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Invariant measures and random attractors of stochastic delay differential equations in Hilbert space

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Abstract. This paper is devoted to a general stochastic delay differential equation with infinite-dimensional diffusions in a Hilbert space. We not only investigate the existence of invariant measures with either Wiener process or Lévy jump process, but also obtain the existence of a pullback attractor under Wiener process. In particular, we prove the existence of a non-trivial stationary solution which is exponentially stable and is generated by the composition of a random variable and the Wiener shift. At last, examples of reaction-diffusion equations with delay and noise are provided to illustrate our results.

Keywords: random dynamical system, delay, invariant measure, random attractor.

2020 Mathematics Subject Classification: 60H15, 60G15, 60H05.

1 Introduction

Delay differential equations arise from evolution phenomena in physical process and biological systems (see e.g. [19, 21, 25]), in which time-delay is used for mathematical modelling to describe the dynamical influence from the past. Recently, the effect of noise on such functional differential equations is increasingly a focus of investigation, in particular, in the combined influence of noise and delay in dynamical systems (see e.g. [5, 6, 13, 35, 37]). In this paper, we consider the following stochastic delay differential equation in a separable Hilbert space H:

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + F(X_t)]\mathrm{d}t + G(X_t)\mathrm{d}Z(t), \quad t > 0, \\ X(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where A: $\mathrm{Dom}(A) \subseteq H \to H$ is the infinitesimal generator of a semigroup, $X_t(s) = X(t+s)$ for $s \in [-\tau, 0]$ and $t \geq 0$. Here, $\mathrm{Dom}(A)$ denotes the domain of A and is a Banach space under the usual graph norm. Let $\mathcal{L} = L^2([-\tau, 0], H)$, and $\|\cdot\|$, $\|\cdot\|_{\mathcal{L}}$ denote the norms in H and $\mathcal{L}$, respectively. For a process $X(t) \in H$, we denote by $\{X_t : t \geq 0\}$ the segment process, which takes values in $\mathcal{L}$ for each t. $Z = \{Z(t), \mathcal{F}_t, t \geq 0\}$ could be an abstract Q-Wiener process or Lévy jump process with values in some separable Hilbert space $\mathcal{U}$, and

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$\varphi = \{\varphi(t) : t \in [-\tau, 0]\}$ is a given real-valued stochastic process, both defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where filtration $\{\mathcal{F}_t : t \geq 0\}$ is the $\mathbb{P}$-completion of the Borel σ-algebra on Ω.

For stochastic delay differential equations, there has been a rather comprehensive mathematical literature on both theories and applications. The existence of invariant measures is well studied in both finite and infinite dimensions by using Krylov–Bogoliubov theorem (see e.g. [7, 17, 18]). Scheutzow [35] formulated a sufficient condition ensuring the existence of an invariant probability measure with additive noise. For a similar approach and connections to stochastic partial differential equations, see Bakhtin and Mattingly [4]. For stochastic delay differential equations driven by Brownian motion, Mohammed [31] investigated the existence and uniqueness of strong or weak solutions under random functional Lipschitz conditions, Mao [30] discussed the method of steps, which provides a unique solution without a regular dependence of the coefficients on values in the past, Liptser and Shiryaev [26] considered weak solutions, Itô and Nisio [23] investigated the existence of weak solutions for equations with finite and infinite delay, Butkovsky and Scheutzow [8] established a general sufficient conditions ensuring the existence of an invariant measure for stochastic functional differential equations and exponential or subexponential convergence to the equilibrium. For stochastic delay differential equations driven by a Lévy process, Gushchin and Küchler [20] established some necessary and sufficient conditions ensuring the existence and uniqueness of stationary solutions, Reiß, Riedle, and van Gaans [32] proved that the segment process is eventually Feller, but in general not eventually strong Feller on the Skorokhod space, and also investigated the existence of an invariant measure by proving the tightness of the segments using semimartingale characteristics and the Krylov-Bogoliubov method. Existence and uniqueness of global solutions have been established under local Lipschitz and linear growth conditions (see e.g. [30, 40]) or weak one-sided local Lipschitz (or monotonicity) conditions. Recently, Liu [28] considered stationary distributions of a class of second-order stochastic delay evolution equations driven by Wiener process or Lévy jump process in Hilbert space. In this paper we shall prove the existence of an invariant measure for (1.1) without boundedness conditions on the diffusion coefficient. Note that the segment process takes values in the infinite dimensional space $\mathcal{L}$, boundedness in probability does not generally imply tightness. In this case, one usually uses compactness of the orbits of the underlying deterministic equation to obtain tightness. However, such a compactness property does not hold for functional differential equation (1.1). For more details, see [7]. In this work, we will study the existence of invariant measures of (1.1) by applying the Krylov–Bogoliubov method.

A criterion for the existence of random attractors for random dynamical systems is established by Crauel and Flandoli [14], who also obtained the invariant Markov measures supported by the random attractor. Caraballo, Kloeden and Keal [10] proved the existence of random attractors of an ordinary differential equation with a random stationary delay. Kloeden and Lorenz [24] pointed out that the classical theory of pathwise random dynamical systems with a skew product (see e.g. [3]) does not apply to nonlocal dynamics such as when the dynamics of a sample path depends on other sample paths through an expectation or when the evolution of random sets depends on nonlocal properties such as the diameter of the sets. In [24], Kloeden and Lorenz showed that such nonlocal random dynamics can be characterized by a deterministic two-parameter process from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology and provided a definition of mean-square random dynamical systems and their attractors. Wu and Kloeden [39] investigated the existence of a random attractor for a mean-square random dynamical system (MS-RDS) generated by a stochastic delay differential equation with random delay for which the drift term is dominated by a nondelay component satisfying a one-sided dissipative Lipschitz condition. The exponential stability of trivial stationary solutions for stochastic partial differential equations has been extensively analyzed (see e.g. [12,22,29]). Caraballo, Kloeden and Schmalfuß [11] obtained the existence of a nontrivial stationary solution and a random fixed point which is exponentially stable. In this paper, we shall generalize the relevant results of Caraballo, Kloeden and Schmalfuß [11] to such a stochastic evolution equation with delay as (1.1). In particular, we shall prove the existence of a random fixed point, which generates the exponentially stable stationary solution of (1.1). Moreover, this stationary solution attracts bounded sets of initial conditions.

In this paper, we first establish a non-autonomous random dynamical system generated by equation (1.1). Then we show the existence of an invariant measure of (1.1) driven by Wiener process. In particular, we obtain a random pullback attractor consisting of a single point which is exponentially stable. Next, the existence of invariant measures of (1.1) driven by Lévy jump process is obtained by using Lévy–Itô decomposition formula. Finally, we apply our results to reaction-diffusion equations with noise and delay.

2 Preliminaries

Throughout this paper, we always assume that H is a separable Hilbert space, and there exists a Gelfand triplet $V \subset H \subset V'$ of separable Hilbert spaces, where V' denotes the dual of V and $V = \mathrm{Dom}(A^{\frac{1}{2}})$ (see page 55 of [38] for more details). The inner product in H is denoted by $\langle \cdot, \cdot \rangle$, and the duality mapping between V' and V by $\langle \cdot, \cdot \rangle_V$. We denote by $a_1 > 0$ the constant of the injection $V \subset H$, i.e.,

$$a_1 \|u\|^2 \le \|u\|_V^2 \quad \text{for } u \in V,$$

and let $-A : V \to V'$ be a positive, linear and continuous operator for which there exists an $a_2 > 0$ such that

$$\langle -Au, u \rangle_V \ge a_2 \|u\|_V^2 \quad \text{for all } u \in V.$$

It is well known (see, for instance, [6,9,15]) that A is the generator of a strongly continuous semigroup $\Phi(t) = e^{tA}$ on H satisfying that

$$\| \exp\{tA\} \|_{\mathscr{L}(H)} \le e^{-\lambda t}, \tag{2.1}$$

where $\lambda = a_1 a_2 > 0$ and $\mathscr{L}(H)$ is a space of bounded linear operators on H.

For any $\varphi \in L^2([-\tau, 0], H)$, the mild solution $X(t, \varphi)$ of (1.1) with the intimal data φ satisfies

$$\begin{cases} X(t,\varphi) = \Phi(t)\varphi(0) + \displaystyle\int_0^t \Phi(t-s) F(X_s(\varphi)) \mathrm{d}s \\ \qquad\qquad + \displaystyle\int_0^t \Phi(t-s) G(X_s(\varphi)) \mathrm{d}Z(s), \qquad t \ge 0, \\ X(t,\varphi) = \varphi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{2.2}$$

where $X_t(\varphi)$ represents $X_t(\varphi)(s) = X(t+s, \varphi)$ for $s \in [-\tau, 0]$ and $t \ge 0$.

Definition 2.1. A measure μ is called an invariant measure for (1.1) if

$$\mu(f) = \mu(\mathbb{P}_t f), \qquad t \ge 0,$$

where

$$\mu(f) = \int_{\mathcal{L}} f(\phi) \mu(d\phi) \quad \text{and} \quad \mathbb{P}_t f(\phi) = \mathbb{E} f(X_t(\phi))$$

for $f \in C_b(\mathcal{L})$, where $\mathbb{P}_t$ is called the transition operator of (1.1) and $C_b(\mathcal{L})$ denotes the set of all bounded and continuous real-valued functions on $\mathcal{L}$. Let $\mu_{X_t(\phi)}$ be the distribution of $X_t(\phi)$, $t \geq 0$. If an $\mathcal{F}_0$-measurable $\phi \in L^2(\Omega, \mathcal{L})$ is such that $\mu_{X_t(\phi)} = \mu_\phi$ for all $t \geq 0$, then μ_ϕ is called a stationary distribution of (1.1) and $X(t, \phi)$ is then called a stationary solution.

It follows from the above definition that an invariant measure μ is a stationary distribution of (1.1) if and only if

$$\int_{\mathcal{L}} \|\phi\|^2_{\mathcal{L}} \mu(\mathrm{d}\phi) < \infty,$$

when $\mathcal{F}_0$ is assumed to be rich enough to allow the existence of an $\mathcal{F}_0$-measurable random variable with distribution μ.

Definition 2.2. Denote by $\mathbb{P}(\mathcal{L})$ the set of Borel probability measures on $\mathcal{L}$ endowed with the topology of weak convergence of measures. For $\mu_1, \mu_2 \in \mathbb{P}(\mathcal{L})$ define a metric on $\mathbb{P}(\mathcal{L})$ by

$$d(\mu_1, \mu_2) = \sup_{f \in \mathcal{M}} \left| \int_{\mathcal{L}} f(\phi)\mu_1(\mathrm{d}\phi) - \int_{\mathcal{L}} f(\phi)\mu_2(\mathrm{d}\phi) \right|,$$

where

$$\mathcal{M} = \{f \in C(\mathcal{L}, \mathbb{R}) : |f(\phi) - f(\psi)| \leq \|\phi - \psi\|_{\mathcal{L}} \text{ for all } \phi, \psi \in \mathcal{L} \text{ and } |f(\cdot)| \leq 1\}.$$

It is well known that $\mathbb{P}(\mathcal{L})$ is complete under the metric $d(\cdot, \cdot)$ (see [16, Theorem 2.4.9]).

In order to show the existence of an invariant measure, we consider the segments of a solution. In contrast to the scalar solution process, the process of segment $\{X_t(\phi) : t \geq 0\}$ is a Markov process [17, 18]. It is shown that the segment process is also Feller and there exists a solution of which the segments are tight (see, for example, [17] for more details). Then we apply the Krylov–Bogoliubov method. In fact, we have the following result.

Lemma 2.3. *Suppose that for any bounded subset U of $\mathcal{L}$,*

(i) $\lim_{t\to\infty} \sup_{\phi,\psi \in U} \mathbb{E}\|X_t(\phi) - X_t(\psi)\|^2_{\mathcal{L}} = 0$;

(ii) $\sup_{t \geq 0} \sup_{\phi \in U} \mathbb{E}\|X_t(\phi)\|^2_{\mathcal{L}} < \infty$.

Then, for any initial condition $\phi \in \mathcal{L}$, the solution of equation (1.1) converges to an invariant measure.

Proof. It suffices to show that for any initial condition $\phi \in \mathcal{L}$, $\{\mathbb{P}(\phi, t, \cdot) : t \geq 0\}$ is Cauchy in the space $\mathbb{P}(\mathcal{L})$ with the metric $d(\cdot, \cdot)$ in Definition 2.2. For this purpose, we only need to show that for any initial data $\phi \in \mathcal{L}$ and $\varepsilon > 0$, there exists a time $T > 0$ such that

$$d(\mathbb{P}(\phi, t+s, \cdot), \mathbb{P}(\phi, t, \cdot)) = \sup_{f \in \mathcal{M}} |\mathbb{E}f(X_{t+s}(\phi)) - \mathbb{E}f(X_t(\phi))| \leq \varepsilon, \qquad \forall t \geq T, s > 0. \tag{2.3}$$

The proof is referred to Lemma 5.1 in [28]. Here we shall provide the details for the sake of completeness. For any $f \in \mathcal{M}$ and $t, s > 0$, note that

$$\begin{aligned}
&|\mathbb{E}f(X_{t+s}(\phi)) - \mathbb{E}f(X_t(\phi))| \\
&\quad = |\mathbb{E}[\mathbb{E}f(X_{t+s}(\phi))|\mathcal{F}_s] - \mathbb{E}f(X_t(\phi))| \\
&\quad = \left| \int_{\mathcal{L}} \mathbb{E}f(X_t(\psi))\mathbb{P}(X_s(\phi), \mathrm{d}\psi) - \mathbb{E}f(X_{t+s}(\phi)) \right| \\
&\quad \leq \int_{\mathcal{L}} |\mathbb{E}f(X_t(\psi)) - \mathbb{E}f(X_t(\phi))| \mathbb{P}(X_s(\phi), \mathrm{d}\psi) \\
&\quad \leq 2\mathbb{P}(X_s(\phi), \mathcal{L}^c_R) + \int_{\mathcal{L}_R} |\mathbb{E}f(X_t(\psi)) - \mathbb{E}f(X_t(\phi))| \mathbb{P}(X_s(\phi), \mathrm{d}\psi),
\end{aligned} \tag{2.4}$$

where $\mathcal{L}_R = \{\phi \in \mathcal{L} : \|\phi\|_{\mathcal{L}} \le R\}$ and $\mathcal{L}_R^c = \mathcal{L} - \mathcal{L}_R$. By virtue of condition (i), there exists a time $T_2 > 0$ such that

$$\sup_{f\in\mathcal{M}} |\mathbb{E}f(X_t(\phi)) - \mathbb{E}f(X_t(\psi))| \le \frac{\varepsilon}{2}, \qquad t \ge T_2.$$

On the other hand, condition (ii) implies that there exists a positive sufficiently large constant R such that

$$\mathbb{P}(X_s(\phi), \mathcal{L}_R^c) \le \frac{\varepsilon}{4}, \qquad \forall s > 0.$$

Hence (2.3) holds and the transition probability $\mathbb{P}(X_s(\phi), \cdot)$ of $X_t(\phi)$ converges weakly to some $\mu \in \mathbb{P}(\mathcal{L})$. For every $f \in C_b(\mathcal{L})$ the Markovian property of $X_t(\phi), t \ge 0$ gives that

$$\mathbb{P}_{t+s} f(\phi) = \mathbb{P}_t \mathbb{P}_s f(\phi) \qquad t, s \ge 0, \phi \in \mathcal{L}.$$

Let $s \to \infty$, it follows that

$$\mu(f) = \mu(\mathbb{P}_t f), \qquad f \in C_b(\mathcal{L}).$$

That is, μ is an invariant measure for $X_t(\phi), t \ge 0$. The proof is completed. $\square$

3 Stochastic systems driven by Wiener process

In this section we consider equation (1.1) with $Z = \{W(t) : t \ge 0\}$, which denotes a $\mathcal{U}$-valued $\{\mathcal{F}_t : t \ge 0\}$-Wiener process defined on $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with covariance operator Q, i.e.,

$$\mathbb{E}\langle W(t), x\rangle\langle W(s), y\rangle = (t \wedge s)\langle Qx, y\rangle \quad \text{for all } x, y \in \mathcal{U},$$

where Q is a linear, symmetric and nonnegative bounded operator on U. In particular, we shall call $\{W(t) : t \ge 0\}$, a $\mathcal{U}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t : t \ge 0\}$.

First, we shall show the solution process is tight. Let $\mathfrak{L}_2^Q(\mathcal{U}, H)$ is the space of all Hilbert–Schmidt operators from $\mathcal{U}$ to H with $\|G\|^2_{\mathfrak{L}_2^Q} := \mathrm{Tr}_H(GQG^*)$. For any $t \ge 0$ and $G(t) \in \mathfrak{L}_2^Q(\mathcal{U}, H)$, let

$$Q_t = \int_0^t \Phi(s)G(s)QG^*(s)\Phi^*(s)\mathrm{d}s,$$

where $G^*(s)$ and $\Phi^*(s)$ are the adjoint operators of $G(s)$ and $\Phi(s)$, respectively. We suppose that

$$\mathrm{Tr}(Q_t) = \int_0^t \mathrm{Tr}[\Phi(s)G(s)QG^*(s)\Phi^*(s)]\mathrm{d}s < \infty \quad \text{for any } t \ge 0. \tag{3.1}$$

Throughout this section, the operator F: $\mathcal{L} \to H$ is supposed to be Lipschitz continuous while the operator G: $\mathcal{L} \to \mathfrak{L}_2^Q(\mathcal{U}, H)$ is supposed to be Lipschitz continuous with respect to the Hilbert-Schmidt norm $\mathfrak{L}_2^Q(\mathcal{U}, H)$ of linear operators from $\mathcal{U}$ to H:

$$\begin{aligned} \|F(x) - F(y)\| + \|G(x) - G(y)\|_{\mathfrak{L}_2^Q} &\le K\|x - y\|_{\mathcal{L}}, \\ \|F(x)\| + \|G(x)\|_{\mathfrak{L}_2^Q} &\le K_1\|x\|_{\mathcal{L}} + K_2 \end{aligned} \tag{3.2}$$

for all $x, y \in \mathcal{L}$, where K, K_1, K_2 are nonpositive constants. Note that under hypotheses (2.1) and (3.2), (1.1) has a unique mild solution of which the segment is a Markov and Feller process (see [33, 34, 39] for more details). In the subsequent two subsections, we investigate the existence of invariant measure and random attractor as well as the exponential stability of stationary solutions.

3.1 Invariant measure

Lemma 3.1. *Assume that $2K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$, Then all trajectories of solution processes (2.2) converge exponentially together in the mean-square sense. In particular, for any bounded subset U of $\mathcal{L}$,*

$$\lim_{t\to\infty}\sup_{\phi,\psi\in U}\mathbb{E}\|X_t(\phi)-X_t(\psi)\|_{\mathcal{L}}^2 = 0.$$

Proof. It follows from $2(1+\varepsilon)K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$ that there exists $\varepsilon > 0$ such that $2(1+\varepsilon)K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$. Note that

$$(A+B+C)^2 \le (1+\varepsilon)(A+B)^2 + \left(1+\frac{1}{\varepsilon}\right)C^2 \le 2(1+\varepsilon)(A^2+B^2) + \left(1+\frac{1}{\varepsilon}\right)C^2$$

for all $A, B, C \ge 0$. Then it follows from (2.2) that for $t > \tau$,

$$\begin{aligned}
&\mathbb{E}\|X_t(\phi)-X_t(\psi)\|_{\mathcal{L}}^2 \\
&\quad\le \mathbb{E}\Bigg\{\left(1+\frac{1}{\varepsilon}\right)\int_{-\tau}^{0}\|\Phi(t+\theta)(\phi(0)-\psi(0))\|^2\mathrm{d}\theta \\
&\qquad + 2(1+\varepsilon)\int_{-\tau}^{0}\left\|\int_0^{t+\theta}\Phi(t+\theta-s)(F(X_s(\phi))-F(X_s(\psi)))\mathrm{d}s\right\|^2\mathrm{d}\theta \\
&\qquad + 2(1+\varepsilon)\int_{-\tau}^{0}\left\|\int_0^{t+\theta}\Phi(t+\theta-s)(G(X_s(\phi))-G(X_s(\psi)))\mathrm{d}W\right\|^2\Bigg\}\mathrm{d}\theta \\
&\quad =: \left(1+\frac{1}{\varepsilon}\right)I_1 + 2(1+\varepsilon)I_2 + 2(1+\varepsilon)I_3.
\end{aligned} \tag{3.3}$$

Following (2.1) we have

$$I_1 \le \frac{e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\lambda}\|\phi(0)-\psi(0)\|^2. \tag{3.4}$$

From (2.1), (3.2) and Hölder's inequality, it follows that for $t > \tau$,

$$\begin{aligned}
I_2 &\le \mathbb{E}\int_{-\tau}^{0}\Bigg[\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}ds \\
&\qquad \int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}\|F(X_s(\phi))-F(X_s(\psi))\|^2\mathrm{d}s\Bigg]\mathrm{d}\theta \\
&\le \frac{K^2}{\lambda}\int_{-\tau}^{0}\int_0^{t+\theta}(1-e^{-\lambda(t+\theta)})e^{-\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\le \frac{K^2e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s.
\end{aligned} \tag{3.5}$$

Using (2.1), (3.2) and the Burkholder–Davis–Gundy inequatlity (see, for example [27, Theorem 6.1]), we get

$$\begin{aligned}
I_3 &\le \mathbb{E}\int_{-\tau}^{0}\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}^2\|G(X_s(\phi))-G(X_s(\psi))\|_{\mathfrak{L}_2^Q}^2\mathrm{d}s\mathrm{d}\theta \\
&\le K^2\int_{-\tau}^{0}\int_0^{t+\theta}e^{-2\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\le K^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s
\end{aligned} \tag{3.6}$$

for $t > \tau$. Then from (3.4), (3.5) and (3.6), we obtain

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|^2 \le \frac{(e^{2\lambda\tau}-1)(1+\varepsilon)}{2\lambda\varepsilon}\|\phi(0)-\psi(0)\|^2 + 2(1+\varepsilon)K^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau})\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2 \mathrm{d}s.$$

Using Gronwall's inequality, we have

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \le \frac{(e^{2\lambda\tau}-1)(1+\varepsilon)}{2\lambda\varepsilon}\|\phi(0)-\psi(0)\|^2 \tag{3.7}$$

The proof is completed. □

Then we will prove the segment process of solution to (1.1) is bounded with Wiener process.

Lemma 3.2. *Assume that $2K_1^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$. Then the solution process (2.2) is ultimately bounded in the mean-square sense, i.e., for any bounded set U of $\mathcal{L}$,*

$$\sup_{t\ge 0}\sup_{\phi\in U}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 < \infty.$$

Proof. It follows from $2K_1^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$ that there exists $\varepsilon > 0$ such that $2K_1^2(e^{2\lambda\tau} + \lambda^{-1}e^{\lambda\tau})(1+\varepsilon)^2 < \lambda$. Similar to (3.3), it follows from (2.2) that for all $t \ge 0$

$$\begin{aligned}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le \mathbb{E}\Big\{ &\left(1+\frac{1}{\varepsilon}\right)\int_{-\tau}^0 \|\Phi(t+\theta)\phi(0)\|^2 \mathrm{d}\theta \\ &+ 2\,(1+\varepsilon)\int_{-\tau}^0 \left\|\int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2 \mathrm{d}\theta \\ &+ 2\,(1+\varepsilon)\int_{-\tau}^0 \left\|\int_0^{t+\theta}\Phi(t+\theta-s)G(X_s(\phi))dW\right\|^2 \mathrm{d}\theta\Big\} \\ =: &\left(1+\frac{1}{\varepsilon}\right)J_1 + 2\,(1+\varepsilon)\,J_2 + 2\,(1+\varepsilon)\,J_3.\end{aligned} \tag{3.8}$$

From (2.1) we have

$$J_1 \le \frac{e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\lambda}\|\phi(0)\|^2. \tag{3.9}$$

Note that

$$(A+B)^2 \le (1+\varepsilon)A^2 + \left(1+\frac{1}{\varepsilon}\right)B^2.$$

Following (2.1), (3.2) and Hölder's inequality we have

$$\begin{aligned}J_2 &\le \mathbb{E}\int_{-\tau}^0 \left\|\int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2 \mathrm{d}\theta \\ &\le \mathbb{E}\int_{-\tau}^0 \left\|\int_0^{t+\theta}\Phi(t+\theta-s)[K_1\|X_s(\phi)\|_{\mathcal{L}}+K_2]\mathrm{d}s\right\|^2 \mathrm{d}\theta \\ &\le \left(1+\frac{1}{\varepsilon}\right)K_2^2\int_{-\tau}^0\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}^2\,\mathrm{d}s\mathrm{d}\theta \\ &\quad + (1+\varepsilon)K_1^2\int_{-\tau}^0\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}\,\mathrm{d}s\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(1+\varepsilon)\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)K_1^2}{\lambda}\int_{-\tau}^{0}\int_{0}^{t+\theta}(1-e^{-\lambda(t+\theta)})e^{-\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)e^{\lambda\tau}K_1^2}{\lambda}\int_{0}^{t} e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s
\end{aligned} \tag{3.10}
$$

for $t > \tau$. It follows from (2.1), (3.2) and the Burkholder–Davis–Gundy inequality that

$$
\begin{aligned}
J_3 &\leq \int_{-\tau}^{0}\int_{0}^{t+\theta}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))\right\|^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\left[(1+\varepsilon)K_1^2\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2 + \left(1+\frac{1}{\varepsilon}\right)K_2^2\right]\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\,K_2^2}{2\varepsilon\lambda}\int_{-\tau}^{0}(1-e^{-2\lambda(t+\theta)})\mathrm{d}\theta + (1+\varepsilon)K_1^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda} + (1+\varepsilon)K_1^2e^{2\lambda\tau}\int_{0}^{t}e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s
\end{aligned} \tag{3.11}
$$

for $t > \tau$. Thus, (3.9), (3.10) and (3.11) together imply that for $t > \tau$,

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\|X_t(\phi)\|^2 &\leq \frac{(1+\varepsilon)\,(e^{2\lambda\tau-1})}{2\varepsilon\lambda}\|\phi(0)\|^2 + \frac{(1+\varepsilon)^2K_2^2\tau}{\varepsilon\lambda}e^{\lambda t} \\
&\quad + 2K_1^2(1+\varepsilon)^2(\lambda^{-1}e^{\lambda\tau}+e^{2\lambda\tau})\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s \\
&= \alpha_1 + \gamma_1 e^{\lambda t} + \beta_1\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{(1+\varepsilon)\,(e^{2\lambda\tau-1})}{2\varepsilon\lambda}\|\phi(0)\|^2, \\
\gamma_1 &= \frac{(1+\varepsilon)^2K_2^2\tau}{\varepsilon\lambda}, \\
\beta_1 &= 2K_1^2(1+\varepsilon)^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau}).
\end{aligned}
$$

Then Gronwall's inequality gives that

$$
e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \leq \alpha_1 + \gamma_1 e^{\lambda t} + \beta_1\int_0^t\left(\gamma_1 e^{\lambda s}+\alpha_1\right)e^{\beta_1(t-s)}\mathrm{d}s
$$

and hence that

$$
\begin{aligned}
\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 &\leq \gamma_1 + \alpha_1 e^{-\lambda t} + \beta_1 e^{(\beta_1-\lambda)t}\int_0^t\left(\gamma_1 e^{\lambda s}+\alpha_1\right)e^{-\beta_1 s}\mathrm{d}s \\
&\leq \gamma_1 + 2\alpha_1 + \frac{\beta_1\gamma_1}{\lambda-\beta_1}
\end{aligned}
$$

for $t > \tau$. This completes the proof. □

By Lemmas 2.3, 3.1 and 3.2, we can have the following result about the existence of invariant measures of equation (1.1) driven by Wiener process. Now we show the uniqueness of invariant measures. If $\mu, \mu' \in \mathbb{P}(\mathcal{L})$ are two different invariant measures for X_t of (1.1), for any $f \in \mathcal{M}$, by virtue of (3.7), Hölder's inequality and the invariance of $\mu(\cdot), \mu'(\cdot)$, it follows that

$$
|\mu(f)-\mu'(f)| \leq \int_{\mathcal{L}\times\mathcal{L}}|\mathbb{P}_t f(\phi)-\mathbb{P}_t f(\psi)|\mu(\mathrm{d}\phi)\mu(\mathrm{d}\psi) \leq K_3 e^{\alpha t}, \qquad t \geq 0,
$$

for some constant $K_3 > 0$, where $\alpha = K^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau}) - \frac{1}{2}\lambda < 0$ under the assumption in Lemma 3.1. We obtain the uniqueness of invariant measures by letting $t \to \infty$.

Theorem 3.3. *Under the assumptions of Lemmas 3.1 and 3.2, equation (1.1) driven by Wiener process has a unique invariant measure.*

3.2 Random attractor

We consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}; H) : \omega(0) = 0\},$$

and $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω (see [3]), while $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega, \mathcal{F}, \mathbb{P})$. Define a shift operators by flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on Ω:

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R},\ \omega \in \Omega.$$

Then, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system, that is, $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t\theta_s$ for all $s, t \in \mathbb{R}$, and $\theta_t(\mathbb{P}) = \mathbb{P}$ for all $t \in \mathbb{R}$. More precisely, $\mathbb{P}$ is ergodic with respect to θ. In addition, with respect to the filtration we have that

$$\theta_s^{-1}\bar{\mathcal{F}}_t = \bar{\mathcal{F}}_{t+s} \tag{3.12}$$

for any $t, s \in \mathbb{R}$, where $\bar{\mathcal{F}}$ is the completion of $\mathcal{F}$, see [3, Definition 2.3.4] for more details. For the sake of convenience, from now on, we will abuse the notation slightly and write the space $\overline{\Omega}$ as Ω.

The Wiener process with covariance Q is adapted to the filtration $\{\bar{\mathcal{F}}_{s+t}\}_{t\geq 0}$. First we define a mean-square random dynamical system referring to [24, 39]. Let

$$\mathbb{R}^2_{\geq} \triangleq \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\},$$

and

$$\Pi \triangleq L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathcal{L}), \qquad \Pi_t \triangleq L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathcal{L})$$

for each $t \in \mathbb{R}$.

Definition 3.4 ([39, Definition 10])**.** A mean-square random dynamical system (MS-RDS) Ψ on $\mathcal{L}$ with probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a family of mappings

$$\Psi(t, t_0, \cdot) : \Pi_{t_0} \to \Pi_t, \qquad (t, t_0) \in \mathbb{R}^2_{\geq},$$

which satisfies

(i) initial value property: $\Psi(t, t_0, \psi) = \psi$ for every $\psi \in \Pi_{t_0}$;

(ii) two-parameter semigroup property: $\Psi(t_2, t_0, \psi) = \Psi(t_2, t_1, \Psi(t_1, t_0, \psi))$ for all $t_2 \geq t_1 \geq t_0$;

(iii) continuity property: $(t, t_0, \psi) \mapsto \Psi(t, t_0, \psi)$ is continuous in the space $\mathbb{R}^2_{\geq} \times \Pi$.

Definition 3.5 ([39, Definition 11])**.** A family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty subsets of Π with $A_t \subset \Pi_t$ is said to be Ψ-invariant if

$$\Psi(t, t_0, A_{t_0}) = A_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq},$$

and Ψ-positively invariant if

$$\Psi(t, t_0, A_{t_0}) \subset A_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq}.$$

Definition 3.6 ([39, Definition 12])**.** A Ψ-invariant family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty compact subsets of $\{\Pi_t\}_{t\in\mathbb{R}}$ is called a mean-square pullback attractor if it pullback attracts all families $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of uniformly bounded subsets of $\{\Pi_t\}_{t\in\mathbb{R}}$, i.e., for any fixed $t \in \mathbb{R}$

$$\mathrm{dist}(\Psi(t, t_0, B_{t_0}), A_t) \to 0 \quad \text{as } t_0 \to -\infty.$$

Let $X(\cdot, t_0, \phi_0)$ be the solution of the following equation with initial value $\phi_0 \in \Pi_{t_0}$,

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + F(X_t)]\mathrm{d}t + G(X_t)\mathrm{d}W(t), & t > t_0, \\ X(t_0 + s) = \phi_0(s), \quad s \in [-\tau, 0]. \end{cases} \tag{3.13}$$

For each $(t, t_0, \phi_0) \in \mathbb{R}^2_{\geq} \times \Pi_{t_0}$, define solution mapping of (3.13):

$$\Psi(t, t_0, \phi_0) = X_t(\cdot, t_0, \phi_0) = X(t + \cdot, t_0, \phi_0).$$

It is easy to see that Ψ satisfies the initial value property,

$$\Psi(t_0, t_0, \phi_0) = X_{t_0}(\cdot, t_0, \phi_0) = X(t_0 + \cdot) = \phi_0$$

for all $(t_0, \phi_0) \in \mathbb{R} \times \Pi_{t_0}$. Existence and uniqueness of solution of (3.13) show that Ψ satisfies the two-parameter semigroup evolution property. Moreover, Ψ is continuous for all $(t, t_0, \phi_0) \in \mathbb{R}^2_{\geq} \times \Pi_{t_0}$ since solution $X_t(\cdot, t_0, \phi_0)$ is continuous with respect to t, ϕ_0. Thus, (3.13) generates a continuous MS-RDS $\Psi = \{\Psi(t, t_0, \cdot), (t, t_0) \in \mathbb{R}^2_{\geq}\}$ with state space $\mathcal{L}$.

It follows from Lemma 3.2 that for any bounded set U of Π there exist constants $B > 0$ and $T_U \geq 0$ such that for all $t \geq t_0 + T_U$ and $\phi_0 \in U \cap \Pi_{t_0}$,

$$\mathbb{E}\|\Psi(t, t_0, \phi_0)\|^2 < B,$$

which can be represented in the pullback sense that

$$\mathbb{E}\|\Psi(t, t_n, \phi_n)\|^2 < B$$

for all $t_n \leq t - T_B$ and $\phi_n \in U \cap \Pi_{t_n}$. Lemma 3.1 shows that any two solutions converge together in the mean-square sense uniformly for different initial conditions at the same starting time. Namely, for any $\phi_0, \psi_0 \in \Pi_{t_0}$,

$$\mathbb{E}\|\Psi(t, t_0, \phi_0) - \Psi(t, t_0, \psi_0)\|^2 \to 0 \quad \text{as } t \to \infty,$$

with the convergence being uniform for initial values in a common bounded subset as well as in the initial time t_0. Let U_B be a bounded ball about the origin of radius B in $\mathcal{L}$. Consider a sequence $t_n \to -\infty$ as $n \to \infty$ with $t_n < -T_U - \tau$ and $t_{n+1} \leq t_n - T_{B_U}$ and define a sequence $\{\chi_n\}_{n=1}^{\infty}$ in $U_B \cap \Pi_0$ by

$$\chi_n \triangleq \Psi(0, t_n, \phi_n) \tag{3.14}$$

for an arbitrary $\phi_n \in U_B \cap \Pi_{t_n}$. Namely,

$$\chi_n(s) = \Psi(s, t_n, \phi_n)$$

for all $s \in [-\tau, 0]$. Then $\{\chi_n\}_{n=1}^{\infty}$ are obviously mean-square bounded by B for all ϕ_n taking values in $U_B \cap \Pi_{t_n}$.

Lemma 3.7. *$\{\chi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with values in $U_B \cap \Pi_0$ and there exists a unique limit $\chi_0^* \in U_B \cap \Pi_0$ such that*

$$\mathbb{E}\|\chi_n - \chi_0^*\|^2 \to 0 \quad \text{as } n \to \infty.$$

Proof. It suffices to prove that for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

$$\mathbb{E}\|\chi_n - \chi_m\|^2 \leq \varepsilon \quad \text{for all } n, m \geq N_\varepsilon. \tag{3.15}$$

Let $t_m < t_n < 0$. Then we have

$$\chi_m = \Psi(0, t_m, \phi_m) = \Psi(0, t_n, \Psi(t_n, t_m, \phi_m)) = \Psi(0, t_n, \widehat{\phi_{n,m}}),$$

where $\widehat{\phi_{n,m}} := \Psi(t_n, t_m, \phi_m) \in U_B \cap \Pi_{t_n}$. Indeed,

$$\mathbb{E}\|\chi_n - \chi_m\|^2 = \mathbb{E}\|\Psi(0, t_n, \phi_n) - \Psi(0, t_m, \phi_m)\|^2 = \mathbb{E}\|\Psi(0, t_n, \phi_n) - \Psi(0, t_n, \widehat{\phi_{n,m}})\|^2.$$

Thus, it follows from Lemma 3.1 that (3.15) holds and all solutions starting in the common bounded subset U_B converge in Π_0. Since Π_0 is complete, the Cauchy sequence has a unique limit $\chi_0^* \in U_B \cap \Pi_0$. The proof is completed. □

From the above process, we can repeat with 0 in (3.14) replaced by -1 to obtain a limit $\chi_{-1}^* \in \Pi_{-1}$. It is easy to see from the construction that $\chi_0^* = \Psi(0, -1, \chi_{-1}^*)$. Follow this way, we can construct a sequence $\{\chi_{-n}^*\}_{n \in \mathbb{N}}$ and hence obtain an entire MS-RDS χ_t^* for all $t \in \mathbb{R}$. Moreover, all other MS-RDS trajectories converge to χ_t^* in the mean-square sense.

Theorem 3.8. *Under the assumptions of Lemmas 3.1 and 3.2, there exists a pullback random attractor for the random dynamical system generated by (1.1) which consists of singleton sets. Furthermore, the random attractor pullback attracts all other solution processes in mean-square sense.*

Proof. The above arguments shows the existence of random attractor consisting of singleton sets $A_t = \{\chi_t^*\}$ and attracts all other solution processes in the mean-square sense. Next we show the random attractor is unique. Suppose there is another entire trajectory $\bar{\chi}_t^* \in A_t$ for all $t \in \mathbb{R}$ and there exists a constant $\varepsilon_0 > 0$ such that

$$\mathbb{E}\|\chi_0^* - \bar{\chi}_0^*\|^2 \geq \varepsilon_0.$$

On the other hand, it follows from the convergence in Lemma 3.1 that there exists $T \geq 0$ such that

$$\mathbb{E}\|\Psi(0, -t, \chi_{-t}^*) - \Psi(0, -t, \bar{\chi}_{-t}^*)\|^2 \leq \frac{\varepsilon_0}{2}$$

for all $t \geq T$. Note that $\chi_0^* = \Psi(0, -t, \chi_{-t}^*)$ and $\bar{\chi}_0^* = \Psi(0, -t, \bar{\chi}_{-t}^*)$. Hence

$$\varepsilon_0 \leq \mathbb{E}\|\chi_0^* - \bar{\chi}_0^*\|^2 = \mathbb{E}\|\Psi(0, -t, \chi_{-t}^*) - \Psi(0, -t, \bar{\chi}_{-t}^*)\|^2 \leq \frac{\varepsilon_0}{2}$$

for $t \geq T$, which is a contradiction. This completes the proof. □

Remark 3.9. If the random attractor $\mathcal{A}(\omega), \omega \in \Omega$ consists of a single point, then $\mathcal{A}$ defines a random fixed point which attracts tempered random sets.

3.3 Exponential stability of stationary solutions

Note that zero is not a solution to the equation (1.1). In this subsection we shall prove that the non-trivial stationary solutions to equation (1.1) with Wiener process are exponentially stable. Consider the process $\theta_s W(\cdot, \omega) = W(\cdot, \theta_s \omega) = W(\cdot + s, \omega) - W(s, \omega)$, for $s \in \mathbb{R}$ which is adapted to the filtration $\{\mathcal{F}_{t+s}\}_{t \geq 0}$. The following equality holds for $\psi_0 \in \Pi_0$, $t \geq 0$, $s \in \mathbb{R}$:

$$\Psi(t, 0, \psi_0)(\theta_s \cdot) = \Psi(t, s, \psi_s)(\cdot) \quad \text{almost surely } t, s \in \mathbb{R}, \tag{3.16}$$

where $\psi_s(\cdot) := \psi_0(\theta_s \cdot)$. It follows from (3.12) that $x_0(\theta_s \cdot)$ is $\bar{\mathcal{F}}_s$-measurable. The following lemma is obvious from the proof of Lemma 3.1.

Lemma 3.10. *For $s \in \mathbb{R}$, $\epsilon \geq 0$, $\psi \in \Pi_s$,*

$$\Psi(\cdot, s+\epsilon, \Psi(\epsilon, s, \psi)) = \Psi(\cdot+\epsilon, s, \psi) \ \textit{almost surely.}$$

Now we can show the existence of the fixed point.

Theorem 3.11. *Under the assumptions in Lemma 3.1, there exists an exponentially attracting fixed point $X^* \in \Pi_0$ which generates an exponentially stable stationary solution for (1.1). In addition, the process $(t, \omega) \to X^*(\theta_t \omega)$ has a continuous version given by $\Psi(\cdot, 0, X^*)$.*

Proof. First we claim that $(\Psi(k, -k, \psi_0(\theta_{-k}\cdot)))_{k\in\mathbb{N}}$ is a Cauchy sequence in Π_0. It follows from Lemma 3.10 and (3.16) that

$$\begin{aligned}
&\mathbb{E}\|\Psi(k, -k, \psi_0(\theta_{-k}\cdot)) - \Psi(k-1, 1-k, \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad = \mathbb{E}\|\Psi(k-1, 1-k, \Psi(k, -k, \psi_0(\theta_{-k}\cdot)))) - \Psi(k-1, 1-k, \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad \leq e^{\sigma(k-1)}\mathbb{E}\|\Psi(1, -k, \psi_0(\theta_{-k}\cdot)) - \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad = e^{\sigma(k-1)}\mathbb{E}\|\Psi(1, 0, \psi_0(\cdot)) - \psi_0(\theta_1\cdot))\|^2.
\end{aligned}$$

It follows from Lemma 3.1 that $\sigma < 0$ and then the Cauchy sequence property holds. Let $X^* \in \Pi_0$ be the limit of this sequence, i.e. in L^2-norm sense,

$$X^*(\theta_t\cdot) = \lim_{k\to\infty} \Psi(k, -k, \psi_0(\theta_{-k}\cdot))(\theta_t\omega),$$

which is equal to

$$X^*(\theta_t\cdot) = \lim_{k\to\infty} \Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))(\omega).$$

For $\psi_0, \phi_0 \in \Pi_0$, we have

$$\begin{aligned}
&\mathbb{E}\|\Psi(k, -k, \psi_0(\theta_{-k}\cdot)) - \Psi(k, -k, \phi_0(\theta_{-k}\cdot))\|^2 \\
&\quad = \mathbb{E}\|\Psi(k, 0, \psi_0(\cdot)) - \Psi(k, 0, \phi_0(\cdot))\|^2 \leq e^{\sigma k}\mathbb{E}\|\psi_0 - \phi_0\|^2_{\mathcal{L}},
\end{aligned}$$

which tends to zero as k goes to infinity. Thus $X^* \in \Pi_0$ is exponentially stable and independent of the choice of $\psi_0 \in \Pi_0$.

Next, we show that X^* is a fixed point, i.e., for any $t \in \mathbb{R}^+$

$$\Psi(t, 0, X^*)(\cdot) = X^*(\theta_t\cdot) \quad \text{almost surely.}$$

Indeed, for any fixed t, from (3.16), Lemma 3.10 and semigroup property we have

$$\begin{aligned}
&\mathbb{E}\|\Psi(t, 0, X^*) - X^*(\theta_t\cdot)\|^2 \\
&\quad = \mathbb{E}\left\|\Psi\left(t, 0, \lim_{k\to\infty}\Psi(k, -k, \psi_0(\theta_{-k}\cdot))\right) - \lim_{k\to\infty}\Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))\right\|^2 \\
&\quad = \lim_{k\to\infty}\mathbb{E}\|\Psi(k, t-k, \Psi(t, -k, \psi_0(\theta_{-k}\cdot)) - \Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))\|^2 \\
&\quad \leq \lim_{k\to\infty} e^{\sigma k}\mathbb{E}\|\Psi(t, -k, \psi_0) - \psi_0(\theta_t\cdot)\|^2 = 0.
\end{aligned}$$

This completes the proof. $\square$

4 Systems driven by Lévy jump process

In this section, we will give the existence of invariant measures of (1.1) with Lévy jump process in separable Hilbert space $\mathcal{U}$. To this end, it suffices to verify the assertions in Lemma 2.3 hold.

Let $Z = \{Z(t) : t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that Z is a *Lévy process* if:

(i) $Z(0) = 0$ (a.s.);

(ii) Z has independent and stationary increments;

(iii) Z is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(\|Z(t) - Z(s)\|_{\mathcal{U}} > a) = 0.$$

We have the following property of Lévy measures on separable Hilbert spaces (see [36]).

Lemma 4.1. *Let $\mathcal{U}$ be a separable Hilbert space. Then a σ-finite measure ν with $\nu(\{0\}) = 0$ is on $\mathcal{U}$ if and only if*

$$\int_{\mathcal{U}} (1 \wedge \|y\|_{\mathcal{U}}^2) \nu(\mathrm{d}y) < \infty.$$

$\nu(\cdot)$ is also called a Lévy measure.

The jump process $\Delta Z = \{\Delta Z(t) : t \geq 0\}$ is defined by

$$\Delta Z(t) = Z(t) - Z(t-)$$

for each $t \geq 0$, where $Z(t-)$ is the left limit at the point t. Furthermore, ΔZ is a Poisson point process. The Poisson process of intensity $\lambda > 0$ is a Lévy process with N taking values in $\mathbb{N} \cup \{0\}$ wherein each $N(t) \sim \pi(\lambda t)$. For $t > 0$ and $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$, define

$$N(t, \Gamma)(\omega) = \sum_{s \in (0,t]} 1_{\Gamma}(\Delta Z(s)(\omega)), \tag{4.1}$$

if $\omega \in \Omega_0$, and $N(t, \Gamma)(\omega) = 0$, if $\omega \in \Omega_0^c$, where $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \to Z(t)(\omega)$ is càdlàg[1] for all $\omega \in \Omega_0^c$. We write $\nu(\cdot) = \mathbb{E}(N(1, \cdot))$ and call it the intensity measure associated with Z. We say that $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ is bounded below if $0 \notin \bar{\Gamma}$. The following results are from [2].

Lemma 4.2. (i) *If Z is a Lévy process, then for fixed $t > 0$, $\Delta Z(t) = 0$ (a.s.);*

(ii) *If Γ is bounded, then $N(t, \Gamma) < \infty$ (a.s.) for all $t \geq 0$;*

(iii) *If Γ is bounded, then $\{N(t, \Gamma) : t \geq 0\}$ is a Poisson process with intensity $\nu(\Gamma)$.*

Let S be a set and $\mathcal{A}$ be a ring of subsets of S. Clearly, if $\mathcal{F}$ is a σ-algebra then it is also a ring. *A random measure* M on $(S, \mathcal{A})$ is a collection of random variables $\{M(B) : B \in \mathcal{A}\}$ such that (i) $M(\varnothing) = 0$, (ii) given any disjoint A, $B \in \mathcal{A}$, $M(A \cup B) = M(A) + M(B)$. A random measure is said to be *independently scattered* if for each disjoint family $\{B_1, \ldots, B_n\}$ in $\mathcal{A}$, the random variables $M(B_1), \ldots, M(B_n)$ are independent.

[1] Let $I = [a, b]$ be an interval in $\mathbb{R}^+$. A mapping $f : I \to \mathbb{R}^d$ is said to be càdlàg if, for all $t \in [a, b]$, f has a left limit at t and f is right-continuous at t.

Let $\mathcal{S}$ be a σ-algebra of subsets of set S. Fix a non-trivial ring $\mathcal{A} \subseteq \mathcal{S}$, an independently scattered σ-finite random measure M on $(S, \mathcal{S})$ is called a *Poisson random measure* if $M(B) < \infty$ for each $B \in \mathcal{A}$ and each $M(B)$ has a Poisson distribution. It follows from (4.1) and Lemma 4.2 that $N(t, \Gamma)$ is a Poisson random measure and $\lambda(\cdot) = t\nu(\cdot)$.

Now introduce the *compensated Poisson process* $\widetilde{N} = \{\widetilde{N}(t) : t \geq 0\}$ where $\widetilde{N}(t) = N(t) - \nu t$. Note that $\mathbb{E}[\widetilde{N}(t)] = 0$ and $\mathbb{E}[\widetilde{N}(t)^2] = \nu t$ for each $t \geq 0$. Then $\widetilde{N}(t)$ is martingale, that is, for all $0 \leq s < t < \infty$, $\mathbb{E}(\widetilde{N}(t)|\mathcal{F}_s) = \widetilde{N}(s)$ a.s. For each $t \geq 0$ and Γ bounded, we define the *compensated Poisson random measure* by

$$\widetilde{N}(t, \Gamma) = N(t, \Gamma) - t\nu(\Gamma).$$

It is easy to see that $\widetilde{N}(t, \Gamma)$ is a σ-finite independently scattered martingale-valued measure.

Let $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$ and $f : \Gamma \to \mathcal{U}$ measurable. Define the following integral

$$\int_\Gamma f(z) N(t, \mathrm{d}z) = \sum_{0<s\leq t} f(\Delta Z(s)) 1_\Gamma(\Delta Z(s)).$$

This is a finite sum $\mathbb{P}$-a.s. since the number of summands is finite $\mathbb{P}$-a.s. For $f \in L^2_\nu \triangleq L^2(\mathcal{U} - \{0\}, \nu|_{\mathcal{U}-\{0\}}; \mathcal{U})$, the next proposition defines the integral with respect to the compensated Poisson random measure (see [36] for more details).

Proposition 4.3. *Let f be strongly square-integrable with respect to $\widetilde{N}(t, \mathrm{d}z)$ and $f \in L^2_\nu$. Then for any $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$ we have*

$$\int_\Gamma f(z) \widetilde{N}(t, \mathrm{d}z) = \sum_{0<s\leq t} f(\Delta Z(s)) 1_\Gamma(\Delta Z(s)) - t \int_\Gamma f(z) \nu(\mathrm{d}z).$$

Proposition 4.4 (cf. [1] and [36])**.** *Let $f \in L^2_\nu$ then for any $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ the integral $\int_\Gamma f(z) \widetilde{N}(t, \mathrm{d}z)$ exists and*

$$\mathbb{E}\left[\left\| \int_\Gamma f(z) \widetilde{N}(t, \mathrm{d}z) \right\|^2\right] = t \int_\Gamma \|f(z)\|^2 \nu(\mathrm{d}z) < \infty.$$

The following is a very important for Lévy processes called Lévy–Itô decomposition (see e.g. [1,2,36]).

Theorem 4.5. *Let $Z = \{Z(t) : t \geq 0\}$ be a Lévy process on a separable Hilbert space $\mathcal{U}$ where the distribution of $Z(t)$ has generating triplet $[tb, tQ, t\nu]$ for each $t \geq 0$,*

$$Z(t) = bt + W_Q(t) + \int_{\|z\|_{\mathcal{U}}<1} z \widetilde{N}(t, \mathrm{d}z) + \int_{\|z\|_{\mathcal{U}}\geq 1} z N(t, \mathrm{d}z),$$

where

$$b = \mathbb{E}\left[Z(1) - \int_{\|z\|_{\mathcal{U}}\geq 1} z N(1, \mathrm{d}z)\right]$$

and $W_Q = \{W_Q(t) : t \geq 0\}$ is a Wiener process with covariance operator Q independent of $N(\cdot, \Gamma)$ for all $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$.

Let Z be a $\mathcal{U}$-valued Lévy process with its Lévy triplet $(0, Q, \nu)$ below. By Lemma 4.1, $\nu(\Gamma)$ is a Lévy measure with $\Gamma \in \mathcal{B}(\Gamma - \{0\})$. Note that an adapted Lévy process with zero mean is martingale, and that a Lévy process is martingale if and only if it is integrable and

$$b + \int_{\|z\|_{\mathcal{U}}\geq 1} z\nu(\mathrm{d}z) = 0.$$

It follows from Lévy–Itô decomposition that the Lévy process can be written as

$$Z(t) = W_Q(t) + \int_{\mathcal{U}-\{0\}} z\widetilde{N}(t, \mathrm{d}z). \tag{4.2}$$

In view of Proposition 4.4, we have

$$K_z \triangleq \int_{\mathcal{U}-\{0\}} \|z\|_{\mathcal{U}}^2 \nu(\mathrm{d}z) < \infty.$$

Throughout this section, we always assume that the operators F and G in (1.1) satisfy

$$\begin{aligned} \|F(x) - F(y)\| + \|G(x) - G(y)\|_{\mathscr{L}(\mathcal{U},H)} &\le K\|x - y\|_{\mathcal{L}}, \\ \|F(x)\| + \|G(x)\|_{\mathscr{L}(\mathcal{U},H)} &\le K_1\|x\|_{\mathcal{L}} + K_2 \end{aligned} \tag{4.3}$$

for all $x, y \in \mathcal{L}$, where K, K_1, K_2 are nonnegative constants, $\mathscr{L}(\mathcal{U}, H)$ is the space of bounded linear operators from $\mathcal{U}$ to H.

The boundedness of solution with Lévy jump process is given as follows.

Lemma 4.6. *Assume that $3K_1^2 e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z\right] < \lambda$. Then for any bounded set U of $\mathcal{L}$,*

$$\sup_{t\ge 0}\sup_{\phi\in U} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 < \infty.$$

Proof. Using the similar arguments as the proof of Lemma 3.2 and the Lévy–Itô decomposition (4.2), we can obtain that for all $t \ge 0$

$$\begin{aligned} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le{}& 3\mathbb{E}\int_{-\tau}^0 \Bigg\{ \left\|\Phi(t+\theta)\phi(0) + \int_0^{t+\theta} \Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2 \\ &+ \left\|\int_0^{t+\theta} \Phi(t+\theta-s)G(X_s(\phi))dW_Q\right\|^2 \\ &+ \left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}} \Phi(t+\theta-s)G(X_s(\phi))z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2 \Bigg\}\mathrm{d}\theta \\ \le{}& 3\mathbb{E}\Bigg\{ (1+\frac{1}{\varepsilon})\int_{-\tau}^0 e^{-2\lambda(t+\theta)}\|\phi\|_{\mathcal{L}}^2\mathrm{d}\theta \\ &+ (1+\varepsilon)\int_{-\tau}^0 \left\|\int_0^{t+\theta} \Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2 \mathrm{d}\theta \Bigg\} \\ &+ 3\mathbb{E}\int_{-\tau}^0 \left\|\int_0^{t+\theta} \Phi(t+\theta-s)G(X_s(\phi))dW_Q\right\|^2 \mathrm{d}\theta \\ &+ 3\mathbb{E}\int_{-\tau}^0 \left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}} \Phi(t+\theta-s)G(X_s(\phi))z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2 \mathrm{d}\theta \\ =:{}& \frac{3(1+\varepsilon)e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)\|^2 + M_1 + M_2 + M_3. \end{aligned} \tag{4.4}$$

By virtue of (3.10), we have

$$M_1 \le 3(1+\varepsilon)\left[\frac{(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)K_1^2 e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\right]. \tag{4.5}$$

Carrying out a similar argument to that of (3.11), we can easily get that

$$\begin{aligned} M_2 &\leq 3\mathrm{Tr}(Q)\int_{-\tau}^{0}\int_{0}^{t+\theta}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))\right\|^2\mathrm{d}s\mathrm{d}\theta \\ &\leq 3\mathrm{Tr}(Q)\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\left[(1+\varepsilon)K_1^2\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2+\left(1+\frac{1}{\varepsilon}\right)K_2^2\right]\mathrm{d}s\mathrm{d}\theta \\ &\leq \frac{3\,(1+\varepsilon)\,\tau K_2^2\mathrm{Tr}(Q)}{2\varepsilon\lambda}+3(1+\varepsilon)\mathrm{Tr}(Q)K_1^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s. \end{aligned} \tag{4.6}$$

Moreover,

$$\begin{aligned} M_3 &\leq 3\int_{-\tau}^{0}\int_{0}^{t+\theta}\int_{\mathcal{U}-\{0\}}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))z\right\|^2\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\mathrm{d}\theta \\ &\leq 3\int_{\mathcal{U}-\{0\}}\|z\|_{\mathcal{U}}^2\nu(\mathrm{d}z)\left[\left(1+\frac{1}{\varepsilon}\right)K_2^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\mathrm{d}s\mathrm{d}\theta\right. \\ &\quad\left.+\,(1+\varepsilon)K_1^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-\lambda(t+\theta-s)}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta\right] \\ &\leq \frac{3\,(1+\varepsilon)\,\tau K_zK_2^2}{2\varepsilon\lambda}+3K_z(1+\varepsilon)K_1^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s. \end{aligned} \tag{4.7}$$

Thus (4.5), (4.6) and (4.7) together imply that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \leq \alpha_2+\gamma_2e^{\lambda t}+\beta_2\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s,$$

where

$$\begin{aligned} \alpha_2 &= \frac{3\,(1+\varepsilon)\,(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)\|, \\ \gamma_2 &= \frac{3\,(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda}(1+\varepsilon+\mathrm{Tr}(Q)+K_z), \\ \beta_2 &= 3K_1^2(1+\varepsilon)\left[(1+\varepsilon)\lambda^{-1}e^{\lambda\tau}+\mathrm{Tr}(Q)e^{2\lambda\tau}+K_ze^{2\lambda\tau}\right]. \end{aligned}$$

Then Gronwall's inequality gives that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \leq \gamma_2e^{\lambda t}+\alpha_2+\beta_2\int_0^t\left(\gamma_2e^{\lambda s}+\alpha_2\right)e^{\beta_2(t-s)}\mathrm{d}s.$$

It follows from $3K_1^2e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau}+\mathrm{Tr}(Q)+K_z\right]<\lambda$ that there exists $\varepsilon>0$ such that

$$3K_1^2e^{2\lambda\tau}(1+\varepsilon)\left[(1+\varepsilon)\lambda^{-1}e^{-\lambda\tau}+\mathrm{Tr}(Q)+K_z\right]<\lambda$$

and hence that

$$\begin{aligned} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 &\leq \gamma_2+\alpha_2e^{-\lambda t}+\beta_2e^{(\beta_2-\lambda)t}\int_0^t\left(\gamma_2e^{\lambda s}+\alpha_2\right)e^{-\beta_3 s}\mathrm{d}s \\ &\leq \gamma_2+2\alpha_2+\frac{\beta_2\gamma_2}{\lambda-\beta_2}. \end{aligned}$$

This completes the proof. □

Now we only need to show the tightness of solution (2.2) with Lévy jump process.

Lemma 4.7. *Suppose that $3K_1^2 e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z\right] < \lambda$. Then for any bounded set U of $\mathcal{L}$,*

$$\lim_{t\to\infty} \sup_{\phi,\psi\in U} \mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 = 0.$$

Proof. Using the similar arguments as the proof of Lemmas 3.1 and 4.6, we have

$$\begin{aligned}
&\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \\
&\le 3\mathbb{E}\int_{-\tau}^{0}\Bigg\{\left\|\Phi(t+\theta)(\phi(0)-\psi(0)) + \int_0^{t+\theta}\Phi(t+\theta-s)(F(X_s(\phi)) - F(X_s(\psi)))\mathrm{d}s\right\|^2 \\
&\quad + \left\|\int_0^{t+\theta}\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]dW_Q\right\|^2 \\
&\quad + \left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\Bigg\}\mathrm{d}\theta \\
&\le 3\mathbb{E}\int_{-\tau}^{0}\Bigg\{\left(1+\frac{1}{\varepsilon}\right)e^{-2\lambda(t+\theta)}\|\phi-\psi\|_{\mathcal{L}}^2 \\
&\quad + (1+\varepsilon)\left\|\int_0^{t+\theta}\Phi(t+\theta-s)(F(X_s(\phi)) - F(X_s(\psi)))\mathrm{d}s\right\|^2 \\
&\quad + \left\|\int_0^{t+\theta}\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]dW_Q\right\|^2 \\
&\quad + \left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\Bigg\}\mathrm{d}\theta \\
&=: \frac{e^{-2\lambda t}3(1+\varepsilon)(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)-\psi(0)\|^2 + N_1 + N_2 + N_3.
\end{aligned} \tag{4.8}$$

Similar to (3.5) we have

$$N_1 \le \frac{3(1+\varepsilon)K^2 e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s. \tag{4.9}$$

Burkholder–Davis–Gundy inequality implies that

$$\begin{aligned}
N_2 &\le 3\mathrm{Tr}(Q)\mathbb{E}\int_{-\tau}^{0}\int_0^{t+\theta}\|\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]\|^2\mathrm{d}s\mathrm{d}\theta \\
&\le 3\mathrm{Tr}(Q)K^2 e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s,
\end{aligned} \tag{4.10}$$

It follows from Proposition 4.4 that there exists $K_z > 0$ such that

$$\begin{aligned}
N_3 &\le 3\int_{-\tau}^{0}\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\mathbb{E}\left\|\Phi(t+\theta-s)\left[G(X_s(\phi)) - G(X_s(\psi))\right]z\right\|^2\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\mathrm{d}\theta \\
&\le 3\int_{\mathcal{U}-\{0\}}\|z\|_{\mathcal{U}}^2\nu(\mathrm{d}z)\int_{-\tau}^{0}\int_0^{t+\theta}e^{-2\lambda(t+\theta-s)}K\mathbb{E}\|X_s(\phi) - X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\le 3K_z K^2 e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s.
\end{aligned} \tag{4.11}$$

Thus (4.9), (4.10) and (4.11) together imply that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \le \alpha_3 + \beta_3\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s,$$

where

$$\alpha_3 = \frac{\left(3(1+\varepsilon)e^{2\lambda\tau} - 1\right)}{2\varepsilon\lambda}\|\phi(0) - \psi(0)\|^2,$$
$$\beta_3 = 3K^2 e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau}(1+\varepsilon) + \mathrm{Tr}(Q) + K_z\right].$$

It follows from $3K^2e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z) < \lambda$ that there exists $\varepsilon > 0$ such that

$$3K^2e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau}(1+\varepsilon) + \mathrm{Tr}(Q) + K_z) < \lambda.$$

Then Gronwall's inequality gives that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \le \alpha_3 e^{\beta_3 t}. \tag{4.12}$$

This completes the proof. □

In view of Lemmas 4.6 and 4.7, it suffices to show the uniqueness of invariant measures of (1.1) driven by Lévy jump process. If $\mu, \tilde{\mu} \in \mathbb{P}(\mathcal{L})$ are two different invariant measures, then for any $f \in \mathcal{M}$, it follows from (4.12) and the invariance of $\mu, \tilde{\mu} \in \mathbb{P}(\mathcal{L})$ that

$$|\mu(f) - \tilde{\mu}(f)| = \int_{\mathcal{L}}\int_{\mathcal{L}} |\mathbb{P}_t f(\phi) - \mathbb{P}_t f(\psi)|\mu(\mathrm{d}\phi)\tilde{\mu}(\mathrm{d}\psi) \le K_4 e^{-\tilde{\alpha}t}, \qquad t \ge 0,$$

for some $K_4 > 0$, where $\tilde{\alpha} = \frac{3}{2}K^2e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z) - \frac{1}{2}\lambda$. Thus, we obtain the following main result immediately.

Theorem 4.8. *Under the assumptions of Lemmas 4.6 and 4.7, equation (1.1) driven by Lévy jump process has a unique invariant measure.*

5 Application

Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ be equipped with the usual Riemannian metric, and let $\mathrm{d}\xi$ denote the Lebesgue measure on $\mathbb{T}$. For any $p \ge 1$, let

$$L^p(\mathbb{T}, \mathbb{R}) = \left\{x : \mathbb{T} \to \mathbb{R};\ \|x\|_p \triangleq \left[\int_{\mathbb{T}} |x(\xi)|^p \mathrm{d}\xi\right]^{1/p} < \infty\right\},$$

and

$$H = \left\{x \in L^2(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} x(\xi)\mathrm{d}\xi = 0\right\}.$$

It is easy to see that H is a real separable Hilbert space with the inner product

$$\langle x, y\rangle = \int_{\mathbb{T}} x(\xi)y(\xi)\mathrm{d}\xi, \qquad x, y \in H,$$

and the norm $\|x\| = \sqrt{\langle x, x\rangle}$. In the following two subsections, we consider two stochastic reaction-diffusion equations on torus $\mathbb{T}$.

5.1 A Brownian motion case

Consider a stochastic reaction-diffusion equation driven by a Brownian motion $\{W(t)\}_{t\geq 0}$ on torus $\mathbb{T}$ as follows:

$$\begin{cases} du(t,\xi) = \left[\dfrac{\partial^2}{\partial\xi^2}u(t,\xi) + f(u(t-1,\xi))\right]dt + g(u(t-1,\xi))dW(t,\xi), & t \geq 0, \\ u(t,\xi) = \phi(t,\xi), \quad t \in [-1,0], \end{cases} \tag{5.1}$$

where $\phi \in \mathcal{C} := C([-1,0], W^{1,2}(\mathbb{T}))$, f: $\mathbb{R} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and satisfy the linear growth, i.e., there exist positive constants K, K_1, and K_2 such that

$$\begin{aligned} f(u) - f(v)| \leq L_f|u-v|, \qquad & |g(u) - g(v)| \leq L_g|u-v|, \\ |f(u)| \leq L_1|u| + L_2, \qquad & |g(u)| \leq L_3|u| + L_4 \end{aligned} \tag{5.2}$$

for all $u, v \in \mathbb{R}$. Obviously, $A = \frac{\partial^2}{\partial\xi^2}$ is a self-adjoint operator on H with the discrete spectral. More precisely, there exist an orthogonal basis $\{\mathbf{e}_k = \exp\{ik(\cdot)\} : k \in \mathbb{Z}_*\}$ with $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$, and a sequence of real numbers $\{\lambda_k = k^2 : k \in \mathbb{Z}_*\}$ such that $-A\mathbf{e}_k = \lambda_k\mathbf{e}_k$. Let V be the domain of the fractional operator $(-A)^{1/2}$, that is,

$$V = \left\{ \sum_{k\in\mathbb{Z}_*} \sqrt{\lambda_k}a_k\mathbf{e}_k : \{a_k\}_{k\in\mathbb{Z}_*} \subset \mathbb{R}, \sum_{k\in\mathbb{Z}_*} a_k^2 < \infty \right\}$$

with the inner product

$$\langle u, v\rangle_V = \langle(-A)^{1/2}u, (-A)^{1/2}v\rangle = \sum_{k\in\mathbb{Z}_*} \lambda_k\langle u, \mathbf{e}_k\rangle\langle v, \mathbf{e}_k\rangle,$$

and with the norm $\|u\|_V = \sqrt{\langle u, u\rangle_V} = \|(-A)^{1/2}u\|$. Clearly, V is densely and compactly embedded in H.

For every $u \in H$, there exists $\{a_k\}_{k\in\mathbb{Z}_*} \subset \mathbb{R}$ such that $u = \sum_{x\in\mathbb{Z}_*} a_k\mathbf{e}_k$. Thus, we have

$$\begin{aligned} \langle -Au, u\rangle_V &= \sum_{k\in\mathbb{Z}_*} \lambda_k\langle -Au, \mathbf{e}_k\rangle\langle u, \mathbf{e}_k\rangle \\ &= \sum_{k\in\mathbb{Z}_*} \lambda_k\langle u, -A\mathbf{e}_k\rangle\langle u, \mathbf{e}_k\rangle = \sum_{k\in\mathbb{Z}_*} a_k^2\lambda_k^2 \geq \lambda_1^2\|u\|^2. \end{aligned}$$

Thus, we obtain (2.1) with $\lambda = \lambda_1^2$.

We consider a symmetric positive linear operator Q in H such that $Q\mathbf{e}_k = q_k\mathbf{e}_k$ for $k \in \mathbb{Z}_*$, where $\{q_k\}_{k\in\mathbb{Z}_*}$ is a bounded sequence of nonnegative real numbers. Thus, $\mathrm{Tr}(Q) \triangleq \sum_{k\in\mathbb{Z}_*}\langle Q\mathbf{e}_k, \mathbf{e}_k\rangle = \sum_{k\in\mathbb{Z}_*} q_k < \infty$, and Q is also called a trace class operator. Let $\{W(t)\}_{t\geq 0}$ be a H-valued Q-Wiener process given by

$$W(t) = \sum_{x\in\mathbb{Z}_*} \sqrt{q_k}W_k(t)\mathbf{e}_k,$$

where $\{W_k(t) : t \geq 0\}_{k\in\mathbb{Z}^*}$ be a sequence of independent standard one-dimensional Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, that is, $W_k(t) \sim \mathcal{N}(0,t)$, $\mathbb{E}W_k(t) = 0$, $\mathbb{E}[W_k(t)]^2 = t$, and $\mathbb{E}[W_k(t)W_k(s)] = \min\{t,s\}$. It is easy to see that the infinite series of $W(t)$ converges in $L^2(\Omega)$, and satisfies

$$\mathbb{E}\langle W(t), W(t)\rangle = t\mathrm{Tr}(Q), \qquad \mathbb{E}\left(\langle W(t), a\rangle\langle W(s), b\rangle\right) = (t \wedge s)\langle a, b\rangle.$$

Then we can rewrite system (5.1) into the abstract form (1.1) with $\tau = 1$, $F(u_t) = f(u(t-1,\cdot))$ and $G(u_t) = g(u(t-1,\cdot))$. Note that the segment process $u_t = u(t+s,\xi), s \in [-1,0]$ is equipped with norm in $\mathcal{C}$, i.e.,

$$\|u_t\|_{\mathcal{C}} = \max_{s\in[-1,0]} \|u(t+s,\xi)\| = \max_{s\in[-1,0]} \left\{ \int_{\mathbb{T}} |u(t+s,\xi)|^2 d\xi \right\}^{\frac{1}{2}}.$$

In what follows, we shall verify that F and G satisfy hypothesis (3.2). In fact, it follows from (5.2) and Minkowski inequality that

$$\begin{aligned}
\|F(u_t) - F(v_t)\| &= \left\{ \int_{\mathbb{T}} [f(u(t-1,\xi)) - f(v(t-1,\xi))]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_f^2 \int_{\mathbb{T}} [u(t-1,\xi) - v(t-1,\xi)]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le L_f \|u_t - v_t\|_{\mathcal{C}}, \\
\|G(u_t) - G(v_t)\|_{\mathfrak{L}_2^Q} &= \left\{ \sum_{k\in\mathbb{Z}_*} \langle (G(u_t) - G(v_t))Q(G(u_t) - G(v_t))^* e_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} L_g^2 \int_{\mathbb{T}} |u(t-1,\xi) - v(t-1,\xi)|^2 d\xi \,\langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} L_g^2 \|u_t - v_t\|_{\mathcal{C}}^2 \,\langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&= L_g \sqrt{\mathrm{Tr}(Q)} \|u_t - v_t\|_{\mathcal{C}},
\end{aligned}$$

and

$$\begin{aligned}
\|F(u_t)\| &= \left\{ \int_{\mathbb{T}} [f(u(t-1,\xi))]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ \int_{\mathbb{T}} (L_1|u(t-1,\xi)| + L_2)^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_1 \int_{\mathbb{T}} |u(t-1,\xi)|^2 d\xi \right\}^{\frac{1}{2}} + \left(\int_{\mathbb{T}} L_2^2 d\xi \right)^{\frac{1}{2}} \le L_1 \|u_t\|_{\mathcal{C}} + L_2, \\
\|G(u_t)\|_{\mathfrak{L}_2^Q} &= \left\{ \sum_{k\in\mathbb{Z}_*} \langle G(u_t)QG(u_t)^* e_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&= \left\{ \sum_{k\in\mathbb{Z}_*} \left\langle \int_{\mathbb{T}} |g(u(t-1,\xi))|^2 d\xi Qe_k, e_k \right\rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} \int_{\mathbb{T}} |L_3 u(t-1,\xi) + L_4|^2 d\xi \,\langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left[\int_{\mathbb{T}} |L_3 u(t-1,\xi) + L_4|^2 d\xi \right]^{\frac{1}{2}} \left[\sum_{k\in\mathbb{Z}_*} q_k \right]^{\frac{1}{2}} \le (L_3 \|u_t\|_{\mathcal{C}} + L_4) \sqrt{\mathrm{Tr}(Q)}.
\end{aligned}$$

Then we can set the parameter values in (3.2) as follows

$$K = L_f + L_g \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, \qquad K_1 = L_1 + L_3 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, \qquad K_2 = L_2 + L_4 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}.$$

Thus, from Theorems 3.3 and 3.8 we have the following result.

Corollary 5.1. *Assume that*

$$\max\left\{ L_f + L_g \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, L_1 + L_3 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}} \right\} \leq (2e^2 + 2e)^{-\frac{1}{2}}.$$

Then equation (5.1) has a unique invariant measure and a pullback attractor.

5.2 A Poisson jumps case

Let $\{N(dt, \mathrm{d}z) : t \in \mathbb{R}^+, z \in \mathbb{R}\}$ is a centered Poisson random measure with parameter $\nu(\mathrm{d}z)dt = 2m(z)\mathrm{d}zdt$, and $\widetilde{N}(dt, \mathrm{d}z) = N(dt, \mathrm{d}z) - \nu(\mathrm{d}z)dt$ be a compensated Poisson random measure, where

$$m(z) = \frac{1}{\sqrt{2\pi}z} \exp\left\{ -\frac{\ln^2 z}{2} \right\}, \qquad 0 \leq z < \infty$$

is the density function of a lognormal random variable. Consider the following stochastic delay differential equations with Poisson jumps on $\mathbb{T}$:

$$\begin{cases} du(t,x) = \left[\frac{\partial^2}{\partial x^2} u(t,x) + f(u(t-1,x)) \right] \mathrm{d}t + \int_{\mathcal{U}} g(u(t-1,x)) z\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \qquad t \geq 0, \\ u(t,x) = \phi(t,x), \qquad t \in [-1, 0], \end{cases} \tag{5.3}$$

where $\phi \in \mathcal{C} := C([-1,0], W^{1,2}(\mathbb{T}))$, $\mathcal{U} = \{z \in \mathbb{R} : 0 < |z| \leq 1\}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy (5.2).

Define $A : H \to H$ by $A = \frac{\partial^2}{\partial x^2}$. It follows from the arguments in Section 5.1 that the space H and operator A is well defined. Note that $\lambda = \lambda_1^2 = 1$. Note that A is the generator of an analytic semigroup $\Phi(t), t \geq 0$, equation (5.3) can be given by the following integral equation

$$\begin{aligned} u(t,x) = \Phi(t)\phi(t,x) &+ \int_0^t \Phi(t-s) f(u(t-1,x)) \mathrm{d}s \\ &+ \int_0^{t+} \int_{\mathcal{U}} \Phi(t-s) g(u(t-1,x)) z\widetilde{N}(\mathrm{d}z, \mathrm{d}s) \end{aligned} \tag{5.4}$$

for $t \in [0, +\infty)$ and $x \in H$. Note that $\nu(\mathcal{U}) = 1$ and

$$K_z = \int_{\mathcal{U}} z^2 \nu(\mathrm{d}z) = \int_0^1 \frac{z}{\sqrt{2\pi}} \exp\left\{ -\frac{\ln^2 z}{2} \right\} \mathrm{d}z \leq e^2.$$

Then we can rewrite system (5.3) into the abstract form (1.1) with $\tau = 1$, $F(u_t) = f(u(t-1,\cdot))$ and $G(u_t) = g(u(t-1,x))$. Note that the segment process $u_t = u(t+s,\xi), s \in [-1,0]$ is equipped with norm in $\mathcal{C}$, i.e.,

$$\|u_t\|_{\mathcal{C}} = \max_{s\in[-1,0]} \|u(t+s,\xi)\| = \max_{s\in[-1,0]} \left\{ \int_{\mathbb{T}} |u(t+s,\xi)|^2 d\xi \right\}^{\frac{1}{2}}.$$

It follows from Section 5.1 that

$$\|F(u_t) - F(v_t)\| \le L_f \|u_t - v_t\|_{\mathcal{C}}$$

and

$$\|F(u_t)\| \le L_1 \|u_t\|_{\mathcal{C}} + L_2.$$

It is easy to check that

$$\begin{aligned}
\|G(u_t) - G(v_t)\|_{\mathscr{L}(\mathbb{R},H)} &= \left\{ \int_{\mathbb{T}} [g(u(t-1,\xi)) - g(v(t-1,\xi))]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_g^2 \int_{\mathbb{T}} [u(t-1,\xi) - v(t-1,\xi)]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le L_g \|u_t - v_t\|_{\mathcal{C}}
\end{aligned}$$

and

$$\begin{aligned}
\|G(u_t)\|_{\mathscr{L}(\mathbb{R},H)} &= \left\{ \int_{\mathbb{T}} [g(u(t-1,\xi))]^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ \int_{\mathbb{T}} (L_3 |u(t-1,\xi)| + L_4)^2 \,\mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_3 \int_{\mathbb{T}} |u(t-1,\xi)|^2 d\xi \right\}^{\frac{1}{2}} + \left(\int_{\mathbb{T}} L_4^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&\le L_3 \|u_t\|_{\mathcal{C}} + L_4.
\end{aligned}$$

Then we set the parameter values in (4.3) as follows

$$K = L_f + L_g, \qquad K_1 = L_1 + L_3, \qquad K_2 = L_2 + L_4.$$

Thus the result of existence of invariant measure of (5.3) follows from Theorems 4.6 and 4.7.

Corollary 5.2. *Assume that*

$$\max\left\{ L_f + L_g, L_1 + L_3 \right\} \le (3e + 3e^4)^{-\frac{1}{2}}.$$

Then equation (5.3) *has a unique invariant measure.*

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