Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **58**, 1–19; https://doi.org/10.14232/ejqtde.2021.1.58 www.math.u-szeged.hu/ejqtde/

Uniqueness and monotonicity of solutions for fractional equations with a gradient term

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Received 4 May 2021, appeared 30 July 2021
Communicated by Patrizia Pucci

Abstract. In this paper, we consider the following fractional equation with a gradient term

$$(-\Delta)^s u(x) = f(x, u(x), \nabla u(x)),$$

in a bounded domain and the upper half space. Firstly, we prove the monotonicity and uniqueness of solutions to the fractional equation in a bounded domain by the sliding method. In order to obtain maximum principle on unbounded domain, we need to estimate the singular integrals define the fractional Laplacians along a sequence of approximate maximum points by using a generalized average inequality. Then we prove monotonicity and uniqueness of solutions to fractional equation in $\mathbb{R}^n_+$ by the sliding method. In order to solve the difficulties caused by the gradient term, some new techniques are developed. The paper may be considered as an extension of Berestycki and Nirenberg [*J. Geom. Phys.* **5**(1988), 237–275].

Keywords: fractional equation with gradient term, monotonicity, uniqueness, sliding method.

2020 Mathematics Subject Classification: 35R11, 35A09, 35B06, 35B09.

1 Introduction

During the last decades, fractional Laplacian has attracted more and more attention due to its various applications. The methods to study the fractional Laplacian are the extension method [6], moving planes method in integral form [11], the method of moving sphere [26] and direct methods of moving planes [9, 22] etc. Recently, to study the monotonicity of the solution, Liu [28], Wu and Chen [35, 36] introduced a direct sliding method for fractional Laplacian and fractional p-Laplacian. Berestycki and Nirenberg [3–5] first developed the sliding method, which was used to establish qualitative properties of solutions for nonlinear elliptic equations involving the regular Laplacian such as monotonicity, nonexistence and uniqueness etc. The essential ingredients are different forms of maximum principles. The main idea lies in comparing values of the solution to the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction, and then the domain

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is slide back to a critical position. While in the method of moving planes, one point is the reflection of the other.

Inspired by the above article, in this article, we show the monotonicity, antisymmetry and uniqueness of solutions for the following fractional equation with a gradient term

$$(-\Delta)^s u(x) = f(x, u(x), \nabla u(x)), \tag{1.1}$$

where ∇u denotes the gradient of u, the fractional Laplacian $(-\Delta)^s$ with $0 < s < 1$ is given by

$$\begin{aligned}(-\Delta)^s u(x) &= C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \\ &= C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.\end{aligned}$$

Define

$$\mathcal{L}_{2s} = \left\{ u : u \in L^1_{loc}(\mathbb{R}^n), \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\},$$

then it is easy to see that for $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathcal{L}_{2s}$, $(-\Delta)^s u$ is well defined.

When $s = 1$, [3] derived the monotonicity, symmetry and uniqueness of (1.1) in a finite cylinder and a bounded domain which is convex in the x_1 direction by the sliding method. In the case $s = 1, f(x, u, \nabla u) = f(u)$, Gidas, Ni, Nirenberg [21] obtained monotonicity and symmetry for positive solutions of (1.1), vanishing on the boundary, using the maximum principle and the method of moving planes; in [2,5], Berestycki, Cafferelli and Nirenberg considered the monotonicity and uniqueness of solution for (1.1) by the sliding method. Recently, in the case $0 < s < 1$, Chen, Li and Li [9] investigated the semilinear equation in the whole space with $f(x, u, \nabla u) = u^p$, $1 < p \leq \frac{n+2s}{n-2s}$, developed a direct method of moving planes for the fractional Laplacian and showed that the nonnegative solution of (1.1) is radially symmetric and monotone decreasing about some point in the critical case $p = \frac{n+2s}{n-2s}$ and nonexistence of positive solutions in the subcritical case $1 < p < \frac{n+2s}{n-2s}$; Dipierro, Soave and Valdinoci [16] proved symmetry, monotonicity and rigidity results to (1.1) in an unbounded domain with the epigraph property.

The purpose of the present paper is to extend the results in [3] to the fractional equation. On the one hand, we extent the case $s = 1$ in [3] to the fractional case $0 < s < 1$, and extend bounded domain to $\mathbb{R}^n_+$. On the other hand, the nonlinear term $f(x, u, \nabla u)$ has a broader form containing nonlinear term $f(u)$ and $f(x, u)$.

In order to solve the difficulty that the nonlinear term at the right side of (1.1) contain the gradient term, in the bounded domain when deriving the contradiction for the minimum point of the function $w^\tau(x)$ (see Section 2 below for definition), for the first time, we use the technique of finding the minimum value of the function $w^\tau(x)$ for the variables τ and x at the same time. This is different from the previous sliding process which only finds the minimum value of the variable x for the fixed τ. In the whole space, we estimate the singular integrals defining the fractional Laplacian along a sequence of approximate maximum, and the estimating is for τ and the sequence of approximate maximum at the same time.

In order to apply the sliding method, we give the exterior condition on u. Let $u(x) = \varphi(x),\ x \in \Omega^c$, and assume that

(C) for any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$, with $y, z \in \Omega^c$, we have

$$\varphi(y) < u(x) < \varphi(z), \quad \text{if } x \in \Omega \tag{1.2}$$

and

$$\varphi(y) \le \varphi(x) \le \varphi(z), \quad \text{if } x \in \Omega^c. \tag{1.3}$$

Remark 1.1. The same monotonicity conditions (1.2) and (1.3) (with Ω^c replaced by $\partial\Omega$) were assumed in [4,5,35].

The main result of this paper is

Theorem 1.2. *Suppose that* $u \in C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})$ *satisfies (C) and is a solution of equation*

$$\begin{cases} (-\Delta)^s u(x) = f(x,u,\nabla u), & x \in \Omega, \\ u(x) = \varphi(x), & x \in \Omega^c, \end{cases} \tag{1.4}$$

where Ω *is a bounded domain which is convex in* x_n *direction. Assume that* f *is continuous in all variables, locally Lipschitz continuous in* $(u,\nabla u)$ *and is nondecreasing in* x_n*. Then* u *is strictly monotone increasing with respect to* x_n *in* Ω*, i.e., for any* $\tau > 0$,

$$u(x', x_n + \tau) > u(x', x_n), \quad \textit{for all } (x', x_n), (x', x_n + \tau) \in \Omega.$$

Furthermore, the solution of (1.4) *is unique.*

Remark 1.3. Theorem (1.2) includes the result of Theorem 2 in [35], and we also prove the uniqueness of solutions in bounded domain. If Ω is the finite cylinder $\mathcal{C} = \{x = (x', x_n) \in \mathbb{R}^n \mid |x_n| < l,\ x' \in \omega\}$, where $l > 0$ and ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary, the result of Theorem 1.2 still holds.

Remark 1.4. The conditions in Theorem 1.2 and Theorem 1 of [14] are different. Neither implies the other. Cheng, Huang and Li [14] studied the positive solution u and obtained that u is strictly increasing in the half of Ω in x_n-direction with $x_n < 0$ by the method of moving planes, but the solution we study can be negative and is strictly increasing with respect to x_n in the whole domain Ω by the sliding method.

We also have a new antisymmetry result for the equation (1.4) if the bounded domain Ω is symmetric about $x_n = 0$.

Corollary 1.5 (Antisymmetry)**.** *Assume that the conditions of Theorem 1.2 are satisfied and in addition that* φ *is odd in* x_n *on* Ω^c*. If* $f(x,u,\nabla u)$ *is odd in* $(x_n, u, \nabla_{x'} u)$*. Then* u *is odd, i.e. antisymmetric in* x_n*:*

$$u(x', -x_n) = -u(x', x_n), \qquad \forall\, x \in \Omega.$$

This follows from the fact that $\bar{u} = -u(x', -x_n)$ is a solution satisfying the same conditions, and so is u.

For the unbounded domain, we give the following result on $\mathbb{R}^n_+$.

Theorem 1.6. *Suppose that* $u \in C^{1,1}_{loc}(\mathbb{R}^n_+) \cap \mathcal{L}_{2s}(\mathbb{R}^n) \cap C(\overline{\mathbb{R}^n_+})$ *is a solution of*

$$\begin{cases} (-\Delta)^s u(x) = f(u,\nabla u), & x \in \mathbb{R}^n_+, \\ 0 < u(x) \le \mu, & x \in \mathbb{R}^n_+, \\ u(x) = 0, & x \notin \mathbb{R}^n_+, \end{cases} \tag{1.5}$$

and

$$\lim_{x_n \to +\infty} u(x', x_n) = \mu, \quad \textit{uniformly for all } x' \in \mathbb{R}^{n-1}. \tag{1.6}$$

Assume that f is bounded, continuous in all variables and nonincreasing in $u \in [\mu-\delta, \mu]$ for some $\delta > 0$. Then u is strictly monotone increasing in x_n direction, and moreover it depends on x_n only.

Furthermore, the solution of (1.5) is unique.

Theorem 1.6 is closely related to the following well-known De Giorgi conjecture [19].

Conjecture (De Giorgi [19])**.** If u is a solution of

$$-\Delta u = u - u^3,$$

such that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \quad \text{for all } x' \in \mathbb{R}^{n-1},$$

and

$$|u(x)| \le 1, \qquad x \in \mathbb{R}^n, \qquad \frac{\partial u}{\partial x_n} > 0.$$

Then there exists a vector $\mu \in \mathbb{R}^{n-1}$ and a function $u_1 : \mathbb{R} \to \mathbb{R}$ such that

$$u(x', x_n) = u_1(\mu x' + x_n) \quad \text{in } \mathbb{R}^n.$$

The other symmetry, uniqueness and monotonicity results on local and nonlocal equations, we also refer readers to [1, 18, 24, 25] for semilinear elliptic equations, [9, 13, 17, 23, 30, 31] for fractional equations, [34, 38] for weighted fractional equation, [14, 37] for fractional equations with a gradient term, [27] for integral system with negative exponents, [12] for weighted Hardy-sobolev type system, [8, 32, 33] for fully nonlinear nonlocal equations with gradient term, [7, 15, 29] for fractional p-Laplace equation, and references therein.

The paper is organized as follows. In Section 2 we prove Theorem 1.2 via the sliding method. In Section 3, we first establish a maximum principle in the unbounded domain, then uniqueness and monotonicity for the fractional equation with a gradient term on $\mathbb{R}^n_+$ are obtained.

2 The proof of Theorem 1.2

For convenience, we list some notations used frequently. For $\tau \in \mathbb{R}$, denote $x = (x', x_n)$, $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. Set

$$u^\tau(x) = u(x', x_n + \tau), \qquad w^\tau(x) = u^\tau(x) - u(x).$$

Proof of Theorem 1.2. For $\tau > 0$, it is defined on the set $\Omega^\tau = \Omega - \tau e_n$ which is obtained from Ω by sliding it downward a distance τ parallel to the x_n axis, where $e_n = (0, \ldots, 0, 1)$. Set

$$D^\tau := \Omega^\tau \cap \Omega, \qquad \tilde{\tau} = \sup\{\tau \mid \tau > 0, D^\tau \neq \emptyset\}$$

and

$$w^\tau(x) = u^\tau(x) - u(x), \qquad x \in D^\tau.$$

We mainly divide the following two steps to prove that u is strictly increased in the x_n direction, *i.e.*

$$w^\tau(x) > 0, \; x \in D^\tau, \quad \text{for any } 0 < \tau < \tilde{\tau}. \tag{2.1}$$

Step 1. *For τ sufficiently close to $\tilde{\tau}$ i.e., D^τ is narrow, we claim that there exists $\delta > 0$ small enough such that*

$$w^\tau(x) \geq 0, \qquad \forall\, x \in D^\tau,\ \forall\, \tau \in (\tilde{\tau} - \delta, \tilde{\tau}). \tag{2.2}$$

Otherwise, we set

$$A_0 = \min_{\substack{x \in \bar{D}^\tau \\ \tilde{\tau} - \delta < \tau < \tilde{\tau}}} w^\tau(x) < 0.$$

From condition (C), A_0 can be obtained for some $(\tau^0, x^0) \in \{(\tau, x) \mid (\tau, x) \in (\tilde{\tau} - \delta, \tilde{\tau}) \times D^\tau\}$. Noticing that $w^{\tau^0}(x) \geq 0, x \in \partial D^{\tau^0}$, we arrive at $x^0 \in D^{\tau^0}$. So $w^{\tau^0}(x^0) = A_0$. Since (τ^0, x^0) is a minimizing point, we have $\nabla w^{\tau^0}(x^0) = 0$, *i.e.*, $\nabla u^{\tau^0}(x^0) = \nabla u(x^0)$. Since u^{τ^0} satisfies the same equation (1.4) in Ω^{τ^0} as u does in Ω, and f is nondecreasing in x_n, so we have

$$\begin{aligned}
(-\Delta)^s w^{\tau^0}(x^0) &= f((x^0)', x_n^0 + \tau^0, u^{\tau^0}(x^0), \nabla u^{\tau^0}(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \\
&\geq f(x^0, u^{\tau^0}(x^0), \nabla u^{\tau^0}(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \\
&= f(x^0, u^{\tau^0}(x^0), \nabla u(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \\
&= -c^{\tau^0}(x^0) w^{\tau^0}(x^0),
\end{aligned} \tag{2.3}$$

where $-c^{\tau^0}(x^0) = \frac{f(x^0, u^{\tau^0}(x^0), \nabla u(x^0)) - f(x^0, u(x^0), \nabla u(x^0))}{u^{\tau^0}(x^0) - u(x^0)}$ is a L^∞ function satisfying

$$|c^{\tau^0}(x^0)| \leq C, \qquad \forall\, x^0 \in D^{\tau^0}.$$

Hence

$$(-\Delta)^s w^{\tau^0}(x^0) + c^{\tau^0}(x^0) w^{\tau^0}(x^0) \geq 0.$$

On the other hand, we obtain

$$\begin{aligned}
&(-\Delta)^s w^{\tau^0}(x^0) + c^{\tau^0}(x^0) w^{\tau^0}(x^0) \\
&= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{w^{\tau^0}(x^0) - w^{\tau^0}(y)}{|x^0 - y|^{n+2s}} dy + c^{\tau^0}(x^0) w^{\tau^0}(x^0) \\
&\leq C_{n,s} w^{\tau^0}(x^0) \int_{(D^{\tau^0})^c} \frac{1}{|x^0 - y|^{n+2s}} dy + \inf_{D^{\tau^0}} c^{\tau^0}(x) w^{\tau^0}(x^0) \\
&\leq w^{\tau^0}(x^0) \left(\frac{C_1}{d_n^{2s}} - C \right) \\
&< 0,
\end{aligned} \tag{2.4}$$

where d_n denotes the width of D^{τ^0} in the x_n direction and D^{τ^0} is narrow. This is a contradiction.

Therefore we derive (2.2) is true for τ sufficiently close to $\tilde{\tau}$.

Step 2. *The inequality (2.2) provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as (2.2) holds to its limiting position.* Define

$$\tau_0 = \inf\{\tau \mid w^\tau(x) \geq 0,\ x \in D^\tau,\ 0 < \tau < \tilde{\tau}\}.$$

We will prove

$$\tau_0 = 0.$$

Otherwise, assume $\tau_0 > 0$, we show that the domain Ω can be slided upward a little bit more and we still have

$$w^\tau(x) \geq 0, \qquad x \in D^\tau, \quad \text{for any } \tau_0 - \varepsilon < \tau \leq \tau_0, \tag{2.5}$$

which contradicts the definition of τ_0.

Since $w^{\tau_0}(x) > 0,\ x \in \Omega \cap \partial D^{\tau_0}$ by condition (C) and $w^{\tau_0}(x) \geq 0,\ x \in D^{\tau_0}$, then

$$w^{\tau_0}(x) \not\equiv 0, \qquad x \in D^{\tau_0}.$$

If there exists a point $\tilde{x} \in D^{\tau_0}$ such that $w^{\tau_0}(\tilde{x}) = 0$, then $\tilde{x}$ is the minimum point. So we have $\nabla w^{\tau_0}(\tilde{x}) = 0$ and

$$(-\Delta)^s w^{\tau_0}(\tilde{x}) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{w^{\tau_0}(\tilde{x}) - w^{\tau_0}(y)}{|\tilde{x} - y|^{n+2s}} dy < 0,$$

which contradicts to

$$\begin{aligned}(-\Delta)^s w^{\tau_0}(\tilde{x}) &= f(\tilde{x}', \tilde{x}_n + \tau_0, u^{\tau_0}(\tilde{x}), \nabla u^{\tau_0}(\tilde{x})) - f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) \\ &\geq f(\tilde{x}, u^{\tau_0}(\tilde{x}), \nabla u^{\tau_0}(\tilde{x})) - f(\tilde{x}, u(\tilde{x}), \nabla u(\tilde{x})) \\ &= 0.\end{aligned}$$

Hence,

$$w^{\tau_0}(x) > 0, \qquad x \in D^{\tau_0}. \tag{2.6}$$

Next we will prove (2.5). Suppose (2.5) is not true, one has

$$A_1 = \min_{\substack{x \in D^\tau \\ \tau_0 - \varepsilon < \tau < \tau_0}} w^\tau(x) < 0.$$

The minimum A_1 can be obtained for some $\mu \in (\tau_0 - \varepsilon, \tau_0), \bar{x} \in D^\mu$ where $w^\mu(\bar{x}) = A_1$ by condition (C). We carve out of D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0} \setminus K$ is narrow. According to (2.6),

$$w^{\tau_0}(x) \geq C_0 > 0, \qquad x \in K.$$

From the continuity of w^τ in τ, we have for small $\varepsilon > 0$,

$$w^\mu(x) \geq 0, \qquad x \in K. \tag{2.7}$$

From (C), it follows

$$w^\mu(x) \geq 0, \qquad x \in (D^\mu)^c.$$

So $\bar{x} \in D^\mu \setminus K$ and $\nabla w^\mu(\bar{x}) = 0$. Since $D^{\tau_0} \subset D^\mu$ and small ε, we obtain that $D^\mu \setminus K$ is a narrow domain. Similar to (2.3), we have

$$(-\Delta)^s w^\mu(\bar{x}) + c(\bar{x}) w^\mu(\bar{x}) \geq 0.$$

Similar to (2.4) and narrow domain $D^\mu \setminus K$, we have

$$(-\Delta)^s w^\mu(\bar{x}) + c(\bar{x}) w^\mu(\bar{x}) < 0.$$

This is a contradiction. Hence we derive (2.5), which contradicts to the definition of τ_0. So $\tau_0 = 0$. Therefore, we have shown that

$$w^\tau(x) \geq 0, \qquad x \in D^\tau, \quad \text{for any } 0 < \tau < \tilde{\tau}. \tag{2.8}$$

Next we prove (2.1). Since

$$w^\tau(x) \not\equiv 0, \qquad x \in D^\tau, \quad \text{for any } 0 < \tau < \tilde{\tau},$$

if there exists a point x^0 for some $\tau_1 \in (0, \tilde{\tau})$ such that $w^{\tau_1}(x^0) = 0$, then x^0 is the minimum point and

$$(-\Delta)^s w^{\tau_1}(x^0) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{w^{\tau_1}(x^0) - w^{\tau_1}(y)}{|x^0 - y|^{n+2s}} dy < 0.$$

This contradicts to

$$(-\Delta)^s w^{\tau_1}(x^0) = f((x^0)', x_n^0 + \tau_1, u^{\tau_1}(x^0), \nabla u^{\tau_1}(x^0)) - f(x^0, u(x^0), \nabla u(x^0)) \geq 0.$$

Therefore, we arrive at (2.1).

Now we prove uniqueness. If $\underline{u}$ is another solution satisfying the same conditions, the same argument as before but replace $w^\tau = u^\tau - u$ with $w^\tau = \underline{u}^\tau - u$. Similarly to (2.8), we have $\underline{u}^\tau(x) \geq u$ in D^τ for any $0 < \tau < \tilde{\tau}$. Hence, $\underline{u} \geq u$. Interchanging the roles of u and $\underline{u}$, we find the opposite inequality. Therefore, $\underline{u} = u$.

This completes the proof of Theorem 1.2. □

3 The uniqueness and monotonicity of solution on $\mathbb{R}^n_+$

In the section, we will prove Theorem 1.6. We first establish a maximum principle in the unbounded domain for the fractional equation with a gradient term.

Lemma 3.1 (Maximum principle)**.** *Let D be an open set in $\mathbb{R}^n$, possibly unbounded and disconnected, suppose that*

$$\underline{\lim}_{k \to \infty} \frac{|D^c \cap (B_{2^{k+1}}(q) \setminus B_{2^k}(q))|}{|(B_{2^{k+1}}(q) \setminus B_{2^k}(q))|} > 0,$$

where q is any point in D. Let $w \in C^{1,1}_{loc}(D) \cap \mathcal{L}_{2s}$ be bounded from above and satisfy

$$\begin{cases} (-\Delta)^s w(x) + c(x) w(x) + \sum_{j=1}^n b_j(x) w_j(x) \leq 0, & x \in D, \\ w(x) \leq 0, & x \in \mathbb{R}^n \setminus D, \end{cases} \tag{3.1}$$

for some nonnegation function $c(x)$. Then

$$w(x) \leq 0, \ x \in D.$$

Furthermore, we have

$$\text{either } w(x) < 0 \text{ in } D \text{ or } w(x) \equiv 0 \text{ in } \mathbb{R}^n. \tag{3.2}$$

Remark 3.2. The proof of Lemma 3.1 is different from Theorem 3 in [35]. Here we mainly use the following generalized average inequality.

Lemma 3.3 ([35] A generalized average inequality)**.** *Suppose that $w \in C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathcal{L}_{2s}$ and $\bar{x}$ is a maximum point of w in $\mathbb{R}^n$. Then for any $r > 0$, we have*

$$\frac{C_0}{C_{n,s}} r^{2s} (-\Delta)^s w(\bar{x}) + C_0 \int_{B_r^c(\bar{x})} \frac{r^{2s}}{|\bar{x} - y|^{n+2s}} w(y) dy \geq w(\bar{x}),$$

where C_0 satisfies

$$C_0 \int_{B_r^c(\bar{x})} \frac{r^{2s}}{|\bar{x} - y|^{n+2s}} dy = 1.$$

Proof of Lemma 3.1. Suppose on the contrary, there is some point x such that $w(x) > 0$ in D, then

$$0 < A := \sup_{x\in\mathbb{R}^n} w(x) < \infty. \tag{3.3}$$

There exists a sequence $\{x^k\} \subset D$ such that

$$w(x^k) \to A > 0, \quad \text{as } k \to \infty. \tag{3.4}$$

Let

$$\eta(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{3.5}$$

where $c > 0$ is a constant, taking $c = e$ such that $\eta(0) = \max_{\mathbb{R}^n} \eta(x) = 1$.

Set

$$\psi_k(x) = \eta(x - x^k). \tag{3.6}$$

From (3.4), there exists a sequence $\{\varepsilon_k\}$ with $\varepsilon_k > 0$ such that

$$w(x^k) + \varepsilon_k\psi_k(x^k) \geq A.$$

Since $w(x) \leq 0$, $x \in \mathbb{R}^n \setminus D$, it follows from (3.4) that x^k is away from ∂D. Without loss of generality, we may assume that $\mathrm{dist}(x^k, \partial D) = 2$. So $B_1(x^k) \subset D$. Since for any $x \in D \setminus B_1(x^k)$, $w(x) \leq A$ and $\psi_k(x) = 0$, hence

$$w(x^k) + \varepsilon_k\psi_k(x^k) \geq w(x) + \varepsilon_k\psi_k(x), \quad \text{for any } x \in \mathbb{R}^n \setminus B_1(x^k).$$

It follows that there exists a point $\bar{x}^k \in B_1(x^k)$ such that

$$w(\bar{x}^k) + \varepsilon_k\psi_k(\bar{x}^k) = \max_{\mathbb{R}^n}(w(x) + \varepsilon_k\psi_k(x)) > A. \tag{3.7}$$

So $(w(\bar{x}^k) + \varepsilon_k\psi_k(\bar{x}^k))_j = 0$ and

$$w_j(\bar{x}^k) \to 0, \quad \text{as } k \to \infty. \tag{3.8}$$

For $w + \varepsilon_k\psi_k$, using Lemma 3.3, we obtain

$$(w + \varepsilon_k\psi_k)(\bar{x}^k) \leq C_1(-\Delta)^s(w + \varepsilon_k\psi_k)(\bar{x}^k) + C_2\int_{B_2^c(\bar{x}^k)} \frac{(w + \varepsilon_k\psi_k)(y)}{|\bar{x}^k - y|^{n+2s}}dy.$$

Let $\varepsilon_k \to 0$, by the first inequality of (3.1), it implies that

$$\begin{aligned} w(\bar{x}^k) &\leq C_1(-\Delta)^s w(\bar{x}^k) + C_2\int_{B_2^c(\bar{x}^k)} \frac{w(y)}{|\bar{x}^k - y|^{n+2s}}dy \\ &\leq -c(\bar{x}^k)w(\bar{x}^k) - \sum_{j=1}^n b_j(x)w_j(\bar{x}^k) + C_2\int_{B_2^c(\bar{x}^k)} \frac{w(y)}{|\bar{x}^k - y|^{n+2s}}dy. \end{aligned} \tag{3.9}$$

Letting $k \to \infty$, combining (3.4), (3.8), (3.9) and nonnegative function $c(x)$, we arrive at

$$0 < (c(x) + 1)A \leftarrow (c(\bar{x}^k) + 1)w(\bar{x}^k) \leq C_2\int_{B_2^c(\bar{x}^k)} \frac{w(y)}{|\bar{x}^k - y|^{n+2s}}dy,$$

this is impossible because of (3.3) and the second inequality of (3.1).

Based on above result, if $w = 0$ at some point $x_0 \in D$, then x_0 is a maximum point of w in D. And we still have $w_j = 0$ in the maximum point. If $w \not\equiv 0$ in $\mathbb{R}^n$, then we have

$$(-\Delta)^s w(x_0) + c(x_0)w(x_0) + \sum_{j=1}^{n} b_j(x_0)w_j(x_0) = C_{n,s}P.V. \int_{\mathbb{R}^n} \frac{-w(y)}{|x_0 - y|^{n+2s}}dy > 0.$$

This is a contradiction with (3.1). So we have either $w < 0$ in D or $w \equiv 0$ in $\mathbb{R}^n$.

This completes the proof Lemma 3.1. □

We also need the following lemma.

Lemma 3.4 ([9], Maximum principle). *Let Γ be a bounded domain in $\mathbb{R}^n$. Assume that $u \in C^{1,1}_{loc}(\Gamma) \cap \mathcal{L}_{2s}$ and u be lower semi-continuous on $\bar{\Gamma}$, and satisfy*

$$\begin{cases} (-\Delta)^s u(x) \geq 0, & x \in \Gamma, \\ u(x) \geq 0, & x \in \mathbb{R}^n \setminus \Gamma. \end{cases}$$

Then

$$u(x) \geq 0, \ x \in \Gamma.$$

If $u(x) = 0$ at some point $x \in \Gamma$, then

$$u(x) = 0 \quad \textit{almost everywhere in } \mathbb{R}^n.$$

Proof of Theorem 1.6. Define $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \mid x_n > 0\}$. Let

$$u^\tau(x) = u(x', x_n + \tau) \quad \text{and} \quad U^\tau(x) = u(x) - u^\tau(x).$$

Outline of the proof: We will use the sliding method to prove the monotonicity and uniqueness of u and divide the proof into three steps.

In Step 1, we will show that for τ sufficiently large, we have $U^\tau(x) \leq 0, \ x \in \mathbb{R}^n$. Especially, since $u \to \mu$ uniformly as $x_n \to +\infty$, for $\delta > 0$, there exists a $M_0 > 0$ such that for $x_n \geq M_0$, $u \in [\mu - \delta, \mu]$ and f is nondecreasing in $u \in [\mu - \delta, \mu]$. Hence we will show that

$$U^\tau(x) \leq 0, \qquad x \in \mathbb{R}^n, \ \forall \ \tau \geq M_0. \tag{3.10}$$

This provides the starting point for the sliding method. Then in Step 2, we decrease τ continuously as long as (3.10) holds to its limiting position. Define

$$\tau_0 := \inf\{\tau \mid U^\tau(x) \leq 0, \ x \in \mathbb{R}^n, \ 0 < \tau < M_0\}. \tag{3.11}$$

We first will show that $\tau_0 = 0$. Then we deduce that the solution u must be strictly monotone increasing in x_n. In Step 3, we obtain that the solution u depends on x_n. Finally we will prove the uniqueness.

Now we show the details in the three steps.

Step 1. Since $u(x) = 0, x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, it yields that

$$U^\tau(x) \leq 0, \qquad \forall \ x \in \mathbb{R}^n \setminus \mathbb{R}^n_+.$$

For $\tau \geq M_0$, suppose (3.10) is violated, there exists a constant $A > 0$ such that

$$\sup_{x \in \mathbb{R}^n_+} U^\tau(x) = A, \tag{3.12}$$

hence for some $\tau_1 \geq M_0$ there exists a sequence $\{x^k\} \subset \mathbb{R}^n_+$ such that

$$U^{\tau_1}(x^k) \to A, \quad \text{as } k \to \infty. \tag{3.13}$$

We will apply Lemma 3.1 to function $U^{\tau_1}(x) - \frac{A}{2}$.

Since $\tau_1 \geq M_0$, we have $u^{\tau_1}(x) \in [\mu - \delta, \mu]$. Let

$$D = \left\{ x \in \mathbb{R}^n \mid U^{\tau_1}(x) - \frac{A}{2} > 0 \right\}.$$

For $x \in D$, we have $u(x) \geq u^{\tau_1} \geq \mu - \delta$. From equation (1.5), $U^{\tau_1}(x)$ satisfied

$$\begin{aligned}(-\Delta)^s U^{\tau_1}(x) &= f(u, \nabla u) - f(u^{\tau_1}, \nabla u^{\tau_1}) \\ &:= -b_j(x)(U^{\tau_1})_j(x) - c(x)U^{\tau_1}(x),\end{aligned}$$

where $c(x) = -\frac{f(u,\nabla u) - f(u^{\tau_1},\nabla u)}{u - u^{\tau_1}} \leq 0$ by the monotonicity of f.

Hence $U^{\tau_1}(x) - \frac{A}{2}$ satisfies

$$\begin{cases} (-\Delta)^s U^{\tau_1}(x) + b_j(x)(U^{\tau_1})_j(x) + c(x)(U^{\tau_1}(x) - \frac{A}{2}) = 0, & x \in D, \\ U^{\tau_1}(x) - \frac{A}{2} \leq 0, & x \in \mathbb{R}^n \setminus D. \end{cases}$$

By Lemma 3.1, we derive

$$U^{\tau_1}(x) - \frac{A}{2} \leq 0, \; x \in \mathbb{R}^n,$$

which contradicts (3.13). Hence we obtain (3.10) and finish the proof of Step 1.

We also give an alternative proof which is an application of the general average inequality (Lemma 3.3), and this idea can be applied to other problems.

For $\tau \geq M_0$, if (3.10) is violated, we have (3.13). Obviously, $U^{\tau_1}(x) \leq 0, x \in \partial\mathbb{R}^n_+$. So by (3.13) we have x^k is away from $\partial\mathbb{R}^n_+$, without loss of generality, assume $\text{dist}(x^k, \partial\mathbb{R}^n_+) > 2$. Thus there exists $0 < \varepsilon_k \to 0$, $\bar{x}^k \in B_1(x^k)$ such that

$$U^{\tau_1}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k) = \max_{\mathbb{R}^n} (U^{\tau_1}(x) + \varepsilon_k \psi_k(x)) \geq A,$$

where $\psi_k(\bar{x}^k)$ is as stated in (3.6). So $\nabla(U^{\tau_1}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k)) = 0$ and

$$\nabla U^{\tau_1}(\bar{x}^k) \to 0, \quad \text{as } k \to \infty. \tag{3.14}$$

Since

$$[U^{\tau_1} + \varepsilon_k \psi_k](\bar{x}^k) \geq [U^{\tau_1} + \varepsilon_k \psi_k](x^k)$$

and $\psi_k(\bar{x}^k) \leq \psi_k(x^k)$, we obtain

$$U^{\tau_1}(\bar{x}^k) \geq U^{\tau_1}(x^k). \tag{3.15}$$

Hence for $\tau_1 \geq M_0$,

$$u(\bar{x}^k) \geq u^{\tau_1}(\bar{x}^k) \geq \mu - \delta.$$

This means $u(\bar{x}^k), u^{\tau_1}(\bar{x}^k)$ are all in the nondecreasing interval of f. So

$$\begin{aligned} &f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u^{\tau_1}(\bar{x}^k), \nabla u^{\tau_1}(\bar{x}^k)) \\ &\quad = f(u, \nabla u(\bar{x}^k)) - f(u, \nabla u^{\tau_1}(\bar{x}^k)) + f(u, \nabla u^{\tau_1}(\bar{x}^k)) - f(u^{\tau_1}, \nabla u^{\tau_1}(\bar{x}^k)) \\ &\quad \leq f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u(\bar{x}^k), \nabla u^{\tau_1}(\bar{x}^k)). \end{aligned} \tag{3.16}$$

Using Lemma 3.3 to the function $U^{\tau_1} + \varepsilon_k \psi_k$ at $\bar{x}^k$, we obtain

$$(U^{\tau_1} + \varepsilon_k \psi_k)(\bar{x}^k) \leq C_1(-\Delta)^s (U^{\tau_1} + \varepsilon_k \psi_k)(\bar{x}^k) + C_2 \int_{B_2^c(\bar{x}^k)} \frac{(U^{\tau_1} + \varepsilon_k \psi_k)(y)}{|\bar{x}^k - y|^{n+2s}} dy.$$

Let $\varepsilon_k \to 0$, by the equation (1.5), it implies that

$$\begin{aligned} U^{\tau_1}(\bar{x}^k) &\leq C_1(-\Delta)^s U^{\tau_1}(\bar{x}^k) + C_2 \int_{B_2^c(\bar{x}^k)} \frac{U^{\tau_1}(y)}{|\bar{x}^k - y|^{n+2s}} dy \\ &= C_1[f(u, \nabla u(\bar{x}^k)) - f(u, \nabla u^{\tau_1}(\bar{x}^k))] + C_2 \int_{B_2^c(\bar{x}^k)} \frac{U^{\tau_1}(y)}{|\bar{x}^k - y|^{n+2s}} dy. \end{aligned} \tag{3.17}$$

From (3.13) and (3.15), we have

$$U^{\tau_1}(\bar{x}^k) \to A > 0, \quad \text{as } k \to \infty. \tag{3.18}$$

Letting $k \to \infty$, combining (3.14), (3.17) and (3.18), we arrive at

$$0 < A \leftarrow U^{\tau_1}(\bar{x}^k) \leq C_2 \int_{B_2^c(\bar{x}^k)} \frac{U^{\tau_1}(y)}{|\bar{x}^k - y|^{n+2s}} dy,$$

this is impossible because of (3.12) and $U^{\tau_1}(y) \leq 0, y \in \mathbb{R}^n \setminus \mathbb{R}^n_+$.

Hence (3.10) is correct and we have finished the proof of Step 1.

Step 2. Firstly, we will check that

$$\tau_0 = 0, \tag{3.19}$$

where τ_0 as defined in (3.11). In fact, suppose on the contrary $\tau_0 > 0$, then τ_0 can be decreased a little bit. To be more rigorously, there exists a $\epsilon > 0$ such that for any $\tau \in (\tau_0 - \epsilon, \tau_0]$, one has

$$U^\tau(x) \leq 0, \quad \text{for any } x \in \mathbb{R}^n_+. \tag{3.20}$$

This is a contradiction with the definition of τ_0. Hence (3.19) is correct. In the sequel, we will prove (3.20).

To do so, we just need to prove

$$\sup_{\mathbb{R}^{n-1} \times (0, M_0+1]} U^\tau(x) < 0, \qquad \forall\, \tau \in (\tau_0 - \epsilon, \tau_0] \tag{3.21}$$

and

$$\sup_{\mathbb{R}^{n-1} \times (M_0+1, +\infty)} U^\tau(x) \leq 0, \qquad \forall\, \tau \in (\tau_0 - \epsilon, \tau_0]. \tag{3.22}$$

In order to prove (3.21) we need to show that

$$\sup_{\mathbb{R}^{n-1} \times (0, M_0+1]} U^{\tau_0}(x) < 0. \tag{3.23}$$

If not, then

$$\sup_{\mathbb{R}^{n-1} \times (0, M_0+1]} U^{\tau_0}(x) = 0.$$

So there exists a sequence $\{x^k\} \subset \mathbb{R}^{n-1} \times (0, M_0 + 1]$ such that

$$U^{\tau_0}(x^k) \to 0, \quad \text{as } k \to \infty. \tag{3.24}$$

We first show that x^k is away from the boundary $\partial\mathbb{R}^n_+$. Suppose that z be a point on $\partial\mathbb{R}^n_+$. Denote $r_z := \text{dist}(z+\tau_0 e_n, \partial\mathbb{R}^n_+),\ e_n = (0,\ldots,0,1)$. For each fixed $\tau_0 > 0$, we have

$$\inf_{x\in\partial\mathbb{R}^n_+} \text{dist}(z+\tau_0 e_n, \partial\mathbb{R}^n_+) := r_0 > 0.$$

For every point z on $\partial\mathbb{R}^n_+$, there exists a ball $B_{r_z}(z+\tau_0 e_n) \subset \mathbb{R}^n_+$ with radius of r_z centered at $z+\tau_0 e_n$. For simplicity of notation, we use B instead of $B_{r_z}(z+\tau_0 e_n)$.

Let

$$E = \{x \in \mathbb{R}^n_+ \mid \text{dist}(x, \partial\mathbb{R}^n_+) \geq 2\}.$$

We construct a subsolution

$$\bar{u}(x) = u_E(x) + \varepsilon\Phi(x), \qquad x \in B,$$

where $\Phi(x) = (1-|x|^2)^s_+$, $u_E := u\cdot\chi_E$ and χ_E is define as

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \in \mathbb{R}^n \setminus E. \end{cases}$$

By $(-\Delta)^s\Phi(x) = C$ [20], for $x \in B$ it yields

$$\begin{aligned} (-\Delta)^s\bar{u}(x) &= (-\Delta)^s(u_E + \varepsilon\Phi)(x) \\ &= \varepsilon(-\Delta)^s\Phi(x) + (-\Delta)^s u_E(x) \\ &\leq \varepsilon C - \varepsilon_1 C_{n,s}\int_E \frac{1}{|x-y|^{n+2s}}dy \\ &\leq \varepsilon C - \varepsilon_1 C C_{n,s}. \end{aligned}$$

We can choose $\varepsilon \leq \varepsilon_1 C_{n,s} C C^{-1} := \varepsilon_0$ such that $(-\Delta)^s\underline{u}(x) \leq 0,\ x \in B$. Then fixing $\varepsilon = \frac{\varepsilon_0}{2}$, combining $u(x) \geq \underline{u}(x),\ x \in B^c$ and Lemma 3.4, we derive

$$u^{\tau_0}(z) = u(z+\tau_0 e_n) \geq \underline{u}(z+\tau_0 e_n) \geq \frac{\varepsilon_0}{2}\Phi(z+\tau_0 e_n) \geq C_{\tau_0} > 0, \qquad \forall\, z \in \partial\mathbb{R}^n_+.$$

Then, we infer that

$$U^{\tau_0}(z) = u^{\tau_0}(z) > C_{\tau_0} > 0, \qquad \forall\, z \in \partial\mathbb{R}^n_+. \tag{3.25}$$

By (3.24) and (3.25), we obtain that x^k is away from the boundary $\partial\mathbb{R}^n_+$. Without loss of generality, we may assume $B_1(x^k) \subset \mathbb{R}^n_+$. Similar to the argument as Lemma 3.1, let $\psi(x) = \eta(x-x^k)$, where η is as stated in (3.5), x^k satisfies $\text{dist}(x^k, \partial\mathbb{R}^n_+) \geq 2$ and $B_1(x^k) \subset \mathbb{R}^n_+$. Then there exists a sequence $\varepsilon_k \to 0$ such that

$$U^{\tau_0}(x^k) + \varepsilon_k\psi(x^k) > 0.$$

Since for $x \in \mathbb{R}^n_+ \setminus B_1(x^k)$, noting that $U^{\tau_0}(x) \leq 0$ and $\psi(x) = 0$, we have

$$U^{\tau_0}(x^k) + \varepsilon_k\psi(x^k) > U^{\tau_0}(x) + \varepsilon_k\psi(x), \quad \text{for any } x \in \mathbb{R}^n \setminus B_1(x^k).$$

Then there exists $\bar{x}^k \in B_1(x^k)$ such that

$$U^{\tau_0}(\bar{x}^k) + \varepsilon_k\psi(\bar{x}^k) = \max_{\mathbb{R}^n}(U^{\tau_0}(x) + \varepsilon_k\psi(x)) > 0. \tag{3.26}$$

It can be seen from

$$U^{\tau_0}(\bar{x}^k) + \varepsilon_k \psi(\bar{x}^k) \geq U^{\tau_0}(x^k) + \varepsilon_k \psi(x^k),$$

and $\psi(\bar{x}^k) \leq \psi(x^k)$ that

$$0 > U^{\tau_0}(\bar{x}^k) \geq U^{\tau_0}(x^k) + \varepsilon_k \psi(x^k) - \varepsilon_k \psi(\bar{x}^k) \geq U^{\tau_0}(x^k) \to 0, \quad \text{as } k \to \infty.$$

Hence

$$U^{\tau_0}(\bar{x}^k) \to 0, \quad \text{as } k \to \infty.$$

Since f is continuous, we have

$$f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u^{\tau_0}(\bar{x}^k), \nabla u^{\tau_0}(\bar{x}^k)) \to 0, \quad \text{as } k \to \infty. \tag{3.27}$$

On one hand, we have

$$\begin{aligned}(-\Delta)^s (U^{\tau_0} + \varepsilon_k \psi)(\bar{x}^k) &= (-\Delta)^s U^{\tau_0}(\bar{x}^k) + (-\Delta)^s (\varepsilon_k \psi)(\bar{x}^k) \\ &= f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u^{\tau_0}(\bar{x}^k), \nabla u^{\tau_0}(\bar{x}^k)) + \varepsilon_k (-\Delta)^s \psi(\bar{x}^k).\end{aligned} \tag{3.28}$$

On the other hand,

$$\begin{aligned}(-\Delta)^s (U^{\tau_0} + \varepsilon_k \psi)(\bar{x}^k) &= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{U^{\tau_0}(\bar{x}^k) + \varepsilon_k \psi(\bar{x}^k) - U^{\tau_0}(y) - \varepsilon_k \psi(y)}{|\bar{x}^k - y|^{n+2s}} dy \\ &\geq C \int_{B_2^c(x^k)} \frac{|U^{\tau_0}(y)|}{|x^k - y|^{n+2s}} dy \\ &= C \int_{B_2^c(0)} \frac{|U^{\tau_0}(z + x^k)|}{|z|^{n+2s}} dz.\end{aligned} \tag{3.29}$$

Denote

$$u_k(x) = u(x + x^k) \quad \text{and} \quad U_k^{\tau_0}(x) = U^{\tau_0}(x + x^k).$$

Since f is bounded, one can derive (see [10]) that $u(x)$ is at least uniformly Hölder continuous, so $u(x)$ is uniformly continuous, by the Arzelà–Ascoli theorem, up to extraction of a subsequence, one has

$$u_k(x) \to u_\infty(x), \qquad x \in \mathbb{R}^n_+, \quad \text{as } k \to \infty.$$

Combining (3.27), (3.28) and (3.29), letting $k \to \infty$, we obtain

$$U_k^{\tau_0}(x) \to 0, \qquad x \in B_2^c(0), \quad \text{uniformly, as } k \to \infty.$$

Therefore,

$$U_k^{\tau_0}(x) \to u_\infty(x) - u_\infty^{\tau_0}(x) \equiv 0, \qquad x \in B_2^c(0). \tag{3.30}$$

Recall that $u > 0$ in $\mathbb{R}^n_+$ while $u(x) \equiv 0$, $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Since $x^k \in \mathbb{R}^{n-1} \times (0, M_0]$, there exists x^0 such that $u_\infty(x^0) = 0$, then by (3.30),

$$\begin{aligned}0 = u_\infty(x^0) &= u_\infty^{\tau_0}(x^0) = u_\infty((x^0)', x_n^0 + \tau_0) = u_\infty^{\tau_0}((x^0)', x_n^0 + \tau_0) \\ &= u_\infty((x^0)', x_n^0 + 2\tau_0) = \cdots = u_\infty((x^0)', x_n^0 + k\tau_0).\end{aligned} \tag{3.31}$$

We obtain from (1.6) that

$$\lim_{x_n \to +\infty} u_\infty(x) = \mu > 0, \quad \text{uniformly in } x' = (x_1, \ldots, x_{n-1}),$$

that is

$$u_\infty((x^0)', x_n^0 + k\tau_0) \to \mu, \quad \text{as } k \to \infty.$$

This is a contradiction with (3.31). Hence (3.23) is correct. Now (3.23) implies immediately that (3.21) holds by the continuity of $U^\tau(x)$ with respect to τ.

Next we prove (3.22). Otherwise, there exists a constant $A > 0$ such that

$$\sup_{x \in \mathbb{R}^{n-1} \times (M_0+1, +\infty)} U^\tau(x) = A > 0, \qquad \forall\, \tau \in (\tau_0 - \epsilon, \tau_0].$$

Then for some $\tau_2 \in (M_0 + 1, +\infty)$ there exists a sequence $\{x^k\} \subset \mathbb{R}^{n-1} \times (M_0 + 1, +\infty)$ such that

$$U^{\tau_2}(x^k) \to A, \quad \text{as } k \to \infty. \tag{3.32}$$

Since $u = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$, it follows that

$$U^{\tau_2}(x) \le 0, \quad \text{for any } x \in \mathbb{R}^n \setminus \mathbb{R}^n_+.$$

Denote $x^k = (x_1^k, \dots, x_n^k)$. Since $U^{\tau_2}(x^k) = u(x^k) - u^{\tau_2}(x^k) \to 0$ as $x_n^k \to +\infty$, then there exists $M_0 > 0$ such that

$$|x_n^k| \le M_0.$$

Set $\psi_k(x) = \eta(x - x^k)$, where η is as stated in (3.5). From (3.32), there exists a sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0$ such that

$$U^{\tau_2}(x^k) + \varepsilon_k \psi_k(x^k) > A.$$

Since for any $x \in \mathbb{R}^n_+ \setminus B_1(x^k)$, $U^{\tau_2}(x) \le A$ and $\psi_k(x) = 0$, hence

$$U^{\tau_2}(x^k) + \varepsilon_k \psi_k(x^k) > U^{\tau_2}(x) + \varepsilon_k \psi_k(x), \quad \text{for any } x \in \mathbb{R}^n_+ \setminus B_1(x^k).$$

It follows that there exists a point $\bar{x}^k \in B_1(x^k)$ *i.e.* $\bar{x}^k \in \mathbb{R}^{n-1} \times (M_0, +\infty)$ such that

$$U^{\tau_2}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k) = \max_{\mathbb{R}^n}(U^{\tau_2}(x) + \varepsilon_k \psi_k(x)) > A. \tag{3.33}$$

On one hand, by the monotonicity of f, we obtain

$$\begin{aligned}(-\Delta)^s(U^{\tau_2} + \varepsilon_k\psi_k)(\bar{x}^k) &= f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u^{\tau_2}(\bar{x}^k), \nabla u^{\tau_2}(\bar{x}^k)) + \varepsilon_k(-\Delta)^s\psi_k(\bar{x}^k) \\ &\le f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u(\bar{x}^k), \nabla u^{\tau_2}(\bar{x}^k)) + \varepsilon_k(-\Delta)^s\psi_k(\bar{x}^k).\end{aligned} \tag{3.34}$$

On the other hand,

$$\begin{aligned}(-\Delta)^s(U^{\tau_2} + \varepsilon_k\psi_k)(\bar{x}^k) &= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{U^{\tau_2}(\bar{x}^k) + \varepsilon_k\psi_k(\bar{x}^k) - (U^{\tau_2}(y) + \varepsilon_k\psi_k(y))}{|\bar{x}^k - y|^{n+2s}} dy \\ &\ge C \int_{D_M} \frac{A - \frac{A}{2}}{|\bar{x}^k - y|^{n+2s}} dy \\ &\ge \frac{CA}{2} \int_{D_M} \frac{1}{|\bar{x}^k - y|^{n+2s}} dy \\ &\ge CA \frac{1}{[\text{dist}(\bar{x}^k, D_M)]^{2s}},\end{aligned} \tag{3.35}$$

where $M > M_0$ and $D_M = \{|x_n| \ge M\}$, in which $U^\tau(y) + \varepsilon_k\psi_k(y) \le \frac{A}{2}$.

Therefore we obtain

$$\begin{aligned} 0 < c \le CA \frac{1}{[\text{dist}(\bar{x}^k, D_M)]^{2s}} \\ \le f(u(\bar{x}^k), \nabla u(\bar{x}^k)) - f(u(\bar{x}^k), \nabla u^{\tau_2}(\bar{x}^k)) + \varepsilon_k (-\Delta)^s \psi_k(\bar{x}^k), \end{aligned} \tag{3.36}$$

from (3.33), so $\nabla(U^{\tau_2}(\bar{x}^k) + \varepsilon_k \psi_k(\bar{x}^k)) = 0$, *i.e.* $\nabla U^{\tau_2}(\bar{x}^k) \to 0$ as $k \to \infty$. Let $k \to \infty$, then the right-hand side of (3.36) is less than or equal to 0, this is impossible. So (3.22) is true, which contradicts to the definition of τ_0. Therefore, $\tau_0 = 0$, we arrive at (3.20).

Secondly, we will show that u is strictly increasing with respect to x_n and $u(x)$ depends on x_n only. We already have

$$U^\tau(x) \le 0, \quad x \in \mathbb{R}^n_+, \quad \forall\, \tau > 0. \tag{3.37}$$

Now we claim that

$$U^\tau(x) < 0, \quad x \in \mathbb{R}^n_+, \quad \forall\, \tau > 0. \tag{3.38}$$

Otherwise, from (3.37) for some $\tau_1 > 0$ there exists $x^0 \in \mathbb{R}^n_+$ such that $U^{\tau_1}(x^0) = 0$, then x^0 is the maximum point of U^{τ_1} in $\mathbb{R}^n_+$. On one hand, since $\nabla U^{\tau_1}(x^0) = 0$ we have

$$(-\Delta)^s U^{\tau_1}(x^0) = f(u(x^0), \nabla u(x^0)) - f(u^{\tau_1}(x^0), \nabla u^{\tau_1}(x^0)) \le 0.$$

On the other hand,

$$(-\Delta)^s U^{\tau_1}(x^0) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{-U^{\tau_1}(y)}{|x^0 - y|^{n+2s}} dy > 0,$$

where the last inequality holds due to $U^{\tau_1}(y) \not\equiv 0$ in $\mathbb{R}^n$.

This is a contradiction. Hence (3.38) must be true.

Step 3. *We will claim that $u(x)$ depends on x_n only and uniqueness.* In fact, it can be seen from the above process that the argument still holds if we replace $u^\tau(x)$ by $u(x + \tau\nu)$, where $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_n > 0$ being an arbitrary vector pointing upward. Applying the similar arguments as in Steps 1 and 2, we can derive that, for each of such ν,

$$u(x + \tau\nu) > u(x), \qquad \forall\, \tau > 0,\ x \in \mathbb{R}^n_+.$$

Letting $\nu_n \to 0$, from the continuity of u, we deduce that for arbitrary ν with $\nu_n = 0$,

$$u(x + \tau\nu) \ge u(x).$$

By replacing ν by $-\nu$, we obtain that

$$u(x + \tau\nu) = u(x)$$

for arbitrary ν with $\nu = 0$. It implies that u is independent of x', hence $u(x) = u(x_n)$.

Finally we prove the uniqueness. Assume that u and v are two bounded solutions of (1.5). For $\tau \ge 0$, denote

$$\tilde{U}^\tau(x) = v(x) - u^\tau(x).$$

We first show that for τ sufficiently large,

$$\tilde{U}^\tau(x) \le 0, \qquad x \in \mathbb{R}^n_+. \tag{3.39}$$

The proof of (3.39) is completely similar to the proof of (3.10), so we omit the details. Note that (3.39) provides a starting point from which we can decrease τ continuously as long as (3.39) holds.

We show that

$$\tilde{U}^{\tau}(x) \leq 0, \qquad \forall\, \tau \geq 0,\ \forall\, x \in \mathbb{R}^n_+. \tag{3.40}$$

Define

$$\tau_0 := \inf\{\tau > 0 \mid \tilde{U}^{\tau}(x) \leq 0,\ \forall\, x \in \mathbb{R}^n_+,\ \ 0 < \tau < M_0\}.$$

Let us prove that

$$\tau_0 = 0. \tag{3.41}$$

Suppose on the contrary $\tau_0 > 0$. Similarly to the argument of monotonicity in Step 2, one can deduce that

$$v_\infty(x) \equiv u_\infty^{\tau_0}(x), \qquad \forall\, x \in \mathbb{R}^n \setminus B_2(0), \tag{3.42}$$

and

$$u^{\tau_0}(z) \geq C_{\tau_0} > 0, \qquad \forall\, z \in \partial\mathbb{R}^n_+. \tag{3.43}$$

Obviously, this property is preserved under translation. Let

$$\mathbb{R}^n_{+k} = \{x \mid x + x^k \in \mathbb{R}^n_+\} \quad \text{and} \quad \mathbb{R}^n_{+\infty} = \lim_{k\to\infty} \mathbb{R}^n_{+k}.$$

Taking a point $x^0 \in \partial\mathbb{R}^n_{+\infty}$, we deduce from (3.43) that

$$u_\infty^{\tau_0}(x^0) > 0, \quad \text{but} \quad v_\infty(x^0) = 0.$$

This contradicts (3.42). Hence we have $\tau_0 = 0$. This verifies that (3.40) is correct, and implies that $v(x) \leq u(x)$. Interchanging u and v, we obtain $u(x) \leq v(x)$. Therefore, we have $u \equiv v$. This yields the uniqueness.

The proof of Theorem 1.6 is completed. □

Acknowledgements

The author would like to thank Professor Wenxiong Chen for suggesting the problem and all useful discussions. The author appreciates the hospitality of Professor Wenxiong Chen. This work was completed when the author was visiting the Department of Mathematical Science, Yeshiva University.

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