_\theta$ such that

$$z(t) \ge bt^{2/\theta} \quad \text{for } t \ge t_3. \tag{2.13}$$

Since case (I) holds, we again arrive at (2.10) for $t \ge t_3$. Using (2.13) in (2.10) gives

$$z'''(t) + b^\beta q(t)\pi_1^\beta(\sigma(t))h^{2\beta/\theta}(t) \le 0 \tag{2.14}$$

for $t \ge t_4$ for some $t_4 \ge t_3$. Integrating (2.14) from t_4 to t yields

$$\int_{t_4}^t q(s)\pi_1^\beta(\sigma(s))h^{2\beta/\theta}(s)ds \le \frac{z''(t_4)}{b^\beta},$$

which contradicts (2.11) and completes the proof. □

Lemma 2.6. *Let $x(t)$ be an eventually positive solution of (1.1) with z(t) satisfying case (II) of Lemma 2.2. Suppose also that there exists a nondecreasing function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that $\sigma(t) \le \eta(t) < \tau(t)$ for $t \ge t_0$. If*

$$\int_{t_0}^{\infty} q(s)\pi_2(\sigma(s))(g(s)-h(s))^{2\beta} ds = \infty, \tag{2.15}$$

then

$$\lim_{t\to\infty} z''(t) = 0. \tag{2.16}$$

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. As in Lemma 2.4, we again see that (2.5) and (2.6) hold. Since $z'(t) < 0$, it follows from (2.6) that

$$z(\tau^{-1}(t)) \ge z(\tau^{-1}(\tau^{-1}(t))),$$

so inequality (2.5) takes the form

$$x(t) \ge \pi_2(t) z(\tau^{-1}(t)). \tag{2.17}$$

Using (2.17) in (1.1) gives

$$z'''(t) + q(t)\pi_2^{\beta}(\sigma(t)) z^{\beta}(h(t)) \le 0 \tag{2.18}$$

for $t \ge t_3$ for some $t_3 \ge t_2$. Since $(-1)^k z^{(k)}(t) > 0$ for $k = 0,1,2$ and $z'''(t) \le 0$, for $t_3 \le u \le v$, it is easy to see that

$$z(u) \ge \frac{(v-u)^2}{2} z''(v). \tag{2.19}$$

Since $\sigma(t) \le \eta(t)$ and τ is increasing, we conclude that $\tau^{-1}(\sigma(t)) \le \tau^{-1}(\eta(t))$, i.e, $h(t) \le g(t)$. Letting $u = h(t)$ and $v = g(t)$ in (2.19), we obtain

$$z(h(t)) \ge \frac{(g(t)-h(t))^2}{2} z''(g(t)).$$

Using the latter inequality in (2.18) gives

$$z'''(t) + \frac{1}{2^{\beta}} q(t)\pi_2^{\beta}(\sigma(t))(g(t)-h(t))^{2\beta} \left(z''(g(t))\right)^{\beta} \le 0. \tag{2.20}$$

Since $\pi_2(t) < 1$, we have $\pi_2^{\beta}(t) \ge \pi_2(t)$. So, inequality (2.20) takes the form

$$z'''(t) + \frac{1}{2^{\beta}} q(t)\pi_2(\sigma(t))(g(t)-h(t))^{2\beta} \left(z''(g(t))\right)^{\beta} \le 0. \tag{2.21}$$

Now, we claim that (2.15) implies $z''(t) \to 0$ as $t \to \infty$. Suppose to the contrary that

$$\lim_{t\to\infty} z''(t) = \ell > 0.$$

Then, $z''(t) \ge \ell$ for $t \ge t_3$ for some $t_3 \ge t_2$. Since $\lim_{t\to\infty} g(t) = \infty$, we can choose $t_4 \ge t_3$ such that $g(t) \ge t_3$ for all $t \ge t_4$. Hence, $z''(g(t)) \ge \ell$ for $t \ge t_4$. Using this in (2.21) gives

$$z'''(t) + \frac{\ell^{\beta}}{2^{\beta}} q(t)\pi_2(\sigma(t))(g(t)-h(t))^{2\beta} \le 0 \quad \text{for } t \ge t_4. \tag{2.22}$$

Integrating (2.22) from t_4 to t yields

$$\int_{t_4}^{t} q(s)\pi_2(\sigma(s))(g(s)-h(s))^{2\beta} ds \le \left(\frac{2}{\ell}\right)^{\beta} z''(t_4),$$

which contradicts (2.15) and completes the proof. □

Now, we are ready to present our main results. Our first result is concerned with equation (1.1) in the case where $\beta = 1$, i.e., equation (1.1) is linear.

Theorem 2.7. *Let (2.3) hold and assume that there exists a nondecreasing function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that $\sigma(t) \le \eta(t) < \tau(t)$ for $t \ge t_0$. If there exist constants $\alpha, \theta \in (0,1)$ such that*

$$\limsup_{t\to\infty}\left(\frac{\alpha\theta h^{1-\frac{2}{\theta}}(t)}{2}\int_{t_0}^{h(t)} sq(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds + \frac{\alpha\theta h^{2-\frac{2}{\theta}}(t)}{2}\int_{h(t)}^{t} q(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds + \frac{\alpha\theta h(t)}{2}\int_t^\infty q(s)\pi_1(\sigma(s))h(s)ds\right) > 1, \tag{2.23}$$

and

$$\limsup_{t\to\infty}\int_{g(t)}^t \frac{1}{2}q(s)\pi_2(\sigma(s))(g(s)-h(s))^2ds > 1, \tag{2.24}$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then, from Lemma 2.2, the corresponding function z satisfies either case (I) or case (II) for $t \ge t_2$ for some $t_2 \ge t_1$.

First, we consider case (I). By Lemma 2.4, we again arrive at (2.10) for $t \ge t_3$, which, for $\beta = 1$, takes the form

$$z'''(t) + q(t)\pi_1(\sigma(t))z(h(t)) \le 0 \quad \text{for } t \ge t_3. \tag{2.25}$$

Integrating (2.25) from t to ∞ yields

$$z''(t) \ge \int_t^\infty q(s)\pi_1(\sigma(s))z(h(s))ds, \tag{2.26}$$

and integrating again from t_3 to t yields

$$\begin{aligned} z'(t) &\ge \int_{t_3}^t\int_u^\infty q(s)\pi_1(\sigma(s))z(h(s))dsdu \\ &= \int_{t_3}^t\int_u^t q(s)\pi_1(\sigma(s))z(h(s))dsdu + \int_{t_3}^t\int_t^\infty q(s)\pi_1(\sigma(s))z(h(s))dsdu \\ &= \int_{t_3}^t (s-t_3)q(s)\pi_1(\sigma(s))z(h(s))ds + (t-t_3)\int_t^\infty q(s)\pi_1(\sigma(s))z(h(s))ds. \end{aligned}$$

For any $\alpha \in (0,1)$ there exists $t_4 \ge t_3$ such that $s - t_3 \ge \alpha s$ and $t - t_3 \ge \alpha t$ for $t \ge s \ge t_4$. Thus, from the last inequality we see that

$$z'(t) \ge \alpha\int_{t_4}^t sq(s)\pi_1(\sigma(s))z(h(s))ds + \alpha t\int_t^\infty q(s)\pi_1(\sigma(s))z(h(s))ds. \tag{2.27}$$

In view of (2.2), it follows that

$$\frac{2z(t)}{\theta t} \ge \alpha\int_{t_4}^t sq(s)\pi_1(\sigma(s))z(h(s))ds + \alpha t\int_t^\infty q(s)\pi_1(\sigma(s))z(h(s))ds. \tag{2.28}$$

From (2.28), we see that

$$\frac{2z(h(t))}{\theta h(t)} \geq \alpha \int_{t_4}^{h(t)} sq(s)\pi_1(\sigma(s))z(h(s))ds + \alpha h(t) \int_{h(t)}^{t} q(s)\pi_1(\sigma(s))z(h(s))ds + \alpha h(t) \int_{t}^{\infty} q(s)\pi_1(\sigma(s))z(h(s))ds. \tag{2.29}$$

Also, for $t \leq s$, we have $h(t) \leq h(s)$. Since $z(t)/t$ is increasing (see Lemma 2.4 (iii)),

$$z(h(s)) \geq \frac{h(s)z(h(t))}{h(t)}. \tag{2.30}$$

For $h(t) \leq s \leq t$, we have $h(h(t)) \leq h(s) \leq h(t)$. Since $z(t)/t^{2/\theta}$ is decreasing (see (2.1)),

$$z(h(s)) \geq h^{2/\theta}(s)\frac{z(h(t))}{h^{2/\theta}(t)}. \tag{2.31}$$

For $t_4 \leq s \leq h(t)$ and $h(t) \leq t$, we have $h(s) \leq h(h(t)) \leq h(t)$. Since $z(t)/t^{2/\theta}$ is decreasing, we again obtain (2.31). Using (2.30) and (2.31) in (2.29) gives

$$\frac{2z(h(t))}{\theta h(t)} \geq \left(\alpha \int_{t_4}^{h(t)} sq(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds\right) \frac{z(h(t))}{(h(t))^{\frac{2}{\theta}}} + \left(\alpha h(t) \int_{h(t)}^{t} q(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds\right) \frac{z(h(t))}{(h(t))^{\frac{2}{\theta}}} + \left(\alpha h(t) \int_{t}^{\infty} q(s)\pi_1(\sigma(s))h(s)ds\right) \frac{z(h(t))}{h(t)}. \tag{2.32}$$

From (2.32), we see that

$$\frac{\alpha\theta h^{1-\frac{2}{\theta}}(t)}{2} \int_{t_4}^{h(t)} sq(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds + \frac{\alpha\theta h^{2-\frac{2}{\theta}}(t)}{2} \int_{h(t)}^{t} q(s)\pi_1(\sigma(s))(h(s))^{2/\theta}ds + \frac{\alpha\theta h(t)}{2} \int_{t}^{\infty} q(s)\pi_1(\sigma(s))h(s)ds \leq 1.$$

Taking the $\limsup_{t\to\infty}$ on both sides of the above inequality, we obtain a contradiction to condition (2.23),

Next, we consider case (II). As in Lemma 2.6, we again arrive at (2.20), which, for $\beta = 1$, takes the form

$$z'''(t) + \frac{1}{2}q(t)\pi_2(\sigma(t))(g(t) - h(t))^2 z''(g(t)) \leq 0. \tag{2.33}$$

Integrating (2.33) from $g(t)$ to t yields

$$z''(t) + \left[\int_{g(t)}^{t} \frac{1}{2}q(s)\pi_2(\sigma(s))(g(s) - h(s))^2 ds - 1\right] z''(g(t)) \leq 0,$$

which, by (2.24), leads to a contradiction. This completes the proof of the theorem. □

Our next results is for equation (1.1) in the case where $\beta < 1$, i.e., equation (1.1) is sublinear.

Theorem 2.8. *Let* (2.3) *and* (2.11) *hold. Assume that there exists a nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t) \le \eta(t) < \tau(t)$ *for* $t \ge t_0$. *If there exists* $\theta \in (0,1)$ *such that*

$$\limsup_{t\to\infty}\Bigg(h^{1-\frac{2}{\theta}}(t)\int_{t_0}^{h(t)} sq(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds + h^{2-\frac{2}{\theta}}(t)\int_{h(t)}^{t} q(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds + \frac{h^{2-\beta}(t)}{h^{2(1-\beta)/\theta}(t)}\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))h^{\beta}(s)ds\Bigg) > 0, \quad (2.34)$$

and

$$\limsup_{t\to\infty}\int_{g(t)}^{t} q(s)\pi_2(\sigma(s))(g(s)-h(s))^{2\beta}ds > 0, \quad (2.35)$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then, by Lemma 2.2, the corresponding function z satisfies either case (I) or case (II) for $t \ge t_2$ for some $t_2 \ge t_1$.

First, we consider case (I). By Lemma 2.4, we again arrive at (2.10) for $t \ge t_3$. Integrating (2.10) from t to ∞ gives

$$z''(t) \ge \int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds. \quad (2.36)$$

Integrating (2.36) from t_3 to t yields

$$\begin{aligned} z'(t) &\ge \int_{t_3}^{t}\int_u^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))dsdu \\ &= \int_{t_3}^{t}\int_u^{t} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))dsdu + \int_{t_3}^{t}\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))dsdu \\ &= \int_{t_3}^{t}(s-t_3)q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds + (t-t_3)\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds. \end{aligned}$$

For any $\alpha \in (0,1)$ there exists $t_4 \ge t_3$ such that $s - t_3 \ge \alpha s$ and $t - t_3 \ge \alpha t$ for $t \ge s \ge t_4$. Thus,

$$z'(t) \ge \alpha\int_{t_4}^{t} sq(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds + \alpha t\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds. \quad (2.37)$$

By (2.2) and (2.37), we observe that

$$\frac{2z(t)}{\theta t} \ge \alpha\int_{t_4}^{t} sq(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds + \alpha t\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds. \quad (2.38)$$

It follows from (2.38) that

$$\frac{2z(h(t))}{\theta h(t)} \ge \alpha\int_{t_4}^{h(t)} sq(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds + \alpha h(t)\int_{h(t)}^{t} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds + \alpha h(t)\int_t^{\infty} q(s)\pi_1^{\beta}(\sigma(s))z^{\beta}(h(s))ds. \quad (2.39)$$

Using (2.30) and (2.31) in (2.39) gives

$$\begin{aligned}\frac{2z(h(t))}{\theta h(t)} \geq &\left(\alpha \int_{t_4}^{h(t)} sq(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds\right)\frac{z^{\beta}(h(t))}{h^{2\beta/\theta}(t)}\\ &+\left(\alpha h(t)\int_{h(t)}^{t} q(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds\right)\frac{z^{\beta}(h(t))}{h^{2\beta/\theta}(t)}\\ &+\left(\alpha h(t)\int_{t}^{\infty} q(s)\pi_1^{\beta}(\sigma(s))h^{\beta}(s)ds\right)\frac{z^{\beta}(h(t))}{h^{\beta}(t)}. \qquad (2.40)\end{aligned}$$

Letting

$$w(t) = \frac{z(h(t))}{(h(t))^{2/\theta}},$$

it follows from (2.40) that

$$\begin{aligned}\frac{2}{\alpha\theta}w^{1-\beta}(t) \geq\, &h^{1-\frac{2}{\theta}}(t)\left(\int_{t_4}^{h(t)} sq(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds\right)\\ &+h^{2-\frac{2}{\theta}}(t)\left(\int_{h(t)}^{t} q(s)\pi_1^{\beta}(\sigma(s))(h(s))^{2\beta/\theta}ds\right)\\ &+\frac{h^{2-\beta}(t)}{h^{2(1-\beta)/\theta}}\left(\int_{t}^{\infty} q(s)\pi_1^{\beta}(\sigma(s))h^{\beta}(s)ds\right). \qquad (2.41)\end{aligned}$$

Taking the $\limsup_{t\to\infty}$ on both sides of the above inequality and using (2.12) , we obtain a contradiction to condition (2.34).

Next, we consider case (II). As in the proof of Lemma 2.6, we again arrive at (2.21). Integrating (2.21) from $g(t)$ to t yields

$$\int_{g(t)}^{t} q(s)\pi_2(\sigma(s))(g(s)-h(s))^{2\beta}ds \leq 2^{\beta}\left(z''(g(t))\right)^{1-\beta}.$$

Noting that (2.35) implies (2.15), we see that (2.16) holds. Taking the $\limsup_{t\to\infty}$ on both sides of the above inequality and using (2.16), we obtain a contradiction to condition (2.35), and this proves the theorem. □

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with an equation with bounded neutral coefficients in the case where p is a constant function; the second example is for an equation with unbounded neutral coefficients in the case where $p(t)\to\infty$ as $t\to\infty$.

Example 2.9. Consider the third-order differential equation of Euler type

$$\left(x(t)+16x\left(\frac{t}{2}\right)\right)''' + \frac{q_0}{t^3}x\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \qquad (2.42)$$

Here $p(t) = 16$, $q(t) = q_0/t^3$, $\beta = 1$, $\tau(t) = t/2$, and $\sigma(t) = t/4$. Then, it is easy to see that conditions (C1)–(C2) hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t,\ h(t) = t/2,\ \text{ and }\ g(t) = 2t/3 \text{ with } \eta(t) = t/3.$$

Choosing $\theta = 2/3$, we see that

$$\left(\frac{t}{\tau(t)}\right)^{2/\theta}\frac{1}{p(t)} = \frac{1}{2},$$

i.e., condition (C3) holds, $\pi_1(t) = 1/32$ and $\pi_2(t) = 15/256$. Letting $\alpha = \theta = 2/3$, by Theorem 2.7, Eq. (2.42) is oscillatory for

$$q_0 > \frac{3 \times 2^{11}}{5 \ln \frac{3}{2}}.$$

Example 2.10. Consider the sublinear equation

$$\left(x(t) + tx\left(\frac{t}{2}\right) \right)''' + \frac{q_0}{t^{6/5}} x^{3/5}\left(\frac{t}{10}\right) = 0, \quad t \geq 16. \tag{2.43}$$

Here $p(t) = t$, $q(t) = q_0/t^{6/5}$, $\beta = 3/5$, $\tau(t) = t/2$, and $\sigma(t) = t/10$. Then, it is easy to see that conditions $(C1)$–$(C2)$ hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t,\ h(t) = t/5, \ \text{ and } \ g(t) = t/4 \text{ with } \eta(t) = t/8.$$

Choosing $\theta = 2/3$, we see that

$$\left(\frac{t}{\tau(t)}\right)^{2/\theta} \frac{1}{p(t)} = \frac{8}{t} \leq \frac{1}{2},$$

i.e., condition (C3) holds. Since $\pi_1(t) \geq 7/16t$ and $\pi_2(t) \geq 63/128t$, by Theorem 2.8, Eq. (2.43) is oscillatory for all $q_0 > 0$.

Remark 2.11. The results of this paper can be extended to the odd-order equation

$$\left(r(t) \left(z^{(n-1)}(t) \right)^{\gamma} \right)' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

under either of the conditions

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty$$

or

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty,$$

where $n \geq 3$ is an odd natural number, $r \in C\left([t_0, \infty), (0, \infty)\right)$, γ is the ratio of odd positive integers, and the other functions in the equation are defined as in this paper.

Remark 2.12. It would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \leq -1$ with $p(t) \not\equiv -1$ for large t.

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