Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **15**, 1–24; https://doi.org/10.14232/ejqtde.2022.1.15 www.math.u-szeged.hu/ejqtde/

Expansion of positivity to a class of doubly nonlinear parabolic equations

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Received 21 May 2021, appeared 4 April 2022
Communicated by Roberto Livrea

Abstract. We establish the *expansion of positivity* of the nonnegative, local, weak solutions to the class of doubly nonlinear parabolic equations

$$\partial_t(u^q) - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \qquad p > 1 \text{ and } q > 0$$

considering separately the two possible cases $q+1-p > 0$ and $q+1-p < 0$. The proof relies on the procedure presented by DiBenedetto, Gianazza and Vespri for both the degenerate and the singular parabolic p-Laplacian equation.

Keywords: doubly nonlinear parabolic equations, expansion of positivity, singular PDE, degenerate PDE, intrinsic scaling.

2020 Mathematics Subject Classification: 35B65, 35K65, 35K67

1 Introduction

In this work we consider the class of doubly nonlinear parabolic equations

$$\partial_t(u^q) - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \quad \text{in } \Omega_T, \quad p > 1 \text{ and } q > 0, \tag{1.1}$$

where $\Omega_T = \Omega \times (0, T]$, being Ω a bounded domain in $\mathbb{R}^N$ and T a real positive number; which models, for instance, the turbulent filtration of non-Newtonian fluids through a porous media (see [4]).

Along the past years many authors have studied this class of evolutionary equations: the simpler case $q = 1$ was widely study (see for instance [2, 3] and the references therein); the Trudinger's equation, corresponding to $q = p-1$, is still object of intensive study (cf. [16,18,19] and more recently [5]; and, for the general case, there are already some results (see [12–14]). In a certain extend, this class of doubly nonlinear equations can be seen as

$$\partial_t u - \operatorname{div}\left(|u|^{m-1}|Du|^{p-2}Du\right) = 0, \quad \text{in } \Omega_T, \quad p > 1$$

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and many are the works concerning its weak solutions (just to mane a few, we refer to [6–10, 15, 20–22]).

The doubly nonlinear equation (1.1) presents several difficulties: for $q > 1$ $(0 < q < 1)$ equation (1.1) has a degeneracy (singularity) in time, since u^{q-1} is zero (explodes) at the points where $u = 0$; while for $1 < p < 2$ $(p > 2)$ (1.1) exhibits a singularity (degeneracy) in space, since the modulus of ellipticity $|Du|^{p-2}$ explodes (is zero) at the points where $Du = 0$. One aspect to always have into consideration is that in order to compensate the degradation of equation's parabolic structure one needs to consider proper cylinders within which the equation behaves as the heat equation – this is known as intrinsic scaling.

The main goal of this work is to give one more contribution to the study of the properties of the weak solutions to this class of doubly nonlinear evolutionary equations (1.1), for $p > 1$ and $q > 0$, namely to present the *expansion of positivity* for its nonnegative bounded weak solutions, which roughly speaking means that the information on the measure of the *positivity set* of u, at a certain time level s over a cuber $K_\rho(y)$, can be expanded to the measure of the *positivity set* of u both in space (say from $K_\rho(y)$ to $K_{2\rho}(y)$) and in time (from the time level s to all further time levels $s + \theta\rho^p$). The proof relies on energy estimates and DeGiorgi-type lemmas and comprehends two steps. The first step consists on the propagation of the positivity information known at a cube located in some time level, say $K_\rho(y) \times \{s\}$, to upper times levels. Not only this is the easiest step but also it holds for all values of $p > 1$ and $q > 0$. As for the second (more demanding and crucial) one, the spacial propagation of positivity is derived from the cube $K_\rho(y)$ to the bigger cube $K_{2\rho}(y)$: the proof is more evolving and requires the cases $q + 1 - p > 0$ and $q + 1 - p < 0$ to be studied separately.

The *expansion of positivity* is the key ingredient to derive Harnack estimates and it can also be an important tool to prove local regularity of the weak solutions. If not only for the mathematical interest per si, these two arguments give extra and relevant reasons for the study at hands.

The paper is organized as follows. In Section 2, we present the notation and some known results needed along the sections to come. In Section 3, we deduce the energy estimates and prove two DeGiorgi-type lemmas which are the core results for the *expansion of positivity*. The proofs of the main results, that is of the *expansion of positivity*, both for $q + 1 - p > 0$ and $q + 1 - p < 0$, are presented in Section 4.

2 Setting the framework

Notation and known results

We start by presenting some notation and some already known results just to keep the text as self contained as possible.

Due to the parabolic nature of our evolutionary equation, we will work with parabolic cylinders and parabolic Sobolev spaces. For that purpose let (x_0, t_0) be an interior point in the space-time domain $\Omega \times (0, T]$. For a cylinder with vertex at (x_0, t_0), of radius $R > 0$ and height τ we can define: the backward cylinder

$$(x_0, t_0) + Q^-(\tau, R) := K_R(x_0) \times (t_0 - \tau, t_0)$$

the forward cylinder

$$(x_0, t_0) + Q^+(\tau, R) := K_R(x_0) \times (t_0, t_0 + \tau),$$

where

$$K_R(x_0) = \{x \in \Omega : \max |x - x_0| < R\}.$$

Let $p \geq 1$. The Sobolev space $H^{1,p}(\Omega)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \left(\int_\Omega (|u|^p + |Du|^p) \right)^{1/p}.$$

A function u belongs to the local Sobolev space $H^{1,p}_{loc}(\Omega)$ if it belongs to $H^{1,p}(K)$ for every compactly contained subset K of Ω. Moreover, the Sobolev space with zero boundary values $H^{1,p}_0(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ with respect to the Sobolev norm.

The parabolic Sobolev space $L^p(t_1, t_2; H^{1,p}(\Omega))$, with $t_1 < t_2$, is the space of functions $u(x,t)$ such that, for almost every $t \in (t_1, t_2)$ the function $u(\cdot, t)$ belongs to $H^{1,p}(\Omega)$ and

$$\int_{t_1}^{t_2} \int_\Omega (|u|^p + |Du|^p) < \infty.$$

The following result establishes an estimate for the gradient of a certain regular function v at the points where $k < v < l$, $k, l \in \mathbb{R}$.

Lemma 2.1. *Let $v \in H^{1,1}(K) \cap C(K)$ and $k, l \in \mathbb{R}$, $k < l$. There exists a positive constant γ, depending only on N and p, such that*

$$(l - k)\, |K \cap [v > l]\,| \leq \gamma \frac{|K|}{|K \cap [v < k]|} \int_{K \cap [k < v < l]} |\nabla v|. \tag{2.1}$$

The result to come establishes a Sobolev embedding.

Proposition 2.2. *There exists a positive constant γ, depending on N, p, and m, such that*

$$\iint_{\Omega_T} |v|^{p\frac{N+m}{N}} \leq \gamma^{p\frac{N+m}{N}} \left\{ \iint_{\Omega_T} |Dv|^p \right\} \times \left\{ \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v|^m \right\}^{\frac{p}{N}}, \tag{2.2}$$

for $v \in L^\infty(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega))$.

The next result concerns algebraic geometric convergence.

Lemma 2.3. *Let $(Y_n)_n$ be a sequence of positive numbers satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$. Then $(Y_n)_n$ converges to zero as $n \to \infty$, provided

$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}.$$

All these results can be found in [2]. The following algebraic result can be found in [1] and in [11], for $0 < q < 1$ and $q > 1$, respectively.

Lemma 2.4. *For any $q > 0$, there exists a positive constant γ, depending on q, such that for all $a, b \in \mathbb{R}$*
$$\frac{1}{\gamma}\left|\mathbf{b}^q - \mathbf{a}^q\right| \le \left(|a| + |b|\right)^{q-1}|b-a| \le \gamma\left|\mathbf{b}^q - \mathbf{a}^q\right|, \tag{2.3}$$
where
$$\mathbf{b}^q = \begin{cases} |b|^{q-1}b, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

As it will be made clearer in the section to come, in order to deduce appropriate energy estimates we'll have to work with the functions
$$g_{\pm}(u,k) = \pm q \int_k^u |s|^{q-1}(s-k)_{\pm}\, ds,$$
for which we need lower and upper bounds. This is the content of the next result (although the proof follows quite closely the one presented in [5] we decided to presented it for the sake of completeness).

Lemma 2.5. *There exists a positive constant γ, depending on q, such that for all $u, k \in \mathbb{R}$, the following holds*
$$\frac{1}{\gamma}\left(|u| + |k|\right)^{q-1}(u-k)^2_{\pm} \le g_{\pm}(u,k) \le \gamma\left(|u| + |k|\right)^{q-1}(u-k)^2_{\pm}. \tag{2.4}$$

Proof. We will present the proof for $g_+(u,k)$, the other case can be treated in an analogous way.

Observe that it is enough to considere $u, k \in \mathbb{R}$, with $u > k$, since otherwise $g_+(u,k) = 0$. So, considering $u > k$, on the one hand
$$\begin{aligned} g_+(u,k) &= q\int_k^u |s|^{q-1}(s-k)_+\, ds \\ &\ge q\int_{\frac{k+u}{2}}^u |s|^{q-1}(s-k)\, ds \\ &\ge q\,\frac{u-k}{2}\int_{\frac{k+u}{2}}^u |s|^{q-1}\, ds \\ &= \frac{u-k}{2}\left(|u|^{q-1}u - \left|\frac{k+u}{2}\right|^{q-1}\frac{k+u}{2}\right) \\ &\ge \frac{(u-k)^2}{\gamma}\left(\left|\frac{k+u}{2}\right| + |u|\right)^{q-1} \\ &\ge \frac{1}{\gamma}(u-k)^2_+\left(|u| + |k|\right)^{q-1}; \end{aligned}$$
on the other hand
$$\begin{aligned} g_+(u,k) &\le (u-k)\, q\int_k^u |s|^{q-1}\, ds \\ &\le \gamma(u-k)^2_+(|u| + |k|)^{q-1}. \end{aligned}$$
The last inequalities in both lower and upper estimates were obtained realizing that
$$\frac{|u| + |k|}{2} \le \left|\frac{k+u}{2}\right| + |u| \le 2(|u| + |k|)$$
and using (2.3). □

Definition of weak solution and Notion of parabolicity

In what follows we state what we mean by a local weak solution to (1.1).

Definition 2.6. A measurable function

$$u \in C(0,T; L^{q+1}_{loc}(\Omega)) \cap L^p_{loc}(0,T; H^{1,p}_{loc}(\Omega))$$

is a weak sub(super)solution to equation (1.1) in $\Omega \times (0,T]$ if, for any compact $K \subset \Omega$ and for almost every $0 < t_1 < t_2 < T$, it satisfies

$$\int_K [\mathbf{u}^q \varphi\ (x,t)]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \left(|Du|^{p-2} Du \cdot D\varphi - \mathbf{u}^q \varphi_t \right) \ \leq (\geq)\, 0, \tag{2.5}$$

for every nonnegative test function

$$\varphi \in H^{1,q+1}_{loc}(0,T; L^{q+1}(K)) \cap L^p_{loc}(0,T; H^{1,p}_0(K)).$$

A weak solution to (1.1) is a function that is both a weak subsolution and a weak supersolution to (1.1).

Remark 2.7. Observe that the regularity assumption on u and on test functions η guarantee the convergence of the integrals appearing in (2.5).

In the case $0 < q < 1$, one can consider, and thereby recover, the regularity assumption on u presented for the p-Laplacian, that is, $u \in C(0,T; L^2_{loc}(\Omega)) \cap L^p_{loc}(0,T; H^{1,p}_{loc}(\Omega))$.

Equation (1.1) presents two interesting and relevant features: one is that the nonlinearity exhibit by the time derivative part does not allow us to add constants to the solution and still have a solution; the other one regards the notion of parabolicity (which does not come directly from the differential equation). Taking this into account, we say that equation (1.1) is *parabolic* if

> for all $k \in \mathbb{R}$, whenever u is a weak sub(super)solution, the function $k \pm (u-k)_\pm$ is a local weak sub(super)solution,

where

$$(u-k)_- = \max\{0, k-u\}, \quad (u-k)_+ = \max\{0, u-k\},$$

and

$$k - (u-k)_- = \min\{u,k\}, \quad \text{and} \quad k + (u-k)_+ = \max\{u,k\}.$$

The following result asserts that equation (1.1) is a parabolic equation. The proof follows closely the one presented in [2], for the p-Laplacian equation, and also the one presented in [5] for the Trudinger's equation.

Lemma 2.8. *Let u be a local weak sub(super)solution to (1.1). Then for all $k \in \mathbb{R}$, the truncated functions $k \pm (u-k)_\pm$ are local weak sub(super)solutions to (1.1).*

Proof. Let (x_0, t_0) be an interior point of Ω_T, which by translation we will assume $(x_0, t_0) = (0,0)$. Let u be a subsolution to (1.1) and consider a real number $k \in \mathbb{R}$ (the case of a supersolution can be treat analogously).

It is well known that the time derivative $\partial_t u$ has to be avoided (its notion may not even exist in a Sobolev sense) and so we use the regularization, proposed by Kinnunen and Lindqvist [17],

$$u^\star(x,t)=\frac{1}{\sigma}\int_0^t e^{\frac{s-t}{\sigma}}u(x,s)\,ds,\quad \sigma>0, \tag{2.6}$$

to overcome this difficulty.

The mollified version of (2.5) is then given by

$$\iint_{\Omega_T}\partial_t\left((\mathbf{u}^q)^\star\right)\varphi+\left(|Du|^{p-2}Du\right)^\star\cdot D\varphi\le\int_\Omega \mathbf{u}^q(x,0)\left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\,ds\right) \tag{2.7}$$

for all $0\le\eta\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^{q+1}(\Omega_T)$.

Consider the test function

$$\varphi(x,t)=\xi^p(x,t)\psi_\epsilon(t)\frac{(u-k)_+}{(u-k)_++h},\qquad h,\epsilon>0,$$

being $\xi\in C^1(Q(\tau,R))$, verifying $0\le\xi\le1$ and vanishing on the lateral boundary of $Q(\tau,R)$; and $\psi_\epsilon(t)$ a piecewise smooth cutoff function defined by

$$\psi_\epsilon(t)=\begin{cases}0, & -\tau\le t\le t_1-\epsilon,\\ 1+\frac{t-t_1}{\epsilon}, & t_1-\epsilon\le t\le t_1,\\ 1, & t_1\le t\le t_2,\\ 1-\frac{t-t_2}{\epsilon}, & t_2\le t\le t_2+\epsilon,\\ 0, & t_2+\epsilon\le t\le 0.\end{cases}$$

Let v_σ be such that $(\mathbf{v}_{\boldsymbol{\sigma}})^q=(\mathbf{u}^q)^\star$.

The parabolic and elliptic terms appearing in (2.7) will be treated separately. As for the parabolic term

$$\begin{aligned}\iint_{\Omega_T}\partial_t\left((\mathbf{u}^q)^\star\right)\varphi=&\iint_{Q(\tau,R)}\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\xi^p\psi_\epsilon\left(\frac{(v_\sigma-k)_+}{(v_\sigma-k)_++h}\right)\\ &+\iint_{Q(\tau,R)}\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\xi^p\psi_\epsilon\left(\frac{(u-k)_+}{(u-k)_++h}-\frac{(v_\sigma-k)_+}{(v_\sigma-k)_++h}\right).\end{aligned}$$

By observing that

$$\partial_t\left(\int_{\mathbf{k}^q}^{(\mathbf{v}_{\boldsymbol{\sigma}})^q}\frac{(\mathbf{s}^{\frac{1}{q}}-k)_+}{(\mathbf{s}^{\frac{1}{q}}-k)_++h}\,ds\right)=\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\frac{(v_\sigma-k)_+}{(v_\sigma-k)_++h}$$

and

$$\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)=\partial_t\left((\mathbf{u}^q)^\star\right)=\frac{\mathbf{u}^q-(\mathbf{v}_{\boldsymbol{\sigma}})^q}{\sigma},$$

the second integral appearing in the right hand side of the previous integral identity is non-negative, since both factors have the same signal due the fact that $f(s)=\frac{(\mathbf{s}^{1/q}-k)_+}{(\mathbf{s}^{1/q}-k)_++h}$ is a monotone nondecreasing function. As for the first integral, note that

$$\begin{aligned}\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\left(\frac{(v_\sigma-k)_+}{(v_\sigma-k)_++h}\right)&=\partial_t\left(\int_{\mathbf{k}^q}^{(\mathbf{v}_{\boldsymbol{\sigma}})^q}\frac{(\mathbf{s}^{\frac{1}{q}}-k)_+}{(\mathbf{s}^{\frac{1}{q}}-k)_++h}\,ds\right)\\ &=\partial_t\left(\mathbf{k}^q+q\int_k^{v_\sigma}\frac{(s-k)_+}{(s-k)_++h}|s|^{q-1}\,ds\right)\\ &\overset{\text{def}}{=}\partial_t\left(\mathcal{I}(v_\sigma,k,h,+)\right).\end{aligned}$$

Gathering these informations we arrive at

$$\begin{aligned}\iint_{\Omega_T} \partial_t\left((u^q)^\star\right)\varphi &\geq \iint_{Q(\tau,R)} \partial_t\left(\mathcal{I}(\mathbf{v}_{\text{œ}},k,h,+)\right)\xi^p\psi_\epsilon \\ &= -\iint_{Q(\tau,R)} \mathcal{I}(\mathbf{v}_{\text{œ}},k,h,+)\left\{\xi^p\psi_\epsilon' + \partial_t\left(\xi^p\right)\psi_\epsilon\right\}.\end{aligned}$$

The regularity assumptions considered allow us to, first pass to limit as $\sigma \to 0$, and then let $\epsilon \to 0$, getting thereby the inferior bound

$$-\iint_{Q(\tau,R)} \mathcal{I}(u,k,h,+)\partial_t\left(\xi^p\right) - \int_{K_R\times\{t_2\}} \mathcal{I}(u,k,h,+)\xi^p + \int_{K_R\times\{t_1\}} \mathcal{I}(u,k,h,+)\xi^p.$$

As for the elliptic term, we start by letting $\sigma \to 0$ and then $\epsilon \to 0$ to arrive at

$$\begin{aligned}\int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot\left(\xi^p\frac{(u-k)_+}{(u-k)_++h}\right) &= \int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot D(\xi^p)\frac{(u-k)_+}{(u-k)_++h} \\ &\quad + \int_{t_1}^{t_2}\int_{K_R} |D(u-k)_+|^p\frac{h}{((u-k)_++h)^2}\xi^p \\ &\geq \int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot D(\xi^p)\frac{(u-k)_+}{(u-k)_++h},\end{aligned}$$

since the last integral appearing in the integral identity is nonnegative. The proof is complete once we let $h \to 0$; just take notice that

$$\mathcal{I}(v_\sigma,k,h,+) \to \left(k^q + q\int_k^u s^{q-1}\,ds\right)\chi_{[u>k]} = (k+(u-k)_+)^q, \quad \text{as } h\to 0 \qquad \square$$

Remark 2.9. The purpose of this work is to present (and prove) the *expansion of positivity* for the nonnegative, local, weak solutions to (1.1). The results to come will be stated in this context (note that for $u \geq 0$, $\mathbf{u}^q = u^q$).

3 Energy estimates and DeGiorgi-type lemmas

The following result comprehends local integral estimates that are the starting point to the proof of the expansion of positivity, the so called energy estimates.

Proposition 3.1. *Let u be a nonnegative, local, weak sub(super)solution to (1.1) in Ω_T in the sense of (2.5). There exists a positive constant C, depending on N, p and q, such that for every cylinder $(x_o,t_o)+Q^-(\tau,R) \subset \Omega_T$, every real number $k \in \mathbb{R}$ and every piecewise smooth nonnegative cutoff function ξ vanishing on the the lateral boundary of $(x_0,t_0)+Q(\tau,R)$ one has*

$$\begin{aligned}&\operatorname*{ess\,sup}_{t_0-\tau<t<t_0}\int_{K_R(x_0)} g_\pm(u,k)\xi^p + \iint_{(x_0,t_0)+Q(\tau,R)} |D(u-k)_\pm|^p\xi^p \\ &\quad\leq \int_{K_R(x_0)\times\{t_0-\tau\}} g_\pm(u,k)\xi^p + C\iint_{(x_0,t_0)+Q(\tau,R)}\left\{(u-k)_\pm^p|D\xi|^p + g_\pm(u,k)\left|\partial_t\left(\xi^p\right)\right|\right\}.\end{aligned} \tag{3.1}$$

Proof. The proof follows quite closely the one presented in Lemma 2.8. In fact, we start by considering a nonnegative, local, weak subsolution u to (1.1) and then work with the mollified version (2.7), taking as test function $\varphi(x,t) = \xi^p(x,t)\psi_\epsilon(t)(u-k)_+$, where ξ and

ψ_ϵ are precisely the same as before. Observe that φ is an admissible test function due to the regularity assumptions on u.

By considering v_σ to be such that $(v_\sigma)^q = (u^q)^\star$ we get

$$\iint_{\Omega_T} \partial_t \left((v_\sigma)^q\right) \varphi + \left(|Du|^{p-2}Du\right)^\star \cdot D\varphi \leq \int_\Omega u^q(x,0) \left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\, ds\right).$$

The left-hand side is estimated as follows. The term evolving the time derivative

$$\begin{aligned}
\iint_{\Omega_T} \partial_t \left((v_\sigma)^q\right) \varphi = & \iint_{Q(\tau,R)} \partial_t \left((v_\sigma)^q\right) \xi^p \psi_\epsilon (v_\sigma - k)_+ \\
& + \iint_{\Omega_T} \partial_t \left((v_\sigma)^q\right) \left((u-k)_+ - (v_\sigma - k)_+\right) \xi^p \psi_\epsilon \\
= & \iint_{\Omega_T} \partial_t \left(g_+(v_\sigma, k)\right) \xi^p \psi_\epsilon \\
& + \iint_{\Omega_T} \frac{u^q - (v_\sigma)^q}{\sigma} \left((u-k)_+ - (v_\sigma - k)_+\right) \xi^p \psi_\epsilon \\
\geq & \iint_{\Omega_T} \partial_t \left(g_+(v_\sigma, k)\right) \xi^p \psi_\epsilon \\
= & -\iint_{\Omega_T} g_+(v_\sigma, k) \partial_t \left(\xi^p\right) \psi_\epsilon - \iint_{\Omega_T} g_+(v_\sigma, k) \xi^p \psi_\epsilon'
\end{aligned}$$

The inequality relies on the fact that $f_1(s) = (s-k)_+$ and $f_2(s) = s^q$, $q > 0$, are monotone increasing functions. We now let $\sigma \to 0$ and thereby get

$$-\int_{t_1-\epsilon}^{t_2+\epsilon}\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right)\psi_\epsilon - \frac{1}{\epsilon}\int_{t_1-\epsilon}^{t_1}\int_{K_R(x_0)} g_+(u,k)\xi^p + \frac{1}{\epsilon}\int_{t_2}^{t_2+\epsilon}\int_{K_R(x_0)} g_+(u,k)\xi^p,$$

since $u \in L^{q+1} \supset L^q$ we have $(v_\sigma) \to u$ in L^q. We then pass to the limit as ϵ goes to zero, obtaining

$$-\int_{t_1}^{t_2}\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right) - \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p$$

and this completes the estimate of the parabolic term. As for the integral evolving the space derivatives, we start by letting $\sigma \to 0$, then we apply Young's inequality to get the inferior estimate

$$\frac{1}{2}\iint_{\Omega_T} |D(u-k)_+|^p \xi^p \psi_\epsilon - (2(p-1))^{p-1} \iint_{\Omega_T} (u-k)_+^p |D\xi|^p \psi_\epsilon$$

and finally, by letting $\epsilon \to 0$, we obtain

$$\frac{1}{2}\int_{t_1}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p \xi^p - (2(p-1))^{p-1} \int_{t_1}^{t_2}\int_{K_R(x_0)} (u-k)_+^p |D\xi|^p.$$

As for the right-hand side,

$$\int_\Omega u^q(x,0)\left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\, ds\right) \to \int_\Omega u^q(x,0)\varphi(x,0) = 0, \quad \text{as } \sigma \to 0.$$

Combining all the previous estimates we have

$$
\begin{aligned}
\int_{K_R(x_0)\times\{t_2\}} & g_+(u,k)\xi^p + \frac{1}{2}\int_{t_1}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \\
&\leq \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{t_1}^{t_2}\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right) \\
&\quad + (2(p-1))^{p-1}\int_{t_1}^{t_2}\int_{K_R(x_0)} (u-k)_+^p|D\xi|^p \\
&\leq \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} g_+(u,k)|\partial_t\left(\xi^p\right)| \\
&\quad + (2(p-1))^{p-1}\int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} (u-k)_+^p|D\xi|^p.
\end{aligned}
$$

By letting $t_1 \to t_0-\tau$ and recalling $u \in C(L^{q+1})$, we have

$$
\int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p \to \int_{K_R(x_0)\times\{t_0-\tau\}} g_+(u,k)\xi^p;
$$

as for the left-hand side of the previous inequality we reason as follows: on the one hand, for $t_0-\tau < t_2 < t_0$,

$$
\int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p + \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \geq \int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p,
$$

and we then take the essential supremum over the set $t_0-\tau < t_2 < t_0$; on the other hand,

$$
\begin{aligned}
\int_{K_R(x_0)\times\{t_2\}} & g_+(u,k)\xi^p + \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \\
&\geq \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \to \frac{1}{2}\int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p,
\end{aligned}
$$

as $t_2 \to t_0$.

A final remark: to prove the estimate for supersolutions it suffices to take $\varphi(x,t) = \xi^p(x,t)\psi_\epsilon(t)(u-k)_-$ and proceed in a similar way. □

The next two results are DeGiorgi-type lemmas, being the second one a variant involving information concerning initial data. They are presented for nonnegative, locally bounded, weak supersolutions to (1.1), however, one can state (and prove) similar results for nonnegative, locally bounded, weak subsolutions to (1.1). We recall that the local boundedness of the nonnegative, local, weak solutions u to (1.1) was obtained in [13] and [14].

To simplify the writing consider

$$
(y,s) + Q^-(\lambda(2\rho)^p, 2\rho) = (y,s) + Q^-_{2\rho}(\lambda)
$$

$$
(y,s) + Q^+(\lambda(2\rho)^p, 2\rho) = (y,s) + Q^+_{2\rho}(\lambda)
$$

Lemma 3.2. *Let u be a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T. Let $\tilde{M}$ be a positive number. There exists a constant $\tilde{\nu}$ depending on the N, p, q and on $\tilde{M}$ and λ, such that, if*

$$
\left|(y,s) + Q^-_{2\rho}(\lambda) \cap \left[u < \tilde{M}\right]\right| \leq \tilde{\nu}\left|Q^-_{2\rho}(\lambda)\right|
$$

then

$$
u \geq \frac{\tilde{M}}{2} \quad \text{a.e. in } (y,s) + Q^-_{\rho}(\lambda).
$$

Proof. Without loss of generality take $(y,s)=(0,0)$. Construct decreasing sequences of numbers (radii and levels, respectively), for $n=0,1,\dots,$

$$\rho_n=\rho\left(1+\frac{1}{2^n}\right),\qquad k_n=\frac{\tilde{M}}{2}\left(1+\frac{1}{2^n}\right)$$

and define $Q_n=Q^-_{\rho_n}(\lambda)$ and $K_n=K_{\rho_n}$. Consider the cutoff function $0\le\xi(x,t)=\xi_1(x)\xi_2(t)\le 1$ defined in Q_n and such that

$$\xi_1(x)=1\quad\text{in } K_{n+1};\qquad \xi_1(x)=0\quad\text{in } \mathbb{R}^N\setminus K_n;\qquad |D\xi|\le\frac{2^{n+1}}{\rho};$$

$$\xi_2(t)=1\quad\text{for } t\ge-\lambda\rho^p_{n+1};\qquad \xi_2(t)=0\quad\text{for } t\le-\lambda\rho^p_n;\qquad 0\le\partial_t\xi_2\le\frac{2^{p(n+1)}}{\lambda\rho^p}.$$

From the energy estimates (3.1) written over Q_n for the truncated functions $(u-k_n)_-$, and recalling the estimates for $g_-(u,k_n)$ given in (2.4), we obtain

$$\begin{aligned}
&\frac{\tilde{M}^{q-1}}{\gamma(q)}\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le0}\int_{K_n\times\{t\}}(u-k_n)_-^2\xi^p+\iint_{Q_n}|D(u-k_n)_-|^p\xi^p\\
&\le\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le0}\int_{K_n\times\{t\}}g_-(u,k_n)\xi^p+\iint_{Q_n}|D(u-k_n)_-|^p\xi^p\\
&\le C(p,q)\,2^{p(n+1)}\,\frac{k_n^p}{\rho^p}\left\{1+\frac{k_n^{q+1-p}}{\lambda}\right\}|A_n|\\
&\le C(p,q)\,2^{p(n+1)}\,\frac{\tilde{M}^p}{\rho^p}\left\{1+\frac{\tilde{M}^{q+1-p}}{\lambda}\right\}|A_n|,
\end{aligned}$$

for $|A_n|=|Q_n\cap[u<k_n]|$.

Observe that, by means of Hölder's inequality together with the Sobolev embedding (2.2),

$$\begin{aligned}
\frac{\tilde{M}}{2^{n+2}}|A_{n+1}|=(k_n-k_{n+1})|A_{n+1}|&\le\iint_{Q_{n+1}}(u-k_n)_-\\
&\le C(N,p)\left(\iint_{Q_n}|D(u-k_n)_-\xi|^p\right)^{\frac{N}{p(N+2)}}\\
&\quad\times\left(\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le0}\int_{K_n\times\{t\}}(u-k_n)_-^2\xi^p\right)^{\frac{1}{N+2}}\quad|A_n|^{1-\frac{N}{p(N+2)}}
\end{aligned}$$

and then, recalling the previous estimates and taking $Y_n=\frac{|A_n|}{|Q_n|}$, we get the recursive algebraic estimate

$$Y_{n+1}\le C(N,p,q)\left(\tilde{M}^{p-(q+1)}\lambda\right)^{\frac{1}{N+2}}\left(1+\frac{\tilde{M}^{q+1-p}}{\lambda}\right)^{\frac{N+p}{p(N+2)}}b^n\,Y_n^{1+\frac{1}{N+2}},\quad\text{for } b=2^{\frac{2N+p+2}{N+2}}.$$

The conclusion follows from the fast convergence Lemma 2.3 once we consider

$$\tilde{\nu}=C(N,p,q)^{-(N+2)}b^{-(N+2)^2}\frac{\frac{\tilde{M}^{q+1-p}}{\lambda}}{\left(1+\frac{\tilde{M}^{q+1-p}}{\lambda}\right)^{\frac{N+p}{p}}}.$$

□

Lemma 3.3. *Let u be a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T. Let $\tilde{M}$ be a positive number such that*

$$u(x,s) \geq \tilde{M} \quad \text{for a.e. } x \in K_{2\rho}(y) \tag{3.2}$$

and

$$\left|(y,s) + Q^+_{2\rho}(\lambda) \cap [u < \tilde{M}]\right| \leq \nu_0 \, \frac{\tilde{M}^{q+1-p}}{\lambda} \, \left|Q^+_{2\rho}(\lambda)\right| \tag{3.3}$$

for ν_0 depending only upon N, p and q. Then

$$u \geq \frac{\tilde{M}}{2} \quad \text{a.e. in } K_\rho(y) + (s, s + \lambda(2\rho)^p].$$

Proof. Take $(y,s) = (0,0)$, construct decreasing sequences of numbers (radii and levels, respectively),

$$\rho_n = \rho\left(1 + \frac{1}{2^n}\right), \quad k_n = \frac{\tilde{M}}{2}\left(1 + \frac{1}{2^n}\right), \quad n = 0, 1, \ldots$$

and take a time independent cutoff function $0 \leq \xi(x) \leq 1$ defined in K_{ρ_n} and, such that, $\xi = 1$ in $K_{\rho_{n+1}}$ and $|D\xi| \leq 2^{n+1}/\rho$.

Keeping in mind that u verifies (3.2), from the energy estimates (3.1) written over $Q_n = K_{\rho_n} \times (0, \lambda(2\rho)^p]$, for the truncated functions $(u - k_n)_-$, we obtain, for all $t \in (0, \lambda(2\rho)^p]$

$$\begin{aligned}
&\frac{\tilde{M}^{q-1}}{\gamma(q)} \int_{K_{\rho_n} \times \{t\}} (u - k_n)^2_- \xi^p + \iint_{Q_n} |D(u - k_n)_-|^p \xi^p \\
&\quad \leq \int_{K_{\rho_n} \times \{t\}} g_-(u, k_n) \xi^p + \iint_{Q_n} |D(u - k_n)_-|^p \xi^p \\
&\quad \leq C(p) \iint_{Q_n} (u - k_n)^p_- |D\xi|^p \leq C(p) \, 2^{np} \, \frac{\tilde{M}^p}{\rho^p} |A_n|,
\end{aligned}$$

for $|A_n| = |Q_n \cap [u < k_n]|$. Arguing in a similar way as in the proof of Lemma 3.2, we deduce

$$\begin{aligned}
\frac{\tilde{M}}{2^{n+2}} |A_{n+1}| &\leq C(N,p) \left(\iint_{Q_n} |D(u - k_n)_- \xi|^p\right)^{\frac{N}{p(N+2)}} \\
&\quad \times \left(\operatorname*{ess\,sup}_{0 < t < \lambda(2\rho)^p} \int_{K_{\rho_n} \times \{t\}} (u - k_n)^2_- \xi^p\right)^{\frac{1}{N+2}} \quad |A_n|^{1 - \frac{N}{p(N+2)}}
\end{aligned}$$

and then, recalling the previous estimates and considering $Y_n = \frac{|A_n|}{|Q_n|}$, we arrive at

$$Y_{n+1} \leq C(N,p,q) \, \tilde{M}^{\frac{p-(q+1)}{N+2}} \, \lambda^{\frac{1}{N+2}} \, b^n \, Y_n^{1 + \frac{1}{N+2}}, \qquad b = 2^{\frac{2N+p+2}{N+2}}.$$

The proof is completed once we take $\nu_0 = C(N,p,q)^{-(N+2)} b^{-(N+2)^2}$. □

4 Expansion of positivity

As it is now well known the *expansion of positivity* is a property of nonnegative supersolutions, to both stationary and evolutionary equations, that allows one to get Harnack inequalities and to prove regularity results. Heuristically speaking it asserts that the information on the measure of the *positivity set* of u, at a certain time level s over a cuber $K_\rho(y)$, can be expanded to the measure of the *positivity set* of u both in space (say from $K_\rho(y)$ to $K_{2\rho}(y)$) and in time (from the time level s to all further time levels till $s+\theta\rho^p$).

The *expansion of positivity* is based on energy estimates and DeGiorgi-type lemmas and comprehends two steps. The first step consists on the propagation of the positivity information on a cube $K_\rho(y)\times\{s\}$ to upper times levels. Not only this is the easiest step but also it holds for all values of $p>1$ and $q>0$. On the second step, one derives spacial propagation of positivity from the cube $K_\rho(y)$ to the bigger cube $K_{2\rho}(y)$. This is much more demanding and the proof has to be performed separately for the cases $q+1-p>0$ and $q+1-p<0$.

In what follows we adopt the technical scheme devised by DiBenedetto, Gianazza and Vespri for degenerate and singular p-Laplacian parabolic equations – the results can be found in [3]: Chapter 4, Sections 4 and 5, respectively.

Along this section we will assume that u is a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T, for $p>1$ and $q>0$.

Lemma 4.1. *Assume that, for some $(y,s)\in\Omega_T$ and some $\rho>0$,*

$$\left|K_\rho(y)\cap[u(\cdot,s)>M]\right|\geq\alpha\left|K_\rho(y)\right|, \tag{4.1}$$

for some $M>0$ and some $0<\alpha<1$. Then there exist $\epsilon,\delta\in(0,1)$, depending on α and on N,p and q, such that

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|, \tag{4.2}$$

for all $t\in(s,s+\delta M^{q+1-p}\rho^p]$.

Proof. Without loss of generality we may take $(y,s)=(0,0)$. Consider the cylinder $Q=K_\rho\times(0,\delta M^{q+1-p}\rho^p]$ and assume $Q\subset\Omega_T$ (take ρ as smaller as needed). Write the energy estimates (3.1) for the cylinder Q, the level $k=M$ and the smooth time independent cutoff function $0\leq\xi=\xi(x)\leq1$ defined in K_ρ, vanishing on the boundary of K_ρ and verifying, for some $\sigma\in(0,1)$,

$$\xi=1\quad\text{in }K_{(1-\sigma)\rho}\quad\text{and}\quad|D\xi|\leq\frac{1}{\sigma\rho}.$$

We then have, for all $t\in(0,\delta M^{q+p-1}\rho^p]$,

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}\times\{t\}}g_-(u,k)&\leq\int_{K_\rho\times\{t\}}g_-(u,k)\xi^p\\&\leq\int_{K_\rho\times\{0\}}g_-(u,k)\xi^p+C(p)\iint_Q(u-k)_-^p|D\xi|^p\\&\leq\int_{K_\rho\times\{0\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\xi^p\chi_{[u<M]}+C(p)\frac{M^p}{\sigma^p\rho^p}|Q|.\end{aligned}$$

The left-hand side is estimated by considering the integration over the smaller cube $K_{(1-\sigma)\rho}\cap[u(\cdot,t)<\epsilon M]$

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}\times\{t\}}g_-(u,k)&=\int_{K_{(1-\sigma)\rho}\times\{t\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\chi_{[u<M]}\\&\geq\int_{K_{(1-\sigma)\rho}\times\{t\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\chi_{[u<\epsilon M]}\end{aligned}$$

$$\geq \int_{K_{(1-\sigma)\rho}\times\{t\}} \left(q \int_{\epsilon M}^{M} s^{q-1}(M-s)\, ds \right) \chi_{[u<\epsilon M]}$$
$$= \left| K_{(1-\sigma)\rho} \cap [u(\cdot,t) < \epsilon M] \right| \times \left(q \int_{\epsilon M}^{M} s^{q-1}(M-s)\, ds \right)$$

and then, for all $t \in (0, \delta M^{q+p-1}\rho^p]$,

$$\begin{aligned}\left| K_{(1-\sigma)\rho} \cap [u(\cdot,t) < \epsilon M] \right| &\leq \frac{\int_{K_\rho \times \{0\}} \left(q \int_u^M s^{q-1}(M-s)\, ds \right) \xi^p \chi_{[u<M]}}{q \int_{\epsilon M}^M s^{q-1}(M-s)\, ds} \\ &\quad + C(p)\, \frac{\delta M^{q+1}}{\sigma^p}\, \frac{1}{q \int_{\epsilon M}^M s^{q-1}(M-s)\, ds}\, |K_\rho| \\ &\leq \left\{ (\gamma(q)\epsilon^q + 1)(1-\alpha) + C(p,q)\frac{\delta}{\sigma^p} \right\} |K_\rho|.\end{aligned}$$

These inequalities were obtained arguing as follows: due to (2.4) and considering $0 < \epsilon < \frac{1}{2}$

$$q \int_{\epsilon M}^M s^{q-1}(M-s)\, ds \geq \frac{1}{2\gamma(q)} M^{q+1}(1-\epsilon)^2(1+\epsilon)^{q-1} \geq \frac{M^{q+1}}{\gamma(q)}\ ;$$

and by making use of the same inequality (2.4) and recalling (4.1)

$$\begin{aligned}&\frac{\int_{K_\rho \times \{0\}} \left(q \int_u^M s^{q-1}(M-s)\, ds \right) \xi^p \chi_{[u<M]}}{q \int_{\epsilon M}^M s^{q-1}(M-s)\, ds} \\ &\quad = \frac{\int_{K_\rho \times \{0\}} \left(q \int_u^{\epsilon M} s^{q-1}(M-s)\, ds \right) \xi^p \chi_{[u<M]}}{q \int_{\epsilon M}^M s^{q-1}(M-s)\, ds} + \int_{K_\rho \times \{0\}} \chi_{[u<M]} \\ &\quad \leq \left\{ \frac{\int_0^{\epsilon M} s^{q-1} M\, ds}{\int_{\epsilon M}^M s^{q-1}(M-s)\, ds} + 1 \right\} (1-\alpha)|K_\rho| \\ &\quad \leq (\gamma(q)\epsilon^q + 1)(1-\alpha)|K_\rho|.\end{aligned}$$

Therefore, for all $t \in (0, \delta M^{q+p-1}\rho^p]$,

$$\left| K_\rho \cap [u(\cdot,t) < \epsilon M] \right| \leq \left\{ (\gamma(q)\epsilon^q + 1)(1-\alpha) + C(p,q)\frac{\delta}{\sigma^p} + \sigma N \right\} |K_\rho|.$$

The proof is complete once we choose $\sigma \in (0,1)$ such that $N\sigma \leq \frac{\alpha}{8}$; then choose

$$\delta \in (0,1) \quad \text{such that} \quad C(p,q)\frac{\delta}{\sigma^p} \leq \frac{\alpha}{8}$$

and finally choose

$$\epsilon \in \left(0, \frac{1}{2}\right) \quad \text{such that} \quad (\gamma(q)\epsilon^q + 1)(1-\alpha) \leq 1 - \frac{3}{4}\alpha.$$

Observe that, with theses choices, the parameters δ and ϵ depend only on α and on N, p, q. $\square$

Expansion of positivity when $q+1-p>0$

Consider a point $(y,s) \in \Omega_T$ and let $\rho > 0$ be such that

$$K_{16\rho}(y) \times (s, s+\delta M^{q+1-p}\rho^p] \subset \Omega_T,$$

where δ and M are the same positive real numbers presented in Lemma 4.1.

Consider that (4.1) holds. In order to obtain the expansion of positivity, we start to consider the change of variables: in space, given by $z = \frac{x-y}{\rho}$ and in time, given by

$$-e^{-\tau} = \frac{t-(s+\delta M^{q+1-p}\rho^p)}{\delta M^{q+1-p}\rho^p},$$

which maps the original cylinder $K_{16\rho}(y) \times (s, s+\delta M^{q+1-p}\rho^p]$ into $K_{16} \times (0,+\infty)$.

Introduce the new function

$$v(z,\tau) = \frac{u(x,t)}{M}\, e^{\frac{\tau}{q+1-p}}$$

which verifies

$$\partial_\tau(v^q) - \operatorname{div}\left(\delta |Dv|^{p-2}Dv\right) = \frac{q}{q+1-p}\, v^q \tag{4.3}$$

where D denotes de space derivates of v with respect to z.

Keeping in mind that δ and ϵ are already determined and depend only on N, p, q and on α, the conclusion of Lemma 4.1 now reads

$$\left| K_1 \cap [v(\cdot,\tau) > \epsilon\, e^{\frac{\tau}{q+1-p}}] \right| \geq \frac{\alpha}{2}\, |K_1|\,, \quad \forall \tau > 0$$

and therefore, once we take $\tau_0 > 0$ and consider the level $k_0 = \epsilon\, e^{\frac{\tau_0}{q+1-p}}$, we have

$$|K_1 \cap [v(\cdot,\tau) \geq k_0]| \geq \frac{\alpha}{2}\, |K_1|\,, \quad \forall \tau \geq \tau_0$$

and then, for all $k \leq k_0$,

$$|K_8 \cap [v(\cdot,\tau) \geq k]| \geq |K_8 \cap [v(\cdot,\tau) \geq k_0]| \geq \frac{\alpha}{2}\, |K_1| = \frac{\alpha}{2^{1+3N}}\, |K_8|\,, \quad \forall \tau \geq \tau_0. \tag{4.4}$$

With the time level τ_0 and the level k_0 we construct the cylinders

$$Q_{\tau_0} = K_8 \times \left(\tau_0 + k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}\right] \subset \tilde{Q}_{\tau_0} = K_8 \times \left(\tau_0, \tau_0 + 2k_0^{q+1-p}\right]$$

and introduce smaller levels

$$k_j = \frac{k_0}{2^j}, \quad \text{for } j = 0, 1, \ldots, s_*,$$

where s_* is to be chosen.

Consider a piecewise smooth cutoff $0 \leq \xi(x,t) = \xi_1(x)\xi_2(t) \leq 1$ defined in $\tilde{Q}_{\tau_0}$ and such that

$$\xi_1(x) = \begin{cases} 1 & \text{in } K_8, \\ 0 & \text{in } \mathbb{R}^N \setminus K_8, \end{cases} \quad \text{and} \quad |D\xi_1| \leq \frac{1}{8}$$

and

$$\xi_2(x) = \begin{cases} 0 & \tau < \tau_0, \\ 1 & \tau \geq \tau_0 + k_0^{q+1-p}, \end{cases} \quad \text{and} \quad 0 \leq \partial_\tau \xi_2 \leq \frac{1}{k_0^{q+1-p}}.$$

At this stage we perform formally (the accurate way to proceed follows the procedure presented before when deducing the energy estimates): we start by multiplying (4.3) by $(v-k_j)_-\xi^p$ and then integrate over $\tilde{Q}_{\tau_0}$. Note that the right-hand side is nonnegative, since $v \geq 0$ and $q+1-p>0$, and therefore we have

$$\iint_{\tilde{Q}_{\tau_0}} \partial_\tau\left(v^q\right)(v-k_j)_-\xi^p + \delta \iint_{\tilde{Q}_{\tau_0}} |Dv|^{p-2}Dv \cdot D\left((v-k_j)_-\xi^p\right) \geq 0.$$

This integral inequality is equivalent to

$$\iint_{\tilde{Q}_{\tau_0}} \partial_\tau\left(g_-(v,k_j)\right)\xi^p + \delta \iint_{\tilde{Q}_{\tau_0}} |D(v-k_j)_-|^{p-2}D(v-k_j)_- \cdot D\left((v-k_j)_-\xi^p\right) \leq 0$$

from which we get, for all $\tau \in (0, \tau_0 + 2k_0^{q+1-p}]$,

$$\begin{aligned}\iint_{\tilde{Q}_{\tau_0}} |D(v-k_j)_-|^p\xi^p &\leq \frac{1}{\delta}\int_{K_8\times\{\tau_0\}} g_-(v,k_j)\xi^p + \frac{C(p)}{\delta}\iint_{\tilde{Q}_{\tau_0}}\left\{(v-k_j)_-^p|D\xi|^p + g_-(v,k_j)\partial_\tau\left(\xi^p\right)\right\}\\ &\leq \frac{C(p,q)}{\delta}\, k_j^p\, |\tilde{Q}_{\tau_0}|.\end{aligned}$$

We have used (2.4) to obtain an upper bound to $g_-(v,k_j)$

$$\begin{aligned}g_-(v,k_j) &\leq \gamma(q)(v-k_j)_-^2(k_j+v)^{q-1} \leq \gamma(q)(k_j+v)^{q+1} \leq \gamma(q)k_j^{q+1}\\ &\leq \gamma(q)\, k_j^p\, (2k_0)^{q+1-p}.\end{aligned}$$

We now apply inequality (2.1), for the levels k_j and k_{j+1} to arrive at

$$\frac{k_j}{2}\left|K_8 \cap [v(\cdot,\tau < k_{j+1}]\right| \quad\leq\quad \frac{\gamma(N)}{\left|K_8\cap[v(\cdot,\tau)>k_j]\right|}\int_{K_8\cap[k_{j+1}<v(\cdot,\tau)<k_j]}|Dv|.$$

By integrating the previous inequality in time over $(\tau_0 + k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}]$, then making use of Hölder's inequality and the estimate obtained previously to $\iint_{\tilde{Q}_{\tau_0}}|D(v-k_j)_-|^p\xi^p$, we get

$$\left|Q_{\tau_0}\cap[v<k_{j+1}]\right| \leq \frac{\gamma(N)}{\alpha}\left(\frac{\gamma(p,q)}{\delta}\right)^{\frac{1}{q}} |Q_{\tau_0}|^{\frac{1}{p}}\ \left|Q_{\tau_0}\cap[k_{j+1}<v<k_j]\right|^{\frac{p-1}{p}}.$$

Take the power $\frac{p}{p-1}$ and add this inequality for $j = 0, 1, \ldots, s_*-1$

$$\left|Q_{\tau_0}\cap\left[v<\frac{k_0}{2^{s_*}}\right]\right| \leq \frac{\gamma(N,p,q)}{\alpha\ \delta^p}\ \frac{1}{s_*^{\frac{p-1}{p}}}\ |Q_{\tau_0}|. \tag{4.5}$$

The "natural" thought now would be to argue in a DeGiorgi fashion (something like what was done in Lemma 3.2) to conclude that, for an appropriate choice of s_*, this measure theoretical information (4.5) implies that

$$v \geq \frac{k_0}{2^{s_*+1}} \quad \text{in a smaller cylinder,}$$

say $K_4 \times (\tau_0 + \lambda k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}]$, for some $\lambda \in (0,1)$. However, the cylinder's length k_0^{q+1-p} is too big for the level $\frac{k_0}{2^{s_*}}$ at hands; and that is the reason why one needs to work within thinner cylinders with length $\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$. That is precisely our purpose in what is to come.

Assume for now that s_* is determined. Consider the cylinder Q_{τ_0} sliced into $(2^{s_*})^{q+1-p}$ subcylinders of length $\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$ (if necessary take a bigger s_* so that $(2^{s_*})^{q+1-p}$ is an integer)

$$Q_{\tau_0} = \bigcup_{i=0}^{(2^{s_*})^{q+1-p}-1} Q_i$$

where

$$Q_i = K_8 \times \left(T_0^i, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} \right]$$

for

$$T_0^i = \tau_0 + k_0^{q+1-p} + i \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} .$$

In at least one of these subcylinders, say Q_i, the previous measure theoretical information (4.5) holds, that is

$$\left| Q_i \cap \left[v < \frac{k_0}{2^{s_*}} \right] \right| \leq \frac{\gamma(N,p,q)}{\alpha\ \delta^p}\ \frac{1}{s_*^{\frac{p-1}{p}}}\ |Q_i|,$$

for some $i = 0, 1, \ldots, (2^{s_*})^{q+1-p} - 1$. Our goal now is to get a lower bound for v in a cylinder contained in the upper half of the cylinder Q_i, via a DeGiorgi argument. For that purpose, we consider decreasing sequences of radii, cylinders and levels given by, for $n = 0, 1, \ldots$

$$4 < R_n = 4\left(1 + \frac{1}{2^n}\right) \leq 8,$$

$$Q_n = K_{R_n} \times \left(T_0^i + \frac{1}{2}\left(1 - \frac{1}{2^n}\right)\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} \right] \subset Q_i,$$

and

$$\frac{k_0}{2^{s_*+1}} < k_n = \frac{k_0}{2^{s_*+1}}\left(1 + \frac{1}{2^n}\right) \leq \frac{k_0}{2^{s_*}}$$

and cutoff function ξ defined in Q_n and such that: $0 \leq \xi \leq 1$, in Q_n, $\xi = 0$ on parabolic boundary of Q_n, $\xi = 1$ in Q_{n+1} and

$$|D\xi| \leq 2^n \quad \text{and} \quad |\partial_\tau \xi| \leq 2^{n+2}\left(\frac{2^{s_*}}{k_0}\right)^{q+1-p} .$$

We then write the energy estimates (3.1) for v, over Q_n, with $k = k_n$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{T_0^i < \tau < T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}} \int_{K_{R_n}\{\tau\}} g_-(v,k_n)\xi^p + \iint_{Q_n} |D(v-k_n)_-|^p \xi^p \\
&\leq C(p,q) \iint_{Q_n} \left\{ (v-k_n)_-^p |D\xi|^p + g_-(v,k_n)\, |\partial_t (\xi^p)| \right\} \\
&\leq C(p,q)\ 2^{np}\ k_n^p\ |A_n|,
\end{aligned}$$

where $|A_n| = |Q_n \cap [v < k_n]$. The last inequality was obtained recalling estimate (2.4). To estimate from below the integral term presenting $g_-(v, k_n)$, we use again (2.4) and the fact that $\frac{k_0}{2^{s_*+1}} < k_n \le \frac{k_0}{2^{s_*}}$ to arrive at

$$\int_{K_{R_n}\{\tau\}} g_-(v,k_n)\xi^p \ge \frac{1}{\gamma(q)} \left(\frac{k_0}{2^{s_*}}\right)^{q-1} \int_{K_{R_n}\{\tau\}} (v-k_n)_-^2 \xi^p.$$

The previous estimate together with the energy estimates, Hölder's inequality and the Sobolev embedding (2.2), with $m = 2$, allows us to get

$$\begin{aligned}(k_n - k_{n+1})|A_{n+1}| &\le \iint_{Q_{n+1}} (v-k_n)_- \le \iint_{Q_n} (v-k_n)_- \xi \\ &\le \left\{ \iint_{Q_n} ((v-k_n)_-\xi)^{p\frac{N+2}{N}} \right\}^{\frac{N}{p(N+2)}} |A_n|^{1-\frac{N}{p(N+2)}} \\ &\le C(N,p,q)\, 2^{\frac{N+p}{(N+2)}n}\, k_n^{\frac{N+p}{N+2}} \left(\frac{2^{s_*}}{k_0}\right)^{\frac{q-1}{N+2}} |A_n|^{1+\frac{1}{N+2}}\end{aligned}$$

and from here, by considering $Y_n = \dfrac{|A_n|}{|Q_n|}$, we deduce the algebraic estimate

$$Y_{n+1} \le C(N,p,q)\, b^n\, Y_n^{1+\frac{1}{N+2}}, \quad \text{for } b = 2^{\frac{2N+p+2}{N+2}} > 1.$$

The algebraic convergence Lemma 2.3 says that

if $Y_0 \le C(N.p,q)^{-(N+2)} b^{-(N+2)^2}$, then $Y_n \to 0$, as $n \to +\infty$;

so we just need to choose s_* such that

$$\frac{\gamma(N,p,q)}{\alpha\, \delta^p} \frac{1}{s_*^{\frac{p}{p-1}}} = C(N,p,q)^{-(N+2)} b^{-(N+2)^2}.$$

Remark 4.2. Observe that with this choice, s_* only depends on N, p, q and α (since δ is already determined and depends on these parameters as well).

The length of the cylinder Q_i is exactly the one needed so that, when arguing in a DeGiorgi fashion, given by Lemma 3.2, there is an independence of ν_0 on the levels $\tilde{M}$ and the cylinder's length λ. In fact, in our case $\tilde{M} = \frac{k_0}{2^{s_*}}$ and $\lambda = \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$.

We thereby obtain the lower bound

$$v \ge \frac{k_0}{2^{s_*+1}} \quad \text{a.e. in } K_4 \times \left(T_0^i + \frac{1}{2}\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} \right];$$

in particular

$$v(, \tau_1) \ge \frac{k_0}{2^{s_*+1}} \quad \text{a.e. in } K_4,$$

for

$$\tau_0 + k_0^{q+1-p} < T_0^i + \frac{1}{2}\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} < \tau_1 \le T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} < \tau_0 + 2k_0^{q+1-p}.$$

Returning to the original coordinates and function, we may conclude that

$$u(, t_1) \ge e^{-\frac{\tau_1}{q+1-p}} \frac{k_0}{2^{s_*+1}} M = \frac{\epsilon}{2^{s_*+1}} e^{\frac{\tau_0-\tau_1}{q+1-p}} M \quad \text{a.e. in } K_{4\rho}(y)$$

where t_1 is defined by

$$-e^{-\tau_1} = \frac{t_1 - (s + \delta M^{q+1-p}\rho^p)}{\delta M^{q+1-p}\rho^p} \Longleftrightarrow t_1 = s + (1 - e^{-\tau_1})\delta M^{q+1-p}\rho^p.$$

We are two steps away to conclude the *expansion of positivity*. First, by considering $\tilde{M} = \frac{\epsilon}{2^{s_*+1}} e^{\frac{\tau_0-\tau_1}{q+1-p}} M$ and choosing $\lambda = \nu_0 \tilde{M}^{q+1-p}$, where $\nu_0 = \nu_0(N,p,q)$, the assumptions of the variant DeGiorgi-type Lemma 3.3 are verified and we may conclude

$$u \geq \frac{\tilde{M}}{2} = \frac{\epsilon}{2^{s_*+2}}\, e^{\frac{\tau_0-\tau_1}{q+1-p}}\, M \quad \text{a.e. in } K_{2\rho}(y)$$

for all times

$$t_1 \leq t \leq t_1 + \lambda(2\rho)^p.$$

Finally, we choose τ_0 such that

$$t_1 + \lambda(2\rho)^p = s + \delta M^{q+1-p}\rho^p,$$

that is, keeping in mind the expressions of t_1 and of λ,

$$e^{\tau_0} = \delta C^{N+2} 2^{(N+2)^2-p} \left(\frac{2^{s_*+1}}{\epsilon}\right)^{q+1-p}$$

and from the range of τ_1 we have

$$t_1 < s + \left(1 - e^{-\tau_0} - 2\epsilon^{q+1-p} e^{\tau_0}\right)\delta M^{q+1-p}\rho^p \leq s + \frac{\delta}{2} M^{q+1-p}\rho^p.$$

Gathering these last estimates, we get

$$u(\cdot,t) \geq \frac{\epsilon}{2^{s_*+2}} e^{-\frac{2k_0^{q+1-p}}{q+1-p}} M \quad \text{a.e. in } K_{2\rho}(y),$$

for all

$$t \in \left(s + (1-\lambda)\delta M^{q+1-p}\rho^p, s + \delta M^{q+1-p}\rho^p\right].$$

We have proved

Proposition 4.3. *Let u is a nonnegative, local, weak supersolution to (1.1) in Ω_T. Assume that, for some $(y,s) \in \Omega_T$ and some $\rho > 0$,*

$$\left|K_\rho(y) \cap [u(\cdot,s) \geq M]\right| \geq \alpha \left|K_\rho(y)\right|,$$

for some $M > 0$ and some $\alpha \in (0,1)$. Then there exist $\delta, \lambda, \eta \in (0,1)$, depending on N, p, q and α, such that

$$u(\cdot,t) \geq \eta M \quad \text{a.e. in } K_{2\rho}(y)$$

and for all $t \in \left(s + (1-\lambda)\delta M^{q+1-p}\rho^p, s + \delta M^{q+1-p}\rho^p\right]$.

Expansion of positivity when $q+1-p<0$

Consider a point $(y,s)\in\Omega_T$ and let $\rho>0$ be such that

$$K_{8\rho}(y)\times\left(s,s+\delta\,\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\,\rho^p\right]\subset\Omega_T,$$

where M is a given positive number and δ,η,b are positive numbers to be determined.

Proposition 4.4. *Let u is a nonnegative, local, weak supersolution to (1.1) in Ω_T. Assume that (4.1) holds, for some $(y,s)\in\Omega_T$, $\rho>0$ and $\alpha\in(0,1)$. Then there exist $\delta,\eta,b\in(0,1)$, depending on N,p,q and α, such that*

$$u(\cdot,t)\geq\eta M\quad \textit{a.e. in } K_{2\rho}(y)$$

and for all $t\in(s+\frac{\delta}{2}\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\rho^p,s+\delta\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\rho^p]$.

Proof. Assume that (4.1) is verified. Then, for all $0<\sigma_0\leq1$, one also has

$$\left|K_\rho(y)\cap[u(\cdot,s)>\sigma_0 M]\right|\geq\alpha\left|K_\rho(y)\right|.$$

Consider the energy estimates written over

$$K_\rho(y)\times(s,s+\delta(\sigma_0 M)^{q+1-p}\rho^p]$$

for the levels $k=\sigma_0 M$. By proceeding as in the proof of Lemma 4.1, we obtain the same parameters ϵ and δ, depending on N,p,q and α, for which

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon\,\sigma_0 M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\tag{4.6}$$

for all $t\in(s,s+\delta\,(\sigma_0 M)^{q+1-p}\rho^p]$.

For $\tau\geq0$, consider the number

$$\sigma_\tau=e^{-\frac{\tau}{p-(q+1)}}\leq1.$$

Since (4.6) holds for all $0<\sigma_0\leq1$, it also holds for σ_τ

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon\,\sigma_\tau M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\quad\forall t\in(s,s+\delta\,(\sigma_\tau M)^{q+1-p}\rho^p]$$

and, in particular,

$$\left|K_\rho(y)\cap[u(\cdot,s+\delta\,(\sigma_\tau M)^{q+1-p}\rho^p)>\epsilon\,\sigma_\tau M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|.$$

Introduce the change of variable

$$e^\tau=(t-s)\,\frac{M^{p-(q+1)}}{\delta\rho^p}$$

and the define the new function

$$v(x,\tau)=\frac{e^{\frac{\tau}{p-(q+1)}}}{M}\,(\delta\rho^p)^{\frac{1}{p-(q+1)}}\,u(x,t).$$

In this new setting, v is a solution to

$$\partial_\tau\left(v^q\right)-\operatorname{div}\left(|Dv|^{p-2}Dv\right)=\frac{q}{p-(q+1)}\,v^q\geq 0$$

and the measure theoretical information on u is translated into the following measure theoretical information on v

$$\left|K_\rho(y)\cap\left[v(\cdot,\tau)>\epsilon\left(\delta\rho^p\right)^{\frac{1}{p-(q+1)}}\right]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\qquad \tau\geq 0.$$

Therefore one gets, for all $\tau\geq 0$

$$\left|K_{4\rho}(y)\cap\left[v(\cdot,\tau)>k_0\right]\right|\geq\frac{\alpha}{2\,4^N}\left|K_{4\rho}(y)\right|,$$

for

$$k_0=\epsilon\left(\delta\rho^p\right)^{\frac{1}{p-(q+1)}}\qquad\text{(completely determined).}$$

Consider the smaller levels

$$k_j=\frac{k_0}{2^j},\qquad\text{for } j=0,1,\dots s^*\quad(s^*\text{ to be chosen}),$$

take the stretching factor θ as

$$\theta=\left(\frac{k_0}{2^{s^*}}\right)^{q+1-p}\geq k_j^{q+1-p},\qquad\text{for all } j=0,1,\dots,s^*,\tag{4.7}$$

construct the cylinders

$$Q=(y,0)+Q^+_{8\rho}(\theta)\quad\text{and}\quad\tilde{Q}=K_{4\rho}(y)\times(\theta(4\rho)^p,\theta(8\rho)^p],\quad\tilde{Q}\subset Q$$

and take $\varphi=(v-k)_-\xi^p$ as a test function, where $\xi\in[0,1]$ is a smooth cutoff function defined in Q, vanishing on its parabolic boundary and verifying

$$\xi=1\quad\text{in }\tilde{Q},\quad|D\xi|\leq\frac{1}{4\rho}\quad\text{and}\quad|\partial_\tau\xi|\leq\frac{1}{\theta(4\rho)^p}.$$

For these choices the arrive at

$$\begin{aligned}\iint_{\tilde{Q}}|D(v-k_j)_-|^p&\leq\iint_Q|D(v-k_j)_-|^p\xi^p\\&\leq C(p)\frac{k_j^p}{(4\rho)^p}\left\{1+\frac{k_j^{q+1-p}}{\theta}\right\}|A_j|\leq C(p)\frac{k_j^p}{(4\rho)^p}|A_j|\end{aligned}$$

due to the definition of θ and taking $|A_j|=|Q\cap[v<k_j]|$. We then proceed in a similar way as in the case $q+1-p>0$, to find out that

$$\left|\tilde{Q}\cap\left[v<\frac{k_0}{2^{s^*}}\right]\right|\leq\frac{C(N,p,q)}{\alpha}\,\frac{1}{(s^*)^{\frac{p-1}{p}}}\,|\tilde{Q}|.$$

This estimate on the measure of the set where v is below the level $\frac{k_0}{2^{s^*}}$ will be the starting point to argument in a DeGiorgi fashion in a backward cylinder, like in Lemma 3.2. Along the way, the length θ of the cylinder will be determined. More precisely, consider the cylinder

$$(y,\tau^*)+Q^-_{4\rho}(\theta)=K_{4\rho}(y)\times(\tau^*-\theta(4\rho)^p,\tau^*]\subset\tilde{Q}\quad\text{for }\tau^*=\theta(8\rho)^p$$

and the sequences of numbers

$$2\rho < \rho_n = 2\rho\left(1+\frac{1}{2^n}\right) \le 4\rho, \quad \frac{k_0}{2^{s^*+1}} < k_n = \frac{k_0}{2^{s^*+1}}\left(1+\frac{1}{2^n}\right) \le \frac{k_0}{2^{s^*}}$$

and of nested and shrinking cylinders

$$Q_n^- = (y,\tau^*) + Q_{\rho_n}^-(\theta),$$

for $n = 0, 1, \ldots$ Take a cutoff function $0 \le \xi \le 1$ defined in Q_n^- and such that: $\xi = 0$ on the parabolic boundary of Q_n^-, $\xi = 1$ in Q_{n+1}^- and

$$|D\xi| \le \frac{2^{n+1}}{\rho} \quad \text{and} \quad |\partial_\tau \xi| \le C\,\frac{2^{np}}{\theta\rho^p}$$

and write the energy estimates (3.1) for v, over Q_n^-, with $k = k_n$. Recalling the estimates (2.4) on $g_-(v,k_n)$ and the definition (4.7) of θ, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau^*-\theta(\rho_n)^p<\tau<\tau^*} \int_{K_{\rho_n}\times\{\tau\}} g_-(v,k_n)\xi^p + \iint_{Q_n^-} |D(v-k_n)_-|^p \xi^p \\
&\quad \le C(p,q) \iint_{Q_n^-} \left\{(v-k_n)_-^p |D\xi|^p + g_-(v,k_n)\,|\partial_\tau(\xi^p)|\right\} \\
&\quad \le C(p,q)\, 2^{np}\, \frac{k_n^p}{\rho^p} \left\{1 + \frac{k_n^{q+1-p}}{\theta}\right\} |A_n| \le C(p,q)\, 2^{np}\, \frac{k_n^p}{\rho^p}\, |A_n|,
\end{aligned}$$

where, as usually, $|A_n| = |Q_n^- \cap [v < k_n]|$. Observe that, on the one hand

$$\iint_{Q_n^-} (v-k_n)_-\xi \ge \iint_{Q_{n+1}^-} (v-k_n)_- \ge (k_n - k_{n+1})|A_{n+1}| = \frac{k_0}{2^{s^*}}\,\frac{1}{2^{n+2}}|A_{n+1}|$$

and, on the other hand, by applying Hölder's inequality with exponent $p\frac{N+2}{N}$, together with Sobolev's embedding, we get

$$\iint_{Q_n^-} (v-k_n)_-\xi \le C(N,p,q)\left(\frac{2^n}{\rho}\right)^{\frac{N+p}{N+2}} k_n^{\frac{N+p+1-q}{N+2}}\, |A_n|^{1+\frac{1}{N+2}}.$$

Consider the numbers $Y_n = \frac{|A_n|}{|Q_n^-|}$. From the previous estimates we deduce

$$Y_{n+1} \le C(N,p,q)\, b^n\, Y_n^{1+\frac{1}{N+2}}, \quad \text{for } b = 2^{\frac{2N+p+2}{N+2}} > 1$$

and we may conclude that Y_n goes to zero as $n \to +\infty$ once we have

$$\frac{\left|(y,\tau^*) + Q_{4\rho}^-(\theta) \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right|}{\left|(y,\tau^*) + Q_{4\rho}^-(\theta)\right|} = Y_0 \le C(N,p,q)^{-(N+2)} 2^{-(2N+p+2)(N+2)}.$$

Recall that, under our hypothesis, we have $(y,\tau^*) + Q_{4\rho}^-(\theta) \subset \tilde{Q}$ and

$$\left|\tilde{Q} \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right| \le \frac{C(N,p,q)}{\alpha}\,\frac{1}{(s^*)^{\frac{p-1}{p}}}\,|\tilde{Q}|$$

and thereby

$$Y_0 \leq \frac{\left|\tilde{Q} \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right|}{|\tilde{Q}|} \frac{|\tilde{Q}|}{\left|(y,\tau^*) + Q^-_{4\rho}(\theta)\right|} \leq \frac{\gamma(N,p,q)}{\alpha} \frac{1}{(s^*)^{\frac{p-1}{p}}}.$$

We determine the parameter s^*, and therefore the length of the cylinder, so that

$$\frac{\gamma(N,p,q)}{\alpha} \frac{1}{(s^*)^{\frac{p-1}{p}}} = C(N,p,q)^{-(N+2)} 2^{-(2N+p+2)(N+2)}.$$

This implies

$$v(\cdot,\tau) \geq \frac{k_0}{2^{s^*}+1} \quad \text{a.e. in } K_{2\rho}$$

for all $\tau \in (\tau^* - \theta(2\rho)^p, \tau^*]$.

Returning to the original time variable t and function $u(x,t)$ we get

$$u(x,t) \geq \eta\ M \quad \text{a.e. in } K_{2\rho}(y)$$

for all $t \in (s + \frac{\delta}{2}\ \frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\ \rho^p, s + \delta \frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\ \rho^p]$, where

$$\eta = \frac{\epsilon}{2^{s^*+1}} e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{8^p}{\delta\ (p-q-1)}}.$$

This time interval was obtained from the previous range of τ and realizing that, on such a range,

$$b_1 = e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{8^p}{\delta\ (p-q-1)}} \leq e^{-\frac{\tau}{p-q-1}} < e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{6^p}{\delta\ (p-q-1)}} = b_2$$

and taking

$$b = \frac{\epsilon}{2^{s^*+1}}.$$

□

Acknowledgements

This research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia – within the Projects UIDB/00013/2020 and UIDP/00013/2020.

References

[1] E. Acerbi, N. Fusco, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, *J. Math. Anal. Appl.* **140**(1989), No. 1, 115–135 https://doi.org/10.1016/0022-247x(89)90098-x

[2] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-0895-2

[3] E. DiBenedetto, U. Gianazza, V. Vespri, *Harnack's inequality for degenerate and singular parabolic equations*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2012. https://doi.org/10.1007/978-1-4614-1584-8

[4] J. I. Diaz, F. Thélin, On a nonlinear parabolic problem arising in some models related to turbulent flows, *SIAM J. Math. Anal.* **25**(1994), No. 4, 1085–1111. https://doi.org/10.1137/s0036141091217731

[5] V. Bögelein, F. Duzaar, N. Liao, On the Hölder regularity of signed solutions to a doubly nonlinear equation, *J. Funct. Anal.* **9**(2021), 109173. https://doi.org/10.1016/j.jfa.2021.109173

[6] S. Fornaro, U. Gianazza, Local properties of non-negative solutions to some doubly non-linear degenerate parabolic equations, *Discrete Contin. Dyn. Syst.* **26**(2010), No. 2, 481–492. https://doi.org/10.3934/dcds.2010.26.481

[7] S. Fornaro, E. Henriques, V. Vespri, Harnack type inequalities for the parabolic logarithmic p-Laplacian equation, *Le Matematiche* **75**(2020), No. 1, 277–311. https://doi.org/10.4418/2020.75.1.13

[8] S. Fornaro, E. Henriques, V. Vespri, Regularity results for a class of doubly nonlinear very singular parabolic equations, *Nonlinear Anal.* **205**(2021), 112213. https://doi.org/10.1016/j.na.2020.112213

[9] S. Fornaro, E. Henriques, V. Vespri, Stability to a class of doubly nonlinear very singular parabolic equations, *Manuscripta Math.*, appeared online, 2021. https://doi.org/10.1007/s00229-021-01302-w

[10] S. Fornaro, M. Sosio, V. Vespri, $L^r_{loc} - L^\infty_{loc}$ estimates and expansion of positivity for a class of doubly non linear singular parabolic equations, *Discrete Contin. Dyn. Syst. Ser. S* **7**(2013), No. 4, 737–760. https://doi.org/10.3934/dcdss.2014.7.737

[11] M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals, *Manuscripta Math.* **57**(1986), No. 1, 55–99. https://doi.org/10.1007/bf01172492

[12] E. Henriques, R. Laleoglu, Local Hölder continuity for some doubly nonlinear parabolic equations in measure spaces, *Nonlinear Anal.* **79**(2013), No. 1, 156–175. https://doi.org/10.1016/j.na.2012.11.022

[13] E. Henriques, R. Laleoglu, Local and global boundedness for some nonlinear parabolic equations exhibiting a time singularity, *Differential Integral Equations* **29**(2016), No. 11–12, 1029–1048. MR3557310; Zbl 1374.35096

[14] E. Henriques, R. Laleoglu, Boundedness for some doubly nonlinear parabolic equations in measure spaces, *J. Dynam. Differential Equations* **30**(2018), No. 3, 1029–1051. https://doi.org/10.1007/s10884-017-9585-3

[15] A. V. Ivanov, Hölder estimates for equations of fast diffusion type, *Algebra i Analiz* **6**(1994), No. 4, 101–142; translation in *St. Petersburg Math. J.* **6**(1995), No. 4, 791–825. MR1304096, Zbl 0839.35067

[16] J. Kinnunen, T. Kuusi, Local behaviour of solutions to doubly nonlinear parabolic equations, *Math. Ann.* **337**(2007), No. 3, 705–728. https://doi.org/10.1007/s00208-006-0053-3

[17] J. Kinnunen, P. Lindqvist, Definition and properties of supersolutions to the porous medium equation, *J. Reine Angew. Math.* **618**(2008), 135–168. https://doi.org/10.1515/CRELLE.2008.035

[18] T. Kuusi, R. Laleoglu, J. Siljander, J. M. Urbano, Hölder continuity for Trudinger's equation in measure spaces, *Calc. Var. Partial Differential Equations* **45**(2012), No. 1–2, 193–229. https://doi.org/10.1007/s00526-011-0456-1

[19] T. Kuusi, J. Siljander, J. M. Urbano, Hölder continuity for doubly nonlinear parabolic equations, *Indiana Univ. Math. J.* **61**(2012), 399–430. https://doi.org/10.1512/iumj.2012.61.4513

[20] M. M. Porzio, V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations* **103**(1993), No. 1, 146–178. https://doi.org/10.1006/jdeq.1993.1045

[21] V. Vespri, On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations, *Manuscripta Math.* **75**(1992), No. 1, 65–80. https://doi.org/10.1007/BF02567072

[22] V. Vespri, M. Vestberg, An extensive study of the regularity of solutions to doubly singular equations, *Adv. Calc. Var.*, published online, 2020. https://doi.org/10.1515/acv-2019-0102