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Stability and Hopf-bifurcation analysis of an unidirectionally coupled system *

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Abstract: In this paper, the stability and existence of periodic solutions for an unidirectionally coupled nonlinear system with delays are investigated by combining the linear stability theory and the embedding technique of asymptotically autonomous semiflows, Hopf bifurcation theory and a continuation theorem of coincidence degree theory, respectively. Some numerical simulations are carried out for illustrating the analytical results.

Keywords: asymptotic stability; Hopf bifurcation; normal form; coincide degree

1 Introduction

It has been shown that coupled nonlinear systems with time delay can exhibit very complex dynamics, such as the appearance of chaotic attractors and chaotic synchronization[1, 2]. Consequently, studies of such systems become very important in order to understand their cooperative dynamics. From recent works [3, 4, 5, 6, 7], the Krasovskii-Lyapunov theory [8] is used to discuss the synchronization in the coupled time-delayed system

$$\begin{aligned} \dot{x}(t) &= -ax(t) + bf(x(t-\tau)), \\ \dot{y}(t) &= -ay(t) + bf(y(t-\tau)) + k(t)[x(t) - y(t)], \end{aligned} \tag{1.1}$$

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where a and b are positive constants, $\tau > 0$ is the time delay, $k(t)$ is the coupling function between the drive and the response system, $f(x)$ is some nonlinear continuous function.

In [7], Senthilkumar et.al. introduced the difference system with the state variable $\Delta = x(t) - y(t)$ for small values of Δ

$$\dot{\Delta} = -[a + k(t)]\Delta + bf'(y(t-\tau))\Delta_\tau, \ \Delta_\tau = \Delta(t-\tau), \tag{1.2}$$

whose coefficients are all time dependent. By use of Krasovskii-Lyapunov theory, they gave the condition

$$a + k(t) > |bf'(y)|, \quad \text{for} \ \ t \geq 0, \ y \in \mathbb{R}$$

under which the zero solution of Eq.(1.2) is stable, which means that the complete synchronization in the coupled time-delayed systems occurs.

The purpose of the present paper is to investigate the coupled system also in view of bifurcation. We choose the coupling function $k(t) = k > 0$, which is the coupling strength. Then Eq.(1.1) becomes

$$\begin{aligned} \dot{x}(t) &= -ax(t) + bf(x(t-\tau)), \\ \dot{y}(t) &= -ay(t) + bf(y(t-\tau)) + k[x(t) - y(t)], \end{aligned} \tag{1.3}$$

and we make the general assumption that

$$f(0) = 0, \quad f'(0) = \varepsilon \neq 0.$$

By use of the results of Wei [12] we get the stability of the zero solution and the existence of Hopf bifurcation when the delay varies for the first equation of (1.3). Then by using the center manifold theory and normal form method introduced by Faria and Magalhães[10, 11], we derive an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. Futhermore we discuss the stability and the existence of periodic solutions of the coupled system (1.3) by using a continuation theorem of coincidence degree theory.

The rest of the present paper is organized as follows: in Section 2, we analyze the stability of the zero solution of the first equation of (1.1) including the special and complex cases under which the corresponding characteristic equation has a simple zero root, and discuss the existence of local Hopf bifurcation. In Section 3, we determine the properties of the bifurcating periodic solution. In Section 4, we discuss the stability of the zero solution of the origin coupled system (1.3). Finally, in Section 5 the existence of periodic solutions for system (1.3) are established by using a continuation theorem of coincidence degree theory, and some numerical simulations are carried out to illustrate the analytic results.

We would like to mention that there are several articles focus on Hopf bifurcation and complexity of dynamics in delayed models, by employing the center manifold theorem and normal forms method, see [21, 22, 23] and the references therein.

2 Stability analysis of the uncoupled system

Notice that the first equation is uncoupled with the other one, therefore we begin with the investigation of the scalar equation

$$\dot{x}(t) = -ax(t) + bf(x(t-\tau)). \tag{2.1}$$

Clearly the origin is the fixed point of the equation and the linearization of Eq.(2.1) around the origin is given by

$$\dot{x}(t) = -ax(t) + b\varepsilon x(t-\tau), \tag{2.2}$$

and the characteristic equation associated with Eq.(2.2) is

$$\Delta(\lambda, \tau) := \lambda + a - b\varepsilon e^{-\lambda\tau} = 0. \tag{2.3}$$

Since Eq.(2.2) and Eq.(2.3) have the same form as that of equations (4) and (5) in [12] except for some constant coefficients, we give the following results without proof.

Theorem 2.1. *For system* (2.1)
(i) *If* $|\varepsilon| < \frac{a}{b}$, *then all the roots of* (2.3) *have negative real parts. Furthermore, the zero solution of* (2.1) *is asymptotically stable for all* $\tau \geq 0$*;*
(ii) *If* $\varepsilon > \frac{a}{b}$, *then Eq.* (2.3) *has at least one positive root, and hence the zero solution of* (2.1) *is unstable for all* $\tau \geq 0$*;*
(iii) *If* $\varepsilon < -\frac{a}{b}$, *then there exists* $\tau_0 < \tau_1 < \cdots < \tau_j < \cdots$, *such that all the roots of* (2.3) *have negative real parts when* $\tau \in [0, \tau_0)$, *and Eq.*(2.3) *has at least a pair of roots with positive real parts when* $\tau > \tau_0$. *Furthermore, the zero solution of* (2.1) *is asymptotically stable when* $\tau \in [0, \tau_0)$, *and unstable when* $\tau > \tau_0$*;*
(iv) *If* $|\varepsilon| > \frac{a}{b}$, *then* (2.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_j, j = 0, 1, 2, \cdots$, *where*

$$\tau_j = \begin{cases} \frac{1}{\omega_0}(\arccos\frac{a}{b\varepsilon} + 2j\pi), & \text{for } \varepsilon < 0, \\ \frac{1}{\omega_0}(2\pi - \arcsin\frac{a}{b\varepsilon} + 2j\pi), & \text{for } \varepsilon > 0, \ \ j = 0, 1, 2, \cdots, \end{cases}$$

and

$$\omega_0 = \sqrt{b^2\varepsilon^2 - a^2}.$$

The theorem above shows that $\varepsilon = \pm\frac{a}{b}$ is a critical value of the stability of the zero solution of Eq.(2.1). A mathematical question is whether the zero solution of Eq.(2.1) is stable at this critical situation.

The following theorem is to describe the stability of the zero solution of Eq.(2.1) when $\varepsilon = \pm\frac{a}{b}$.

Theorem 2.2. (i) *If* $\varepsilon = -\frac{a}{b}$, *then the zero solution of system* (2.1) *is asymptotically stable;*
(ii) *If* $\varepsilon = \frac{a}{b}$, *then system* (2.1) *undergoes a fixed point bifurcation; and if* $f''(0) \neq 0$, *the zero solution of system* (2.1) *is unstable; if* $f''(0) = 0$, *the zero solution of system* (2.1) *is asymptotically stable when* $f'''(0) < 0$, *and unstable when* $f'''(0) > 0$.

Proof. (i) When $\varepsilon = -\frac{a}{b}$, the characteristic equation (2.3) becomes

$$\lambda + a + ae^{-\lambda\tau} = 0 \tag{2.4}$$

Obviously, $\lambda = 0$ is not a root of Eq.(2.4), and $\lambda = -2a < 0$ When $\tau = 0$. Suppose $\lambda = \mathrm{i}\omega$ is a root of Eq.(2.4), then we have

$$\mathrm{i}\omega + a + a\cos\omega\tau - \mathrm{i}a\sin\omega\tau = 0,$$

separating the real and imaginary parts yields

$$a + a\cos\omega\tau = 0,\ \omega - a\sin\omega\tau = 0,$$

which implies that $a^2 + \omega^2 = a^2$. Furthermore, we get $\lambda = 0$, which is a contradiction since we have known that $\lambda = 0$ is not a root of Eq.(2.4). This shows that all roots of Eq.(2.4) have negative real part.

(ii) Clearly, $\lambda = 0$ is a simple root of the characteristic equation (2.3) with $\varepsilon = \frac{a}{b}$, since

$$\frac{\mathrm{d}\Delta(0,\tau)}{\mathrm{d}\lambda} = 1 + a\tau > 0.$$

From theorem 2.1 we know that all roots of characteristic equation (2.3) have negative real parts except $\lambda = 0$ when $\varepsilon = \frac{a}{b}$. In order to study the stability of the zero solution of system (2.1), similar to the method in [13, 14], we employ the center manifold theory and normal form method for FDE introduced by Faria et al. [10, 11].

Let $\Lambda = \{0\}$ and $B = 0$, clearly the non-resonance conditions relative to Λ are satisfied. Therefore there exists a 1-dimensional ODE which governs the dynamics Eq.(2.1) near the origin.

Firstly, we re-scale the time delay by $t \mapsto (t/\tau)$ to normalize the delay so that Eq.(2.1) can be written in the form:

$$\dot{x}(t) = -a\tau x(t) + b\tau f(x(t-1)). \tag{2.5}$$

Clearly, the phase space for Eq.(2.5) is $C := C([-1,0],\mathbb{R})$. For $\varphi \in C$, define

$$L(\varphi) = -a\tau\varphi(0) + a\tau\varphi(-1).$$

and

$$F(\varphi) = b\tau\left[\frac{f''(0)}{2!}\varphi^2(-1) + \frac{f'''(0)}{3!}\varphi^3(-1)\right] + O(\varphi^4(0), \varphi^4(-1)).$$

Then Eq. (2.5) can be rewritten in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = L(x_t) + F(x_t).$$

Choosing

$$\eta(\theta) = \begin{cases} 0, & \theta = -1 \quad or \quad 0 \\ a\tau, \theta \in (-1,0), \end{cases}$$

we obtain

$$L\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\varphi(\theta).$$

Using the formal adjoint theory for FDEs[8], we decompose C by Λ as $C = P \oplus Q$, where $P = \mathrm{span}\Phi(\theta)$ with $\Phi(\theta) = 1$ being the center space for

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = L(x_t).$$

Choosing a basis Ψ for the adjoint space P^* such that $<\Psi, \Phi>= 1$, where $<\cdot,\cdot>$ is the bilinear form on $C^* \times C$ defined by

$$<\psi, \varphi>= \psi(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi(\xi-\theta)\mathrm{d}\eta(\theta)\varphi(\theta)\mathrm{d}\xi.$$

Thus $\Psi(s) = (1 + a\tau)^{-1}$. Taking the enlarged phase space

$$BC = \{\varphi : [-1,0) \mapsto C, \varphi \text{ is continuous on } [-1,0) \text{ and } \lim_{\theta\to 0}\varphi(\theta) \text{ exists}\},$$

we obtain the abstract ODE with the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax_t + X_0F(x_t). \tag{2.6}$$

Here for any $\varphi \in C$,

$$A\varphi = \dot{\varphi}(\theta) + X_0(L_0\varphi - \dot{\varphi}(0)),$$

and X_0 is given by

$$X_0 = \begin{cases} I, \theta = 0 \\ 0, \theta \in [-1, 0). \end{cases}$$

The definition of the continuous projection

$$\pi : BC \mapsto P, \quad \pi(\varphi + X_0\alpha) = \Phi[< \Psi, \varphi > + \Psi(0)\alpha]$$

allows us to decompose the enlarged space by Λ as $BC = C \oplus \mathrm{Ker}\pi$. Since π commutes with A in C^1, and using the decomposition

$$x_t = \Phi x(t) + y, x(t) \in \mathbb{C}, y = y(\theta) \in Q^1,$$

the abstract ODE (2.6) is therefore decomposed as the system

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)F(\Phi x + y), \\ \dot{y} &= A_{Q^1} + (I - \pi)X_0F(\Phi x + y). \end{aligned} \tag{2.7}$$

Since

$$\Psi(0)F(\Phi x + y) = \frac{b\tau}{1 + a\tau}\left[\frac{f''(0)}{2!}(x + y(-1))^2 + \frac{f'''(0)}{3!}(x + y(-1))^3\right] + O(4),$$

therefore the local invariant manifold of system (2.1) tangent to P at the origin satisfying $y(\theta) = 0$ and the flow on this manifold is given by the following 1-dimensional ODE:

$$\dot{x}(t) = \frac{b\tau}{1 + a\tau}\left[\frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3\right] + O(4). \tag{2.8}$$

Since a, b and τ are all positive, the zero solution of ODE (2.8) is unstable when $f''(0) \neq 0$; and if $f''(0) = 0$, the zero solution of ODE (2.8) is asymptotically stable when $f'''(0) < 0$, and unstable when $f'''(0) > 0$, and so is the zero solution of system (2.1). The proof is completed. □

3 Hopf bifurcation analysis

In section 2, we obtain the conditions under which Eq. (2.1) undergoes a Hopf bifurcation at some critical values of τ. In this section we shall study the direction and stability of the bifurcating periodic solutions. The method we use here is based on the normal form method and center manifold theory introduced by Faria et al. [10, 11].

For convenience, let $\tau = \tau_j + \mu$, $\mu \in \mathbb{R}$, then $\mu = 0$ is the bifurcation value of Eq. (2.1). Defining

$$\begin{aligned} L_\mu(\varphi) &= -a(\tau_j + \mu)\varphi(0) + b\varepsilon(\tau_j + \mu)\varphi(-1), \\ F_\mu(\varphi) &= b(\tau_j + \mu)\left[\frac{f''(0)}{2!}\varphi^2(-1) + \frac{f'''(0)}{3!}\varphi^3(-1)\right] + O(\varphi^4(0), \varphi^4(-1)). \end{aligned}$$

Then we can rewrite Eq.(2.1) in the following form:

$$\dot{x}(t) = L_\mu(x_t) + F_\mu(x_t). \tag{3.1}$$

Using the normal form method introduced by Farai directly, we get

$$\begin{aligned} L_0(\varphi(\theta)) &= -a\tau_j\varphi(0) + b\varepsilon\tau_j\varphi(-1), \\ L_0(\theta e^{i\omega_0\theta}) &= -b\varepsilon\tau_j e^{-i\omega_0} = -(a\tau_j + \mathrm{i}\tau_j\omega_0), \\ L_0(1) &= -a\tau_j + b\varepsilon\tau_j, \\ L_0(e^{2i\omega_0\theta}) &= -a\tau_j + b\varepsilon\tau_j e^{-2i\omega_0} = -a\tau_j + \frac{(a\tau_j + \mathrm{i}\tau_j\omega_0)^2}{b\varepsilon\tau_j}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & F(x_1 e^{i\omega_0\theta} + x_2 e^{-i\omega_0\theta} + x_3 1 + x_4 e^{2i\omega_0\theta}, 0) \\ &= \frac{b\tau_j}{2!} f''(0)(x_1 e^{-i\omega_0} + x_2 e^{i\omega_0} + x_3 + x_4 e^{-2i\omega_0})^2 \\ &\quad + \frac{b\tau_j}{3!} f'''(0)(x_1 e^{-i\omega_0} + x_2 e^{i\omega_0} + x_3 + x_4 e^{-2i\omega_0})^3 + O(4) \\ &= B_{(2,0,0,0)}x_1^2 + B_{(1,1,0,0)}x_1x_2 + B_{(1,0,1,0)}x_1x_3 + B_{(0,1,0,1)}x_2x_4 \\ &\quad + B_{(2,1,0,0)}x_1^2x_2 + \cdots . \end{aligned}$$

Comparing the coefficients, we have

$$\begin{aligned} B_{(2,0,0,0)} &= \frac{b\tau_j}{2} f''(0)e^{-2i\omega_0} = \frac{1}{2b\tau_j\varepsilon^2} f''(0)(a\tau_j + \mathrm{i}\tau_j\omega_0)^2, \\ B_{(1,1,0,0)} &= b\tau_j f''(0), \\ B_{(1,0,1,0)} &= b\tau_j f''(0)e^{-i\omega_0} = \frac{1}{\varepsilon} f''(0)(a\tau_j + \mathrm{i}\tau_j\omega_0), \\ B_{(0,1,0,1)} &= b\tau_j f''(0)e^{-i\omega_0} = \frac{1}{\varepsilon} f''(0)(a\tau_j + \mathrm{i}\tau_j\omega_0), \\ B_{(2,1,0,0)} &= \frac{b\tau_j}{2} f'''(0)e^{-i\omega_0} = \frac{1}{2\varepsilon} f'''(0)(a\tau_j + \mathrm{i}\tau_j\omega_0). \end{aligned} \tag{3.3}$$

By Faria's normal form theory, the flow of Eq.(3.1) on the center manifold of the origin is given in polar coordinates (ρ, ξ) by equation

$$\begin{cases} \dot{\rho} = \mu\alpha'(0)\rho + K\rho^3 + O(\mu^2\rho + |(\rho,\mu)|^4), \\ \dot{\xi} = -\omega + O(|(\rho,\mu)|), \end{cases}$$

where

$$K = Re\left[\frac{1}{1 - L_0(\theta e^{i\omega\theta})}\left(B_{(2,1,0,0)} - \frac{B_{(1,1,0,0)}B_{(1,0,1,0)}}{L_0(1)} + \frac{B_{(2,0,0,0)}B_{(0,1,0,1)}}{2i\omega - L_0(e^{2i\omega\theta})}\right)\right]. \tag{3.4}$$

Substituting (3.2) and (3.3) into (3.4), and since we have that $\alpha'(0) > 0$, we obtain the following theorem.

Theorem 3.1. *If $K < 0$ (respectively, $K > 0$), there exists a unique nontrivial periodic orbit in the neighborhood of $\rho = 0$ for $\mu > 0$ (respectively, $\mu < 0$) and no nontrivial periodic orbits for $\mu < 0$ (respectively, $\mu > 0$); and the bifurcating periodic solution on the center manifold is orbitally asymptotically stable (respectively, unstable). Particularly, the stability of the bifurcating periodic solutions of (2.1) and that on the center manifold are coincident at the first critical value τ_0.*

Corollary 3.2. *If $f''(0) = 0$, then the nontrivial periodic orbit exists in the neighborhood of $\rho = 0$ for $\mu > 0$ ($\mu < 0$) and the bifurcating periodic solutions on the center manifold are orbitally asymptotically stable (unstable) when $\varepsilon f'''(0) < 0(> 0)$.*

In fact, from (3.3) it follows that

$$B_{(1,1,0,0)} = B_{(1,0,1,0)} = B_{(2,0,0,0)} = B_{(0,1,0,1)} = 0,$$

and

$$B_{(2,1,0,0)} = \frac{1}{2\varepsilon} f'''(0)(a\tau_j + \mathrm{i}\tau_j\omega_0).$$

Substituting the coefficients above into (3.4), it follows that

$$K = \frac{f'''(0)}{2\varepsilon}\frac{a\tau_j + a^2\tau_j^2 + \tau_j^2\omega_0^2}{(1 + a\tau_j)^2 + \tau_j^2\omega_0^2}.$$

And hence from (3.4) and theorem 3.1, the conclusion is reached.

4 Stability analysis of the coupled system

So far, we have investigated the stability and Hopf bifurcation for the first equation of system (1.3). In the following we will focus on the stability of the zero solution of the coupled system. The theory we use here is from[15, 16], and its notions are discussed below.

Lemma 4.1. *([16]) Let e be a locally asymptotically stable equilibrium of Θ and $W_s(e) = \{x \in X : \Theta(t, x) \to e, t \to \infty\}$ its basin of attraction(or stable set). Then every pre-compact Φ orbit whose $\omega-\Phi-$limit set intersects $W_s(e)$ converges to e.*

To make use of the lemma above, similar to [17], we introduce some notations first.

Define $C := C([-\tau, 0], \mathbb{R})$, and

$$d(x, y) = \max_{\theta \in [-\tau, 0]} |x(\theta) - y(\theta)|,$$

for any $x, y \in C$. Suppose Ω is an open subset of C, $F \in C(\Omega, \mathbb{R})$, and F is Lipschitz in each compact set in Ω. Consider the following equation

$$\begin{aligned} &\dot{x}(t) = F(x_t) + G(t), \quad t \geq \sigma, \\ &x_\sigma = \phi, \end{aligned} \tag{4.1}$$

with $(\sigma, \phi) \in \mathbb{R} \times C$.

E.q.(4.1) is a 1-dimensional non-autonomous retarded differential equation and has a unique solution through any given initial function. Let $x(s, \phi)(t)$ be the unique solution through (σ, ϕ) and well-defined for all $t \in [\sigma - \tau, \sigma + \alpha]$. Here we assume further that the maximal existence interval of the solutions is $[\sigma - \tau, \infty)$. From [8], we know that $x(s, \phi)(t)$ is continuous in σ, ϕ, t for $\sigma \in \mathbb{R}, \phi \in C$ and $t \in [\sigma - \tau, \infty)$.

Consider a mapping $\Phi : \Delta \times C \to C$, which is defined as $\Phi(t, s, \phi) = x_t(s, \phi) \in C$, with $x_t(s, \phi)(\theta) = x(x, \phi)(t + \theta)$. It can be verified that Φ is continuous non-autonomous semiflow on $\Delta \times C$ by the existence and uniqueness of solutions.

Further we consider the corresponding autonomous equation

$$\begin{aligned} \dot{x}(t) &= F(x_t), \\ x_0 &= \psi, \end{aligned} \tag{4.2}$$

for $\psi \in C$. Under the same assumptions, let $y(\psi)(t)$ be the unique solution through $(0, \psi)$, and define $\Theta : [0, \infty) \times C \to C$ as $\Theta(t, \psi) = y(\psi)$, with $y_t(\psi)(\theta) = y(\psi)(t + \theta)$. Similarly we can verify that Θ is a continuous autonomous semiflow. Before stating the main theorem, we give the following property of Φ and Ψ defined above.

Lemma 4.2. *[17] If $G(t) \to 0, t \to \infty$, then Φ is the asymptotically autonomous -with limit-semiflow Θ.*

Theorem 4.3. *If $-\frac{a}{b} \leq \varepsilon < \frac{a}{b}$, then the zero solution of the coupled system* (1.3) *is asymptotically stable.*

Proof. Consider the second equation of (1.3) as follows

$$\begin{aligned} \dot{y}(t) &= -(a+k)y(t) + bf(y(t-\tau)) + kx(t) \\ &= F(y_t) + G(t) \end{aligned} \tag{4.3}$$

and the corresponding autonomous equation

$$\begin{aligned} \dot{y}(t) &= -(a+k)y(t) + bf(y(t-\tau)) \\ &= F(y_t) \end{aligned} \tag{4.4}$$

From theorem 2.1,we know that the zero solution of (2.1) is asymptotically stable, which implies that $G(t) \to 0$ as $t \to \infty$. Therefore Φ defined by (4.3) is asymptotically autonomous with limit-semiflow Θ defined by (4.4), where lemma 4.2 is used. Next we begin to investigate the asymptotic behavior of the autonomous system (4.4). Similar to theorem 2.1, it can be proved that the zero solution of (4.4) is asymptotically stable when $-\frac{a+k}{b} \leq \varepsilon < \frac{a+k}{b}$. Furthermore, let $e = 0$ be the stable equilibrium of Θ on C and then the intersection of C and e's basin of attraction is nonempty. On the other hand, it is known that every Φ-orbit is pre-compact by Ascoli-Arzela theorem. Therefore, lemma 4.1 implies that the zero solution of Eq.(4.3) is asymptotically stable. Notice that $[-\frac{a}{b}, \frac{a}{b}) \subset [-\frac{a+k}{b}, \frac{a+k}{b})$, this completes the proof. $\square$

5 Existence of periodic solutions in the coupled system

Results in section 3 show that under certain conditions, there exist nonconstant periodic solutions to (2.1) due to Hopf bifurcation when τ lies in some neighborhood of each bifurcation value. In the following we assume that these conditions ensuring the appearance of Hopf bifurcation are met. For convenience, we denote by D the region where τ lies and bifurcating periodic solutions for (2.1) exist. Our purpose is to obtain sufficient conditions for the existence of the periodic solutions to the original coupled system

$$\begin{aligned} \dot{x}(t) &= -ax(t) + bf(x(t-\tau)), \\ \dot{y}(t) &= -ay(t) + bf(y(t-\tau)) + k[x(t) - y(t)], \end{aligned} \tag{5.1}$$

by employing the coincide degree theory from Gaines and Mawhin[18].

To make use of the continuation theorem of coincidence degree theory, similar to [17, 19, 20], we need to introduce following notations.

Let X, Y be real Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ be a Fredholm mapping of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$ and $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Denote by L_P the restriction of L to $\mathrm{Dom}L \cap \mathrm{Ker}P$. Denote by $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ the inverse of L_P, and denote by $J : \mathrm{Im}Q \to \mathrm{Ker}L$ an isomorphism of $\mathrm{Im}Q$ onto $\mathrm{Ker}L$.

For convenience, we also cite below the continuation theorem.

Lemma 5.1. *[18] Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is $L-$compact on $\overline{\Omega}$(i.e.,$QN : \overline{\Omega} \to Y$ and $K_P(I-Q)N : \overline{\Omega} \to Y$ are compact). Assume*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;*

(ii) *for each $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$, and* $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap \mathrm{Dom}l$.

To use the continuation theorem of coincidence degree theory, we take $X = Y = \{y(t) \in C(\mathbb{R}, \mathbb{R}) : y(t+\omega) = y(t)\}$. With the maximal norm $|\cdot|_0$, X and Y are Banach spaces. Set

$$L : \mathrm{Dom}L \cap X, \quad Ly = \dot{y},$$

where $\mathrm{Dom}L = \{y(t) \in C^1(\mathbb{R}, \mathbb{R})\}$. Define two projectors P and Q as

$$Py = Qy = \frac{1}{\omega}\int_0^\omega y(t)dt, \quad y \in X.$$

Clearly, $\mathrm{Ker}L = \mathbb{R}$, $\mathrm{Im}L = \{y \in X : \int_0^\omega y(t)dt = 0\}$ is closed in X and $\mathrm{dimKer}L = \mathrm{Im}L = 1$. Hence, L is a Fredholm mapping of index 0. Furthermore, through an easy computation, we find that the inverse K_P of L_P has the form

$$K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P,$$
$$K_P(y) = \int_0^t y(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^u y(s)dsdu, \quad t \in [0, \omega].$$

We are now in a position to state and prove our main result.

Theorem 5.2. *If $|\varepsilon| > \frac{a}{b}$, then system* (1.3) *has at least one periodic solution for $\tau \in D$.*

Proof. Consider the second equation of (5.1)

$$\dot{y}(t) = -(a+k)y(t) + bf(y(t-\tau)) + kx(t). \tag{5.2}$$

Then $x(t)$ is a ω-periodic solution of (2.1) when $\tau \in D$. To complete the proof, it suffices to show that (5.2) has an ω-periodic solution. Define $N : X \to Y$ as

$$Ny = -(a+k)y(t) + bf(y(t-\tau)) + kx(t).$$

Notice that $QN : X \to X$ takes the form

$$QN(y) = \frac{1}{\omega}\int_0^\omega [-(a+k)y(t) + bf(y(t-\tau)) + kx(t)]dt.$$

By some computation, we can show that $K_P(I-Q)N : X \to Y$ takes the form

$$\begin{aligned}
K_P(I-Q)N(y) = &\int_0^t [-(a+k)y(s) + bf(y(s-\tau)) + kx(s)]ds \\
&-\frac{1}{\omega}\int_0^\omega \int_0^u [-(a+k)y(s) + bf(y(s-\tau)) + kx(s)]dsdu \\
&+(\frac{1}{2} - \frac{t}{\omega})\int_0^\omega [-(a+k)y(t) + bf(y(t-\tau)) + kx(t)]dt.
\end{aligned}$$

The integration form of the terms of both QN and $K_P(I-Q)N$ imply that they are continuously differentiable with respect to t and that they map bounded continuous functions to bounded continuous functions. By the Ascoli-Arzela theorem, we see that $QN(\overline{\Omega}), K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Therefore, N is $L-$compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. Corresponding to the operator equation $Ly = \lambda N_y, \lambda \in (0,1)$, we have

$$\dot{y}(t) = \lambda[-(a+k)y(t) + bf(y(t-\tau)) + kx(t)]. \tag{5.3}$$

Suppose that $y(t) \in X$ is a solution of (5.3) for some $\lambda \in (0,1)$. On one hand, we choose $t_M \in [0,\omega]$ and $t_m \in [0,\omega]$ such that

$$y(t_M) = \max_{t\in[0,\omega]} y(t), \quad y(t_m) = \min_{t\in[0,\omega]} y(t).$$

Then it is clear that $\dot{y}(t_M) = \dot{y}(t_m) = 0$. From this and (5.3), we obtain

$$\begin{aligned}
y(t) \geq y(t_m) &= \frac{1}{a+k}[bf(y(t_m-\tau)) + kx(t_m)], \\
y(t) \leq y(t_M) &= \frac{1}{a+k}[bf(y(t_M-\tau)) + kx(t_M)],
\end{aligned}$$

which implies that

$$\frac{-1}{a+k}(bM + k|x|_0) \leq y(t) \leq \frac{1}{a+k}(bM + k|x|_0),$$

where $M > 0$ satisfies $|f(u)| \leq M, \forall u \in X$ since f is continuous. On the other hand, suppose $y \in \mathrm{Ker}L$ satisfies $QNy = 0$, and notice that $\mathrm{Ker}L = \mathbb{R}$, which means that $y(t) = y$ is a constant, we obtain

$$\begin{aligned} QNy &= \frac{1}{\omega}\int_0^{\omega}[-(a+k)y + bf(y) + kx(t)]dt \\ &= -(a+k)y + bf(y) + \frac{1}{\omega}\int_0^{\omega} kx(t)dt = 0. \end{aligned}$$

This implies that

$$|y| = \left|\frac{1}{a+k}\left[bf(y) + \frac{1}{\omega}\int_0^{\omega} kx(t)dt\right]\right| \leq \frac{1}{a+k}(bM + k|x|_0).$$

Taking $A = \frac{1}{a+k}(bM + k|x|_0 + 1)$ and $\Omega = \{y(t) \in X : |y|_0 < A\}$, then it is clear that Ω satisfies condition (i) and the first part of condition (ii) in Lemma 5.1.

Furthermore, take $J = I : \mathrm{Im}Q \to \mathrm{Ker}L,\ x \mapsto x$ and by a straightforward computation, we see that

$$\deg[JQN, \mathrm{Ker}L \cap \Omega, 0] \neq 0.$$

The conclusion now follows from Lemma (5.1). This completes the proof. □

6 An example

Taking $f(x) = -\sin x$ as an example, then Eq.(2.1) has the form:

$$\dot{x}(t) = -ax(t) - b\sin(x(t-\tau)). \tag{6.1}$$

Obviously $x = 0$ is the fixed point of (6.1), and

$$f(0) = 0,\ \varepsilon := f'(0) = -1,\ f''(0) = 0,\ f'''(0) = 1.$$

Basing on the discussion above, we get the following conclusions directly.

Theorem 6.1. *For Eq.* (6.1)
(i) *If* $0 < b < a$, *then the zero solution of* (6.1) *is asymptotically stable for all* $\tau \geq 0$;

(ii) *If* $b > a$, *then there exists* $\tau_0 < \tau_1 < \cdots < \tau_j < \cdots$ *such that the zero solution of* (6.1) *is stable when* $\tau \in [0, \tau_0)$, *and unstable for* $\tau > \tau_0$, *and Eq.*(6.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_j, j = 0, 1, 2, \cdots$, *where* τ_j *is defined as*

$$\tau_j = \frac{1}{\omega_0}\left(\arccos(-\frac{a}{b}) + 2j\pi\right), j = 0, 1, 2 \cdots .$$

and

$$\omega_0 = \sqrt{b^2 - a^2}.$$

And the bifurcating periodic solution exists for $\tau > \tau_j$, *and the bifurcating periodic solutions on the center manifold are stable. Particularly, the bifurcating periodic solutions from the first bifurcation vale* τ_0 *are asymptotically stable.*

If we take $a = 1,\ b = 3,\ k = 1$, then (5.1) becomes

$$\begin{aligned} \dot{x}(t) &= -x(t) - 3\sin(x(t-\tau)), \\ \dot{y}(t) &= -y(t) - 3\sin(y(t-\tau)) + [x(t) - y(t)]. \end{aligned} \tag{6.2}$$

Clearly, $f(0) = f''(0) = 0,\ \varepsilon := f'(0) = -1,\ f'''(0) = 1, \omega_0 = \sqrt{b^2 - a^2} = 2.828,\ \tau_0 = 0.6756$. Then by the discussion above, the zero solution of (6.2) is asymptotically stable when $\tau \in [0, 0.6756)$, and has periodic solutions in the right neighborhood of τ_j.

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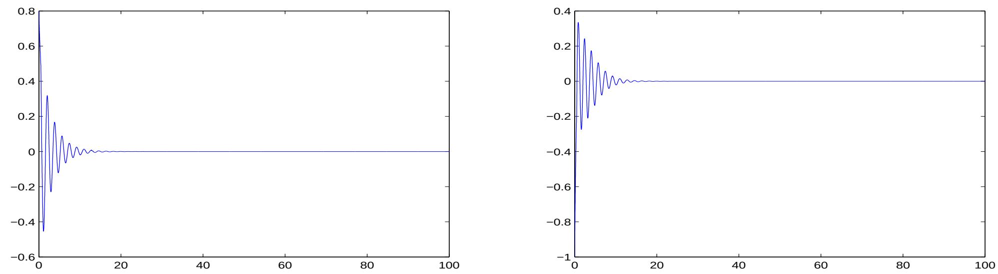

Figure 1: Numerical simulations for system (6.2) with $\tau = 0.5 \in [0, 0.6756)$ shows that the origin is asymptotically stable.

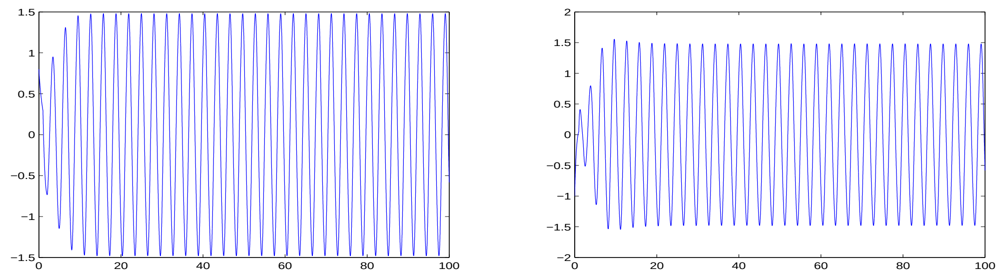

Figure 2: Wave plot for (6.2) with $\tau = 0.7 > \tau_0$ shows that the bifurcating periodic solutions are stable.

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