Solitary waves for a fractional Klein–Gordon–Maxwell equation

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Abstract. We investigate existence of solutions for a fractional Klein–Gordon coupled with Maxwell's equation. On the basis of overcoming the lack of compactness, we obtain that there is a radially symmetric solution for the critical system by means of variational methods.

Keywords: variational methods, Klein–Gordon–Maxwell equations, solitary waves.

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1 Introduction and preliminaries

Recently, a great attention has been focused on the study of non-linear problems involving the fractional Laplacian, in view of concrete real-world applications. For instance, this type of operators arises in the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, materials science and water waves, see [13]. Moreover fractional Laplace equations can be applied to many subjects, such as anomalous diffusion, elliptic problems with measure data, gradient potential theory, minimal surfaces, non-uniformly elliptic problems, optimization, phase transitions, quasigeostrophic flows, singular set of minima of variational functionals and water waves (see [2, 5–7, 13, 16–21] and the references therein). In present paper, we consider the following fractional system

$$\begin{cases} (-\Delta)^s u + \left[m^2 - (\omega + \phi)^2 \right] u = \mu |u|^{q-2} u + |u|^{2_s^* - 2} u, & x \in \mathbb{R}^3 \\ (\Delta)^s \phi = (\omega + \phi) u^2, & x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\frac{3}{4} < s < 1$, $\mu > 0$ and $4 \leq q < 2_s^* = \frac{2n}{n-2s} = \frac{6}{3-2s}$, m and ω are real constants, $u \in H^s(\mathbb{R}^3), \phi \in D^{s,2}(\mathbb{R}^3)$, $(-\Delta)^s$ stands for the fractional Laplacian, 2_s^* is the fractional Sobolev critical exponent.

The Klein–Gordon–Maxwell equations have been introduced in [3] as a model describing solitary waves for the non-linear stationary Klein–Gordon equation coupled with Maxwell

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equation in the three dimensional space interacting with the eletrostatic field. In recent years, some existence and nonexistence results for the Klein–Gordon–Maxwell equations have been proved. In [3,4,12], the authors investigated the existence of infinitely many radially symmetric solutions (u, ϕ) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. In [1] the existence of a ground state solution (u, ϕ) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ was established; In [11], the nonexistence results for system related to Klein–Gordon–Maxwell system were obtained.

Cassani in [8] investigated the following system when $n = 3$ and $s = 1$

$$\begin{cases} -\Delta u + \left[m_0^2 - (\omega + \phi)^2 \right] u = \mu |u|^{q-2} u + |u|^{2^*-2} u, & x \in \mathbb{R}^3 \\ \Delta \phi = (\omega + \phi) u^2, & x \in \mathbb{R}^3 \end{cases}$$

where $\mu > 0$ and $4 \leq q < 6 = 2^*$. Cassani proved that the system has at least a radially symmetric (nontrivial) solution.

In [20], Servadei and Valdinoci showed the non-local fractional counterpart of the Laplace equation involving critical non-linearities studied in the famous paper of Brezis and Nirenberg (1983) by the following system

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*_s - 2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

and the authors firstly studied the problem in a general framework

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*_s - 2} + f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where $\mathcal{L}_K$ is a general non-local integrodifferential operator of order s, f is a lower order perturbation of the critical power $|u|^{2^*_s - 2}$. In this setting they proved an existence result through variational techniques. Then, as a concrete example, they derived a Brezis–Nirenberg type result for the problem.

The authors in [15] explored the problem

$$\begin{cases} (-\Delta)^s u + V(x) u - (2\omega + \phi) \phi u = K(x) f(u), & \text{in } \mathbb{R}^3, \\ (\Delta)^s \phi = (\omega + \phi) u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $K : \mathbb{R}^3 \to \mathbb{R}$ is a function satisfying some decay condition, $V : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous function, $\phi, u : \mathbb{R}^3 \to \mathbb{R}$ are functions. Furthermore, they showed the existence and positivity of the ground state solution with zero mass potential for the problem, that is, when the potential $V(x) \to 0$, as $|x| \to \infty$ and they also studied the case when V is bounded and considered carefully the weight $K(x)$. In addition, they treated the problem using the fractional Laplace operator instead of classical Laplace operator.

Next there are two ways to define fractional Sobolev space. One is via Gagliardo seminorm

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{3}{2} + s}} \in L^2 \left(\mathbb{R}^3 \times \mathbb{R}^3 \right) \right\},$$

the other is via Fourier transformation

$$\widehat{H}^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(1 + |\xi|^{2s} \right) |\mathcal{F} u(\xi)|^2 d\xi < +\infty \right\},$$

and $H^s(\mathbb{R}^3) = \hat{H}^s(\mathbb{R}^3)$. In the present paper, as the norm of fractional Sobolev space, we define

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^3} \left(m^2 - \omega^2\right) u^2 \mathrm{d}x + \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

The fractional Laplacian is defined by

$$
\begin{aligned}
(-\Delta)^s u(x) &= C_{3,s} \text{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{3+2s}} \mathrm{d}y \\
&= C_{3,s} \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - u(y)}{|x-y|^{3+2s}} \mathrm{d}y \\
&= -\frac{1}{2} C_{3,s} \int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2s}} \mathrm{d}y \\
&= \mathcal{F}^{-1}\left(|\xi|^{2s} \mathcal{F}u(\xi)\right),
\end{aligned}
$$

where

$$C_{3,s} = \left(\int_{\mathbb{R}^3} \frac{1 - \cos(\zeta_1)}{|\zeta|^{3+2s}} \mathrm{d}\zeta\right)^{-1},$$

and P.V. is the principle value defined by the latter formula.

Consider the Sobolev space

$$D^{s,2}(\mathbb{R}^3) := \left\{ u \in L^{2^*_s}(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2\left(\mathbb{R}^3 \times \mathbb{R}^3\right) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{s,2}}^2 := \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

Theorem 1.1. *If $|m| > |\omega|$ and $4 < q < 2^*_s$, then the problem (1.1) has a radially symmetric solution $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$ for each $\mu > 0$.*

Theorem 1.2. *If $|m| > |\omega|$ and $q = 4$, system (1.1) still possesses a radially symmetric solution provided that μ is sufficiently large.*

According to system (1.1), one obtains the functional

$$F(u, \phi) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{2}\|\phi\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} \left(2\omega\phi + \phi^2\right) u^2 \mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} \mathrm{d}x. \quad (1.4)$$

It's easy to know that $F(u, \phi)$ exhibits a strong indefiniteness, namely it is unbounded from below and from above on infinite dimensional subspaces. This indefiniteness can be removed by using the reduction methods. For u and ϕ defined above, we have the following lemmas.

Lemma 1.3. *Let $u \in H^s(\mathbb{R}^3)$, then there exists a unique solution $\Phi(u)$ of the second equation for problem (1.1) such that $\phi = \Phi(u) \in D^{s,2}(\mathbb{R}^3)$.*

Proof. The proof is similar to the proof of in Reference [15, Lemma 2.1], so we omit its proof. $\square$

Remark 1.4. Define the map $\Phi : H^s(\mathbb{R}^3) \to D^{s,2}(\mathbb{R}^3)$. We can get that for each $u \in H^s(\mathbb{R}^3)$, the map Φ gives the unique solution $\Phi(u) = \phi$, i.e., $\Phi(u) = \left((\Delta)^s - u^2\right)^{-1} \omega u^2$.

Next we state some properties of problem (1.1) as follows.

Lemma 1.5. *For any $u \in H^s(\mathbb{R}^3)$, it results in $\Phi(u) \leq 0$. Moreover, $\Phi(u)(x) \geq -\omega$ if $u(x) \neq 0$ and $\omega > 0$.*

Proof. Multiplying the second equation of problem (1.1) by $\Phi^+(u) = \max\{\Phi(u), 0\}$, we get

$$-\|\Phi^+(u)\|_{D^{s,2}}^2 = \omega \int_{\mathbb{R}^3} \Phi^+(u)u^2 dx + \int_{\mathbb{R}^3} u^2 \left(\Phi^+(u)\right)^2 dx \geq 0,$$

so that $\Phi^+(u) \equiv 0$.

If we multiply the second equation of problem (1.1) by $(\omega + \Phi(u))^-$, one has

$$\int_{\{x:\Phi(u)<-\omega\}} \left| (-\Delta)^{\frac{s}{2}} \Phi(u) \right|^2 dx = -\int_{\{x:\Phi(u)<-\omega\}} (\omega + \Phi(u))^2 u^2 dx,$$

so that $(\omega + \Phi(u))^- = 0$ where $u(x) \neq 0$. □

Lemma 1.6. *The map Φ is C^1 and*

$$G_\phi = \left\{ (u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3) \mid F'_\phi(u, \phi) = 0 \right\}.$$

Proof. Noticing that $\Phi(u)$ is a solution of the second equation in problem (1.1), we have

$$-\|\Phi(u)\|_{D^{s,2}}^2 = \int_{\mathbb{R}^3} (\omega + \Phi(u)) \Phi(u)u^2 dx = \int_{\mathbb{R}^3} \omega\Phi(u)u^2 dx + \int_{\mathbb{R}^3} \Phi^2(u)u^2 dx. \quad (1.5)$$

In addition,

$$F(u, \Phi(u)) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{2}\|\Phi(u)\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} \left(2\omega\Phi(u) + \Phi^2(u)\right) u^2 dx$$
$$- \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx$$

and

$$F'_\phi(u, \Phi(u)) = -\|\Phi(u)\|_{D^{s,2}}^2 - \int_{\mathbb{R}^3} \omega\Phi(u)u^2 dx - \int_{\mathbb{R}^3} \Phi^2(u)u^2 dx,$$

according to (1.5), one gets that $F'_\phi(u, \Phi(u)) = 0$ for any $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$. Thus

$$F'(u, \Phi(u)) = F'_u(u, \Phi(u)) + F'_\phi(u, \Phi(u))\Phi'(u) = F'_u(u, \Phi(u)). \quad □$$

Define $I(u) := F(u, \Phi(u))$ and if $u, v \in H^s(\mathbb{R}^3)$, one gets that

$$I'(u)v = \langle u, v \rangle_{H^s} + \int_{\mathbb{R}^3} \left[\left(m^2 - (\omega + \Phi(u))^2\right) uv - \mu|u|^{q-2}uv - |u|^{2_s^*-2}uv \right] dx. \quad (1.6)$$

Lemma 1.7. *The following statements are equivalent:*

(i) $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$ *is a solution of problem (1.1).*

(ii) u *is a critical point of I and $\phi = \Phi(u)$.*

Proof. $(ii) \implies (i)$ Obviously.

$(i) \implies (ii)$ Let $F'_u(u, \phi)$ and $F'_\phi(u, \phi)$ denote the partial derivatives of F at $(u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$. Then for every $v \in H^s(\mathbb{R}^3)$ and $\psi \in D^{s,2}(\mathbb{R}^3)$, one obtains that

$$F'_u(u, \phi)[v] = \langle u, v \rangle_{H^s} + \int_{\mathbb{R}^3} \left[\left(m^2 - (\omega + \phi)^2 \right) uv - \mu |u|^{q-2} uv - |u|^{2^*_s - 2} uv \right] dx, \qquad (1.7)$$

$$F'_\phi(u, \phi)[\psi] = -\langle \phi, \psi \rangle_{D^{s,2}} - \int_{\mathbb{R}^3} \left(\omega \psi u^2 + \phi \psi u^2 \right) dx. \qquad (1.8)$$

By standard computations, we can prove that $F'_u(u, \phi)$ and $F'_\phi(u, \phi)$ are continuous. From (1.7) and (1.8), it is easy to obtain that its critical points are solutions of problem (1.1), moreover, by Lemma 1.3, one has $\phi = \Phi(u)$. $\qquad \square$

2   Proof of Theorem 1.1

Lemma 2.1. *For $u \in H^s_r(\mathbb{R}^3)$, if $|m| > |\omega|$, then there exist some constants $\rho, \alpha > 0$ such that $I(u)\big|_{\|u\|_{H^s} = \rho} \geq \alpha > 0$.*

Proof. From (1.4) and (1.5), $I(u)$ can be written in the following form

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2_{H^s} - \frac{1}{2}\|\phi\|^2_{D^{s,2}} - \frac{1}{2}\int_{\mathbb{R}^3} \left(2\omega\phi + \phi^2 \right) u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} dx \\ &= \frac{1}{2}\|u\|^2_{H^s} + \frac{1}{2}\|\phi\|^2_{D^{s,2}} + \frac{1}{2}\int_{\mathbb{R}^3} \phi^2 u^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} dx. \end{aligned} \qquad (2.1)$$

Then by the Sobolev inequality, we have

$$I(u) \geq \frac{1}{2}\|u\|^2_{H^s} - C_1\|u\|^q_{H^s} - C_2\|u\|^{2^*_s}_{H^s} \geq \alpha > 0, \ \text{for } u \in H^s(\mathbb{R}^3), \ \|u\|_{H^s} = \rho.$$

Thus

$$I(u)\big|_{\|u\|_{H^s} = \rho} \geq \alpha > 0$$

and the proof is completed. $\qquad \square$

Lemma 2.2. *Under the assumptions of Theorem 1.1, there exists a function $e \in H^s(\mathbb{R}^3)$ with $\|e\|_{H^s} > \rho$ such that $I(e) < 0$.*

Proof. For any $u \in H^s(\mathbb{R}^3) \setminus \{0\}$, in view of (1.4), it is easy to obtain that

$$\begin{aligned} \lim_{t \to +\infty} I(tu) &= \frac{t^2}{2}\|u\|^2_{H^s} - \frac{1}{2}\|\Phi(tu)\|^2_{D^{s,2}} - \frac{t^2}{2}\int_{\mathbb{R}^3} \left(2\omega\Phi(tu) + \Phi^2(tu) \right) u^2 dx \\ &\quad - \frac{t^q \mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} dx \\ &\leq \frac{t^2}{2}\left(\|u\|^2_{H^s} + \int_{\mathbb{R}^3} 2\omega^2 u^2 dx \right) - \frac{t^q \mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} dx \\ &\to -\infty, \end{aligned}$$

which implies that $I(u) \to -\infty$, as $\|u\|_{H^s} \to \infty$.

The lemma is proved by taking $e = tu$ with $t > 0$ large enough and $u \neq 0$. Therefore we know that there exists $e \in H^s(\mathbb{R}^3)$, $\|e\|_{H^s} > \rho$ such that $I(e) < 0$. $\qquad \square$

Define

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)), \tag{2.2}$$

where $\Gamma = \{\gamma \in C\left([0,1], H^s(\mathbb{R}^3)\right) \big| \gamma(0) = 0, \gamma(1) = e\}$ is the *MP* level. Obviously, $c \ge \alpha > 0$. There exists a $(PS)_c$ sequence $\{u_k\} \subset E$ such that

$$\begin{aligned} I(u_k) &\to c, \\ I'(u_k) &\to 0, \quad k \to \infty. \end{aligned} \tag{2.3}$$

Lemma 2.3. *The $(PS)_c$ sequence $\{u_k\} \subset E$ given in (2.3) is bounded.*

Proof. There is a positive constant M such that

$$\begin{aligned} M + o(1)\|u_k\| &\ge I(u_k) - \frac{1}{q}(I'(u_k), u_k) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_k\|_{H^s}^2 + \frac{1}{2}\|\Phi(u_k)\|_{D^{s,2}}^2 + \left(\frac{1}{2} + \frac{1}{q}\right)\int_{\mathbb{R}^3}\Phi^2(u_k)u_k^2\mathrm{d}x \\ &\quad + \frac{2}{q}\int_{\mathbb{R}^3}\omega\Phi(u_k)u_k^2\mathrm{d}x + \left(\frac{1}{q} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_k|^{2_s^*}\mathrm{d}x. \end{aligned} \tag{2.4}$$

Substituting (1.5) into (2.4), we get that

$$\begin{aligned} M + o(1)\|u_k\| &\ge I(u_k) - \frac{1}{q}(I'(u_k), u_k) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_k\|_{H^s}^2 + \left(\frac{1}{2} - \frac{2}{q}\right)\|\Phi(u_k)\|_{D^{s,2}}^2 \\ &\quad + \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^3}\Phi^2(u_k)u_k^2\mathrm{d}x + \left(\frac{1}{q} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}|u_k|^{2_s^*}\mathrm{d}x \\ &\ge C_4\|u_k\|_{H^s}^2. \end{aligned}$$

Since $4 < q < 2_s^*$, as a consequence of the above inequality, $\{u_k\}$ is bounded in $H^s(\mathbb{R}^3)$. $\square$

Furthermore, according to (1.5), one gets that

$$\|\Phi(u)\|_{D^{s,2}}^2 = -\int_{\mathbb{R}^3}(\omega + \Phi(u))\Phi(u)u^2\mathrm{d}x = -\int_{\mathbb{R}^3}\omega\Phi(u)u^2\mathrm{d}x - \int_{\mathbb{R}^3}\Phi^2(u)u^2\mathrm{d}x. \tag{2.5}$$

Then by Hölder inequality and Sobolev inequality, one obtains that

$$\begin{aligned} \|\Phi(u_k)\|_{D^{s,2}}^2 &\le -\int_{\mathbb{R}^3}\omega\Phi(u_k)u_k^2\mathrm{d}x \\ &\le |\omega|\left(\int_{\mathbb{R}^3}|\Phi(u_k)|^{2_s^*}\mathrm{d}x\right)^{\frac{1}{2_s^*}}\left(\int_{\mathbb{R}^3}|u_k|^{\frac{2\cdot 2_s^*}{2_s^*-1}}\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}} \\ &= |\omega|\left(\int_{\mathbb{R}^3}|\Phi(u_k)|^{\frac{6}{3-2s}}\mathrm{d}x\right)^{\frac{3-2s}{6}}\left(\int_{\mathbb{R}^3}|u_k|^{\frac{12}{3+2s}}\mathrm{d}x\right)^{\frac{3+2s}{6}} \\ &\le C_5\|\Phi(u_k)\|_{D^{s,2}}\|u_k\|_{H^s}^2. \end{aligned}$$

Thus $\{\Phi(u_k)\}$ is bounded (even uniformly).

Due to the presence of the unbounded domain, the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ $\left(2 \le q \le \frac{2n}{n-2s} = \frac{6}{3-2s}\right)$ is not compact. In order to overcome this kind, we restrict I to radial

functions, namely, $H_r^s(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) | u(x) = u(|x|)\}$ is compactly embedded in $L_r^q(\mathbb{R}^3)$ for $2 < q < \frac{2n}{n-2s} = \frac{6}{3-2s}$ (see [9] and [14]). By the standard arguments we know that a critical point $u \in H_r^s(\mathbb{R}^3)$ for the functional $I|_{H_r^s(\mathbb{R}^3)}$ is also a critical point for I.

Up to a subsequence, we may assume that there exist $u \in H_r^s(\mathbb{R}^3)$ and $\varphi \in D_r^{s,2}$ such that

$$u_k \rightharpoonup u \quad \text{in } H_r^s(\mathbb{R}^3), \tag{2.6}$$

$$u_k \to u \quad \text{in } L_r^q(\mathbb{R}^3) \text{ for } 2 < q < 2_s^*, \tag{2.7}$$

$$\Phi(u_k) \rightharpoonup \varphi \quad \text{in } D_r^{s,2}(\mathbb{R}^3). \tag{2.8}$$

Lemma 2.4. $\varphi = \Phi(u)$ and $\Phi(u_k) \to \Phi(u)$ in $D_r^{s,2}(\mathbb{R}^3)$.

Proof. First we prove the uniqueness. For every fixed $u \in H_r^s(\mathbb{R}^3)$, we consider the following minimizing problem

$$\inf_{\phi \in D_r^{s,2}} E_u(\phi),$$

where $E_u : D_r^{s,2} \to \mathbb{R}$ defined as energy functional of the second equation in system (1.1).

$$E_u(\phi) = \frac{1}{2}\|\phi\|_{D_r^{s,2}}^2 + \int_{\mathbb{R}^3} \omega\phi u^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi^2 u^2 \mathrm{d}x.$$

In fact, by the proof of [22, Lemma 2.1], we know that

$$\Phi(u_k) \to \varphi, \text{ locally uniformly in } \mathbb{R}^3,$$

so we obtain that

$$\int_{\mathbb{R}^3} \Phi(u_k)u_k^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \varphi u^2 \mathrm{d}x, \qquad \int_{\mathbb{R}^3} \Phi^2(u_k)u_k^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \varphi^2 u^2 \mathrm{d}x.$$

From the weak lower semicontinuity of the norm in $D_r^{s,2}(\mathbb{R}^3)$ and the convergence above, one has

$$E_u(\varphi) \leq \liminf_{k\to\infty} E_{u_k}(\Phi(u_k)) \leq \liminf_{k\to\infty} E_{u_k}(\Phi(u)) = E_u(\Phi(u)),$$

then by Lemma 1.3, $\varphi = \Phi(u)$.

Next we prove that $\{\Phi(u_k)\}$ converges strongly in $D_r^{s,2}(\mathbb{R}^3)$. Since $\Phi(u_k)$ and $\Phi(u)$ satisfy the second equation in problem (1.1).

$$\begin{cases} \langle \Phi(u_k), \psi \rangle_{D_r^{s,2}} = -\int_{\mathbb{R}^3} \left[\omega u_k^2 \psi + \Phi(u_k)u_k^2\psi\right] \mathrm{d}x, \\ \langle \Phi(u), \psi \rangle_{D_r^{s,2}} = -\int_{\mathbb{R}^3} \left[\omega u^2 \psi + \Phi(u)u^2\psi\right] \mathrm{d}x, \end{cases}$$

then we take the difference for Φ, one obtains that

$$\langle \Phi(u_k) - \Phi(u), \psi \rangle_{D_r^{s,2}} = -\int_{\mathbb{R}^3} \left[\omega\left(u_k^2 - u^2\right)\psi + \left(\Phi(u_k)u_k^2 - \Phi(u)u^2\right)\psi\right] \mathrm{d}x, \quad \psi \in D_r^{s,2}(\mathbb{R}^3). \tag{2.9}$$

Thus

$$\langle \Phi(u_k) - \Phi(u), \psi \rangle_{D_r^{s,2}} + \int_{\mathbb{R}^3} \left[u_k^2\left(\Phi(u_k) - \Phi(u)\right)\psi\right]\mathrm{d}x + \int_{\mathbb{R}^3} \left(u_k^2 - u^2\right)\Phi(u)\psi\mathrm{d}x$$

$$= -\omega\int_{\mathbb{R}^3} \left(u_k^2 - u^2\right)\psi\mathrm{d}x, \quad \psi \in D_r^{s,2}(\mathbb{R}^3). \tag{2.10}$$

By the Hölder inequality and the Sobolev inequality, testing with $\psi = (\Phi(u_k) - \Phi(u))$, the following holds:

$$\|\Phi(u_k) - \Phi(u)\|^2_{D_r^{s,2}}$$

$$= -\omega \int_{\mathbb{R}^3} \left(u_k^2 - u^2 \right) \left(\Phi(u_k) - \Phi(u) \right) dx$$

$$\quad - \int_{\mathbb{R}^3} u_k^2 \left(\Phi(u_k) - \Phi(u) \right)^2 dx - \int_{\mathbb{R}^3} \left(u_k^2 - u^2 \right) \Phi(u) \left(\Phi(u_k) - \Phi(u) \right) dx$$

$$\leq |\omega| \int_{\mathbb{R}^3} |u_k^2 - u^2||\Phi(u_k) - \Phi(u)| dx + \int_{\mathbb{R}^3} |u_k^2 - u^2||\Phi(u)||\Phi(u_k) - \Phi(u)| dx$$

$$\leq |\omega||\Phi(u_k) - \Phi(u)|_{\frac{6}{3-2s}} |u_k^2 - u^2|_{\frac{6}{3+2s}} + |u_k^2 - u^2|_{\frac{3}{2s}} |\Phi(u)|_{\frac{6}{3-2s}} |\Phi(u_k) - \Phi(u)|_{\frac{6}{3-2s}}$$

$$\leq C_6 |u_k - u|_{\frac{12}{3+2s}} + C_7 |u_k - u|_{\frac{3}{s}}.$$

Since $u_k \rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$, $u_k \to u$ in $L_r^q(\mathbb{R}^3)(2 < q < 2_s^*)$, one has $\Phi(u_k) \to \Phi(u)$ strongly in $D_r^{s,2}(\mathbb{R}^3)$. $\qquad\square$

Lemma 2.5. *The weak limit $(u, \Phi(u))$ solves problem* (1.1).

Proof. From (1.6), we know that

$$(I'(u_k), v) = \langle u_k, v \rangle_{H^s} + \int_{\mathbb{R}^3} \left[\left(m^2 - (\omega + \Phi(u_k))^2 \right) u_k v \right] dx$$

$$\quad - \int_{\mathbb{R}^3} \left[\mu |u_k|^{q-2} u_k v + |u_k|^{2_s^*-2} u_k v \right] dx, \quad v \in H_r^s(\mathbb{R}^3). \tag{2.11}$$

All convergences in the sequel must be understood passing to a subsequence if necessary. Since $\{u_k\}$ is bounded in $L_r^{2_s^*}(\mathbb{R}^3)$,

$$|u_k|^{2_s^*-2} u_k \rightharpoonup |u|^{2_s^*-2} u, \quad \text{in } (L_r^{2_s^*}(\mathbb{R}^3))^*.$$

Moreover by Lemma 2.4, for any $v \in H_r^s(\mathbb{R}^3)$, one gets that

$$\int_{\mathbb{R}^3} u_k \Phi^2(u_k) v dx + 2\omega \int_{\mathbb{R}^3} \Phi(u_k) u_k v dx \to \int_{\mathbb{R}^3} u \Phi^2(u) v dx + 2\omega \int_{\mathbb{R}^3} \Phi(u) u v dx.$$

In fact one obtains that

$$\int_{\mathbb{R}^3} |\Phi(u)u - \Phi(u_k)u_k| \, |v| dx$$

$$\quad \leq |\Phi(u) - \Phi(u_k)|_{\frac{6}{3-2s}} |u|_{\frac{3}{2s}} |v|_{\frac{6}{3-2s}} + |\Phi(u_k)|_{\frac{6}{3-2s}} |v|_{\frac{6}{3-2s}} |u_k - u|_{\frac{3}{2s}} \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} |u_k \Phi^2(u_k) - u \Phi^2(u)||v| dx$$

$$\quad \leq |u_k - u|_{\frac{3}{2s}} |\Phi(u_k)|^2_{\frac{12}{3-2s}} |v|_{\frac{6}{3-2s}} + |\Phi(u_k) - \Phi(u)|_{\frac{6}{3-2s}} |\Phi(u_k) + \Phi(u)|_{\frac{6}{3-2s}} |u|_{\frac{3}{s}} |v|_{\frac{3}{s}}. \tag{2.13}$$

The compactness of the embedding $H_r^s(\mathbb{R}^3) \hookrightarrow L_r^q(\mathbb{R}^3)$ the lemma follows. $\qquad\square$

In the following we will prove $u \neq 0$, so we assume that c denotes the *MP* level.

Claim 2.6. $c < \frac{s}{3}S_s^{\frac{3}{2s}}$, where S_s corresponds to the best constant for the fractional Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, precisely,

$$S_s := \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{D^{s,2}}^2}{|u|_{2_s^*}^2}. \tag{2.14}$$

Proof. By [10], S_s is attained by

$$\widetilde{u}(x) = \kappa \left(\varepsilon^2 + |x - x_0|\right)^{-\frac{3-2s}{2}},$$

i.e., $S_s = \frac{\|\widetilde{u}\|_{D^{s,2}}^2}{|\widetilde{u}|_{2_s^*}^2}$, normalizing $\widetilde{u}$ by $|\widetilde{u}|_{2_s^*}$, one obtains that $\overline{u} = \frac{\widetilde{u}}{|\widetilde{u}|_{2_s^*}}$. Thus

$$S_s = \inf_{u \in D^{s,2}(\mathbb{R}^3), |u|_{2_s^*}=1} \|u\|_{D^{s,2}}^2 = \|\overline{u}\|_{D^{s,2}}^2.$$

Moreover $u_1 = S_s^{\frac{1}{2_s^*-2}}\overline{u}$ is a positive ground state solution of $(-\Delta)^s = |u|^{2_s^*-2}$ in $\mathbb{R}^3$ and

$$\|u_1\|_{D^{s,2}}^2 = |u_1|_{2_s^*}^{2_s^*} = S_s^{\frac{3}{2s}}.$$

Now according to Reference [20], given $\varepsilon > 0$, we consider the function

$$U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} u_1\left(\frac{x}{\varepsilon}\right), \quad U_\varepsilon \in D^{s,2}(\mathbb{R}^3). \tag{2.15}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^3$, $\varphi \equiv 1$ in $B_\delta(\delta > 0)$ and $\varphi \equiv 0$ in $CB_{2\delta}$, where $B_\delta = B(0, \delta)$ and $CB_\delta = \mathbb{R}^3 \setminus B_\delta$. For every $\varepsilon > 0$ we denote u_ε by the following function: $u_\varepsilon = \varphi(x)U_\varepsilon(x), x \in \mathbb{R}^3$ and

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon(x)|_{2_s^*}}.$$

Let $e > 0$ and $\mu > 0$, if $x \in CB_e$, then

$$|\nabla u_\varepsilon(x)| \le C\varepsilon^{\frac{3-2s}{2}} \quad \text{for any } \varepsilon > 0$$

and for some positive constant C, possibly depending on μ, e and s. Suppose $s \in (\frac{3}{4}, 1)$. Then according to [20], the following estimate holds true:

$$X_\varepsilon := \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x-y|^{3+2s}} dx dy \le S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right), \quad \text{as } \varepsilon \to 0. \tag{2.16}$$

Since as $t \to +\infty, I(tv_\varepsilon) \to -\infty$, we may assume that

$$\sup_{t \ge 0} I(tv_\varepsilon) = I(t_\varepsilon v_\varepsilon)$$

and without loss of generality that $t_\varepsilon \ge C_0 > 0$, for all $\varepsilon > 0$. Otherwise, there exists a sequence ε_n such that

$$\lim_{n\to\infty} t_{\varepsilon_n} = 0$$

and then

$$0 < c \le \lim_{n\to\infty} I(t_{\varepsilon_n} v_{\varepsilon_n}) = 0.$$

Next we will prove the above bound of t_ε, that is, for any $\varepsilon > 0$ small enough

$$t_\varepsilon \leq \left(X_\varepsilon + \int_{\mathbb{R}^3} m^2 v_\varepsilon^2 dx \right)^{\frac{1}{2_s^* - 2}} = T. \tag{2.17}$$

Set $f(t) = I(t v_\varepsilon)$ and compute

$$\begin{aligned}
f'(t) &= \left(I'(t v_\varepsilon), v_\varepsilon \right) \\
&= t T^{2_s^* - 2} - t^{2_s^* - 1} - t \int_{\mathbb{R}^3} (\omega + \Phi(t v_\varepsilon))^2 v_\varepsilon^2 dx - \mu t^{q-1} \int_{\mathbb{R}^3} |v_\varepsilon|^q dx \leq 0, \quad t \geq T.
\end{aligned}$$

Hence, $f'(t) \leq 0$ if $t \geq T$ and (2.17) holds.

Since the function $t \mapsto \frac{1}{2} t^2 T^{2_s^* - 2} - \frac{1}{2_s^*} t^{2_s^*}$ is increasing in the internal $[0, T)$, by (2.16), one obtains that

$$\begin{aligned}
I(t_\varepsilon v_\varepsilon) &= \frac{t_\varepsilon^2}{2} \left(\frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^2}{|x - y|^{3 + 2s}} dx dy + \int_{\mathbb{R}^3} m^2 v_\varepsilon^2 dx \right) - \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} (\omega + \Phi(t_\varepsilon v_\varepsilon))^2 v_\varepsilon^2 dx \\
&\quad - \frac{1}{2} \|\Phi(t_\varepsilon v_\varepsilon)\|_{D_r^{s,2}}^2 - \frac{\mu t_\varepsilon^q}{q} \int_{\mathbb{R}^3} |v_\varepsilon|^q dx - \int_{\mathbb{R}^3} \frac{t_\varepsilon^{2_s^*}}{2_s^*} |v_\varepsilon|^{2_s^*} dx \\
&\leq \frac{s}{3} \left(S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s} \right) + \int_{\mathbb{R}^3} m^2 v_\varepsilon^2 dx \right)^{\frac{3}{2s}} + \int_{\mathbb{R}^3} \frac{t_\varepsilon^2}{2} \omega^2 v_\varepsilon^2 dx - \frac{\mu t_\varepsilon^q}{q} \int_{\mathbb{R}^3} |v_\varepsilon|^q dx.
\end{aligned}$$

Then using the inequality $(a + b)^\sigma \leq a^\sigma + \sigma (a + b)^{\sigma - 1} b$, for all $\sigma \geq 1, a, b \geq 0$, we get that

$$I(t_\varepsilon v_\varepsilon) \leq \frac{s}{3} S_s^{\frac{9}{4s^2}} + O\left(\varepsilon^{3-2s} \right) + C_1(\varepsilon) \int_{\mathbb{R}^3} v_\varepsilon^2 dx - C_2(\varepsilon) \int_{\mathbb{R}^3} |v_\varepsilon|^q dx,$$

with constants $C_i(\varepsilon) > 0 \, (i = 1, 2)$. On the other hand, we may get the conclusion that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2s}} \int_{\mathbb{R}^3} \left(v_\varepsilon^2 - \mu |v_\varepsilon|^q \right) dx = -\infty \quad \text{for } \varepsilon \text{ small enough.} \tag{2.18}$$

In fact, by the definition of u_ε, since for $\varepsilon \to 0$, as in [20],

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx = S_s^{\frac{3}{2s}} + O\left(\varepsilon^3 \right), \tag{2.19}$$

it suffices to evaluate (2.18) with u_ε in place of v_ε. For $p \geq 1$, one has

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_\varepsilon(x)|^p dx &= \int_{B_\delta} |U_\varepsilon(x)|^p dx + \int_{B_{2\delta} \setminus B_\delta} |\varphi(x) U_\varepsilon(x)|^p dx \\
&= C_8 \varepsilon^{-\frac{p(3-2s)}{2}} \int_{B_\delta} |u_1(\frac{x}{\varepsilon})|^p dx \\
&= C_8 \varepsilon^{\frac{6 - 3p + 2ps}{2}} \int_R^{\frac{\delta}{\varepsilon}} |u_1(r)|^p r^2 dr \\
&= C_8 \varepsilon^{\frac{6 - 3p + 2ps}{2}} \int_R^{\frac{\delta}{\varepsilon}} r^{-3p + 2ps + 2} dr
\end{aligned} \tag{2.20}$$

for any $0 < R < \frac{\delta}{\varepsilon}$ and therefore, one has for $4 < q < 2_s^*$, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^3} u_\varepsilon^2 dx - \mu \int_{\mathbb{R}^3} u_\varepsilon^q dx \leq C_9 \varepsilon^{2s} - C_{10} \mu \varepsilon^{\frac{6 - 3q + 2qs}{2}}, \tag{2.21}$$

where $C_i > 0 \, (i = 9, 10)$ are independent from ε. According to (2.19) and (2.21), we complete the proof of (2.18). $\qquad \square$

Claim 2.7. *The solution u is nontrivial.*

Proof. By contradiction, suppose that $u \equiv 0$. It follows that $\Phi(u) = 0$ and as $k \to \infty$,

$$
\begin{aligned}
\left(I'(u_k), u_k\right) = {} & \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} \left(m^2 - \omega^2\right) u_k^2 dx \\
& - \int_{\mathbb{R}^3} \left(2\omega\Phi(u_k) + \Phi^2(u_k)\right) u_k^2 dx - \mu \int_{\mathbb{R}^3} |u_k|^q dx - \int_{\mathbb{R}^3} |u_k|^{2_s^*} dx \\
& \to 0
\end{aligned}
$$

and

$$
u_k \to 0 \text{ in } L_r^q(\mathbb{R}^3).
$$

Thus one obtains that

$$
\int_{\mathbb{R}^3} |u_k|^q dx \to 0
$$

and

$$
\int_{\mathbb{R}^3} \left(2\omega\Phi(u_k) + \Phi^2(u_k)\right) u_k^2 dx \to 0.
$$

Hence, up to a subsequence, if necessary, we can assume that

$$
\frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} \left(m^2 - \omega^2\right) u_k^2 dx \to L, \tag{2.22}
$$

and

$$
\int_{\mathbb{R}^3} |u_k(x)|^{2_s^*} dx \to L, \qquad L \geq 0. \tag{2.23}
$$

Furthermore, $I(u_k) \to c$, it follows that

$$
c = \left(\frac{1}{2} - \frac{1}{2_s^*}\right) L = \frac{s}{3} L. \tag{2.24}
$$

Since $c \geq \alpha > 0$, it is easily seen that $L > 0$. In addition,

$$
\frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{3+2s}} dx dy \geq S_s |u_k|_{2_s^*}^2,
$$

so that taking into account (2.22) and (2.23), we get $L \geq S_s L^{\frac{2}{2_s^*}}$, which combined with (2.24) gives

$$
c \geq \frac{s}{3} S_s^{\frac{2_s^*}{2_s^* - 2}} = \frac{s}{3} S_s^{\frac{3}{2s}},
$$

this contradicts Claim 2.6. Hence u is nontrivial. $\qquad\square$

3 Proof of Theorem 1.2

We can observe if $q = 4$, in (2.21) one can stress the parameter choosing $\mu = \varepsilon^{-\sigma}$, $\sigma > 0$, then to get (2.18), the rest proof of Theorem 1.2 is similar to proof of Theorem 1.1.

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