Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **12**, 1–19; https://doi.org/10.14232/ejqtde.2022.1.12 www.math.u-szeged.hu/ejqtde/

Global smooth linearization of nonautonomous contractions on Banach spaces

Davor Dragičević[✉]

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, Rijeka, 51 000, Croatia

Received 25 August 2021, appeared 6 March 2022
Communicated by Mihály Pituk

Abstract. The main purpose of this paper is to establish a global smooth linearization result for two classes of nonautonomous dynamics with discrete time. More precisely, we consider a nonlinear and nonautonomous dynamics given by a two-sided sequence of maps as well as variational systems whose linear part is contractive, and under suitable assumptions we construct C^1 conjugacies between the original dynamics and its linear part. We stress that our dynamics acts on a arbitrary Banach space. Our arguments rely on related results dealing with autonomous dynamics.

Keywords: nonautonomous contractions, linearization.

2020 Mathematics Subject Classification: 34A34, 34D20,34D08.

1 Introduction

One of the basic strategies when analysing a complex nonlinear system near its equilibrium is to linearize it, i.e. to study its linear part. Such a procedure is natural since linear systems are much easier to study. However, this strategy is useful only if one knows that a system and the associated linear part have similar behaviour near the equilibrium since only in that case we can conclude something meaningful about the original system by studying its linearization.

Many works have been devoted to the problem of formulating conditions under which the system and its linear part are C^r-conjugated (or equivalent). The first contributions deal with complex dynamics. Indeed, Poincaré [23] proved that an analytic diffeomorphism can be analytically conjugated to its linear part near a fixed point if all eigenvalues of the linear part lie inside the unit circle S^1 (or outside S^1) and satisfy the nonresonant condition. Later, Siegel [32], Brjuno [6] and Yoccoz [37] made contributions to the case of eigenvalues on S^1, in which the small divisor problem is involved.

In the context of real dynamics, the most important result is the famous Hartman–Grobman Theorem [17], which asserts that a C^1-diffeomorphism on $\mathbb{R}^n$ can be C^0-linearized near the hyperbolic fixed point. Later this result was generalized (with simplified proofs) to Banach spaces independently by Palis [21] and Pugh [24].

✉ Email: ddragicevic@math.uniri.hr

It is well-known that in general the conjugacy in the Grobman–Hartman theorem is only locally Hölder continuous and that it may fail to be locally Lipschitz even for C^∞ dynamic (see [3] and references therein). However, C^0-linearization is often not sufficient since for example it can fail to distinguish a node from a focus as pointed out by van Strien [35].

Sternberg [33, 34] proved that C^k ($k \geq 1$) diffeomorphisms can be C^r linearized near the hyperbolic fixed points, where the integer $r > k$ depends on k and the nonresonant conditions. Hence, in order to obtain C^r-linearization, we need to require that our dynamics exhibits higher regularity. Later Belitskii [4, 5] gave conditions for C^k linearization of $C^{k,1}$ ($k \geq 1$) diffeomorphisms under appropriate nonresonant conditions. His results was partially generalized to infinite-dimensional setting in [16, 27, 40].

The C^1 linearization in the case when the linear part is a contraction (its spectrum is contained in the unit circle) was discussed in [15, 26, 27] in the infinite-dimensional setting as well as in [38, 39] in the finite-dimensional setting. We in particularly mention the recent paper by H. M. Rodrigues and J. Solà-Morales [29], in which the global C^1-linearization result for contractions on Banach spaces has been established.

We emphasize that all the above mentioned results deal with *autonomous* dynamics. The first contributions dealing with the linearization of nonautonomous dynamics with continuous time are due to Palmer [22] and to Aulbach and Wanner [2] for dynamics with discrete time. For more recent results we refer to the works of Jiang [18] and Lopez-Fenner and Pinto [19]. The first results related to C^r-linearization ($r \geq 1$) in the framework of nonautonomous dynamics were obtained only quite recently. More precisely, a Sternberg type theorem for nonautonomous dynamics with continuous time was established in [10]. The C^1-linearization for nonautonomous dynamics with discrete time was discussed in [13] (see also [14] for related results for continuous time). Moreover, results related to differentiable linearization of nonautonomous contractions were obtained in interesting papers [7, 8].

The main purpose of this paper is to formulate new conditions for C^1-linearization of nonautonomous contractions on an arbitrary Banach space. Our strategy follows very closely the arguments developed in [13] and consist of passing from nonautonomous to the associated autonomous dynamics acting on a larger space. Then, for the autonomous dynamics we apply results from [29] and after that we return back to the framework of our original nonautonomous dynamics. However, we emphasize that the results from [13] don't imply the results in the present paper. Indeed, the conditions for the linear part that ensure C^1-linearization are given in terms of the spectrum of the associated Mather operator which are difficult to verify in practice, while in the present paper the conditions are given directly in terms of the constants in the notion of an exponential contraction (we refer to Remark 2.15 for a detailed explanation). Furthermore, our results differ from those in [7, 8]. Indeed, besides considering discrete (and not continuous) dynamics on an arbitrary Banach space we also don't require the boundedness for the nonlinearities. Furthermore, we use completely different techniques from those developed in [7, 8].

Following similar ideas (but with substantial changes), we also discuss C^1-linearization of variational contractive dynamical systems with discrete time. We refer to [9, 30, 31] and references therein for a detailed explanation of the relevance of variational systems in the investigation of qualitative properties of nonautonomous dynamics (and to [1] for the exposition of the theory of closely related random dynamical systems).

We note that the main results of the present paper can be viewed as a generalization of Hartman's work [17] to nonautonomous contractions acting on Banach spaces.

2 Nonuniform exponential contractions

Throughout this paper, $X = (X, \|\cdot\|)$ will be an arbitrary Banach space and $B(X)$ will denote the space of all bounded linear operators on X. For a sequence $(A_n)_{n\in\mathbb{Z}} \subset B(X)$ of invertible operators, we set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n; \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{for } m < n. \end{cases}$$

Let us introduce a notion of a nonuniform exponential contraction.

Definition 2.1. We say that $(A_n)_{n\in\mathbb{Z}}$ admits a *nonuniform exponential contraction* if there exist $0 < \lambda \le \mu$ and a map $\mathcal{D}\colon \mathbb{Z} \to (0,\infty)$ such that

$$\|\mathcal{A}(m,n)\| \le \mathcal{D}(n)e^{-\lambda(m-n)}, \quad \text{for } m \ge n, \tag{2.1}$$

and

$$\|\mathcal{A}(m,n)\| \le \mathcal{D}(n)e^{\mu(n-m)}, \quad \text{for } m \le n. \tag{2.2}$$

The following example is taken from [11].

Example 2.2. Let $X = \mathbb{R}$ and consider a sequence $(A_n)_{n\in\mathbb{Z}}$ given by

$$A_n = e^{\omega+\epsilon[(-1)^n-\frac{1}{2}]} \quad n \in \mathbb{Z},$$

where $\omega < 0$ and $\epsilon \ge 0$ are some fixed numbers. Then, $(A_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential contraction with $\mathcal{D}$ being a scalar multiple of the map $n \mapsto e^{\epsilon|n|}$.

Moreover, the notion of a nonuniform exponential contraction is ubiquitous from the ergodic theory point of view (see Remark 3.13 for details).

A sequence $(A_n)_{n\in\mathbb{Z}}$ gives rise to a linear nonautonomous dynamics given by

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{2.3}$$

Assume that $(f_n)_{n\in\mathbb{Z}}$ is a sequence of (nonlinear) maps $f_n\colon X \to X$, $n \in \mathbb{Z}$. We consider also the associated nonautonomous dynamics

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z}. \tag{2.4}$$

The following is our first result. It gives conditions under which nonlinear dynamics (2.4) can be C^1-linearized. We stress that the proof of Theorem 2.3 will follow closely the proof of [13, Theorem 2.], but we include all details for the sake of completeness.

Theorem 2.3. *Assume that $(A_n)_{n\in\mathbb{Z}} \subset B(X)$ is a sequence of invertible operators that admits a nonuniform exponential contraction and assume that $0 < \lambda \le \mu$ and $\mathcal{D}\colon \mathbb{Z} \to (0,\infty)$ are such that (2.1) and (2.2) hold. Furthermore, suppose that*

$$\mu < 2\lambda. \tag{2.5}$$

In addition, assume that:

- *f_n is differentiable for each $n \in \mathbb{Z}$;*

- *for every $n \in \mathbb{Z}$,*
$$f_n(0) = 0 \quad \text{and} \quad Df_n(0) = 0; \tag{2.6}$$
- *there exists $\gamma > 0$ such that*
$$\|Df_n(x) - Df_n(y)\| \le \frac{\gamma}{\mathcal{D}(n+1)}\|x-y\|, \quad \text{for } n \in \mathbb{Z} \text{ and } x, y \in X; \tag{2.7}$$
- *there exists $\eta > 0$ such that*
$$\|Df_n(x)\| \le \frac{\eta}{\mathcal{D}(n+1)}, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X. \tag{2.8}$$

Then, if η is sufficiently small, there exists a sequence $(h_n)_{n\in\mathbb{Z}}$ of C^1 diffeomorphisms on X such that
$$h_{n+1} \circ (A_n + f_n) = A_n \circ h_n, \quad \text{for } n \in \mathbb{Z}. \tag{2.9}$$

Proof. Choose $\epsilon > 0$ such that
$$\epsilon < \lambda \quad \text{and} \quad \mu - 2\lambda + 3\epsilon < 0. \tag{2.10}$$
Observe that such ϵ can be chosen since (2.5) holds. For $n \in \mathbb{Z}$ and $x \in X$, we define
$$\|x\|_n = \sum_{m=n}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n)} + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)}.$$
Observe that $\|x\| \le \|x\|_n$. Moreover, (2.1) and (2.2) imply that
$$\|x\|_n \le \mathcal{D}(n)\left(\sum_{m=n}^{\infty} e^{-\epsilon(m-n)} + \sum_{m=-\infty}^{n-1} e^{-\epsilon(n-m)}\right)\|x\|.$$
We conclude that
$$\|x\| \le \|x\|_n \le c\mathcal{D}(n)\|x\| \quad \text{for } x \in X \text{ and } n \in \mathbb{Z}, \tag{2.11}$$
where
$$c = \frac{1+e^{-\epsilon}}{1-e^{-\epsilon}} > 0. \tag{2.12}$$

Lemma 2.4. *We have that*
$$\|A_n x\|_{n+1} \le e^{-(\lambda-\epsilon)}\|x\|_n \quad \text{and} \quad \|A_n^{-1}x\|_n \le e^{\mu+\epsilon}\|x\|_{n+1},$$
for $x \in X$ and $n \in \mathbb{Z}$.

Proof of the lemma. We have that
$$\begin{aligned}
\|A_n x\|_{n+1} &= \sum_{m=n+1}^{\infty} \|\mathcal{A}(m,n+1)A_n x\| e^{(\lambda-\epsilon)(m-n-1)} + \sum_{m=-\infty}^{n} \|\mathcal{A}(m,n+1)A_n x\| e^{-(\mu+\epsilon)(n+1-m)} \\
&= \sum_{m=n+1}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n-1)} + \sum_{m=-\infty}^{n} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n+1-m)} \\
&= e^{-(\lambda-\epsilon)} \sum_{m=n}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n)} - e^{-(\lambda-\epsilon)}\|x\| \\
&\quad + e^{-(\mu+\epsilon)}\|x\| + e^{-(\mu+\epsilon)} \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)} \\
&= e^{-(\lambda-\epsilon)}\|x\|_n + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)}) \cdot \left(\|x\| + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)}\right).
\end{aligned}$$

Since $\lambda - \epsilon < \mu + \epsilon$, we have that $e^{-(\mu+\epsilon)} < e^{-(\lambda-\epsilon)}$ and thus we obtain the first inequality in the statement of the lemma. Moreover, since

$$\|x\| + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)} \leq \|x\|_n,$$

we have that

$$\|A_n x\|_{n+1} \geq e^{-(\lambda-\epsilon)}\|x\|_n + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)})\|x\|_n = e^{-(\mu+\epsilon)}\|x\|_n,$$

which readily implies the second inequality in the statement of the lemma. □

Set

$$Y_\infty := \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \|\mathbf{x}\|_\infty := \sup_{n\in\mathbb{Z}} \|x_n\|_n < \infty \right\}.$$

Then, it is easy to verify that $(Y_\infty, \|\cdot\|_\infty)$ is a Banach space. We define a linear operator $\mathbb{A}\colon Y_\infty \to Y_\infty$ by

$$(\mathbb{A}\mathbf{x})_n = A_{n-1}x_{n-1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Lemma 2.5. *$\mathbb{A}$ is a bounded operator and*

$$\|\mathbb{A}^m\| \leq e^{-(\lambda-\epsilon)m}, \quad \textit{for } m \in \mathbb{N}.$$

In particular, we have that $r(\mathbb{A}) < 1$, where $r(\mathbb{A})$ denotes the spectral radius of $\mathbb{A}$.

Proof of the lemma. It follows from Lemma 2.4 that

$$\begin{aligned}
\|\mathbb{A}^m\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}} \|(\mathbb{A}^m\mathbf{x})_n\|_n &= \sup_{n\in\mathbb{Z}} \|\mathcal{A}(n, n-m)x_{n-m}\|_n \\
&\leq e^{-(\lambda-\epsilon)m} \sup_{n\in\mathbb{Z}} \|x_{n-m}\|_{n-m} \\
&= e^{-(\lambda-\epsilon)m} \|\mathbf{x}\|_\infty,
\end{aligned}$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$, which yields the desired conclusion. □

Lemma 2.6. *$\mathbb{A}$ is invertible and $\|\mathbb{A}^{-1}\| \leq e^{\mu+\epsilon}$.*

Proof of the lemma. It is easy to verify that $\mathbb{A}$ is invertible and that its inverse is given by

$$(\mathbb{A}^{-1}\mathbf{x})_n = A_n^{-1}x_{n+1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Moreover, it follows from Lemma 2.5 that

$$\begin{aligned}
\|\mathbb{A}^{-1}\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}} \|(\mathbb{A}^{-1}\mathbf{x})_n\|_n &= \sup_{n\in\mathbb{Z}} \|A_n^{-1}x_{n+1}\|_n \\
&\leq e^{\mu+\epsilon} \sup_{n\in\mathbb{Z}} \|x_n\|_n \\
&= e^{\mu+\epsilon} \|\mathbf{x}\|_\infty,
\end{aligned}$$

for each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$, which yields the desired result. □

Lemma 2.7. *We have that*

$$\|\mathbb{A}\|^2 \cdot \|\mathbb{A}^{-1}\| < 1.$$

Proof of the lemma. Observe that it follows from Lemmas 2.5 and 2.6 that

$$\|\mathbb{A}\|^2 \cdot \|\mathbb{A}^{-1}\| \leq e^{-2(\lambda-\epsilon)} \cdot e^{\mu+\epsilon}.$$

Hence, the conclusion of the lemma follows from the second inequality in (2.10). □

We now define $F\colon Y_\infty \to Y_\infty$ by

$$(F(\mathbf{x}))_n = f_{n-1}(x_{n-1}), \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Lemma 2.8. *F is well-defined.*

Proof of the lemma. Observe that (2.6) and (2.7) imply that

$$\|f_n(x)\| \leq \frac{\gamma}{\mathcal{D}(n+1)}\|x\|^2, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X. \tag{2.13}$$

By (2.11) and (2.13), we have that

$$\begin{aligned}\|(F(\mathbf{x}))_n\|_n = \|f_{n-1}(x_{n-1})\|_n &\leq c\mathcal{D}(n)\|f_{n-1}(x_{n-1})\| \\ &\leq c\mathcal{D}(n)\frac{\gamma}{\mathcal{D}(n)}\|x_{n-1}\|^2 \\ &\leq c\gamma\|x_{n-1}\|_{n-1}^2,\end{aligned}$$

for $n \in \mathbb{Z}$ and therefore

$$\|F(\mathbf{x})\|_\infty \leq c\gamma\|\mathbf{x}\|_\infty^2,$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$. We conclude that F is well-defined. □

Lemma 2.9. *F is differentiable and*

$$(DF(\mathbf{x})\mathbf{y})_n = Df_{n-1}(x_{n-1})y_{n-1},$$

for $n \in \mathbb{Z}$, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$.

Proof of the lemma. Let us fix $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$. We define an operator $L\colon Y_\infty \to Y_\infty$ by

$$(L\mathbf{y})_n = Df_{n-1}(x_{n-1})y_{n-1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Observe that (2.8) and (2.11) imply that

$$\begin{aligned}\|(L\mathbf{y})_n\|_n = \|Df_{n-1}(x_{n-1})y_{n-1}\|_n &\leq c\mathcal{D}(n)\|Df_{n-1}(x_{n-1})y_{n-1}\| \\ &\leq c\mathcal{D}(n)\frac{\eta}{\mathcal{D}(n)}\|y_{n-1}\| \\ &\leq c\eta\|y_{n-1}\|_{n-1},\end{aligned}$$

for $n \in \mathbb{Z}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$. Hence,

$$\|L\mathbf{y}\|_\infty \leq c\eta\|\mathbf{y}\|_\infty,$$

and we conclude that L is a bounded linear operator. Furthermore, for each $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in Y_\infty$, we have that

$$\begin{aligned}(F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x}) - L\mathbf{h})_n &= f_{n-1}(x_{n-1}+h_{n-1}) - f_{n-1}(x_{n-1}) - Df_{n-1}(x_{n-1})h_{n-1} \\ &= \int_0^1 Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}\,dt - Df_{n-1}(x_{n-1})h_{n-1} \\ &= \int_0^1 \big(Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1} - Df_{n-1}(x_{n-1})h_{n-1}\big)\,dt.\end{aligned}$$

Then, (2.7) and (2.11) imply that

$$\begin{aligned}\|(F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h})_n\|_n &\le \int_0^1 \|Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}-Df_{n-1}(x_{n-1})h_{n-1}\|_n\,dt\\ &\le c\mathcal{D}(n)\int_0^1 \|Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}-Df_{n-1}(x_{n-1})h_{n-1}\|\,dt\\ &\le c\gamma\|h_{n-1}\|^2 \le c\gamma\|h_{n-1}\|_{n-1}^2,\end{aligned}$$

for $n\in\mathbb{Z}$ and $\mathbf{h}=(h_n)_{n\in\mathbb{Z}}\in Y_\infty$, and consequently

$$\|F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h}\|_\infty \le c\gamma\|\mathbf{h}\|_\infty^2.$$

We conclude that

$$\lim_{\mathbf{h}\to 0}\frac{\|F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h}\|_\infty}{\|\mathbf{h}\|_\infty}=0,$$

which implies the desired conclusion. □

Lemma 2.10. *We have that DF is uniformly continuous. Moreover,*

$$\sup_{\mathbf{x}\in Y_\infty\setminus\{0\}}\frac{\|DF(\mathbf{x})\|}{\|\mathbf{x}\|_\infty}<\infty.$$

Proof of the lemma. For $\mathbf{x}^i=(x_n^i)_{n\in\mathbb{Z}}$, $i=1,2$ and $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$, it follows from (2.7), (2.11) and Lemma 2.9 that

$$\begin{aligned}\|(DF(\mathbf{x}^1)\mathbf{y})_n-(DF(\mathbf{x}^2)\mathbf{y})_n\|_n &= \|Df_{n-1}(x_{n-1}^1)y_{n-1}-Df_{n-1}(x_{n-1}^2)y_{n-1}\|_n\\ &\le c\mathcal{D}(n)\|Df_{n-1}(x_{n-1}^1)y_{n-1}-Df_{n-1}(x_{n-1}^2)y_{n-1}\|\\ &\le c\gamma\|x_{n-1}^1-x_{n-1}^2\|\cdot\|y_{n-1}\|\\ &\le c\gamma\|x_{n-1}^1-x_{n-1}^2\|_{n-1}\cdot\|y_{n-1}\|_{n-1},\end{aligned}$$

for each $n\in\mathbb{Z}$. Thus,

$$\|DF(\mathbf{x}^1)-DF(\mathbf{x}^2)\|_\infty \le c\gamma\|\mathbf{x}^1-\mathbf{x}^2\|_\infty,$$

which implies that DF is uniformly continuous.

In addition, by (2.6), (2.7) and (2.11) we have that

$$\begin{aligned}\|(DF(\mathbf{x})\mathbf{y})_n\|_n &= \|Df_{n-1}(x_{n-1})y_{n-1}\|_n\\ &\le c\mathcal{D}(n)\|Df_{n-1}(x_{n-1})y_{n-1}\|\\ &\le c\gamma\|x_{n-1}\|\cdot\|y_{n-1}\|\\ &\le c\gamma\|x_{n-1}\|_{n-1}\cdot\|y_{n-1}\|_{n-1},\end{aligned}$$

for $n\in\mathbb{Z}$ and thus

$$\|DF(\mathbf{x})\|\le c\gamma\|\mathbf{x}\|_\infty.$$

Consequently,

$$\sup_{\mathbf{x}\in Y_\infty\setminus\{0\}}\frac{\|DF(\mathbf{x})\|}{\|\mathbf{x}\|_\infty}\le c\gamma<\infty.$$

The proof of the lemma is completed. □

Lemma 2.11. *We have that*

$$\sup_{\mathbf{x}\in Y_\infty} \|DF(\mathbf{x})\| \le c\eta.$$

Proof of the lemma. By (2.8), (2.11) and Lemma 2.9, we have that

$$\|(DF(\mathbf{x})\mathbf{y})_n\|_n = \|Df_{n-1}(x_{n-1})y_{n-1}\|_n \le c\eta \|y_{n-1}\|_{n-1},$$

for $n\in \mathbb Z$, $\mathbf{x}=(x_n)_{n\in \mathbb Z}$ and $\mathbf{y}=(y_n)_{n\in \mathbb Z}\in Y_\infty$. Hence,

$$\sup_{\mathbf{x}\in Y_\infty} \|DF(\mathbf{x})\| \le c\eta. \qquad \square$$

It follows from Lemmas 2.5, 2.7, 2.10, 2.11 and [29, Theorem 3] that for η is sufficiently small, there exists a C^1-diffeomorphism $H\colon Y_\infty \to Y_\infty$ such that

$$H\circ (\mathbb A+F)=\mathbb A\circ H.$$

For $n\in \mathbb Z$ and $v\in X$, we define

$$h_n(v)=(H(\mathbf{v}^n))_n,$$

where $\mathbf{v}^n=(v_m^n)_{m\in \mathbb Z}\in Y_\infty$ is given by

$$v_m^n=\begin{cases} v & \text{if } m=n; \\ 0 & \text{of } m\neq n. \end{cases}$$

Proceeding as in the proof of [13, Theorem 2.], one can conclude that h_n is a C^1-diffeomorphism for each $n\in \mathbb Z$ and that (2.9) holds. $\square$

Remark 2.12. Note that (2.9) implies the following: if $(x_n)_{n\in \mathbb Z}$ solves (2.3), then $(y_n)_{n\in \mathbb Z}$ given by $y_n=h_n^{-1}(x_n)$, $n\in \mathbb Z$ solves (2.4). Hence, the condition (2.9) can be interpreted as a linearization of the nonlinear dynamics (2.4).

Let us now give an interpretation of Theorem 2.3 in the particular case of uniform exponential contractions.

Definition 2.13. We say that a sequence $(A_n)_{n\in \mathbb Z}\subset B(X)$ of invertible operators admits a *uniform exponential contraction* if it admits a nonuniform exponential contraction with a constant function $\mathcal D\colon \mathbb Z\to (0,\infty)$.

The following result is a direct consequence of Theorem 2.3.

Corollary 2.14. *Assume that $(A_n)_{n\in \mathbb Z}$ is a sequence of bounded and invertible linear operators that admits a uniform exponential contraction and assume that $0<\lambda\le \mu$ are such that (2.1) and (2.2) hold (with $\mathcal D$ being a constant function). Furthermore, suppose that (2.5) holds. In addition, assume that:*

- *f_n is differentiable for each $n\in \mathbb Z$;*
- *for every $n\in \mathbb Z$, (2.6) holds;*
- *there exists $\gamma>0$ such that*

$$\|Df_n(x)-Df_n(y)\|\le \gamma\|x-y\|, \quad \text{for } n\in \mathbb Z \text{ and } x,y\in X;$$

- *there exists $\eta > 0$ such that*

$$\|Df_n(x)\| \le \eta, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X.$$

Then, if η is sufficiently small, there exists a sequence $(h_n)_{n\in\mathbb{Z}}$ of C^1 diffeomorphisms on X such that (2.9) holds.

Remark 2.15. We are now in a position to elaborate on how Theorem 2.3 and Corollary 2.14 differ from [13, Theorem 2.]. Firstly, we emphasize that [13, Theorem 2.] works under a more general assumption that (2.3) admits an exponential dichotomy and thus is particularly applicable to our setting when (2.3) is contractive. However, the conditions for the smooth linearization in [13] are given in terms of the spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$. More precisely, if $X = \mathbb{R}^d$ one can show that

$$\sigma(\mathbb{A}) \cap (0, \infty) = [a_1, b_1] \cup \cdots \cup [a_k, b_k],$$

with $0 < a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$. When (2.3) is contractive, we have that $b_k < 1$. Then [13, Theorem 2.] is applicable under suitable conditions for the quotients b_j/a_j, $1 \le j \le k$. However, it is very difficult to verify these conditions in practice since numbers a_j, b_j are difficult to compute (or even estimate). On the other hand, Theorem 2.3 and Corollary 2.14 do not require this as (2.5) is concerned only with the relationship between λ and μ.

Remark 2.16. We note that the condition (2.7) implies that Df_n is a Lipschitz map for each $n \in \mathbb{Z}$. For certain smooth linearization results which do not require that the derivative of the nonlinear part is Lipschitz, we refer to [25,36].

3 Nonuniform exponential contractions for variational systems

The purpose of this section is to established result analogous to Theorem 2.3 for variational contractive systems with discrete time. We will begin by recalling some necessary terminology.

Assume that Θ be a metric space and let $\sigma \colon \Theta \to \Theta$ be a homeomorphism.

Definition 3.1. A map $\mathcal{A} \colon \Theta \times \mathbb{Z} \to B(X)$ is said to be a *linear cocycle* over σ if:

- $\mathcal{A}(q, 0) = \mathrm{Id}$ for $q \in \Theta$;
- $\mathcal{A}(q, n+m) = \mathcal{A}(\sigma^n q, m)\mathcal{A}(q, n)$ for $q \in \Theta$ and $n, m \in \mathbb{Z}$.

The map $A \colon \Theta \to B(X)$ defined by $A(q) = \mathcal{A}(q, 1)$, $q \in \Theta$ is said to be a *generator* of $\mathcal{A}$.

Remark 3.2. Let $\mathcal{A}$ be a linear cocycle over σ with generator A. We consider the *discrete variational system* given by

$$x_q(n+1) = A(\sigma^n q)x_q(n), \quad (q, n) \in \Theta \times \mathbb{Z}.$$

Observe that its solution satisfies

$$x_q(m) = \mathcal{A}(\sigma^n q, m-n)x_q(n), \quad \text{for } q \in \Theta \text{ and } m \ge n.$$

Definition 3.3. Let $\Theta_0 \subset \Theta$ be σ-invariant, i.e. that $\sigma(\Theta_0) = \Theta_0$. A linear cocycle $\mathcal{A}$ is said to be *nonuniformly exponentially contractive* on Θ_0 if there exist a map $K \colon \Theta_0 \to (0, \infty)$ and $0 < \lambda \le \mu$ such that:

- $$\|\mathcal{A}(q,n)\| \le K(q)e^{-\lambda n} \quad \text{for } q \in \Theta_0 \text{ and } n \ge 0; \tag{3.1}$$
- $$\|\mathcal{A}(q,-n)\| \le K(q)e^{\mu n} \quad \text{for } q \in \Theta_0 \text{ and } n \ge 0. \tag{3.2}$$

The following is a version of Theorem 2.3 for discrete variational systems.

Theorem 3.4. *Assume that $\mathcal{A}$ is a nonuniformly exponentially contractive linear cocycle on a σ-invariant set $\Theta_0 \subset \Theta$. Furthermore, let $K\colon \Theta_0 \to (0,\infty)$ and $0 < \lambda \le \mu$ be such that (3.1) and (3.2) hold. In addition, suppose that (2.5) holds. Finally, assume that $(f_q)_{q\in\Theta_0}$ is a family of maps $f_q\colon X \to X$ such that:*

- *f_q is differentiable for each $q \in \Theta_0$;*
- *for every $q \in \Theta_0$,*
$$f_q(0) = 0 \quad \text{and} \quad Df_q(0) = 0; \tag{3.3}$$
- *there exists $\gamma > 0$ such that*
$$\|Df_q(x) - Df_q(y)\| \le \frac{\gamma}{K(\sigma q)}\|x-y\|, \quad \text{for } q \in \Theta_0 \text{ and } x, y \in X; \tag{3.4}$$
- *there exists $\eta > 0$ such that*
$$\|Df_q(x)\| \le \frac{\eta}{K(\sigma q)}, \quad \text{for } q \in \Theta_0 \text{ and } x \in X. \tag{3.5}$$

Then, if η is sufficiently small, there exists a family $(h_q)_{q\in\Theta_0}$ of C^1 diffeomorphisms on X such that
$$h_{\sigma q} \circ (A(q) + f_q) = A(q) \circ h_q, \quad \text{for } q \in \Theta_0. \tag{3.6}$$

Proof. We choose $\epsilon > 0$ such that (2.10) holds. For $q \in \Theta_0$ and $x \in X$, set
$$\|x\|_q := \sum_{n=0}^{\infty} \|\mathcal{A}(q,n)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n}.$$
It follows easily from (3.1) and (3.2) that
$$\|x\| \le \|x\|_q \le cK(q)\|x\| \quad \text{for } q \in \Theta_0 \text{ and } x \in X, \tag{3.7}$$
with c as in (2.12).

Lemma 3.5. *We have that*
$$\|A(q)x\|_{\sigma q} \le e^{-(\lambda-\epsilon)}\|x\|_q \quad \text{and} \quad \|A(q)^{-1}x\|_q \le e^{\mu+\epsilon}\|x\|_{\sigma q},$$
for $x \in X$ and $q \in \Theta_0$.

Proof of the lemma. We have that
$$\begin{aligned}
\|A(q)x\|_{\sigma q} &= \sum_{n=0}^{\infty} \|\mathcal{A}(\sigma q,n)A(q)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(\sigma q,-n)A(q)x\| e^{-(\mu+\epsilon)n} \\
&= \sum_{n=0}^{\infty} \|\mathcal{A}(q,n+1)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-(n-1))x\| e^{-(\mu+\epsilon)n} \\
&= e^{-(\lambda-\epsilon)} \sum_{n=0}^{\infty} \|\mathcal{A}(q,n)x\| e^{(\lambda-\epsilon)n} - e^{-(\lambda-\epsilon)}\|x\| \\
&\quad + e^{-(\mu+\epsilon)}\|x\| + e^{-(\mu+\epsilon)} \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n} \\
&= e^{-(\lambda-\epsilon)}\|x\|_q + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)}) \cdot \left(\|x\| + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n}\right).
\end{aligned}$$

Since $\lambda - \epsilon < \mu + \epsilon$, we have that $e^{-(\mu+\epsilon)} < e^{-(\lambda-\epsilon)}$ and thus we obtain the first inequality in the statement of the lemma. Moreover, since

$$\|x\| + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n} \leq \|x\|_q$$

we have that

$$\|A(q)x\|_{\sigma q} \geq e^{-(\lambda-\epsilon)}\|x\|_q + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)})\|x\|_q = e^{-(\mu+\epsilon)}\|x\|_q,$$

which readily implies the second inequality in the statement of the lemma. $\square$

Set

$$Z_\infty := \left\{ v \colon \Theta_0 \to X : \|v\|_\infty := \sup_{q\in\Theta_0} \|v(q)\|_q < \infty \right\}.$$

Then, $(Z_\infty, \|\cdot\|_\infty)$ is a Banach space. We define a linear operator $\mathbb{B}\colon Z_\infty \to Z_\infty$ by

$$(\mathbb{B}v)(q) = A(\sigma^{-1}q)v(\sigma^{-1}q), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Lemma 3.6. $\mathbb{B}$ *is a bounded operator and*

$$\|\mathbb{B}^m\| \leq e^{-(\lambda-\epsilon)m}, \quad \textit{for } m \in \mathbb{N}.$$

In particular, we have that $r(\mathbb{B}) < 1$.

Proof of the lemma. It follows from Lemma 3.5 that

$$\begin{aligned}
\|\mathbb{B}^m v\|_\infty = \sup_{q\in\Theta_0} \|(\mathbb{B}^m v)(q)\|_q &= \sup_{q\in\Theta_0} \|\mathcal{A}(\sigma^{-m}q, m)v(\sigma^{-m}q)\|_q \\
&\leq e^{-(\lambda-\epsilon)m} \sup_{q\in\Theta_0} \|v(\sigma^{-m}q)\|_{\sigma^{-m}q} \\
&= e^{-(\lambda-\epsilon)m}\|v\|_\infty,
\end{aligned}$$

for $v \in Z_\infty$, which yields the desired conclusion. $\square$

Lemma 3.7. $\mathbb{B}$ *is invertible and* $\|\mathbb{B}^{-1}\| \leq e^{\mu+\epsilon}$.

Proof of the lemma. It is easy to verify that $\mathbb{B}$ is invertible and that its inverse is given by

$$(\mathbb{B}^{-1}v)(q) = A(q)^{-1}v(\sigma q), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Moreover, it follows from Lemma 3.5 that

$$\begin{aligned}
\|\mathbb{B}^{-1}v\|_\infty = \sup_{q\in\Theta_0} \|(\mathbb{B}^{-1}v)(q)\|_q &= \sup_{q\in\Theta_0} \|A(q)^{-1}v(\sigma q)\|_q \\
&\leq e^{\mu+\epsilon} \sup_{q\in\Theta_0} \|v(\sigma q)\|_{\sigma q} \\
&= e^{\mu+\epsilon}\|v\|_\infty,
\end{aligned}$$

for each $v \in Z_\infty$, which yields the desired result. $\square$

As in the proof of Lemma 2.7, it follows from (2.10) and Lemmas 3.6 and 3.7 that

$$\|\mathbb{B}\|^2 \cdot \|\mathbb{B}^{-1}\| < 1. \tag{3.8}$$

We define $G \colon Z_\infty \to Z_\infty$ by

$$(G(v))(q) = f_{\sigma^{-1}q}(v(\sigma^{-1}q)), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Lemma 3.8. *G is well-defined.*

Proof of the lemma. Observe that (3.3) and (3.4) imply that

$$\|f_q(x)\| \le \frac{\gamma}{K(\sigma q)}\|x\|^2, \quad \text{for } q \in \Theta_0 \text{ and } x \in X. \tag{3.9}$$

By (3.7) and (3.9), we have that

$$\begin{aligned}\|(G(v))(q)\|_q = \|f_{\sigma^{-1}q}(v(\sigma^{-1}q))\|_q &\le cK(q)\|f_{\sigma^{-1}q}(v(\sigma^{-1}q))\| \\ &\le cK(q)\frac{\gamma}{K(q)}\|v(\sigma^{-1}q)\|^2 \\ &\le c\gamma\|v(\sigma^{-1}q)\|^2_{\sigma^{-1}q},\end{aligned}$$

for $q \in \Theta_0$ and $v \in Z_\infty$. Hence,

$$\|G(v)\|_\infty \le c\gamma\|v\|_\infty^2 \quad \text{for every } v \in Z_\infty,$$

and therefore G is well-defined. □

Lemma 3.9. *G is differentiable and*

$$(DG(v)w)(q) = Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q),$$

for $q \in \Theta_0$ and $v, w \in Z_\infty$.

Proof of the lemma. Let us fix $v \in Z_\infty$. We define an operator $L \colon Z_\infty \to Z_\infty$ by

$$(Lw)(q) = Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q), \quad \text{for } q \in \Theta_0 \text{ and } w \in Z_\infty.$$

Observe that (3.5) and (3.7) imply that

$$\begin{aligned}\|(Lw)(q)\|_q &= \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\|_q \\ &\le cK(q)\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\| \\ &\le cK(q)\frac{\eta}{K(q)}\|w(\sigma^{-1}q)\| \\ &\le c\eta\|w(\sigma^{-1}q)\|_{\sigma^{-1}q},\end{aligned}$$

for $q \in \Theta_0$ and thus

$$\|Lw\|_\infty \le c\eta\|w\|_\infty,$$

for every $w \in Z_\infty$. Hence, L is a bounded linear operator.

Furthermore, for each $h \in Z_\infty$, we have that

$$\begin{aligned}
&(G(v+h) - G(v) - Lh)(q) \\
&\quad = f_{\sigma^{-1}q}(v(\sigma^{-1}q) + h(\sigma^{-1}q)) - f_{\sigma^{-1}q}(v(\sigma^{-1}q)) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q) \\
&\quad = \int_0^1 Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q)\, dt - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q) \\
&\quad = \int_0^1 \big(Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\big)\, dt.
\end{aligned}$$

Then, (3.4) and (3.7) imply that

$$\begin{aligned}
&\|(G(v+h) - G(v) - Lh)(\omega)\|_q \\
&\quad \le \int_0^1 \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|_q\, dt \\
&\quad \le cK(q) \int_0^1 \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|\, dt \\
&\quad \le c\gamma \|h(\sigma^{-1}q)\|^2 \le c\gamma \|h(\sigma^{-1}q)\|^2_{\sigma^{-1}q},
\end{aligned}$$

for $q \in \Theta_0$ and $h \in Z_\infty$, and consequently

$$\|G(v+h) - G(v) - Lh\|_\infty \le c\gamma \|h\|_\infty^2,$$

which implies the desired conclusion. □

Lemma 3.10. *DG is uniformly continuous. Moreover,*

$$\sup_{v \in Z_\infty \setminus \{0\}} \frac{\|DG(v)\|}{\|v\|_\infty} < \infty.$$

Proof of the lemma. For v_i, $i = 1, 2$ and $h \in Z_\infty$, it follows from (3.4), (3.7) and Lemma 3.9 that

$$\begin{aligned}
&\|(DG(v_1)h)(q) - (DG(v_2)h)(q)\|_q \\
&\quad = \|Df_{\sigma^{-1}q}(v_1(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v_2(\sigma^{-1}q))h(\sigma^{-1}q)\|_q \\
&\quad \le cK(q)\|Df_{\sigma^{-1}q}(v_1(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v_2(\sigma^{-1}q))h(\sigma^{-1}q)\| \\
&\quad \le c\gamma \|v_1(\sigma^{-1}q) - v_2(\sigma^{-1}q)\| \cdot \|h(\sigma^{-1}q)\| \\
&\quad \le c\gamma \|v_1(\sigma^{-1}q) - v_2(\sigma^{-1}q)\|_{\sigma^{-1}q} \cdot \|h(\sigma^{-1}q)\|_{\sigma^{-1}q},
\end{aligned}$$

for each $q \in \Theta_0$. Therefore,

$$\|DG(v_1) - DG(v_2)\|_\infty \le c\gamma \|v_1 - v_2\|_\infty.$$

In addition, by (3.3), (3.4) and (3.7) we have that

$$\begin{aligned}
\|(DG(v)h)(q)\|_q &= \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|_q \\
&\le cK(q)\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\| \\
&\le c\gamma \|v(\sigma^{-1}q)\| \cdot \|h(\sigma^{-1}q)\| \\
&\le c\gamma \|v(\sigma^{-1}q)\|_{\sigma^{-1}q} \cdot \|h(\sigma^{-1}q)\|_{\sigma^{-1}q},
\end{aligned}$$

for $q \in \Theta_0$ and thus

$$\|DG(v)\| \le c\gamma \|v\|_\infty,$$

which completes the proof of the lemma. □

Lemma 3.11. *We have that*

$$\sup_{v\in Z_\infty}\|DG(v)\|\le c\eta.$$

Proof of the lemma. By (3.5), (3.7) and Lemma 3.9, we have that

$$\|(DG(v)w)(q)\|_q=\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\|_q\le c\eta\|w(\sigma^{-1}q)\|_{\sigma^{-1}q},$$

for $q\in\Theta_0$, $v,w\in Z_\infty$. Hence,

$$\sup_{v\in Z_\infty}\|DG(v)\|\le c\eta. \qquad\square$$

It follows from (3.8), Lemmas 3.6, 3.10, 3.11 and [29, Theorem 3] that for η is sufficiently small, there exists a C^1-diffeomorphism $H\colon Z_\infty\to Z_\infty$ such that

$$H\circ(\mathbb{B}+G)=\mathbb{B}\circ H. \tag{3.10}$$

Take now $q_0\in\Theta_0$ and $x_0\in X$. We define $v_{q_0,x_0}\in Z_\infty$ by

$$v_{q_0,x_0}(q)=\begin{cases}x_0 & \text{if } q=q_0;\\ 0 & \text{if } q\neq q_0.\end{cases}$$

Finally, we define $h_{q_0}\colon X\to X$ by $h_{q_0}(x_0)=H(v_{q_0,x_0})(q_0)$. Observe that

$$(\mathbb{B}+G)(v_{q_0,x_0})(q)=\begin{cases}A(q_0)x_0+f_{q_0}(x_0) & \text{if } q=\sigma q_0;\\ 0 & \text{if } q\neq \sigma q_0,\end{cases}$$

and thus

$$(\mathbb{B}+G)(v_{q_0,x_0})=v_{\sigma q_0,A(q_0)x_0+f_{q_0}(x_0)}.$$

Now we observe that it follows from (3.10) that

$$((H\circ(\mathbb{B}+G))(v_{q,x_0}))(\sigma q_0)=((\mathbb{B}\circ H)(v_{q_0,x_0}))(\sigma q_0),$$

and thus

$$(h_{\sigma q_0}\circ(A(q_0)+f_{q_0}))(x_0)=(A(q_0)\circ h_{q_0})(x_0).$$

Since $q_0\in\Theta_0$ and $x_0\in X$ were arbitrary, we conclude that (3.6) holds.

Furthermore, we claim that h_{q_0} is differentiable and that

$$Dh_{q_0}(x_0)y=(DH(v_{q_0,x_0})v_{q_0,y})(q_0). \tag{3.11}$$

Indeed, we have that

$$\begin{aligned}&\frac{\|h_{q_0}(x_0+y)-h_{q_0}(x_0)-(DH(v_{q_0,x_0})v_{q_0,y})(q_0)\|}{\|y\|}\\ &\le cK(q_0)\frac{\|H(v_{q_0,x_0}+v_{q_0,y})(q_0)-H(v_{q_0,x_0})(q_0)-(DH(v_{q_0,x_0})v_{q_0,y})(q_0)\|_{q_0}}{\|y\|_{q_0}}\\ &\le cK(q_0)\frac{\|H(v_{q_0,x_0}+v_{q_0,y})-H(v_{q_0,x_0})-(DH(v_{q_0,x_0})v_{q_0,y})\|_\infty}{\|v_{q_0,y}\|_\infty}.\end{aligned}$$

Letting $\|y\|\to 0$, we have that $\|v_{q_0,y}\|_\infty\to 0$ and we conclude that (3.11) holds.

Let us show that Dh_{q_0} is continuous. For x_0 and $\tilde{x}_0 \in X$, we have that

$$\begin{aligned}\|Dh_{q_0}(x_0) - Dh_{\omega_0}(\tilde{x}_0)\| &= \sup_{\|y\|\le 1} \|Dh_{q_0}(x_0)y - Dh_{q_0}(\tilde{x}_0)y\| \\ &= \sup_{\|y\|\le 1} \|(DH(v_{q_0,x_0})v_{q_0,y})(q_0) - (DH(v_{\omega_0,\tilde{x}_0})v_{q_0,y})(q_0)\| \\ &\le \sup_{\|y\|\le 1} \|(DH(v_{q_0,x_0})v_{q_0,y})(q_0) - (DH(v_{q_0,\tilde{x}_0})v_{q_0,y})(q_0)\|_{q_0} \\ &\le \sup_{\|y\|\le 1} \|DH(v_{q_0,x_0})v_{q_0,y} - DH(v_{q_0,\tilde{x}_0})v_{q_0,y}\|_\infty \\ &\le cK(q_0)\|DH(v_{q_0,x_0}) - DH(v_{q_0,\tilde{x}_0})\|.\end{aligned}$$

Letting $\tilde{x}_0 \to x_0$, we have that $v_{q_0,\tilde{x}_0} \to v_{q_0,x_0}$ in Z_∞ and thus since H is of class C^1 we conclude that $Dh_{q_0}(\tilde{x}_0) \to Dh_{q_0}(x_0)$.

Finally, it is easy to show that

$$h_{q_0}^{-1}(x_0) = H^{-1}(v_{q_0,x_0})(q_0),$$

and proceeding as above, one can show that $h_{q_0}^{-1}$ is of class C^1. □

Let us now discuss the applicability of Theorem 3.4 in the setting when we can apply the version of the Oseledets multiplicative ergodic theorem [20] for the cocycle $\mathcal{A}$.

Remark 3.12. Assume that $X = \mathbb{R}^d$ and that on Θ we have a Borel probability measure $\mathbb{P}$ such that σ preserves $\mathbb{P}$. Moreover, suppose that $\mathbb{P}$ is ergodic and that

$$\int_\Theta \log^+\|A(q)\|\, d\mathbb{P}(q) < \infty.$$

Hence, we can apply the Oseledets multiplicative ergodic theorem [20] to conclude that there exist Lyapunov exponents

$$-\infty < \lambda_r < \cdots < \lambda_2 < \lambda_1 < +\infty, \quad 1 \le r \le d,$$

σ-invariant Borel set $\Theta_0 \subset \Theta$, $\mathbb{P}(\Theta_0) = 1$ and for $q \in \Theta_0$, the corresponding Oseledets splitting

$$\mathbb{R}^d = \bigoplus_{i=1}^r E_i(q)$$

such that:

- $A(q)E_i(q) = E_i(\sigma q)$ for $q \in \Theta_0$ and $1 \le i \le r$;
- for $q \in \Theta_0$, $v \in E_i(q) \setminus \{0\}$ and $1 \le i \le r$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)v\| = \lambda_i.$$

Assume now that $\lambda_1 < 0$. It follows from [12, Theorem 2] that $\mathcal{A}$ is nonuniformly exponentially contractive on Θ_0. Furthermore, (3.1) and (3.2) hold with

$$\lambda = -\lambda_1 - \epsilon \quad \text{and} \quad \mu = -\lambda_r + 2\epsilon,$$

with any sufficiently small $\epsilon > 0$. Hence, we conclude that in this case (2.5) holds if $2\lambda_1 < \lambda_r$.

As we promised, we now explain why nonuniform contractions introduced in previous section are ubiquitous from the ergodic theory point of view.

Remark 3.13. Let X and $\mathcal{A}$ be as in Remark 3.12. Then, it follows from [12, Theorem 3] that for $q \in \Theta_0$, the sequence $(A_n)_{n\in\mathbb{Z}}$ given by

$$A_n = A(\sigma^n q) \quad n \in \mathbb{Z},$$

admits a nonuniform exponential contraction, where $\mathcal{D}$ is the scalar multiple of $n \mapsto e^{\epsilon|n|}$ and $\epsilon > 0$ is arbitrary.

As in previous section, we will now formulate a direct consequence of Theorem 3.4 dealing with uniformly exponentially contractive variational systems.

Definition 3.14. Let $\Theta_0 \subset \Theta$ be σ-invariant. A linear cocycle $\mathcal{A}$ over σ is said to be *uniformly exponentially contractive* on Θ_0 if there exist $K > 0$ and $0 < \lambda \le \mu$ such that:

- for $q \in \Theta_0$ and $n \ge 0$,
$$\|\mathcal{A}(q,n)\| \le Ke^{-\lambda n}; \tag{3.12}$$
- for $q \in \Theta_0$ and $n \ge 0$,
$$\|\mathcal{A}(q,-n)\| \le Ke^{\mu n}. \tag{3.13}$$

The following is a consequence of Theorem 3.4.

Corollary 3.15. *Assume that $\mathcal{A}$ is a uniformly exponentially contractive linear cocycle on a σ-invariant set $\Theta_0 \subset \Theta$ and suppose that $K > 0$ and $0 < \lambda \le \mu$ are such that (3.12) and (3.13) hold. Furthermore, suppose that (2.5) holds. Finally, assume that $(f_q)_{q\in\Theta_0}$ is a family of maps $f_\omega \colon X \to X$ such that:*

- *f_q is differentiable for each $q \in \Theta_0$;*
- *for every $q \in \Theta_0$,*
$$f_q(0) = 0 \quad \text{and} \quad Df_q(0) = 0;$$
- *there exists $\gamma > 0$ such that*
$$\|Df_q(x) - Df_q(y)\| \le \gamma\|x-y\|, \quad \text{for } q \in \Theta_0 \text{ and } x, y \in X;$$
- *there exists $\eta > 0$ such that*
$$\|Df_q(x)\| \le \eta, \quad \text{for } q \in \Theta_0 \text{ and } x \in X.$$

Then, if η is sufficiently small, there exists a family $(h_q)_{q\in\Theta_0}$ of C^1 diffeomorphisms on X such that

$$h_{\sigma q} \circ (A(q) + f_q) = A(q) \circ h_q, \quad \text{for } q \in \Theta_0.$$

Acknowledgements

D. D. was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-prirod-18-9 and uniri-prprirod-19-16. He would like to thank Weinian Zhang and Wenmeng Zhang for fruitful collaboration that inspired many arguments in the present paper.

References

[1] L. Arnold, Random dynamical systems, Springer-Verlag, Berlin, 1998. https://doi.org/10.1007/978-3-662-12878-7; MR1723992; Zbl 0906.34001

[2] B. Aulbach, T. Wanner, Topological simplification of nonautonomous difference equations, *J. Difference Equ. Appl.* **12**(2006), No. 3–4, 283–296. https://doi.org/10.1080/10236190500489384; MR2215845; Zbl 1104.39001

[3] L. Backes, D. Dragičević, K. J. Palmer, Linearization and Hölder continuity for nonautonomous systems, *J. Differential Equations* **297**(2021), 536–574. https://doi.org/10.1016/j.jde.2021.06.035; MR4282665; Zbl 07373384

[4] G. R. Belitskii, Functional equations and the conjugacy of diffeomorphism of finite smoothness class, *Funct. Anal. Appl.* **7**(1973), 268–277. https://doi.org/10.1007/BF01075731; MR0331437; Zbl 0293.39005

[5] G. R. Belitskii, Equivalence and normal forms of germs of smooth mappings, *Uspekhi Mat. Nauk* **33**(1978), 95–115. https://doi.org/10.1070/RM1978v033n01ABEH002237; MR0490708; Zbl 0385.58007

[6] A. D. Brjuno, Analytical form of differential equations I, II, *Trudy Moskov. Mat. Obšč.* **25**(1971) 119–262. MR0377192; Zbl 0272.34018

[7] A. Castañeda, G. Robledo, Differentiability of Palmer's linearization theorem and converse result for density function, *J. Differential Equations* **259**(2015), 4634–4650. https://doi.org/10.1016/j.jde.2015.06.004; MR3373416; Zbl 1339.34050

[8] A. Castañeda, P. Monzon, G. Robledo, Nonuniform contractions and density stability results via a smooth topological equivalence, preprint, available at https://arxiv.org/abs/1808.07568

[9] S.-N. Chow, H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces, *J. Differential Equations* **120**(1995), 429–477. https://doi.org/10.1006/jdeq.1995.1117; MR1347351; Zbl 0831.34067

[10] L. V. Cuong, T. S. Doan, S. Siegmund, A Sternberg theorem for nonautonomous differential equations, *J. Dynam. Differential Equations* **31**(2019), 1279–1299. https://doi.org/10.1007/s10884-017-9629-8; MR3992072; Zbl 1423.34039

[11] D. Dragičević, Strong nonuniform behaviour: A Datko type characterization, *J. Math. Anal. Appl.* **459**(2018), No. 1, 266–290. https://doi.org/10.1016/j.jmaa.2017.10.056; MR3730440; Zbl 6817543

[12] D. Dragičević, G. Froyland, Hölder continuity of Oseledets splitting for semi-invertible operator cocycles, *Ergodic Theory Dynam. Syst.* **38**(2018), 961–981. https://doi.org/10.1017/etds.2016.55; MR3784250; Zbl 1390.37093

[13] D. Dragičević, W. Zhang, W. Zhang, Smooth linearization of nonautonomous difference equations with a nonuniform dichotomy, *Math. Z.* **292**(2019), 1175–1193. https://doi.org/10.1007/s00209-018-2134-x; MR3980287; Zbl 1428.37021

[14] D. Dragičević, W. Zhang, W. Zhang, Smooth linearization of nonautonomous differential equations with a nonuniform dichotomy, *Proc. Lond. Math. Soc.* **121**(2020), 32–50. https://doi.org/10.1112/plms.12315; MR4048734; Zbl 1450.37025

[15] M. S. ElBialy, Local contractions of Banach spaces and spectral gap conditions, *J. Funct. Anal.* **182**(2001), No. 1, 108–150. https://doi.org/10.1006/jfan.2000.3723; MR1829244; Zbl 1024.46026

[16] M. S. ElBialy, Smooth conjugacy and linearization near resonant fixed points in Hilbert spaces, *Houston J. Math.* **40**(2014), 467–509. MR3248650; Zbl 1355.37039

[17] P. Hartman, On local homeomorphisms of Euclidean spaces, *Bol. Soc. Mat. Mexicana* **5**(1960), 220–241. MR0141856; Zbl 0127.30202

[18] L. Jiang, Generalized exponential dichotomy and global linearization, *J. Math. Anal. Appl.* **315**(2006), 474–490. https://doi.org/10.1016/j.jmaa.2005.05.042; MR2202594; Zbl 1098.34042

[19] J. Lopez-Fenner, M. Pinto, On a Hartman linearization theorem for a class of ODE with impulse effect, *Nonlinear Anal.* **38**(1999) 307–325. https://doi.org/10.1016/S0362-546X(98)00198-9; MR1705785; Zbl 0931.34007

[20] V. Oseledets, A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems, *Trudy Moskov. Mat. Obšč.* **19**(1968), 179–210. MR0240280; Zbl 0236.93034

[21] J. Palis, On the local structure of hyperbolic points in Banach spaces, *An. Acad. Brasil. Ciênc.* **40**(1968), 263–266. MR0246331; Zbl 0184.17803

[22] K. J. Palmer, A generalization of Hartman's linearization theorem, *J. Math. Anal. Appl.* **41**(1973), 753–758. https://doi.org/10.1016/0022-247X(73)90245-X; MR0330650; Zbl 0272.34056

[23] H. Poincaré, Sur le problème des trois corps et les équations de la dynamique (in French) [On the three-body problem and the equations of dynamics], *Acta Math.* **13**(1890), 1–270. https://doi.org/10.1007/BF02392505

[24] C. Pugh, On a theorem of P. Hartman, *Amer. J. Math.* **91**(1969), 363–367. https://doi.org/10.2307/2373513; MR0257533; Zbl 0197.20701

[25] A. Reinfelds, Smooth linearization of a node, *Differencial'nye Uravnenija* **13**(1977), 1885–1887. MR0461580; Zbl 0379.34031

[26] H. M. Rodrigues, J. Solà-Morales, Linearization of class C^1 for contractions on Banach spaces, *J. Differential Equations* **201**(2004), 351–382. https://doi.org/10.1016/j.jde.2004.02.013; MR2059612; Zbl 1058.37061

[27] H. M. Rodrigues, J. Solà-Morales, Smooth linearization for a saddle on Banach spaces, *J. Dynam. Differential Equations* **16**(2004), 767–793. https://doi.org/10.1007/s10884-004-6116-9; MR2109165; Zbl 1173.37316

[28] H. M. Rodrigues, J. Solà-Morales, Invertible contractions and asymptotically stable ODE's that are not C^1-linearizable, *J. Dynam. Differential Equations* **18**(2006), 961–974. https://doi.org/10.1007/s10884-006-9050-1; MR2263410; Zbl 1103.37010

[29] H. M. Rodrigues, J. Solà-Morales, Differentiability with respect to parameters in global smooth linearization, *J. Differential Equations* **262**(2017), 3583–3596. https://doi.org/10.1016/j.jde.2016.11.038; MR3592651; Zbl 1371.58008

[30] A. L. Sasu, B. Sasu, Admissibility and exponential trichotomy of dynamical systems described by skew-product flows, *J. Differential Equations* **260**(2016), 1656–1689. https://doi.org/10.1016/j.jde.2015.09.042; MR3419742; Zbl 1334.47073

[31] B. Sasu, Input–output control systems and dichotomy of variational difference equations, *J. Difference Equa. Appl.* **17**(2011), No. 6, 889–913. https://doi.org/10.1080/10236190903456071; MR2810916; Zbl 1221.93169

[32] C. L. Siegel, Iteration of analytic functions, *Ann. Math.* **43**(1942), 607–612. https://doi.org/10.2307/1968952; MR0007044; Zbl 0061.14904

[33] S. Sternberg, Local contractions and a theorem of Poincaré, *Amer. J. Math.* **79**(1957), 809–824. https://doi.org/10.2307/2372437; MR0096853; Zbl 0080.29902

[34] S. Sternberg, On the structure of local homeomorphisms of Euclidean n-space, *Amer. J. Math.* **80**(1958), 623–631. https://doi.org/10.2307/2372774; MR0096854; Zbl 0083.31406

[35] S. van Strien, Smooth linearization of hyperbolic fixed points without resonance conditions, *J. Differential Equations* **85**(1990), 66–90. https://doi.org/10.1016/0022-0396(90)90089-8; MR1052328; Zbl 0726.58039

[36] R. J. Venti, Linear normal forms of differential equations, *J. Differential Equations* **2**(1966), 182–194. https://doi.org/10.1016/0022-0396(66)90042-8; MR0188518; Zbl 0135.30902

[37] J. C. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes $\mathrm{d}(\mathbb{C},0)$ (in French) [Linearization of germs of holomorphic diffeomorphisms of $(\mathbf{C},0)$], *C. R. Acad. Sci. Paris* **36**(1988), 55–58. MR0929279; Zbl 0668.58010

[38] W. M. Zhang, W. N. Zhang, C^1 linearization for planar contractions, *J. Funct. Anal.* **260**(2011), No. 7, 2043–2063. https://doi.org/10.1016/j.jfa.2010.12.029; MR2756149; Zbl 1217.37019

[39] W. M. Zhang, W. N. Zhang, Sharpness for C^1 linearization of planar hyperbolic diffeomorphisms, *J. Differential Equations* **257**(2014), 4470–4502. https://doi.org/10.1016/j.jde.2014.08.014; MR3268732; Zbl 1300.37023

[40] W. M. Zhang, W. N. Zhang, W. Jarczyk, Sharp regularity of linearization for $C^{1,1}$ hyperbolic diffeomorphisms, *Math. Ann.* **358**(2014), 69–113. https://doi.org/10.1007/s00208-013-0954-x; MR3157992; Zbl 1288.37026