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Cauchy problem for nonlocal diffusion equations modelling Lévy flights

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Abstract. In the present paper, we study the time-space fractional diffusion equation involving the Caputo differential operator and the fractional Laplacian. This equation describes the Lévy flight with the Brownian motion component and the drift component. First, the asymptotic behavior of the fundamental solution of the fractional diffusion equation is considered. Then, we use the fundamental solution to obtain the representation formula of solutions of the Cauchy problem. In the last, the L^2-decay estimates for solutions are proved by employing the Fourier analysis technique.

Keywords: Caputo differential operator, fractional Laplacian, Cauchy problem, nonlocal diffusion equation, representation of solution, asymptotic behavior, Lévy flight.

2020 Mathematics Subject Classification: 35R11, 35A08, 35B40, 35C15, 45K05, 47G20.

1 Introduction

1.1 Statement of the problem

In this paper, we consider the Cauchy problem for the fractional diffusion equation

$$\partial_t^\alpha u(t,x) = -(-\triangle)^\beta u(t,x) + b\cdot\nabla u(t,x) + h\triangle u(t,x), \qquad t>0,\ x\in\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x\in\mathbb{R}^n, \tag{1.2}$$

where $n\in\mathbb{N}, \alpha\in(0,1], \beta\in(0,1), h>0$ and $b\in\mathbb{R}^n$. Here ∂_t^α denotes the Caputo fractional differential operator defined by [16] ∂_t^1 being the classical differential operator and

$$\partial_t^\alpha v(t) = \frac{d}{dt}J_t^{1-\alpha}(v-v(0))(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}(v(s)-v(0))ds, \qquad t>0$$

for $\alpha\in(0,1)$, where J_t^a is the Riemann–Liouville fractional integral operator of order $a\geq 0$ defined by [16] J_t^0 being the identity operator and

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$$J_t^a v(t) = \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} v(s) ds$$

for $a > 0$. Also, $(-\triangle)^\beta$ is the fractional Laplacian defined by

$$(-\triangle)^\beta v(x) = F^{-1}(|\xi|^{2\beta} F v(\xi))(x), \qquad x \in \mathbb{R}^n$$

for $\beta \in (0,1)$. Here $\mathscr{F}$ and $\mathscr{F}^{-1}$ are respectively Fourier transform and inverse Fourier transform defined by [16,27]

$$\mathscr{F} v(\xi) = \tilde{v}(\xi) = \int_{\mathbb{R}^n} v(x) e^{-ix\xi} dx, \qquad \xi \in \mathbb{R}^n,$$
$$\mathscr{F}^{-1} v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} v(\xi) e^{ix\xi} d\xi, \qquad x \in \mathbb{R}^n.$$

1.2 Physical background

The equation (1.1) is derived from the continuous time random walk (CTRW in short) theory, which is characterized by the waiting time probability density function (PDF in short) $\psi(t)$ and the jump length PDF $\omega(x)$. The famous Montroll–Weiss equation is given in the Fourier–Laplace space by [21]

$$\hat{\tilde{u}}(s,\xi) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{1-\hat{\psi}(s)\tilde{\omega}(\xi)}, \tag{1.3}$$

where $\hat{\psi}(s)$ means the Laplace transform of $\psi(t)$ defined by [16]

$$\hat{\psi}(s) = \mathscr{L}\psi(s) = \int_0^\infty e^{-ts} \psi(t) dt$$

and $\hat{\tilde{u}}(s,\xi)$ stands for the Fourier–Laplace transform of the PDF $u(t,x)$ of being at position x at time t.

Take

$$\hat{\psi}(s) = \frac{1}{1+s^\alpha}, \qquad \alpha \in (0,1], \tag{1.4}$$
$$\tilde{\omega}(\xi) = e^{-\zeta_\beta(\xi)}, \tag{1.5}$$

where ζ_β is defined by

$$\zeta_\beta(\xi) = |\xi|^{2\beta} + h|\xi|^2 - ib \cdot \xi. \tag{1.6}$$

Since $\frac{1}{1+s^\alpha}$ is a Stieltjes function, $\psi(t)$ becomes a PDF. In fact, $\psi(t)$ is the Mittag-Leffler PDF given by [11]

$$\psi(t) = \mathscr{L}^{-1}\left(\frac{1}{1+s^\alpha}\right)(t) = t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha). \tag{1.7}$$

When $\alpha = 1$, $\psi(t)$ becomes a Poisson PDF. By Lemma 6.9 in [17], the function ζ_β is negative definite. It follows from the negative definiteness of ζ_β that $e^{-\zeta_\beta(\xi)}$ is a positive definite function. For details of the Bernstein function theory, see [28]. Then, by the Bochner theorem, $\omega(x)$ becomes a PDF. Also, we have

$$e^{-\zeta_\beta(\xi)} \to 1 - \zeta_\beta(\xi), \qquad \xi \to 0. \tag{1.8}$$

Combining (1.8) with (1.3) and (1.4), we deduce

$$\hat{u}(s,\xi) = \frac{s^{\alpha-1}}{s^{\alpha} + \zeta_{\beta}(\xi)}. \tag{1.9}$$

We can rewrite (1.9) as

$$s^{\alpha}\hat{u}(s,\xi) - s^{\alpha-1} = -\zeta_{\beta}(\xi)\hat{u}(s,\xi).$$

Taking the inverse Fourier–Laplace transform, we obtain the equation (1.1).

In [3,7], Cartea and del-Castillo-Negrete use the jump length PDF in the Lévy–Khintchine representation to derive a general time-space nonlocal diffusion equation including (1.1). Also, employing the general nonlocal diffusion equation, they described the tempered Lévy flight which looks like a Lévy process in a small time and behaves like a Brownian random walk in a large time. Meerschaert et al. also considered general nonlocal diffusion equations including (1.1) in [20]. In [6], del-Castillo-Negrete and Cartea modelled the resistive pressure-gradient-driven plasma turbulence by employing the time-space fractional diffusion equation.

The equation (1.1) captures the Lévy flight with the Brownian motion component and the drift component. In (1.1), the fractional Laplacian and the classical Laplacian mean the jump component and the Brownian motion component respectively. Also, the gradient in (1.1) stands for the drift component. For details of the Lévy process, see [27].

1.3 State of the art

In [2], Blumenthal and Getoor established the following estimate for the transition density of the 2β-stable Lévy process

$$u(t,x) \sim \min\{t^{-\frac{n}{2\beta}}, t|x|^{-n-2\beta}\}. \tag{1.10}$$

The transition density corresponds to the fundamental solution of the equation (1.1) of the case: $\alpha = 1, \beta \in (0,1)$, $b = h = 0$. In [12], Ignat and Rossi used the energy method to obtain the decay estimate results for solutions of the space fractional diffusion equation. Kaleta and Sztonyk [13] studied the asymptotic behavior of transition density and its derivatives for the tempered Lévy flight.

Eidelman and Kochubei [8] obtained the various estimates for fundamental solutions of the time fractional diffusion equation. In [26], the existence of solutions and the large time behavior for the initial boundary value problem of the Caputo time fractional diffusion-wave equation were investigated. In [14,31,32], the optimal decay estimates for solutions of time nonlocal diffusion equations were established.

In [18], Mainardi, Luchko and Pagnini studied the analytical properties of fundamental solutions of the time-space fractional diffusion-wave equation involving the Caputo differential operator and the Riesz–Feller operator. Chen, Meerschaert and Nane [4] established the probabilistic representations for solutions of the equation (1.1) of the case: $\alpha \in (0,1], \beta \in (0,1]$, $b = h = 0$. In [15], Kemppainen, Siljander and Zacher used the properties of the Fox H-function and the Fourier analysis technique to prove the results for the asymptotic behavior of solutions of the equation

$$\partial_t^{\alpha} u(t,x) = -(-\triangle)^{\beta} u(t,x) + f(t,x), \qquad t > 0,\ x \in \mathbb{R}^n. \tag{1.11}$$

Allen, Caffarelli and Vasseur [1] studied the Hölder regularity for the nonlocal diffusion equation with the Caputo fractional derivative and a generalization of the fractional Laplacian. By employing the Laplace transform, Cheng, Li and Yamamoto [5] obtained the large time behavior result for initial value problem and initial boundary value problem of the time-space fractional diffusion-reaction equation. In [30], the author proved the existence of solutions of the time-space nonlocal diffusion equation involving the generalized Caputo-type differential operator and the generalized fractional Laplacian introduced in [29]. We mention also [10, 22–25], where analytical solutions of several time-space fractional diffusion-wave equations were established.

The goal of this paper is to obtain the asymptotic behavior of solutions of the Cauchy problem (1.1)–(1.2).

1.4 Outline

This paper is organized as follows.

In Section 2, we give necessary concepts and lemmas for obtaining the main results of the paper.

In Section 3, we study the asymptotic behavior of fundamental solutions of the equation (1.1). The asymptotic behavior result for the fundamental solution shows that the equation (1.1) captures the Lévy flight which looks like a Brownian random walk in a short time and behaves like a Lévy process in a long time.

In Section 4, we prove the representation formula for solutions of the Cauchy problem (1.1)–(1.2) by using the fundamental solution and the properties of the Wright function.

In Section 5, we obtain the L^2-decay estimates for solutions of the Cauchy problem (1.1)–(1.2) by employing the Fourier analysis method.

2 Preliminaries

First of all, we introduce some basic notations. Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will mean the sets of natural, real and complex numbers respectively. $C > 0$ stands for a universal positive constant which can be different at different places. Also, $a \lesssim b$ means $a \le Cb$ for some constant $C > 0$ and $a \gtrsim b$ denotes $a \ge Cb$ for some constant $C > 0$. In addition, we write $a \sim b$ if $a \lesssim b \lesssim a$.

Let $a, b \in \mathbb{C}$ and $Re(a) > 0$. The two parameter Mittag-Leffler function is defined by [16]

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}, \qquad z \in \mathbb{C}.$$

Lemma 2.1 ([16,32])**.** *Let $k \in \mathbb{C}, a > 0$ and $j \in \mathbb{N}$. Then*

$$\partial_t^a E_{a,1}(kt^a) = kE_{a,1}(kt^a), \qquad t > 0. \tag{2.1}$$

Proof. The relation (2.1) was proved in [16]. □

Lemma 2.2. *Let $a \in (0,2)$ and $b \in \mathbb{R}$. Suppose that μ is such that $\pi a/2 < \mu < \min\{\pi, \pi a\}$. Then there exists a constant $C = C(a,b,\mu) > 0$ such that*

$$|E_{a,b}(z)| \le \frac{C}{1+|z|}, \qquad \mu \le |\arg(z)| \le \pi. \tag{2.2}$$

Let $a > -1$ and $b \in \mathbb{C}$. The Wright function $W_{a,b}$ is defined by [16]

$$W_{a,b}(t) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(aj+b)}, \qquad z \in \mathbb{C}. \tag{2.3}$$

Let $r \in (0,1)$. The functions F_r and M_r are special cases of the Wright function defined by [19]

$$F_r(z) = W_{-r,0}(-z) = \sum_{j=1}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj)}, \qquad z \in \mathbb{C}, \tag{2.4}$$

$$M_r(z) = W_{-r,1-r}(-z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj+1-r)}, \qquad z \in \mathbb{C}. \tag{2.5}$$

The functions F_r and M_r are related through

$$F_r(z) = rzM_r(z), \qquad z \in \mathbb{C}. \tag{2.6}$$

By the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, the following equality holds.

$$F_r(z) = \frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{j-1}z^j\frac{\Gamma(j\alpha+1)}{j!}\sin(j\pi\alpha), \qquad z \in \mathbb{C}. \tag{2.7}$$

Also, the following relations hold [19].

$$M_r\Big(\frac{t}{r}\Big) \approx \frac{1}{\sqrt{2\pi(1-r)}}t^{\frac{r-\frac{1}{2}}{1-r}}e^{-\frac{1-r}{r}t^{\frac{1}{1-r}}}, \qquad t \to +\infty. \tag{2.8}$$

$$\int_0^{\infty} M_r(t)dt = 1. \tag{2.9}$$

$$\frac{dW_{a,b}(t)}{dt} = W_{a,a+b}(t). \tag{2.10}$$

By (2.8) and (2.6), we can easily see the asymptotic behavior of the function $F_r(t)$.

$$e^{-\tau s^{\alpha}} = \int_0^{\infty} e^{-st}\theta_\alpha(t,\tau)dt, \qquad \tau, s > 0, \tag{2.11}$$

where

$$\theta_\alpha(t,\tau) = \frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{j-1}\tau^j t^{-\alpha j-1}\frac{\Gamma(j\alpha+1)}{j!}\sin(j\pi\alpha), \qquad t,\tau > 0. \tag{2.12}$$

By (2.7) and (2.4),

$$\theta_\alpha(t,\tau) = \frac{1}{t}F_\alpha\Big(\frac{\tau}{t^\alpha}\Big) = \frac{1}{t}W_{-\alpha,0}\Big(-\frac{\tau}{t^\alpha}\Big), \qquad t,\tau > 0. \tag{2.13}$$

From (2.11), we obtain

$$\lim_{t\to 0}\theta_\alpha(t,\tau) = \lim_{s\to\infty} se^{-\tau s^\alpha} = 0, \qquad \tau > 0. \tag{2.14}$$

Let $Z_{\alpha,\beta}$ stand for the fundamental solution of the following time-space fractional diffusion equation

$$\partial_t^\alpha u(t,x) = -(-\triangle)^\beta u(t,x), \qquad t > 0,\ x \in \mathbb{R}^n. \tag{2.15}$$

Let $n \in \mathbb{N}$ and $b > 0$. Denote

$$\bar{p}(n,b) := \begin{cases} \frac{n}{n-2b}, & n > 2b, \\ \infty, & \text{otherwise.} \end{cases}$$

3 Fundamental solution of fractional diffusion equation

In this section, we consider the fundamental solution of the fractional diffusion equation (1.1).

3.1 Fundamental solution of space fractional diffusion equation

In this subsection, we discuss the space fractional diffusion equation of the form

$$\frac{\partial u(t,x)}{\partial t} = -(-\triangle)^{\beta}u(t,x) + b\cdot\nabla u(t,x) + h\triangle u(t,x), \qquad t>0,\ x\in\mathbb{R}^n, \tag{3.1}$$

which is corresponding to the equation (1.1) when $\alpha = 1$ and $\beta \in (0,1)$. Applying the Fourier transform to (3.1) with respect to the space variable x, we obtain

$$\frac{\partial \tilde{u}(t,\xi)}{\partial t} = -(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)\tilde{u}(t,\xi), \qquad t>0,\ \xi\in\mathbb{R}^n. \tag{3.2}$$

The solution of the equation (3.2) with the condition $\tilde{u}(0,\xi) = 1$ has the form:

$$\tilde{u}(t,\xi) = e^{-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t}.$$

For convenience, we write $\zeta_\beta(\xi) = |\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi$. The fundamental solution $A_{1,\beta}$ of the equation (3.1) is represented by

$$A_{1,\beta}(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)t}\cos(x\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-(|\xi|^{2\beta}+h|\xi|^2)t}\cos((x+bt)\xi)d\xi. \tag{3.3}$$

Since $e^{-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t}$ is positive definite, it follows from the Bochner theorem that $A_{1,\beta}(t,x)\geq 0$. Also, the following relation holds.

$$e^{-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t} = \int_{\mathbb{R}^n} A_{1,\beta}(t,x)\cos(x\xi)dx = \tilde{A}_{1,\beta}(t,\xi). \tag{3.4}$$

Moreover,

$$\int_{\mathbb{R}^n} A_{1,\beta}(t,x)dx = 1, \qquad t>0.$$

If $h = 0$, then

$$A_{1,\beta}(t,x) = Z_{1,\beta}(t,x+bt). \tag{3.5}$$

If $h \neq 0$, then

$$A_{1,\beta}(t,x) = \frac{1}{(4\pi ht)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x+bt-y|^2}{4ht}} Z_{1,\beta}(t,y)dy. \tag{3.6}$$

Theorem 3.1. *Let $n\in\mathbb{N}, \beta\in(0,1), h>0$ and $b\in\mathbb{R}^n$. Then the following relations hold.*

$$A_{1,\beta}(t,x) \lesssim \begin{cases} t^{-\frac{n}{2}}, & \text{if } x\in\mathbb{R}^n \text{ and } t\in(0,1], \\ t^{-\frac{n}{2\beta}}, & \text{if } x\in\mathbb{R}^n \text{ and } t\in(1,\infty). \end{cases}$$

$$A_{1,\beta}(t,x-bt) \lesssim \begin{cases} t\left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t\in(0,1], \\ t\left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t\in(1,\infty). \end{cases}$$

Proof. The asymptotic behavior (1.10) of $Z_{1,\beta}(t,x)$ is the following.

$$Z_{1,\beta}(t,x) \sim \min\{t^{-\frac{n}{2\beta}}, t|x|^{-n-2\beta}\}. \tag{3.7}$$

By (3.7), we have

$$\begin{aligned} A_{1,\beta}(t,x-bt) &= \frac{1}{(4\pi ht)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(|x-y|)^2}{4ht}} Z_{1,\beta}(t,y)dy \\ &\sim \frac{1}{(4\pi ht)^{\frac{n}{2}}} \int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} t^{-\frac{n}{2\beta}} dy + \frac{1}{(4\pi ht)^{\frac{n}{2}}} \int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} t|y|^{-n-2\beta} dy \\ &\sim t^{-\frac{n}{2\beta}-\frac{n}{2}} \int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy + t^{1-\frac{n}{2}} \int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy. \end{aligned} \tag{3.8}$$

First, we estimate the integral

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy.$$

For $x \in \mathbb{R}^n$, we have

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{\frac{n}{2\beta}}.$$

For $x \in \mathbb{R}^n$, we obtain

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim (ht)^{\frac{n}{2}} \lesssim t^{\frac{n}{2}}.$$

For $|x| > t^{\frac{1}{2\beta}}$, we estimate

$$\int_{|x-z|<t^{\frac{1}{2\beta}}} e^{-\frac{|z|^2}{4ht}} dz \le e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}} \int_{|x-z|<t^{\frac{1}{2\beta}}} dz \lesssim t^{\frac{n}{2\beta}} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}}.$$

For $|x| > t^{\frac{1}{2\beta}}$ and $m > 0$, we deduce

$$\begin{aligned} \int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy &\lesssim \int_{|y|<t^{\frac{1}{2\beta}}} \frac{t^{\frac{m}{2}}}{|x-y|^m} dy \lesssim \int_0^{t^{\frac{1}{2\beta}}} \frac{t^{\frac{m}{2}}}{(|x|-r)^m} r^{n-1} dr \\ &\lesssim t^{\frac{m}{2}} \left(|x| - t^{\frac{1}{2\beta}}\right)^{-m} \int_0^{t^{\frac{1}{2\beta}}} r^{n-1} dr \lesssim t^{\frac{m}{2}} t^{\frac{n}{2\beta}} \left(|x| - t^{\frac{1}{2\beta}}\right)^{-m}. \end{aligned}$$

Setting $m = n+2$,

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{\frac{n}{2\beta}+\frac{n}{2}+1} \left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2}.$$

Setting $m = n+2\beta$,

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{\frac{n}{2\beta}+\frac{n}{2}+\beta} \left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2\beta}.$$

Next, we estimate the integral

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy.$$

For $x \in \mathbb{R}^n$, we obtain

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \lesssim t^{\frac{-n-2\beta}{2\beta}} \int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{\frac{-n-2\beta}{2\beta}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{-\frac{n}{2\beta}+\frac{n}{2}-1}.$$

For $x \in \mathbb{R}^n$, we have

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \le \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \lesssim t^{-1}.$$

If $|x| > t^{\frac{1}{2\beta}}$, then

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy = &\int_{|y-x|<\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \\ &+ \int_{|y-x|>\frac{|x|-t^{\frac{1}{2\beta}}}{2},|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy.\end{aligned}$$

Then, for $|x| > t^{\frac{1}{2\beta}}$,

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy &\le \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2\beta} \int_0^{\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{r^2}{4ht}} r^{n-1} dr \\ &\quad + e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \\ &\lesssim (4ht)^{\frac{n}{2}} \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2\beta} + \frac{1}{2\beta t} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \\ &\lesssim t^{\frac{n}{2}} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2\beta} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}}.\end{aligned}$$

Also, for $|x| > t^{\frac{1}{2\beta}}$,

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy &\le \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2} \int_0^{\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{r^2}{4ht}} r^{2-2\beta+n-1} dr \\ &\quad + e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \\ &\lesssim (4ht)^{\frac{n}{2}+1-\beta} \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2} + \frac{1}{2\beta t} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \\ &\lesssim t^{\frac{n}{2}+1-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}}.\end{aligned}$$

Combining the previous estimates, we obtain the following results.
If $|x| > t^{\frac{1}{2\beta}}$ and $t \in (0,1]$, then, by (3.8), for $m > 0$,

$$\begin{aligned}A_{1,\beta}(t, x-bt) &\lesssim t^{-\frac{n}{2\beta}-\frac{n}{2}} t^{\frac{n}{2\beta}} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}} + t^{1-\frac{n}{2}} \left(t^{\frac{n}{2}+1-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \right) \\ &\lesssim t^{-\frac{n}{2}} t^{\frac{m}{2}} \left(|x|-t^{\frac{1}{2\beta}}\right)^{-m} + t^{2-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-\frac{n}{2}} t^{\frac{m}{2}} \left(|x|-t^{\frac{1}{2\beta}}\right)^{-m}.\end{aligned}$$

Setting $m = n+2$, we deduce

$$A_{1,\beta}(t,x-bt) \lesssim t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}.$$

If $x \in \mathbb{R}^n$ and $t \in (0,1]$, then

$$A_{1,\beta}(t,x-bt) \lesssim t^{-\frac{n}{2\beta}-\frac{n}{2}} t^{\frac{n}{2\beta}} + t^{1-\frac{n}{2}} t^{-1} \lesssim t^{-\frac{n}{2}}.$$

If $|x| > t^{\frac{1}{2\beta}}$ and $t > 1$, then, by (3.8), for $m > 0$,

$$\begin{aligned} A_{1,\beta}(t,x-bt) &\lesssim t^{-\frac{n}{2\beta}-\frac{n}{2}} t^{\frac{n}{2\beta}} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}} + t^{1-\frac{n}{2}}\left(t^{\frac{n}{2}}\big(|x| + t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}}\right) \\ &\lesssim t^{-\frac{n}{2}} t^{\frac{m}{2}} \big(|x| - t^{\frac{1}{2\beta}}\big)^{-m} + t\big(|x| + t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t^{-\frac{n}{2}} t^{\frac{m}{2}} \big(|x| - t^{\frac{1}{2\beta}}\big)^{-m}. \end{aligned}$$

Setting $m = n+2\beta$, we deduce

$$A_{1,\beta}(t,x-bt) \lesssim t^{\beta}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta} \lesssim t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}.$$

If $x \in \mathbb{R}^n$ and $t > 1$, then

$$A_{1,\beta}(t,x-bt) \lesssim t^{-\frac{n}{2\beta}-\frac{n}{2}} t^{\frac{n}{2}} + t^{1-\frac{n}{2}} t^{-1-\frac{n}{2\beta}+\frac{n}{2}} \lesssim t^{-\frac{n}{2\beta}}. \qquad \square$$

Remark 3.2. From the relation (3.7), we can easily see

$$\frac{Z_{1,\beta}(t,x)}{Z_{1,\beta}(t,x+bt)} \lesssim \max\{1, t^{n+2\beta-\frac{n}{2\beta}-1}\}, \quad \text{if } t > 0 \text{ and } x \in \mathbb{R}^n.$$

Then, from (3.6) and Theorem 3.1, we obtain the following result.

If $\beta \in [\frac{1}{2}, 1)$, then

$$A_{1,\beta}(t,x) \lesssim \begin{cases} t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (0,1], \\ t^{n+2\beta-\frac{n}{2\beta}}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (1,\infty). \end{cases}$$

If $\beta \in (0, \frac{1}{2})$, then

$$A_{1,\beta}(t,x) \lesssim \begin{cases} t^{n+2\beta-\frac{n}{2\beta}}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (0,1], \\ t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (1,\infty). \end{cases}$$

We consider the L^p-estimate of the fundamental solution $A_{1,\beta}$.

Theorem 3.3. *Let $n \in \mathbb{N}, \beta \in (0,1), h > 0, b \in \mathbb{R}^n$ and $p \in [1,\infty]$. Then the following relation holds.*

$$\|A_{1,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{2}(1-\frac{1}{p})}, & t \in (0,1], \\ t^{-\frac{n}{2\beta}(1-\frac{1}{p})}, & t \in (1,\infty). \end{cases}$$

Proof. Using Theorem 3.1, we can easily obtain the result.
For $t \in (0,1]$, we obtain

$$\begin{aligned}\|A_{1,\beta}(t,\cdot)\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} A_{1,\beta}(t,x)^p dx = \int_{|x|<t^{\frac{1}{2}}} A_{1,\beta}(t,x)^p dx + \int_{|x|>t^{\frac{1}{2}}} A_{1,\beta}(t,x)^p dx \\ &\lesssim t^{-\frac{np}{2}+\frac{n}{2}} + t^p \int_{t^{\frac{1}{2}}}^{\infty} \frac{r^{n-1}}{(r-t^{\frac{1}{2\beta}})^{np+2p}} dr \\ &\lesssim t^{-\frac{np}{2}+\frac{n}{2}} + t^p t^{\frac{n}{2}-\frac{np}{2}-\frac{2p}{2}} \int_1^{\infty} \frac{s^{n-1}}{(s-t^{\frac{1}{2\beta}-\frac{1}{2}})^{np+2p}} ds \lesssim t^{-\frac{np}{2}+\frac{n}{2}}.\end{aligned}$$

For $t > 1$, we have

$$\begin{aligned}\|A_{1,\beta}(t,\cdot)\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} A_{1,\beta}(t,x)^p dx = \int_{|x|<2t^{\frac{1}{2\beta}}} A_{1,\beta}(t,x)^p dx + \int_{|x|>2t^{\frac{1}{2\beta}}} A_{1,\beta}(t,x)^p dx \\ &\lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}} + t^p \int_{2t^{\frac{1}{2\beta}}}^{\infty} \frac{r^{n-1}}{(r-t^{\frac{1}{2\beta}})^{np+2\beta p}} dr \\ &\lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}} + t^p t^{\frac{n}{2\beta}-\frac{np}{2\beta}-\frac{2\beta p}{2\beta}} \int_2^{\infty} \frac{s^{n-1}}{(s-1)^{np+2\beta p}} ds \lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}}.\end{aligned}$$

□

3.2 Fundamental solution of time-space fractional diffusion equation

In this subsection, we consider the time-space fractional diffusion equation (1.1) of the case: $\alpha \in (0,1)$ and $\beta \in (0,1)$. Let $A_{\alpha,\beta}$ denote the fundamental solution of the equation (1.1). Applying the Fourier transform to (1.1) with respect to the space variable x, we obtain

$$\partial_t^\alpha \tilde{u}(t,\xi) = -\zeta_\beta(\xi)\tilde{u}(t,\xi), \qquad t>0,\ \xi \in \mathbb{R}^n. \tag{3.9}$$

The solution of the equation (3.9) with the condition $\tilde{u}(0,\xi) = 1$ is of the form

$$\tilde{u}(t,\xi) = E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha).$$

By the Laplace transform, we obtain

$$\int_0^\infty E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)e^{-st}dt = \frac{s^{\alpha-1}}{s^\alpha + \zeta_\beta(\xi)}. \tag{3.10}$$

Now we define the function $\phi(t,\tau)$ by

$$\phi(t,\tau) = J_t^{1-\alpha}\theta_\alpha(t,\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\theta_\alpha(s,\tau)ds, \qquad t,\tau>0. \tag{3.11}$$

Then

$$\int_0^\infty \phi(t,\tau)e^{-ts}dt = s^{\alpha-1}e^{-\tau s^\alpha}, \qquad s,\tau>0$$

and

$$\lim_{t\to 0}\phi(t,\tau) = \lim_{s\to\infty} s^\alpha e^{-\tau s^\alpha} = 0, \qquad \tau>0.$$

From (4.26) in [18], we obtain

$$\phi(t,\tau) = \frac{1}{t^\alpha} M_\alpha\Big(\frac{\tau}{t^\alpha}\Big) = \frac{1}{t^\alpha} W_{-\alpha,1-\alpha}\Big(-\frac{\tau}{t^\alpha}\Big), \qquad t,\tau>0. \tag{3.12}$$

By the asymptotic behavior of M_α, we have

$$\lim_{\tau\to 0}\phi(t,\tau)=\lim_{\tau\to\infty}\phi(t,\tau)=0,\qquad t>0.$$

Also,

$$\begin{aligned}\int_0^\infty\int_0^\infty \phi(t,\tau)e^{-\zeta_\beta(\xi)\tau}d\tau e^{-st}dt&=\int_0^\infty\int_0^\infty \phi(t,\tau)e^{-st}dte^{-\zeta_\beta(\xi)\tau}d\tau\\&=\int_0^\infty s^{\alpha-1}e^{-\tau s^\alpha}e^{-\zeta_\beta(\xi)\tau}d\tau=\frac{s^{\alpha-1}}{s^\alpha+\zeta_\beta(\xi)}.\end{aligned}\tag{3.13}$$

It follows from the uniqueness of the Laplace transform and (3.10) that

$$\tilde u(t,\xi)=E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)=\int_0^\infty\phi(t,\tau)e^{-\zeta_\beta(\xi)\tau}d\tau,\qquad t>0,\ \xi\in\mathbb{R}^n.\tag{3.14}$$

From (3.7) and (3.12), we deduce

$$\begin{aligned}u(t,x)&=\frac{1}{(2\pi)^n}\int_0^\infty\phi(t,\tau)\int_{\mathbb{R}^n}e^{-\zeta_\beta(\xi)\tau}\cos(\xi x)d\xi d\tau=\int_0^\infty\phi(t,\tau)A_{1,\beta}(\tau,x)d\tau\\&=\frac{1}{t^\alpha}\int_0^\infty M_\alpha\Big(\frac{\tau}{t^\alpha}\Big)A_{1,\beta}(\tau,x)d\tau=\int_0^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds,\qquad t>0,\ x\in\mathbb{R}^n.\end{aligned}$$

Therefore the fundamental solution of the equation (1.1) is represented by

$$A_{\alpha,\beta}(t,x)=\int_0^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds,\qquad t>0,\ x\in\mathbb{R}^n.\tag{3.15}$$

It follows from $A_{1,\beta}(s,x)\ge 0$ that $A_{\alpha,\beta}(t,x)\ge 0$. By the relation (2.9), we obtain

$$\int_{\mathbb{R}^n}A_{\alpha,\beta}(t,x)dx=\int_0^\infty M_\alpha(s)\int_{\mathbb{R}^n}A_{1,\beta}(st^\alpha,x)dxds=\int_0^\infty M_\alpha(s)ds=1,\qquad t>0.$$

Now we consider the asymptotic behavior of $A_{\alpha,\beta}(t,x)$ when $|x|>t^{\frac{\alpha}{2\beta}}$.

Lemma 3.4. *Let $n\in\mathbb{N},\alpha\in(0,1),\beta\in(0,1),h>0,b\in\mathbb{R}^n,t\in(0,1)$ and $|x|>t^{\frac{\alpha}{2\beta}}$. Then the following relation holds.*

$$A_{\alpha,\beta}(t,x)\lesssim\begin{cases}t^\alpha|x|^{-n-2}, & |x|\le 2,\\ t^\alpha|x|^{-n-2\beta}, & |x|>2.\end{cases}\tag{3.16}$$

Proof. Now we will prove the relation (3.16) when of $\beta\in[\frac12,1)$. If $\beta\in(0,\frac12)$, then the relation can be proved similarly.

First, we consider the case of $|x|\le 2$. It follows from Theorem 3.1 and Remark 3.2 that

$$\begin{aligned}A_{\alpha,\beta}(t,x)=&\int_0^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds+\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}}M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds\\&+\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds\\\le&\int_0^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}M_\alpha(s)st^\alpha\big(|x|-(st^\alpha)^{\frac{1}{2\beta}}\big)^{-n-2}ds+\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}}M_\alpha(s)(st^\alpha)^{-\frac{n}{2}}ds\\&+\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds.\end{aligned}$$

We obtain the following relations:

$$\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}} M_\alpha(s)s^{-\frac{n}{2}}ds = \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}} sM_\alpha(s)s^{-\frac{n}{2}-1}ds \le \left(\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}\right)^{-\frac{n}{2}-1}\int_1^\infty M_\alpha(s)sds,$$

$$\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds \le \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)\left(\frac{1}{t^\alpha}t^\alpha\right)^{-\frac{n}{2\beta}}ds \le \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)ds.$$

By the asymptotic behavior of M_α, the function

$$r\int_r^\infty M_\alpha(s)ds$$

has a maximum value in $[1,\infty)$. Then we have

$$A_{\alpha,\beta}(t,x) \lesssim t^\alpha|x|^{-n-2} + t^{\alpha+\frac{n\alpha}{2}}|x|^{-n\beta-2\beta} + t^\alpha \lesssim t^\alpha|x|^{-n-2}.$$

Next, we consider the case of $|x|>2$. By Theorem 3.1 and Remark 3.2, we have

$$\begin{aligned} A_{\alpha,\beta}(t,x) = &\int_0^{\frac{1}{t^\alpha}} M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds + \int_{\frac{1}{t^\alpha}}^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}} M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds \\ &+ \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x-bt)ds \\ \lesssim &\int_0^{\frac{1}{t^\alpha}} M_\alpha(s)st^\alpha\left(|x|-(st^\alpha)^{\frac{1}{2\beta}}\right)^{-n-2}ds \\ &+ \int_{\frac{1}{t^\alpha}}^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}} M_\alpha(s)(st^\alpha)^{n+2\beta-\frac{n}{2\beta}}\left(|x|-(st^\alpha)^{\frac{1}{2\beta}}\right)^{-n-2\beta}ds \\ &+ \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds. \end{aligned}$$

Since

$$\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds \le \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)\left(\frac{|x|^{2\beta}t^\alpha}{2^{2\beta}t^\alpha}\right)^{-\frac{n}{2\beta}}ds \lesssim \frac{|x|^{-n}t^\alpha}{|x|^{2\beta}}\frac{|x|^{2\beta}}{t^\alpha}\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)ds \lesssim t^\alpha|x|^{-n-2\beta},$$

we have

$$A_{\alpha,\beta}(t,x) \lesssim t^\alpha(|x|-1)^{-n-2} + t^{\alpha(n+2\beta-\frac{n}{2\beta})}|x|^{-n-2\beta} + t^\alpha|x|^{-n-2\beta} \lesssim t^\alpha|x|^{-n-2\beta}. \qquad \square$$

Now we obtain the L^p-decay estimate for the fundamental solution $A_{\alpha,\beta}(t,x)$.

Theorem 3.5. *Let $n\in\mathbb{N}$, $\alpha\in(0,1)$, $\beta\in(0,1)$, $h>0$ and $b\in\mathbb{R}^n$. Then,*

$$\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})}, \qquad t\in(0,1] \tag{3.17}$$

for $p\in[1,\bar{p}(n,1))$. Also,

$$\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}, \qquad t\in(1,\infty) \tag{3.18}$$

for $p\in[1,\bar{p}(n,\beta))$.

Proof. If $p \in [1, \bar{p}(n,1))$, then the integral

$$\int_0^\infty M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds$$

is finite. Then, for $t > 0$ and $p \in [1, \bar{p}(n,1))$, we have

$$\begin{aligned}
\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} &\leq \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)\|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)\|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})} \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds + t^{-\frac{\alpha n}{2}(1-\frac{1}{p})} \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})}.
\end{aligned}$$

If $p \in [1, \bar{p}(n,\beta))$, then the integral

$$\int_0^\infty M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds$$

is finite. Then, for $t > 0$ and $p \in [1, \bar{p}(n,\beta))$, we have

$$\begin{aligned}
\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} &\leq \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)\|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)\|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})} \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})} \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}.
\end{aligned}$$

□

Remark 3.6. Comparing Theorem 3.3 and Theorem 3.5 with Lemma 5.1 in [15], we can see that the equation (1.1) describes the Lévy flight which looks like a Brownian random walk in a small time and behaves like a Lévy process in a large time.

4 Representation formula of solutions

In this section, we establish a representation formula of the fractional diffusion equation (1.1) with the initial condition (1.2).

4.1 Classical solution

In this subsection, we discuss a classical solution of the problem (1.1)–(1.2).

Definition 4.1. We call $u \in C([0,\infty) \times \mathbb{R}^n)$ a classical solution of the Cauchy problem (1.1)–(1.2) if

(P1) $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of x for any $t > 0$,

(P2) for any $x \in \mathbb{R}^n$, $J_t^{1-\alpha}u(t,x)$ is continuously differentiable with respect to $t > 0$,

(P3) $u(t,x)$ satisfies the equation (1.1) for any $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and the initial condition (1.2) for any $x \in \mathbb{R}^n$.

Theorem 4.2. *Let $n \in \mathbb{N}, \alpha = 1, \beta \in (0,1), h > 0$ and $b \in \mathbb{R}^n$. Let $u_0 \in C(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n)$ be a function such that $\tilde{u}_0 \in L^1(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a classical solution represented by*

$$u(t,x) = \int_{\mathbb{R}^n} A_{1,\beta}(t,x-y)u_0(y)dy. \tag{4.1}$$

Proof. First, we prove that the function (4.1) satisfies the condition (P1). Using (3.4), we have

$$\zeta_\beta(\xi)\tilde{u}(t,\xi) = \zeta_\beta(\xi)\tilde{A}_{1,\beta}(t,\xi)\tilde{u}_0(\xi) = \zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\tilde{u}_0(\xi).$$

Then it follows from the condition $\tilde{u}_0 \in L^1(\mathbb{R}^n)$ that $\zeta_\beta(\cdot)\tilde{u}(t,\cdot) \in L^1(\mathbb{R}^n)$ for any $t > 0$. By the Riemann–Lebesgue lemma, $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of x for any $t > 0$.

Next, we show that the function (4.1) satisfies the condition (P2). We have

$$\frac{\partial A_{1,\beta}(t,x)}{\partial t} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\cos(x\xi)d\xi, \tag{4.2}$$

which implies that $\frac{\partial A_{1,\beta}(t,x)}{\partial t}$ is continuous with respect to t and x. Since the function

$$\frac{\partial A_{1,\beta}(t,x)}{\partial t}$$

is a bounded continuous function of x for any $t > 0$ and $u_0 \in L^1(\mathbb{R}^n)$,

$$\frac{\partial A_{1,\beta}(t,x-\cdot)}{\partial t}u_0(\cdot) \in L^1(\mathbb{R}^n),$$

which implies that the function

$$\int_{\mathbb{R}^n} \frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy$$

is a continuous function of x for any $t > 0$. Then, we have

$$\frac{\partial u(t,x)}{\partial t} = \int_{\mathbb{R}^n} \frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy. \tag{4.3}$$

In the last, we deduce that the function (4.1) satisfies the condition (P3). For $(t,x) \in (0,\infty) \times \mathbb{R}^n$, we deduce

$$\mathscr{F}^{-1}(\zeta_\beta(\xi)\tilde{u}(t,\xi))(x) = \mathscr{F}^{-1}\Big(\zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\tilde{u}_0(\xi)\Big)(x) = \int_{\mathbb{R}^n} \frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy = \frac{\partial u(t,x)}{\partial t}.$$

For any $\epsilon > 0$, there exists a $\delta > 0$ such that $|u_0(y) - u_0(x)| < \epsilon$ for $x, y \in \mathbb{R}^n$ satisfying the relation $|x - y| < 2\delta$. By the asymptotic behavior of $A_{1,\beta}(t, x)$, for any $x \in \mathbb{R}^n$ and $t \in (0, \min\{\delta/|b|, \delta^{2\beta}/2^{2\beta}\})$, we have

$$\begin{aligned}
|u(t,x) - u_0(x)| &= \left| \int_{\mathbb{R}^n} A_{1,\beta}(t, x - y - bt)(u_0(y + bt) - u_0(x))dy \right| \\
&\leq \int_{|x-y|<\delta} A_{1,\beta}(t, x - y - bt)|u_0(y + bt) - u_0(x)|dy \\
&\quad + \int_{|x-y|>\delta} A_{1,\beta}(t, x - y - bt)|u_0(y + bt) - u_0(x)|dy \\
&\lesssim \epsilon \int_{|x-y|<\delta} A_{1,\beta}(t, x - y - bt)dy + 2\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y|>\delta} t(|x - y| - t^{\frac{1}{2\beta}})^{-n-2}dy \\
&\lesssim \epsilon + 2t\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_\delta^\infty (r - t^{\frac{1}{2\beta}})^{-n-2} r^{n-1} dr \\
&\lesssim \epsilon + 2t\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_\delta^\infty \left(r - \frac{\delta}{2}\right)^{-n-2} r^{n-1} dr.
\end{aligned}$$

If t is sufficiently small, $|u(t, x) - u_0(x)| < 2\epsilon$ for $x \in \mathbb{R}^n$. Since ϵ is arbitrary, for any $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} |u(t, x) - u_0(x)| = 0. \qquad \square$$

Theorem 4.3. *Let $n \in \mathbb{N}, \alpha \in (0, 1), \beta \in (0, 1), h > 0$ and $b \in \mathbb{R}^n$. Let $u_0 \in C(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n)$ be a function such that $\tilde{u}_0 \in L^1(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a classical solution represented by*

$$u(t, x) = \int_{\mathbb{R}^n} A_{\alpha,\beta}(t, x - y)u_0(y)dy. \tag{4.4}$$

Proof. First, we prove that the function (4.1) satisfies the condition (P1). Using (3.14) and (2.2), we have

$$\zeta_\beta(\xi)\tilde{u}(t, \xi) = \zeta_\beta(\xi)\tilde{A}_{\alpha,\beta}(t, \xi)\tilde{u}_0(\xi) = \zeta_\beta(\xi)E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)\tilde{u}_0(\xi) \lesssim \frac{1}{t^\alpha}\tilde{u}_0(\xi), \quad t > 0,\ \xi \in \mathbb{R}^n.$$

Then it follows from the condition $\tilde{u}_0 \in L^1(\mathbb{R}^n)$ that $\zeta_\beta(\cdot)\tilde{u}(t, \cdot) \in L^1(\mathbb{R}^n)$ for any $t > 0$. By the Riemann–Lebesgue lemma, $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t, \cdot))(x)$ is a continuous function of x for any $t > 0$.

Next, we show that the function (4.1) satisfies the condition (P2). From the relation

$$\int_0^\infty J_t^{1-\alpha}\phi(t, \tau)e^{-st}dt = s^{2\alpha-2}e^{-\tau s^\alpha}, \qquad s, \tau > 0,$$

we obtain

$$\lim_{t\to 0} J_t^{1-\alpha}\phi(t, \tau) = \lim_{s\to\infty} s^{2\alpha-1}e^{-\tau s^\alpha} = 0, \qquad \tau > 0.$$

From the relations (2.13) and (2.10), we deduce

$$\frac{\partial\theta_\alpha(t, \tau)}{\partial t} = -\frac{1}{t^2}F_\alpha\left(\frac{\tau}{t^\alpha}\right) - \frac{1}{t^{1+\alpha}}F'_\alpha\left(\frac{\tau}{t^\alpha}\right) = -\frac{1}{t^2}F_\alpha\left(\frac{\tau}{t^\alpha}\right) - \frac{1}{t^{1+\alpha}}W_{-\alpha,-\alpha}\left(\frac{\tau}{t^\alpha}\right). \tag{4.5}$$

By the formula (2.3) and the asymptotic behavior of the Wright function given by (1.11.8) in [16], we have

$$\frac{\partial\theta_\alpha(t, \tau)}{\partial t} \to 0, \qquad \tau \to 0 \text{ or } \tau \to \infty. \tag{4.6}$$

Since

$$\int_0^\infty \frac{\partial \theta_\alpha(t,\tau)}{\partial t} e^{-ts} dt = s e^{-\tau s^\alpha}, \qquad s, \tau > 0,$$

we obtain

$$\lim_{t\to 0} \frac{\partial \theta_\alpha(t,\tau)}{\partial t} = \lim_{s\to\infty} s^2 e^{-\tau s^\alpha} = 0, \qquad \tau > 0, \tag{4.7}$$

$$\lim_{t\to \infty} \frac{\partial \theta_\alpha(t,\tau)}{\partial t} = \lim_{s\to 0} s^2 e^{-\tau s^\alpha} = 0, \qquad \tau > 0. \tag{4.8}$$

Then

$$\partial_t^\alpha \phi(t,\tau) = \frac{\partial}{\partial t} J_t^{1-\alpha} \phi(t,\tau) = \frac{\partial}{\partial t} J_t^{2-2\alpha} \theta_\alpha(t,\tau) = J_t^{2-2\alpha} \frac{\partial \theta_\alpha(t,\tau)}{\partial t}, \qquad t, \tau > 0. \tag{4.9}$$

Meanwhile, we have

$$J_t^{1-\alpha} A_{\alpha,\beta}(t,x) = \int_0^\infty J_t^{1-\alpha} \phi(t,\tau) A_{1,\beta}(\tau,x) d\tau = \frac{1}{(2\pi)^n} \int_0^\infty J_t^{1-\alpha} \phi(t,\tau) \int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)\tau} \cos(\xi x) d\xi d\tau.$$

By (4.6) and (4.9), $J_t^{1-\alpha} A_{\alpha,\beta}(t,x)$ is continuously differentiable with respect to $t > 0$. From Theorem 3.1 and (3.3), for $t > 0$ and $x \in \mathbb{R}^n$, we obtain

$$\partial_t^\alpha A_{\alpha,\beta}(t,x) = \int_0^\infty \partial_t^\alpha \phi(t,\tau) A_{1,\beta}(\tau,x) d\tau = \frac{1}{(2\pi)^n} \int_0^\infty \partial_t^\alpha \phi(t,\tau) \int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)\tau} \cos(x\xi) d\xi d\tau. \tag{4.10}$$

Then the function $\partial_t^\alpha A_{\alpha,\beta}(t,x)$ is a continuous function of x for any $t > 0$. It follows from Theorem 3.3 that $\partial_t^\alpha A_{\alpha,\beta}(t,\cdot) \in L^1(\mathbb{R}^n)$ for any $t > 0$. Also, the function

$$\int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy$$

is a continuous function of x for any $t > 0$. Therefore, we have

$$\partial_t^\alpha u(t,x) = \int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy. \tag{4.11}$$

In the last, we deduce that the function (4.4) satisfies the condition (P3). For $t > 0$ and $x \in \mathbb{R}^n$, we deduce

$$\begin{aligned}
\mathscr{F}^{-1}((\zeta_\beta(\xi)\tilde{u}(t,\xi))(x) &= \mathscr{F}^{-1}\big(\zeta_\beta(\xi) E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)\tilde{u}_0(\xi)\big)(x) \\
&= \mathscr{F}^{-1}\big(\partial_t^\alpha (E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha))\tilde{u}_0(\xi)\big)(x) \\
&= \mathscr{F}^{-1}\left(\int_0^\infty \partial_t^\alpha \phi(t,\tau) e^{-\zeta_\beta(\xi)\tau^\alpha} \tilde{u}_0(\xi) d\tau\right)(x) \\
&= \int_0^\infty \partial_t^\alpha \phi(t,\tau) \mathscr{F}^{-1}\big(e^{-\zeta_\beta(\xi)\tau^\alpha}\tilde{u}_0(\xi)\big)(x) d\tau \\
&= \int_0^\infty \partial_t^\alpha \phi(t,\tau) \int_{\mathbb{R}^n} A_{1,\beta}(\tau, x-y) u_0(y) dy d\tau \\
&= \int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy = \partial_t^\alpha u(t,x).
\end{aligned}$$

As in the proof of Theorem 4.2, using the asymptotic behavior of $A_{\alpha,\beta}(t,x)$ obtained in Lemma 3.4, we can prove the initial condition $\lim_{t\to 0} |u(t,x) - u_0(x)| = 0$ for any $x \in \mathbb{R}^n$. □

4.2 Mild solution

In this subsection, we consider a mild solution of the Cauchy problem (1.1)–(1.2). Now we give a rigorous definition of the solution of the equation (1.1)–(1.2).

Definition 4.4. We call u a mild solution to (1.1)–(1.2) if (1.1) holds in $L^2(\mathbb{R}^n)$ and $u(t,\cdot) \in H^2(\mathbb{R}^n)$ for $t>0$ and $u \in C([0,\infty); L^2(\mathbb{R}^n))$, $\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = 0$.

Theorem 4.5. *Let $\alpha \in (0,1], \beta \in (0,1), h > 0, b \in \mathbb{R}^n$ and $u_0 \in L^2(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a unique mild solution u represented by (4.4). Moreover, $u \in C((0,\infty); H^2(\mathbb{R}^n))$ and the following relations hold.*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)}, \qquad t \ge 0, \tag{4.12}$$
$$\|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}(1 + t^{-\alpha}), \qquad t > 0, \tag{4.13}$$
$$\|u(t,\cdot)\|_{H^1(\mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}(1 + t^{-\frac{\alpha}{2}}), \qquad t > 0. \tag{4.14}$$

If $n < 4\beta$, then

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n\alpha}{4}}, & t \in (0,1), \\ t^{-\frac{n\alpha}{4\beta}}, & t \in [1,\infty). \end{cases}$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned} ||\xi|^2\tilde{u}(t,\xi)| &\le |\tilde{u}_0(\xi)||\xi|^2 \big|E_{\alpha,1}(-(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)t^\alpha)\big| \\ &\lesssim |\tilde{u}_0(\xi)||\xi|^2 \frac{1}{1+|\xi|^2 t^\alpha} \lesssim \frac{|\tilde{u}_0(\xi)|}{t^\alpha}, \qquad t > 0, \xi \in \mathbb{R}^n, \\ ||\xi|\tilde{u}(t,\xi)| &\le |\tilde{u}_0(\xi)||\xi| \big|E_{\alpha,1}(-(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)t^\alpha)\big| \\ &\lesssim |\tilde{u}_0(\xi)||\xi| \frac{1}{1+|\xi|^2 t^\alpha} \lesssim \frac{|\tilde{u}_0(\xi)|}{t^{\frac{\alpha}{2}}}, \qquad t > 0, \xi \in \mathbb{R}^n. \end{aligned}$$

By using the Plancherel theorem, for $t > 0$, we deduce

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0 \tilde{A}_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|\tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Also, for $t > 0$, we have

$$\begin{aligned} \|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} &= \|(1 + |\cdot|^2)\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \left(1 + \frac{1}{t^\alpha}\right)\|u_0\|_{L^2(\mathbb{R}^n)}, \\ \|u(t,\cdot)\|_{H^1(\mathbb{R}^n)} &= \|(1 + |\cdot|)\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \left(1 + \frac{1}{t^{\frac{\alpha}{2}}}\right)\|u_0\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

For $t > 0$, we estimate

$$\|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}(t,\cdot) - \tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0(1 - \tilde{A}_{\alpha,\beta}(t,\cdot))\|_{L^2(\mathbb{R}^n)} \le 2\|u_0\|_{L^2(\mathbb{R}^n)}.$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = \lim_{t\to 0} \|\tilde{u}(t,\cdot) - \tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0 \lim_{t\to 0}(1 - \tilde{A}_{\alpha,\beta}(t,\cdot))\|_{L^2(\mathbb{R}^n)} = 0.$$

For $t_1, t_2 > 0$, we deduce

$$\begin{aligned}\|u(t_1,\cdot)-u(t_2,\cdot)\|_{H^2(\mathbb{R}^n)} &= \|(1+|\cdot|^2)(\tilde{u}(t_1,\cdot)-\tilde{u}(t_2,\cdot)\|_{L^2(\mathbb{R}^n)}\\ &= \|(1+|\cdot|^2)\tilde{u}_0(\tilde{A}_{\alpha,\beta}(t_1,\cdot)-\tilde{A}_{\alpha,\beta}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}\\ &\lesssim \frac{1}{t_1^\alpha}\|u_0\|_{L^2(\mathbb{R}^n)}+\frac{1}{t_2^\alpha}\|u_0\|_{L^2(\mathbb{R}^n)}+2\|u_0\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we estimate

$$\begin{aligned}\lim_{t_1\to t_2}\|u(t_1,\cdot)-u(t_2,\cdot)\|_{H^2(\mathbb{R}^n)} &= \lim_{t\to 0}\|(1+|\cdot|^2)(\tilde{u}(t_1,\cdot)-\tilde{u}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}\\ &= \|(1+|\cdot|^2)\tilde{u}_0(\cdot)\lim_{t_1\to t_2}(\tilde{A}_{\alpha,\beta}(t_1,\cdot)-\tilde{A}_{\alpha,\beta}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}=0.\end{aligned}$$

Similarly, we can prove $u \in C((0,\infty); H^1(\mathbb{R}^n))$.

Using the Plancherel theorem, (2.1) and (3.9), for $t > 0$, we have

$$\begin{aligned}&\left\|\frac{\partial^\alpha u(t,\cdot)}{\partial t^\alpha}+(-\triangle)^\beta u(t,x)-b\cdot\nabla u(t,x)-h\triangle u(t,x)\right\|_{L^2(\mathbb{R}^n)}\\ &=\left\|\frac{\partial^\alpha \tilde{u}(t,\cdot)}{\partial t^\alpha}-\tilde{u}_0(\xi)(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)E_{\alpha,1}(-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t^\alpha)\right\|_{L^2(\mathbb{R}^n)}=0.\end{aligned}$$

In the case of $n < 4\beta$, using Young inequality for convolution and Theorem 3.5, we obtain

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)}\|A_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n\alpha}{4}}, & t\in(0,1),\\ t^{-\frac{n\alpha}{4\beta}}, & t\in[1,\infty).\end{cases}$$

□

5 Decay behavior of solutions

In this section, we consider the L^2-decay of solutions of the nonlocal diffusion equation (1.1) with the initial condition (1.2).

Theorem 5.1. *Let $n \in \mathbb{N}$, $\alpha = 1, \beta \in (0,1), h > 0, b \in \mathbb{R}^n$ and $u_0 \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then the mild solution u of the Cauchy problem* (1.1)–(1.2) *satisfies the following relation.*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(0,1],\\ t^{-\frac{n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(1,\infty).\end{cases}$$

Proof. Using Theorem 3.3, Young's inequality for convolution and the Plancherel theorem, for $t > 0$, we obtain

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|A_{1,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(0,1],\\ t^{-\frac{n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(1,\infty).\end{cases}$$

□

Theorem 5.2. *Let $n \in \mathbb{N}$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $h > 0$, $b \in \mathbb{R}^n$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then the mild solution u of the Cauchy problem (1.1)–(1.2) satisfies the following relations.*

If $n \neq 4\beta$, then

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha}{2}\min\{\frac{n}{2},1\}}, & t \in (0,1], \\ t^{-\alpha\min\{\frac{n}{4\beta},1\}}, & t \in (1,\infty). \end{cases}$$

If $n = 4\beta$, then

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha}{2}\min\{\frac{n}{2},1\}}, & t \in (0,1], \\ t^{-\frac{\alpha}{2}\max\{\frac{n}{2},1\}}, & t \in (1,\infty). \end{cases}$$

Proof. Using Theorem 3.5 and Young inequality for convolution, we have

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|A_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & \text{for } n < 4 \text{ and } t \in (0,1], \\ t^{-\frac{\alpha n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & \text{for } n < 4\beta \text{ and } t \in (1,\infty). \end{cases}$$

If $n > 4\beta$, then, from the Plancherel theorem, Lemma 2.2 and the Hardy–Littlewood–Sobolev theorem [9], we deduce

$$\begin{aligned} (2\pi)^n\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &= \|\tilde{u}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\tilde{A}_{\alpha,\beta}(t,\xi)|^2|\tilde{u}_0(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)|^2|\tilde{u}_0(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2} d\xi \\ &\lesssim t^{-2\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^{4\beta}t^{2\alpha}}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-2\beta}\tilde{u}_0(\xi)|^2 d\xi \lesssim t^{-2\alpha}\|(-\triangle)^{-\beta}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-2\alpha}\|u_0\|^2_{L^{\frac{2n}{n+4\beta}}(\mathbb{R}^n)}. \end{aligned}$$

If $n > 2\beta$, then we obtain

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2} d\xi \lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^{2\beta}t^\alpha}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-\beta}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{||\xi|^{-\beta}\tilde{u}_0(\xi)|^2}{(|\xi|^{-\beta}t^{-\frac{\alpha}{2}}+|\xi|^\beta t^{\frac{\alpha}{2}})^2} d\xi \lesssim t^{-\alpha}\|(-\triangle)^{-\frac{\beta}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-\alpha}\|u_0\|^2_{L^{\frac{2n}{n+2\beta}}(\mathbb{R}^n)}. \end{aligned}$$

If $n > 4$, then we have

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2} d\xi \lesssim t^{-2\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^4 t^{2\alpha}}{(1+|\xi|^2 t^\alpha)^2}||\xi|^{-2}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-2\alpha}\|(-\triangle)^{-1}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-2\alpha}\|u_0\|^2_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}. \end{aligned}$$

If $n > 2$, then we estimate

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2} d\xi \lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^2 t^\alpha}{(1+|\xi|^2 t^\alpha)^2}||\xi|^{-1}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{||\xi|^{-1}\tilde{u}_0(\xi)|^2}{(|\xi|^{-1}t^{-\frac{\alpha}{2}}+|\xi|t^{\frac{\alpha}{2}})^2} d\xi \lesssim t^{-\alpha}\|(-\triangle)^{-\frac{1}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-\alpha}\|u_0\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}. \end{aligned}$$

If $n < 4$, then we deduce

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\xi|^2 t^\alpha)^2} d\xi \lesssim \|\tilde{u}_0\|^2_{L^\infty(\mathbb{R}^n)}\int_0^\infty \frac{r^{n-1}}{(1+r^2 t^\alpha)^2} dr \\ &= t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)}\int_0^\infty \frac{w^{n-1}}{(1+w^2)^2} dw. \end{aligned}$$

Combining the previous estimates, we obtain the desired result. □

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