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Existence, nonexistence and multiplicity of positive solutions for singular quasilinear problems

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Abstract. In the present paper we deal with a quasilinear problem involving a singular term and a parametric superlinear perturbation. We are interested in the existence, nonexistence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. In our first result, the superlinear perturbation has an arbitrary growth and we obtain the existence of a solution for the problem by using the sub-supersolution method. For the second result, the superlinear perturbation has subcritical growth and we employ the Mountain Pass Theorem to show the existence of a second solution.

Keywords: extended functional, sub-supersolution method, singular problem, variational methods.

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1 Introduction

This paper is concerned with the existence, nonexistence and multiplicity of solutions for the family of quasilinear problems with singular nonlinearity

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $0 < \gamma, 3 \le p < \infty, 0 \le \lambda$ is a parameter, $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a bounded smooth domain and $a(x)$ is a positive measurable function.

We say that a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution (solution, for short) of (P_λ) if $u > 0$ a.e. in Ω, and, for every $\psi \in H_0^1(\Omega)$,

$$au^{-\gamma}\psi, u^p\psi \in L^1(\Omega)$$

and

$$\int_\Omega [(1+2u^2)\nabla u \nabla\psi + 2u|\nabla u|^2\psi] = \int_\Omega a(x)u^{-\gamma}\psi + \lambda\int_\Omega u^p\psi.$$

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Solutions of this type are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + V(x)z + \eta(|z|^2)z - \kappa\Delta\rho(|z|^2)\rho'(|z|^2)z, \tag{1.1}$$

where $z : \mathbb{R} \times \Omega \to \mathbb{C}$, $V(x)$ is a given potential, $\kappa > 0$ is a constant and η, ρ are real functions. Quasilinear equations of form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ρ. The case of $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara [20] (cf. [21]). In the case $\rho(s) = (1+s)^{1/2}$, equation (1.1) models the self-channeling of a high-power ultrashort laser in matter, see [7,9,11,28] and the references in [8].

Consider the following quasilinear Schrödinger equation

$$-\Delta u - \Delta(u^2)u = g(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. When g is a singular nonlinearity, problems of type (1.2) was studied by Do Ó–Moameni [25], Liu–Liu–Zhao [23], Wang [32], Dos Santos–Figueiredo–Severo [29], Alves–Reis [2] and Bal–Garain–Mandal–Sreenadh [6]. In particular, the authors in [23] considered the problem

$$\begin{cases} -\Delta_s u - \frac{s}{2^{s-1}}\Delta(u^2)u = a(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $N \geq 3$, Δ_s is the s-Laplacian operator, $2 < 2s < p+1 < \infty$, $0 < \gamma$ and $a \geq 0$ is a nontrivial measurable function satisfying the following condition:

(H) There are $\varphi \in C_0^1(\overline{\Omega})$ and $q > N$ such that $\varphi > 0$ on Ω and $a\varphi^{-\gamma} \in L^q(\Omega)$.

The authors used sub-supersolution method, truncation arguments and variational methods to prove the existence of solutions for (1.3) provided $\lambda > 0$ is small enough.

In [29], Dos Santos–Figueiredo–Severo studied the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\gamma} + z(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $N \geq 3$, the function a satisfies the hypothesis (H) and the nonlinearity $z : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies (among other conditions):
There exist $C > 0, r \geq 1$ and $b \in L^\infty(\Omega), b \geq 0$ almost everywhere in Ω, such that

$$|z(x,t)| \leq C(1 + b(x)|t|^{r-1}), \qquad \forall t \in \mathbb{R} \text{ and a.e. in } \Omega.$$

By using sub-supersolution method, truncation arguments and the Mountain Pass Theorem they showed the existence of solutions provided $\|b\|_\infty$ is small enough. When $z(x,t) = \lambda|t|^{r-2}t$ this is equivalent to the existence of solutions for $\lambda > 0$ small enough.

In this paper, our first goal is to show the existence and nonexistence of solutions for (P_λ) without restriction on the parameter λ and exponent $p \geq 3$ (see Remark 1.4). We would like to emphasize that for $0 < p < 3$ the arguments carried out in [1,2] can be adapted to prove that problem (P_λ) has at least one solution for all $\lambda \in \mathbb{R}$ (see Remark 4.1).

It is worth pointing out that to prove our main results, we use the method of changing variables developed in Colin–Jeanjean [13]. Thus, we reformulate problem (P_λ) into a new one, denoted with (Q_λ) (cfr. Section 2), which finds its natural setting in the Sobolev space $H_0^1(\Omega)$.

Our first result is the following.

Theorem 1.1. *Under the assumptions (H) and $p \geq 3$ there exists $0 < \lambda_* < \infty$ such that problem (P_λ) has at least one solution v_λ for $0 < \lambda < \lambda_*$ and no solution for $\lambda > \lambda_*$. Moreover, λ_* is characterized variationally by (3.1) and $v_\lambda \in C_0^1(\overline{\Omega})$.*

The proof of Theorem 1.1 is based on the method of sub-supersolutions. However, by virtue of the arbitrary growth of the singular and superlinear terms that appear in problem (Q_λ) we cannot use directly the method of sub-supersolutions here. An additional difficulty comes from the fact that these singular and superlinear terms are nonhomogeneous. To overcome this difficulty we develop new arguments and a regularity result that allows us to obtain a subsolution to problem (Q_λ) for all $\lambda > 0$. In particular, we establish some preliminary results and we prove a sub-supersolution theorem (see Theorem 2.8).

To prove the multiplicity of solutions for (P_λ), with $\lambda \in (0, \lambda_*)$, we need a refinement of hypotheses (H). We introduce the following assumption:

$(H)_\infty$ There exists $\varphi \in C_0^1(\overline{\Omega})$ such that $\varphi > 0$ on Ω and $a\varphi^{-1-\gamma} \in L^\infty(\Omega)$.

If the function φ satisfies $(H)_\infty$ then it satisfies (H), too (see Section 4).

We denote by $2^* = 2N/(N-2)$ the critical Sobolev exponent. Now we state our second result.

Theorem 1.2. *Under the assumptions $(H)_\infty$ and $3 < p < 22^* - 1$, problem (P_λ) has at least two solutions for $0 < \lambda < \lambda_*$ and no solution for $\lambda > \lambda_*$.*

Example 1.3. When Ω is the unit ball, the functions $a(x) = (1 - |x|^2)^\sigma, \sigma \geq \gamma + 1$ and $\varphi(x) = 1 - |x|^2$ satisfy assumption $(H)_\infty$.

Remark 1.4. The results obtained in Theorems 1.1 and 1.2 are almost global, that is, they do not show the existence and multiplicity of solutions only for $\lambda = \lambda_*$, with parameter λ_* having the property that problem (P_λ) has at least one solution for $\lambda \in (0, \lambda_*)$ and no solutions for $\lambda > \lambda_*$. Thus, in our main results we do not assume the restriction that λ is small enough to guarantee the existence of solutions, because we prove the existence of a solution for all $\lambda \in (0, \lambda_*)$. Furthermore, when $0 < \gamma < 1$ (weak singularity), combining Theorems 1.1, 1.2 and Proposition 4.8 we have a global result:

a) Problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$. Namely, $\mathcal{L} = (0, \lambda_*]$ (see Section 3 for definition of $\mathcal{L}$).

b) Problem (P_λ) has at least two solutions for $\lambda \in (0, \lambda_*)$ and at least one solution for $\lambda = \lambda_*$ and no solution for $\lambda > \lambda_*$.

Let us highlight that the hypotheses $(H)_\infty$ plays a crucial role in the proof of Theorem 1.2. Indeed, it allows us to show that v_λ is a local minimum of the functional J_λ in the topology of $C_0^1(\overline{\Omega})$ and that the modified functional $\mathcal{J}_\lambda$ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ and satisfies the assumptions of Theorem 1 in Brezis–Nirenberg [10] (see (4.2) and (4.7) in Section 4 for definition of J_λ and $\mathcal{J}_\lambda$, respectively). In particular, we get that $v = 0$ is a local minimum of the functional $\mathcal{J}_\lambda$ in the $H_0^1(\Omega)$ topology. Then, after fine arguments we apply the Mountain Pass Theorem to obtain a second solution of (P_λ). It is worth pointing out that under the assumption (H) we are not able to show Lemma 4.2 and that $\mathcal{J}_\lambda$ satisfies the assumptions of Theorem 1 in [10].

We emphasize that Theorem 1.1 improve the works [23,29] in the sense that we show the existence and nonexistence of solutions for (P_λ) without restriction on the parameter λ (that is, our results are almost global and we do not assume that λ is small enough to obtain a solution, see Remark 1.4). They also did not prove a result of nonexistence of solutions. As far as we know, Theorem 1.2 is the first result of multiplicity of $H_0^1(\Omega)$-solutions for singular problems with strong singularity $\gamma > 1$ and without restriction on the parameter λ, that is, we do not assume λ small enough. Notice that no restriction on the $\gamma > 0$ is assumed.

Let us compare our parameter λ_* and results with the parameters and results obtained in [6,29].

- Let ϵ_0 and ϵ_1 be the parameters obtained in Theorem 1.2 and 1.3, respectively, in [29] when $h(x,t) = \lambda|t|^{r-2}t$. Then, we will prove in Remark 3.3 that $\epsilon_1 \leq \epsilon_0 < \lambda_*$.

- When $0 < \gamma < 1$ and $3 < p < 22^* - 1$ problem (P_λ) was also studied in [6]. As mentioned by the authors of that work, using the Nehari manifold method they proved the existence of two solutions for λ sufficiently small. More precisely, they proved that there is a parameter $\nu > 0$ such that problem (P_λ) has two solutions for $0 < \lambda < \nu$ and $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \lambda < \nu$ (here we use ν to avoid confusion with our Λ of Theorem 3.2 and see page 4 of [6] for the definition of $\mathcal{N}_\lambda^0$). In our work we are assuming arbitrary $\gamma > 0$, unlike [6] which assumes $0 < \gamma < 1$. Furthermore the technique used in [6] cannot be used when $\gamma \geq 1$, because they need the continuity of the energy functional associated with the problem and use that $0 < 1 - \gamma < 1$ to get estimates (at this point they need Sobolev embeddings and therefore it is very important that $0 < 1 - \gamma < 1$). For $\gamma \geq 1$ these facts are not true and therefore for $\gamma \geq 1$ the results obtained in [6] cannot be compared with our results obtained here (in particular with Theorem 1.1 where we assume that p can be supercritical, that is, $22^* - 1 < p$).

- When $0 < \gamma < 1$ and $p < 22^* - 1$, combining Theorem 1.1 and Proposition 4.8 of our work we have that problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$. Thus, our result is global in this case. As a consequence $\nu \leq \lambda_*$ (see previous paragraph for the definition of ν). In [6] they did not prove the existence of a solution for $\lambda = \nu$, that is, they obtained a local result. Therefore, even in this case our work improves the result of [6] in the sense that we prove that problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$ (in particular for $\lambda = \lambda_*$), and therefore we do not have the restriction that λ is small enough as in [6]. Finally, we emphasize that a similar problem, but without the term $\Delta(u^2)u$, was studied in [3] using the Nehari manifold method and in that work the authors obtained solutions for parameters $\lambda > 0$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. This result suggests that parameter ν obtained in [6] is small enough and satisfies $\nu < \lambda_*$.

There is a wide literature dealing with existence and multiplicity results for problems involving both the p-Laplacian operator and singular nonlinearities. The reader who wishes to broaden his/her knowledge on these topics is referred to [3,15–18,26,27], and to the references therein.

The paper is structured as follows: In Section 2, we reformulate problem (P_λ) into a new one which finds its natural setting in the Sobolev space $H_0^1(\Omega)$ and we present some results that will be important for our work. In particular, we prove a nonexistence result and a sub-supersolution theorem. In Section 3, we prove Theorem 1.1 and Section 4 is devoted to prove Theorem 1.2.

Notation. Throughout this paper, we make use of the following notations:

- $L^q(\Omega)$, for $1 \leq q \leq \infty$, denotes the Lebesgue space with usual norm denoted by $\|u\|_q$.
- $H_0^1(\Omega)$ denotes the Sobolev space endowed with inner product
$$(u,v) = \int_\Omega \nabla u \nabla v, \qquad \forall u,v \in H_0^1(\Omega).$$
The norm associated with this inner product will be denoted by $\|\ \ \|$.
- $W_0^{2,q}(\Omega)$ denotes the Sobolev space with norm
$$\|u\| = \left(\sum_{|\alpha|\leq 2} \|D^\alpha u\|_q^q \right)^{1/q}.$$
- Let us consider the space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ equipped with the norm $\|u\|_{C^1} = \max_{x\in\Omega} |u(x)| + \max_{x\in\Omega} |\nabla u(x)|$. If on $C_0^1(\overline{\Omega})$ we consider the pointwise partial ordering (i.e., $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in \overline{\Omega}$), which is induced by the positive cone
$$C_0^1(\overline{\Omega})_+ = \left\{ u \in C_0^1(\overline{\Omega}) : u \geq 0 \text{ for all } x \in \Omega \right\},$$
then this cone has a nonempty interior given by
$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) < 0 \text{ for all } x \in \partial\Omega \right\},$$
where ν is the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.
- $B_r(v)$ denotes the ball centered at $v \in C_0^1(\overline{\Omega})$ with radius $r > 0$ (with respect to the topology of $C_0^1(\overline{\Omega})$).
- The function $d(x) = d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.
- We denote by ϕ_1 the $L^\infty(\Omega)$-normalized (that is, $\|\phi_1\|_\infty = 1$) positive eigenfunction for the smallest eigenvalue $\lambda_1 > 0$ of $\left(-\Delta, H_0^1(\Omega)\right)$.
- If u is a measurable function, we denote the positive and negative parts by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, respectively.
- If A is a measurable set in $\mathbb{R}^N$, we denote by $|A|$ the Lebesgue measure of A.
- $k, c, c_1, c_2, \ldots$ and C denote (possibly different from line to line) positive constants.
- The arrow $\rightharpoonup$ (respectively, $\to$) denotes weak (respectively strong) convergence.

2 Preliminaries

In this section, we will establish some preliminaries which will be important for our work. We reduce the study of the existence of positive solutions for (P_λ) to the existence of positive solutions of a singular elliptic problem. In particular, we will prove a nonexistence result and a sub-supersolution theorem.

We denote by ϕ_1 the $L^\infty(\Omega)$-normalized positive eigenfunction for the smallest eigenvalue $\lambda_1 > 0$ of $\left(-\Delta, H_0^1(\Omega)\right)$. We start by proving that ϕ_1 satisfies the assumption (H). We consider the following assumption.

(H') There is $q > N$ such that $a\phi_1^{-\gamma} \in L^q(\Omega)$.

Lemma 2.1. *Assumptions (H) and (H') are equivalent.*

Proof. Suppose that (H) holds. One has $\phi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and $\varphi \in C_0^1(\overline{\Omega})_+$. Then, from Proposition 1 in [24] there exists $k > 0$ such that $\phi_1 \geq k\varphi$ in Ω and hence $a\phi_1^{-\gamma} \leq k^{-\gamma}a\varphi^{-\gamma} \in L^q(\Omega)$, proving (H').

If (H') holds, then the function $\varphi = \phi_1$ and q satisfy (H). This concludes the proof. □

Remark 2.2.

a) The arguments in the proof of Lemma 2.1 can be used to prove that if (H) holds, then any function $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ satisfies the assumption (H), too.

b) If φ satisfies the assumption (H) then $a\varphi^{1-\gamma}, a \in L^q(\Omega)$. Indeed, $a = a\varphi^{-\gamma}\varphi^{\gamma} \leq \|\varphi\|_\infty^\gamma a\varphi^{-\gamma} \in L^q(\Omega)$ and $a\varphi^{1-\gamma} \leq \|\varphi\|_\infty a\varphi^{-\gamma} \in L^q(\Omega)$.

c) It is well known that $\phi_1 \in C^1(\overline{\Omega})$ and satisfies $cd(x) \leq \phi_1(x) \leq Cd(x)$, $x \in \Omega$, for some constants $c, C > 0$ (see [31]).

Now, we observe that the natural energy functional corresponding to the problem (P_λ) is the following:

$$Q(u) = \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 + \frac{1}{\gamma - 1}\int_\Omega a(x)F(|u|) - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}, \qquad u \in A(Q),$$

where

$$A(Q) = \left\{u \in H_0^1(\Omega) : \int_\Omega a(x)F(|u|) < \infty \text{ and } \int_\Omega |u|^{p+1} < \infty\right\}$$

and the function $F : [0,\infty) \to [0,\infty]$ satisfies $F'(s) = s^{-\gamma}$ for $s > 0$ (see [1] for a complete definition of F).

However, this functional is not well defined, because $\int_\Omega u^2|\nabla u|^2dx$ is not finite for all $u \in H_0^1(\Omega)$, hence it is difficult to apply variational methods directly. In order to overcome this difficulty, we use the following change of variables introduced by [13], namely, $v := g^{-1}(u)$, where g is defined by

$$\begin{cases} g'(t) = \frac{1}{(1+2|g(t)|^2)^{\frac{1}{2}}} & \text{in } [0,\infty), \\ g(t) = -g(-t) & \text{in } (-\infty, 0]. \end{cases} \tag{2.1}$$

We list some properties of g, whose proofs can be found in [2,13,22,30].

Lemma 2.3. *The function g satisfies the following properties:*

(1) *g is uniquely defined, C^∞ and invertible;*

(2) $g(0) = 0$*;*

(3) $0 < g'(t) \le 1$ *for all* $t \in \mathbb{R}$*;*

(4) $\frac{1}{2}g(t) \le tg'(t) \le g(t)$ *for all* $t > 0$*;*

(5) $|g(t)| \le |t|$ *for all* $t \in \mathbb{R}$*;*

(6) $|g(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*

(7) $(g(t))^2 - g(t)g'(t)t \ge 0$ *for all* $t \in \mathbb{R}$*;*

(8) *There exists a positive constant C such that $|g(t)| \ge C|t|$ for $|t| \le 1$ and $|g(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$;*

(9) $g''(t) < 0$ *when* $t > 0$ *and* $g''(t) > 0$ *when* $t < 0$*;*

(10) *the function $(g(t))^{1-\gamma}$ for $\gamma > 1$ is decreasing for all $t > 0$;*

(11) *the function $(g(t))^{-\gamma}$ is decreasing for all $t > 0$;*

(12) $|g(t)g'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*;*

(13) $g^2(ts) \ge tg^2(s)$ *for all* $t \ge 1$ *and* $s \ge 0$*.*

Corollary 2.4. *For each $s > 0$ there exists a constant $K > 0$ such that $|t^\gamma \ln(g(t))| \le K$ for all $0 < t \le s$.*

Proof. Since $h(t) = t^\gamma \ln(g(t))$, $t > 0$, is a continuous function it is sufficient to show that $\lim_{t\to 0} t^\gamma \ln(g(t)) = 0$. From Lemma 2.3 (8) one has

$$|t^\gamma \ln(g(t))| \le |C^{-\gamma} g^\gamma(t) \ln(g(t))|,$$

for all $0 < t \le 1$, which implies that $\lim_{t\to 0} t^\gamma \ln(g(t)) = 0$, because $\lim_{t\to 0} t^\gamma \ln(t) = 0$ and $\lim_{t\to 0} g(t) = 0$. $\square$

After a change of variable $v = g^{-1}(u)$, we define an associated problem

$$\begin{cases} -\Delta v = [a(x)(g(v))^{-\gamma} + \lambda(g(v))^p]\, g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{Q_λ}$$

We say that a function $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution (solution, for short) of (Q_λ) if $v > 0$ a.e. in Ω, and, for every $\psi \in H_0^1(\Omega)$,

$$a(x)(g(v))^{-\gamma} g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$$

and

$$\int_\Omega \nabla v \nabla \psi = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi.$$

It is easy to see that problem (Q_λ) is equivalent to our problem (P_λ), which takes $u = g(v)$ as its solutions. Thus, our goal is reduced to proving the existence, nonexistence and multiplicity of solutions for the family of problems (Q_λ).

In order to study problem (Q_λ), one introduces the assumption:

$(H)_d$ There are $\varphi \in C_0^1(\overline{\Omega})$ and $q > N$ such that $\varphi > 0$ on Ω and $ag^{-\gamma}(\varphi)g'(\varphi) \in L^q(\Omega)$.

The following lemma show the relation between the assumptions (H) and $(H)_d$.

Lemma 2.5. *Suppose that the function φ satisfies (H). Then φ satisfies $(H)_d$. Moreover, $ag^{1-\gamma}(\varphi) \in L^q(\Omega)$ if $\gamma \neq 1$ and $a(x)\ln(g(\varphi)) \in L^q(\Omega)$ if $\gamma = 1$.*

Proof. Let $0 < \epsilon < 1$ such that $\epsilon\|\varphi\|_\infty < 1$ holds. By using (3), (8), (9) and (11) of Lemma 2.3 and Corollary 2.4 (if $\gamma = 1$) we find

$$ag^{-\gamma}(\varphi)g'(\varphi) \leq ag^{-\gamma}(\epsilon\varphi)g'(\epsilon\varphi) \leq C^{-\gamma}\epsilon^{-\gamma}a\varphi^{-\gamma} \in L^q(\Omega),$$

$$ag^{1-\gamma}(\varphi) \leq g(\|\varphi\|_\infty)ag^{-\gamma}(\epsilon\varphi) \leq g(\|\varphi\|_\infty)\epsilon^{-\gamma}C^{-\gamma}a\varphi^{-\gamma} \in L^q(\Omega)$$

and

$$|a(x)\ln(g(\varphi))| = |a(x)\varphi^{-\gamma}\varphi^{\gamma}\ln(g(\varphi))| \leq Ka(x)\varphi^{-\gamma} \in L^q(\Omega),$$

namely, $ag^{-\gamma}(\varphi)g'(\varphi) \in L^q(\Omega)$ and $ag^{1-\gamma}(\varphi) \in L^q(\Omega)$ and $a(x)\ln(g(\varphi)) \in L^q(\Omega)$ if $\gamma = 1$. $\square$

To prove the nonexistence of solutions for (Q_λ) we define the function $m(x) = \min\{a(x), 1\} \in L^\infty(\Omega)$ and we will denote by $\lambda_1[m]$ the principal eigenvalue of

$$\begin{cases} -\Delta u = \lambda m(x)u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{A}$$

It is known that $\lambda_1[m]$ is simples, $\lambda_1[m] > 0$, and the associated eigenfunction $\tilde{\phi}_1$ can be chosen such that $\tilde{\phi}_1 > 0$ in Ω (see [14, Theorem 6.2.9]).

Next, we prove the nonexistence of positive solutions for (Q_λ).

Lemma 2.6. *There exists a constant $\lambda^* > 0$ such that problem (Q_λ) has no solution for all $\lambda \in (\lambda^*, \infty)$.*

Proof. Let us start by defining the function $j_\lambda(t) = (g^{-\gamma}(t)g'(t) + \lambda g^p(t)g'(t))/t$ for $t > 0$. Using (4) of Lemma 2.3 we have that

$$j_\lambda(t) \geq \frac{g^{1-\gamma}(t)}{2t^2} + \lambda\frac{g^{p+1}(t)}{2t^2}, \qquad t > 0.$$

We now distinguish two cases:

Case $\gamma > 1$. From (5) and (8) of Lemma 2.3 we get

$$j_\lambda(t) \geq \begin{cases} \dfrac{t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{p-1}}{2} & \text{if } 0 < t \leq 1, \\ \dfrac{t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{(p-3)/2}}{2} & \text{if } t \geq 1. \end{cases} \tag{2.2}$$

In order to find a lower bound for the function j_λ we observe that the function

$$\tilde{f}(t) = \frac{t^{-1-\gamma}}{2} + \lambda\frac{C^{p+1}t^{p-1}}{2}, \qquad t > 0,$$

has a global minimizer

$$t_\lambda = \left[\frac{(1+\gamma)}{\lambda(p-1)C^{p+1}}\right]^{\dfrac{1}{p+\gamma}},$$

such that $t_\lambda < 1$ for λ large enough and

$$\min_{t>0} \tilde{f}(t) = \tilde{f}(t_\lambda) = \frac{1}{2}\left[\frac{\lambda(p-1)C^{p+1}}{1+\gamma}\right]^{\dfrac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right). \tag{2.3}$$

Hereafter, we fix λ such that $t_\lambda < 1$. Then, by using (2.2) and (2.3), we infer that

$$\min_{t>0} j_\lambda(t) \geq \min\left\{\frac{1}{2}\left[\frac{\lambda(p-1)C^{p+1}}{1+\gamma}\right]^{\dfrac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right), \lambda\frac{C^{p+1}}{2}\right\}$$

and as a consequence there exists λ^* such that

$$j_{\lambda^*}(t_{\lambda^*}) := \min_{t>0} j_{\lambda^*}(t) \geq \lambda_1[m]. \tag{2.4}$$

Case $\gamma \leq 1$. From (8) of Lemma 2.3 we get

$$j_\lambda(t) \geq \begin{cases} \dfrac{C^{1-\gamma}t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{p-1}}{2} & \text{if } 0 < t \leq 1, \\ \dfrac{C^{1-\gamma}t^{(-3-\gamma)/2}}{2} + \lambda\dfrac{C^{p+1}t^{(p-3)/2}}{2} & \text{if } t \geq 1. \end{cases} \tag{2.5}$$

In order to find a lower bound for the function j_λ we observe that the function

$$\tilde{h}(t) = \frac{C^{1-\gamma}t^{-1-\gamma}}{2} + \lambda\frac{C^{p+1}t^{p-1}}{2}, \qquad t > 0,$$

has a global minimizer

$$t_\lambda = \left[\frac{(1+\gamma)}{\lambda(p-1)C^{p+\gamma}}\right]^{\dfrac{1}{p+\gamma}},$$

such that $t_\lambda < 1$ for λ large enough and

$$\min_{t>0} \tilde{h}(t) = \tilde{h}(t_\lambda) = \frac{C^2}{2}\left[\frac{\lambda(p-1)}{1+\gamma}\right]^{\dfrac{1+\gamma}{p+\gamma}} \left[\frac{p+\gamma}{p-1}\right]. \tag{2.6}$$

Hereafter, we fix λ such that $t_\lambda < 1$. Then, by using (2.5) and (2.6), we infer that

$$\min_{t>0} j_\lambda(t) \geq \min\left\{\frac{C^2}{2}\left[\frac{\lambda(p-1)}{1+\gamma}\right]^{\dfrac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right), \lambda\frac{C^{p+1}}{2}\right\}$$

and as a consequence there exists λ^* such that

$$j_{\lambda^*}(t_{\lambda^*}) := \min_{t>0} j_{\lambda^*}(t) \geq \lambda_1[m]. \tag{2.7}$$

Now, arguing by contradiction, we suppose that for some $\lambda > \lambda^*$ problem (Q_λ) has a solution v_λ, where λ^* is defined in (2.4) (if $\gamma > 1$) and (2.7) (if $\gamma \leq 1$). By taking $\tilde{\phi}_1$ as a test function in the equation satisfied by v_λ and v_λ in the equation satisfied by $\tilde{\phi}_1$ we obtain

$$\begin{aligned}\int (a(x)g^{-\gamma}(v_\lambda) + \lambda^* g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1 &\geq \int m(x)(g^{-\gamma}(v_\lambda) + \lambda^* g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1 \\ &\geq \int m(x) j_{\lambda^*}(t_{\lambda^*}) v_\lambda \tilde{\phi}_1 \\ &\geq \int \lambda_1[m] m(x) v_\lambda \tilde{\phi}_1 \\ &= \int \nabla\tilde{\phi}_1 \nabla v_\lambda \\ &= \int (a(x)g^{-\gamma}(v_\lambda) + \lambda g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1\end{aligned}$$

and hence $\lambda^* \geq \lambda$, which is impossible by the choice of λ. By virtue of the relation between (P_λ) and (Q_λ) we deduce that problem (P_λ) has no solution for $\lambda > \lambda^*$. □

Now, we define the notions of subsolution and supersolution and prove a sub-supersolution theorem.

Definition 2.7. We say that v is a subsolution of problem (Q_λ) if $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v > 0$ in Ω, $a(x)(g(v))^{-\gamma}g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$ and

$$\int_\Omega \nabla v \nabla \psi \leq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω. Similarly, $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v > 0$ in Ω, is a supersolution of (Q_λ) if $a(x)(g(v))^{-\gamma}g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$ and

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω.

Theorem 2.8. *Let $\underline{v}$ and $\overline{v}$ be a subsolution respectively a supersolution of problem (Q_λ) such that $\underline{v} \leq \overline{v}$ in Ω. Then there exists a solution $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (Q_λ) such that $\underline{v} \leq v \leq \overline{v}$ in Ω.*

Proof. We define a truncated function $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by letting,

$$\tilde{g}(x,t) = \begin{cases} g^p(\underline{v}(x))g'(\underline{v}(x)) & \text{if } t \leq \underline{v}(x), \\ g^p(t)g'(t) & \text{if } \underline{v}(x) \leq t \leq \overline{v}(x), \\ g^p(\overline{v}(x))g'(\overline{v}(x)) & \text{if } \overline{v}(x) \leq t. \end{cases}$$

Clearly, $\tilde{g}$ is a Carathéodory function. Moreover, (3) and (5) of Lemma 2.3 imply that

$$|\tilde{g}(x,t)| \leq |\overline{v}(x)|^p \leq \|\overline{v}\|_\infty^p =: c, \tag{2.8}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. We denote by $\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds$ the primitive of $\tilde{g}$ such that $\tilde{G}(x,0) = 0$.

Now, we consider the auxiliary singular elliptic problem

$$\begin{cases} -\Delta v = a(x)(g(v))^{-\gamma}g'(v) + \lambda\tilde{g}(x,v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{A_λ}$$

We will show that problem (A_λ) has a solution v such that $\underline{v} \leq v \leq \overline{v}$ in Ω. Thus, from definition of $\tilde{g}$ we obtain that v is a solution of (Q_λ). Define the function G as it follows:
if $0 < \gamma < 1$, $G(t) = \frac{g^{1-\gamma}(|t|)}{1-\gamma}$ and $t \in \mathbb{R}$,
if $\gamma = 1$,

$$G(t) = \begin{cases} \ln g(t), & \text{if } t > 0, \\ +\infty, & \text{if } t = 0, \end{cases}$$

if $\gamma > 1$,

$$G(t) = \begin{cases} \frac{g^{1-\gamma}(t)}{1-\gamma}, & \text{if } t > 0, \\ +\infty, & \text{if } t = 0. \end{cases}$$

We can associate to problem (A_λ) the following energy functional

$$I_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - \lambda \int_\Omega \tilde{G}(x,v), \tag{2.9}$$

for every $v \in D$, where

$$D = \left\{ v \in H_0^1(\Omega) : \int_\Omega a(x)G(|v|) \in \mathbb{R} \right\} \tag{2.10}$$

is the effective domain of I_λ. As we known, the functional I_λ fails to be Gâteaux differentiable because of the singular term, then we can not apply the critical point theory for functionals of class C^1.

The assumption (H) and Lemmas 2.1 and 2.5 imply that $aG(\phi_1) \in L^q(\Omega)$. In particular, one has $\phi_1 \in D$ and hence $D \neq \emptyset$. Then, using (2.8) and arguing as in the proof of Theorems 1.1 and 1.2 of [1] we can show that there exists a solution v of (A_λ) and it satisfies

$$I_\lambda(v) = \inf_{z \in D} I_\lambda(z).$$

It remains to check that $\underline{v} \leq v \leq \overline{v}$ in Ω. We set $(v - \underline{v})^- = \max\{-(v-\underline{v}), 0\}$. Using that $\underline{v}$ is a subsolution and v is a solution, we have

$$\int_\Omega \nabla \underline{v} \nabla (v-\underline{v})^- \leq \int_\Omega a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v})(v-\underline{v})^- + \lambda \int_\Omega (g(\underline{v}))^p g'(\underline{v})(v-\underline{v})^-,$$

$$\int_\Omega \nabla v \nabla (v-\underline{v})^- = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)(v-\underline{v})^- + \lambda \int_\Omega \tilde{g}(x,v)(v-\underline{v})^-,$$

and applying (9), (10) and (11) of Lemma 2.3, we find

$$\begin{aligned} -\int_\Omega |\nabla (v-\underline{v})^-|^2 &\geq \int_{\{v<\underline{v}\}} a(x)((g(v))^{-\gamma} g'(v) - (g(\underline{v}))^{-\gamma} g'(\underline{v}))(v-\underline{v})^- \\ &\quad + \lambda \int_{\{v<\underline{v}\}} (\tilde{g}(x,v) - (g(\underline{v}))^p g'(\underline{v}))(v-\underline{v})^- \\ &\geq \lambda \int_{\{v<\underline{v}\}} (\tilde{g}(x,v) - (g(\underline{v}))^p g'(\underline{v}))(v-\underline{v})^- \\ &= \lambda \int_{\{v<\underline{v}\}} ((g(\underline{v}))^p g'(\underline{v}) - (g(\underline{v}))^p g'(\underline{v}))(v-\underline{v})^- \\ &= 0, \end{aligned}$$

namely $\|(v-\underline{v})^-\| = 0$, which means that $\underline{v} \leq v$ in Ω.

Similarly, setting $(v-\overline{v})^+ = \max\{v-\overline{v},0\}$ and using that $\overline{v}$ is a supersolution and v is a solution, jointly with (9), (10) and (11) of Lemma 2.3, we get

$$\begin{aligned}\int_\Omega |\nabla(v-\overline{v})^+|^2 &\leq \int_{\{\overline{v}<v\}} a(x)((g(v))^{-\gamma}g'(v) - (g(\overline{v}))^{-\gamma}g'(\overline{v}))(v-\overline{v})^+ \\ &\quad + \lambda\int_{\{\overline{v}<v\}} (\tilde{g}(x,v) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &\leq \lambda\int_{\{\overline{v}<v\}} (\tilde{g}(x,v) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &= \lambda\int_{\{\overline{v}<v\}} ((g(\overline{v}))^p g'(\overline{v}) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &= 0,\end{aligned}$$

namely $\|(v-\overline{v})^+\| = 0$, which means that $v \leq \overline{v}$ in Ω. This completes the proof of the theorem. □

Remark 2.9.

a) Arguing as in the proof of Lemmas 2.1 and 2.5 we can show that $\mathrm{int}(C_0^1(\overline{\Omega})_+) \subset D$ (see (2.10)). Hence it makes sense to consider the local minimum obtained in Lemma 4.2, because $v_\lambda \in \mathrm{int}(C_0^1(\overline{\Omega})_+) \subset D$.

b) If $0<\gamma<1$ holds, then $I_\lambda(v) < 0$. Indeed, applying Lemma 2.6 (8) we obtain

$$I_\lambda(v) \leq I_\lambda(t\phi_1) \leq \frac{t^2}{2}\|\phi_1\|^2 - \frac{C^{1-\gamma}t^{1-\gamma}}{1-\gamma}\int_\Omega a(x)\phi_1^{1-\gamma} < 0,$$

provided $0<t<1$ is small enough.

The following lemma shows the existence of a subsolution of (Q_λ) for all $\lambda>0$.

Lemma 2.10. *If $v_0 \in H_0^1(\Omega)$ is the unique weak solution of (Q_0), then $v_0 \in C_0^1(\overline{\Omega})$ and $v_0(x) \geq Cd(x)$ in Ω for some constant $C>0$. Moreover, $a(x)(g(v_0))^{-\gamma}g'(v_0) \in L^q(\Omega)$ and v_0 is a subsolution of (Q_λ) for all $\lambda > 0$.*

Proof. From Lemma 2.1 and Remark 2.2 $b)$ one has $a(x)\phi_1^{1-\gamma} \in L^q(\Omega), q>1$, and hence, the existence of a unique weak solution $v_0 \in H_0^1(\Omega)$ of (Q_0) follows from Theorem 1.3 in [2]. Now we want to show that $v_0 \in C_0^1(\overline{\Omega})$. Using Theorem 3 of Brezis–Nirenberg [10] there exist constants $c_1, c_2 > 0$ such that $v_0(x) \geq c_2 d(x) \geq c_1\phi_1(x)$ in Ω and $c_1\phi_1(x) < 1$ in Ω. By Lemma 2.3 (3), (8), (11) and Lemma 2.1,

$$a(x)(g(v_0))^{-\gamma}g'(v_0) \leq C^{-\gamma}c_1^{-\gamma}a(x)\phi_1^{-\gamma} \in L^q(\Omega),$$

that is, $a(x)(g(v_0))^{-\gamma}g'(v_0) \in L^q(\Omega)$ with $q>N$. Thus, by elliptic regularity, $v_0 \in W_0^{2,q}(\Omega)$, and then by the Sobolev embedding theorem we have $v_0 \in C_0^1(\overline{\Omega})$. Finally, from the fact that v_0 is a solution of (Q_0) and $v_0 \in C_0^1(\overline{\Omega})$ one deduces that v_0 is a subsolution of (Q_λ) for all $\lambda > 0$. This completes the proof. □

We end this section with the following lemma.

Lemma 2.11. *Let $v \in H_0^1(\Omega), v > 0$ in Ω, and suppose that*

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in C_0^1(\overline{\Omega}), \psi \geq 0$, holds. Then

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, holds. In particular, $v \geq v_0$ in Ω, where v_0 is the unique solution of (Q_0).

Proof. Let $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, then from the proof of Theorem 4.4 of [12] there exists $\psi_n \in C_0^\infty(\overline{\Omega}), \psi_n \geq 0$ such that $\psi_n \to \psi$ in $H_0^1(\Omega)$ and $\psi_n \to \psi$ a.e. in Ω. Hence,

$$\int_\Omega \nabla v \nabla \psi_n \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi_n + \lambda \int_\Omega (g(v))^p g'(v)\psi_n,$$

and using the Fatou lemma we deduce that

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

proving the first statement of the lemma.

It remains to show that $v \geq v_0$ in Ω. For this, we take $(v - v_0)^-$ as a test function in the equation satisfied by v_0 and in the inequality satisfied by v, and arguing as in Theorem 2.8 one finds $v \geq v_0$ in Ω. The proof is complete. □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In the rest of this paper we will use the same notation introduced in the previous section.

Let us define

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{problem } (Q_\lambda) \text{ has at least one solution}\} \\ &= \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has at least one solution}\} \end{aligned}$$

and set

$$\Lambda = \sup \mathcal{L}.$$

We start by proving the following lemma.

Lemma 3.1. *The set $\mathcal{L}$ is nonempty and Λ is finite.*

Proof. Let $\underline{v} = v_0$ and consider the problem

$$\begin{cases} -\Delta v = a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{T}$$

By using Lemma 2.10 we infer that $a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) + 1 \in L^q(\Omega)$. Therefore problem (T) has a solution $\overline{v} \in W^{2,q}(\Omega)$ and by the Sobolev embedding theorem, $\overline{v} \in C_0^1(\overline{\Omega})$. Moreover,

$$-\Delta \overline{v} \geq a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) = -\Delta \underline{v} \quad \text{in } \Omega,$$

which implies that $\overline{v} \geq \underline{v}$ in Ω. From this and Lemmas 2.3 (9), (11) and 2.10 we get that

$$\int_\Omega \nabla\overline{v}\nabla\psi \geq \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi + \lambda\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, and for $\lambda > 0$ satisfying $\lambda\|(g(\overline{v}))^p g'(\overline{v})\|_\infty \leq 1$. For such values of λ, we can apply Theorem 2.8 to deduce the existence of a solution v of (Q_λ) such that $\underline{v} \leq v \leq \overline{v}$ in Ω (and consequently $v \in L^\infty(\Omega)$). Therefore $\mathcal{L} \neq \emptyset$.

By Lemma 2.6 we obtain that Λ is finite. The proof is complete. □

Following [19] we introduce

$$\lambda_* = \sup_{v\in S}\inf_{\psi\in\Phi}\{L(v,\psi)\} \tag{3.1}$$

where

$$L(v,\psi) := \frac{\int_\Omega \nabla v\nabla\psi - \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi}{\int_\Omega (g(v))^p g'(v)\psi}$$

is the extended functional and

$$\Phi = \left\{\psi \in C_0^1(\overline{\Omega})\backslash\{0\} : \psi \geq 0 \text{ in } \Omega\right\},$$

$$S = \left\{v \in H_0^1(\Omega)\cap L^\infty(\Omega) : v \geq C(v)d(x) \text{ in } \Omega\right\},$$

where $0 < C(v) < \infty$ is a positive constant which can depend on v. If $v \in S$ then $v \geq k\phi_1$ in Ω for some $k > 0$ (see Remark 2.2 c)), and from Lemmas 2.1, 2.3 and 2.5 it follows that L is well defined.

Some properties of λ_* are stated in the following theorem.

Theorem 3.2. *The following properties hold true:*

a) $0 < \lambda_* < \infty$.

b) $\lambda_* = \Lambda$.

Proof. *a*) From Lemma 3.1 there exist $\lambda > 0$ and $v \in H_0^1(\Omega)\cap L^\infty(\Omega), v > 0$ in Ω, such that

$$\int_\Omega \nabla v\nabla\psi = \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega)$, which together with Theorem 3 of Brezis–Nirenberg [10] implies that $v \in S$ and $0 < \lambda = L(v,\psi)$ for all $\psi \in \Phi$. As a consequence we get

$$0 < \lambda = \inf_{\psi\in\Phi}\{L(v,\psi)\} \leq \lambda_*.$$

To prove that $\lambda_* < \infty$, we argue by contradiction. Assume that $\lambda_* = \infty$. Then, by the definition of λ_* there exists $v \in S$ such that $\Lambda < \lambda := \inf_{\psi\in\Phi}\{L(v,\psi)\}$, that is,

$$\int_\Omega \nabla v\nabla\psi \geq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in \Phi$. By using Lemma 2.11 we deduce that v is a supersolution of (Q_λ) and $v \geq v_0$ in Ω. Moreover, from Lemma 2.10 one has that v_0 is a subsolution of (Q_λ). As a consequence we can apply Theorem 2.8, with $\underline{v} = v_0$ and $\overline{v} = v$, to deduce the existence of a solution of (Q_λ), which implies $\lambda \le \Lambda$, contradicting the fact that $\lambda > \Lambda$. Therefore $\lambda_* < \infty$.

b) Let $v \in S$ such that $0 < \lambda = \inf_{\psi\in\Phi} \{L(v,\psi)\}$. Arguing as in *a*) we can prove that problem (Q_λ) has a solution, namely, $\lambda \in \mathcal{L}$ and since λ is arbitrary, we have $\lambda_* = \sup_{v\in S} \inf_{\psi\in\Phi} \{L(v,\psi)\} \le \Lambda$. We claim that $\lambda_* = \Lambda$. Otherwise, $\lambda_* < \Lambda$ and by the definition of Λ there exists $\lambda > \lambda_*$ such that problem (Q_λ) has a solution v. Again, arguing as in *a*) we find that $v \in S$ and $\lambda = \inf_{\psi\in\Phi} \{L(v,\psi)\} \le \lambda_*$, contradicting the fact that $\lambda > \lambda_*$. Therefore $\lambda_* = \Lambda$. This finishes the proof. □

Remark 3.3. We will compare parameter λ_* with parameters ϵ_0 and ϵ_1 obtained in Theorems 1.2 and 1.3 of [29]. First, note that when $h(x,t) = \lambda|t|^{r-1}t$ the hypothesis (h_2) in [29] is satisfied with $b(x) := \lambda, C = 1$ and in our notation $r - 1 = p$. Hence $b(x) := \lambda$ is the parameter and problem (Q_λ) (or equivalently problem (P_λ)) has a solution for all $0 < \lambda \le \epsilon_0$. As a consequence, by the definition of $\Lambda = \lambda_*$ (see Theorem 3.2), one has $\epsilon_0 \le \lambda_*$. Let us remark that $\epsilon_1 \le \epsilon_0$, because one of the solutions obtained in Theorem 1.3 is the same as in Theorem 1.2 (both theorems mentioned here are from [29]).

We claim that $\epsilon_0 < \lambda_*$. To show this, we will use some notations and results obtained in Lemma 2.3 of [29]. Let $\underline{v}, \overline{v} \in C_0^1(\overline{\Omega})$ be the sub and supersolution, respectively, obtained in Lemma 2.3 of [29]. Then, $0 < \underline{v} \le \overline{v}$ in Ω and $\overline{v}$ satisfies

$$\begin{cases} -\Delta\overline{v} = a(x)(g(\underline{v}))^{-\gamma}g'(\underline{v}) + 2C & \text{in } \Omega, \\ \overline{v} > 0 & \text{in } \Omega, \\ \overline{v}(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{F}$$

which together with Theorem 3 of Brezis–Nirenberg [10] implies that $\overline{v} \in S$.

Moreover, ϵ_0 satisfies (see end of proof Lemma 2.3 of [29])

$$1 - \epsilon_0(g(\overline{v}))^p \ge 0$$

and since $0 < g'(t) \le 1$ for all $t > 0$, this implies

$$\epsilon_0 \|(g(\overline{v}))^p g'(\overline{v})\|_\infty \le 1. \tag{3.2}$$

Let us evaluate $L(\overline{v},\psi), \psi \in \Phi$. From ($F$), $\underline{v} \le \overline{v}$ in Ω and Lemma 2.3 (9), (11) we get

$$\begin{aligned} L(\overline{v},\psi) &= \frac{\int_\Omega \nabla\overline{v}\nabla\psi - \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi}{\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi} \\ &= \frac{\int_\Omega a(x)(g(\underline{v}))^{-\gamma}g'(\underline{v})\psi - \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi + 2C\int_\Omega \psi}{\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi} \\ &\ge \frac{2C\int_\Omega \psi}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty \int_\Omega \psi}, \end{aligned}$$

whence

$$L(\overline{v},\psi) \ge \frac{2C}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty}.$$

Since $C = 1$ and $\overline{v} \in S$, this implies

$$\lambda_* = \sup_{v\in S} \inf_{\psi\in\Phi} \{L(v,\psi)\} \ge \inf_{\psi\in\Phi} \{L(\overline{v},\psi)\} \ge \frac{2}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty}$$

and therefore

$$\lambda_* \|(g(\overline{v}))^p g'(\overline{v})\|_\infty > 1. \tag{3.3}$$

From $\lambda_* \geq \epsilon_0$, (3.2) and (3.3) one deduces that $\epsilon_1 \leq \epsilon_0 < \lambda_*$.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us show that problem (Q_λ) has a solution for $\lambda \in (0, \lambda_*)$ and no solution for $\lambda \in (\lambda_*, \infty)$, where λ_* is defined in (3.1). Let $\lambda \in (0, \lambda_*)$. Then, by the definition of λ_*, there exists $z \in S$ such that $\lambda \leq L(z, \psi)$ for all $\psi \in \Phi$. We deduce from this inequality and Lemma 2.11 that z is a supersolution of (Q_λ) with $z \geq v_0$ in Ω. Applying Theorem 2.8 with $\underline{v} = v_0$ and $\overline{v} = z$ we obtain that problem (Q_λ) has a solution v_λ with $\underline{v} \leq v_\lambda \leq \overline{v}$ in Ω. To show that $v_\lambda \in C^1_0(\overline{\Omega})$ we follow [2]. By Lemma 2.3 $(3), (5), (9), (11)$ and Lemma 2.10 we infer

$$a(x) g^{-\gamma}(v_\lambda) g'(v) \leq a(x) g^{-\gamma}(\underline{v}) g'(\underline{v}) \in L^q(\Omega)$$

and

$$g^p(v_\lambda) g'(v) \leq |\overline{v}|^p \leq \|\overline{v}\|^p_\infty \in L^\infty(\Omega)$$

and as a consequence there exist $z_1, z_2 \in C^{1,\alpha}_0(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfying

$$\int_\Omega \nabla z_1 \nabla \psi = \int_\Omega a(x)(g(v_\lambda))^{-\gamma} g'(v_\lambda)\psi \quad \text{and} \quad \int_\Omega \nabla z_2 \nabla \psi = \lambda \int_\Omega (g(v_\lambda))^p g'(v_\lambda)\psi,$$

for all $\psi \in H^1_0(\Omega)$. From this we get

$$\int_\Omega \nabla v_\lambda \nabla \psi = \int_\Omega \nabla z_1 \nabla \psi + \int_\Omega \nabla z_2 \nabla \psi,$$

for all $\psi \in H^1_0(\Omega)$, which implies $v_\lambda = z_1 + z_2$, and hence $v_\lambda \in C^{1,\alpha}_0(\overline{\Omega})$. Furthermore, by the strong maximum principle and the Hopf lemma we find that $v_\lambda \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$.

Finally, from Theorem 3.2 we have $\lambda_* = \Lambda$ and by the definition of Λ problem (Q_λ) has no solution for $\lambda > \lambda_* = \Lambda$. This completes the proof of the theorem. □

4 Proof of Theorem 1.2

In this section we are going to prove Theorem 1.2. In order to do this, we adapt the arguments carried out in [4]. From now on, we will assume $(H)_\infty$ and $3 < p < 22^* - 1$ hold. Proceeding as in Section 1 we can prove that:

- $a\phi_1^{-1-\gamma}, a\phi_1^{-\gamma} \in L^\infty(\Omega)$.
- $ag^{-1-\gamma}(\varphi)g'(\varphi), ag^{-1-\gamma}(\phi_1)g'(\phi_1) \in L^\infty(\Omega)$ and $ag^{1-\gamma}(\varphi) \in L^\infty(\Omega)$ if $\gamma \neq 1$, and $a(x)\ln(g(\varphi)) \in L^\infty(\Omega)$ if $\gamma = 1$.
- if v_λ is the solution obtained in Theorem 1.1, then
$$a(x) g^{-\gamma}(v_\lambda) g'(v_\lambda) \in L^\infty(\Omega). \tag{4.1}$$

We start by defining the functional

$$J_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega a(x) G(|v|) - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(v), \qquad v \in D. \tag{4.2}$$

It is worth recalling that $\operatorname{int}(C^1_0(\overline{\Omega})_+) \subset D$ (see (2.10) and Remark 2.9). The functional J_λ fails to be Fréchet differentiable in $H^1_0(\Omega)$ because of the singular term, then critical point theory could not be applied to obtain the existence of solutions directly.

Remark 4.1. Let (H) and $0 < p < 3$ hold. Denote by $D^+ = \{u \in D : u \geq 0 \text{ a.e. in } \Omega\}$. Then, by Lemma 2.5 one has $D^+ \neq \emptyset$. The arguments carried out in [1,2] can be adapted to prove that problem (P_λ) has at least one solution for all $\lambda \in \mathbb{R}$.

a) Assume that $\lambda \geq 0$. By Lemma 2.3 (6)

$$J_\lambda(v) \geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - \frac{\lambda 2^{\frac{p+1}{4}}}{p+1}\int_\Omega |v|^{\frac{p+1}{2}}$$
$$\geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - C\|v\|^{\frac{p+1}{2}},$$

for all $v \in D^+$ and for some constant $C > 0$. Hence, since $1/2 < (p+1)/2 < 2$, we argue in a similar way to the proof of Lemma 2.1 of [1] to show that J_λ is coercive on D^+ and there exists $v_\lambda \in D^+$ such that

$$J_\lambda(v_\lambda) = \inf_{v \in D^+} J_\lambda(v).$$

Finally, considering the cases $\gamma \geq 1$ and $0 < \gamma < 1$ respectively, we argue in a similar way to the first part of the proof of Theorems 1.1 and 1.2 of [1] to show that v_λ is a solution of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

For $\lambda \leq 0$ we get

$$J_\lambda(v) \geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|)$$

for all $v \in D^+$. We argue in the same way as in the case $\lambda \geq 0$ to show that there exists a solution v_λ of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

b) We define the following constraint sets

$$\mathcal{N}_1 = \left\{ v \in D^+ : \|v\|^2 - \int_\Omega (g(v))^p g'(v)v \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)v \right\}$$

and

$$\mathcal{N}_2 = \left\{ v \in D^+ : \|v\|^2 - \int_\Omega (g(v))^p g'(v)v = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)v \right\}.$$

Since $1 < p+1 < 4$, by Lemma 2.3 (6) we have $\lim_{t\to\infty} J_\lambda(tv) = \infty$ for all $v \in D^+$. Moreover, $\lim_{t\to 0^+} J_\lambda(tv) = \infty$ if $\gamma \geq 1$ and $\lim_{t\to 0^+} J_\lambda(tv) = 0$ if $0 < \gamma < 1$. Therefore, for all $v \in D^+$ there exists a $t(v) > 0$ such that $J_\lambda(t(v)v) = \inf_{t>0} J_\lambda(tv)$ and $t(v)v \in \mathcal{N}_2$. Using this fact, Lemma 2.3 (6) and that $1 < p+1 < 4$ we show that J_λ is coercive on $\mathcal{N}_1$ and there exists $v_\lambda \in \mathcal{N}_1$ such that $J_\lambda(v_\lambda) = \inf_{v\in\mathcal{N}_1} J_\lambda(v) = \inf_{v\in\mathcal{N}_2} J_\lambda(v)$. Finally, we argue in a similar way to the first part of the proof of Theorem 1.1 of [2] to show that v_λ is a solution of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

In this section, we denote by I_λ the functional defined in (2.9) of Theorem 2.8.

An important property of the solution obtained in Theorem 1.1 is the following.

Lemma 4.2. *Let $0 < \lambda < \lambda_*$. If v_λ is the solution of (Q_λ) obtained in Theorem 1.1, then v_λ is a local minimum of J_λ in the $C_0^1(\overline{\Omega})$ topology.*

Proof. Without loss of generality, we can assume that $\overline{v}$ is a solution of (Q_μ) for some $\mu \in (\lambda, \lambda^*)$. Hence, arguing as in Theorem 1.1 one has $\overline{v} \in C_0^1(\overline{\Omega})$ and by the strong maximum principle and the Hopf lemma we infer that $\overline{v} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Now, the proof is based on the following claims.

Claim 1. $\overline{v} - v_\lambda \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. We have

$$-\Delta(\overline{v} - v_\lambda) \geq a(x)((g(\overline{v}))^{-\gamma} g'(\overline{v}) - (g(v_\lambda))^{-\gamma} g'(v_\lambda)) + \lambda(g(\overline{v}))^p g'(\overline{v}) - (g(v_\lambda))^p g'(v_\lambda))$$

and by the mean value theorem there exist measurable functions $\theta_1(x)$ and $\theta_2(x)$ such that $v_\lambda(x) \leq \theta_1(x), \theta_2(x) \leq \overline{v}(x)$ for all $x \in \Omega$ and

$$\begin{aligned} -\Delta(\overline{v} - v_\lambda) \geq\ & a(x)((g(\theta_1(x)))^{-\gamma} g'(\theta_1(x)))'(\overline{v}(x) - v_\lambda(x)) \\ & + \lambda((g(\theta_2(x))))^p g'(\theta_2(x))))'(\overline{v}(x) - v_\lambda(x)). \end{aligned} \tag{4.3}$$

From the definition of g' and Lemma 2.3 (3), it follows that

$$\begin{aligned} &(g^{-\gamma}(t) g'(t))' \geq -g^{-1-\gamma}(t)(\gamma + 2g^2(t)), && t > 0, \\ &|(g^p(t) g'(t))'| \leq p g^{p-1}(t) + 2g^{p+1}(t), && t > 0, \end{aligned} \tag{4.4}$$

hold. Then, again by Lemma 2.3 (3), (11) one has

$$\begin{aligned} &(g^{-\gamma}(\theta_1(x)) g'(\theta_1(x)))' \geq -g^{-1-\gamma}(v_\lambda(x))(\gamma + 2g^2(\|\overline{v}\|_\infty)), \\ &|(g^p(\theta_2(x)) g'(\theta_2(x)))'| \leq p g^{p-1}(\|\overline{v}\|_\infty) + 2g^{p+1}(\|\overline{v}\|_\infty), \end{aligned}$$

for all $x \in \Omega$. We set

$$c_1 = \|a g^{-1-\gamma}(v_\lambda)\|_\infty(\gamma + 2g^2(\|\overline{v}\|_\infty)), \qquad c_2 = p\lambda g^{p-1}(\|\overline{v}\|_\infty) + 2\lambda g^{p+1}(\|\overline{v}\|_\infty)$$

and $c = c_1 + c_2$. With these estimates and definitions, in view of (4.3), we get

$$-\Delta(\overline{v} - v_\lambda) \geq (-c_1 - c_2)(\overline{v} - v_\lambda) = -c(\overline{v} - v_\lambda)$$

that is

$$-\Delta(\overline{v} - v_\lambda) + c(\overline{v} - v_\lambda) \geq 0 \text{ in } \Omega,$$

and since $\overline{v} - v_\lambda \neq 0$, we can apply Theorem 3 of [10] to deduce the existence of constants $c_3, c_4 > 0$ such that

$$\overline{v} - v_\lambda \geq c_3 d(x) \geq c_4 \phi_1(x) \quad \text{in } \Omega.$$

As a consequence we obtain

$$\frac{\partial(\overline{v} - v_\lambda)}{\partial \nu} \leq c_4 \frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

which jointly with $\overline{v} - v_\lambda > 0$ in Ω means that $\overline{v} - v_\lambda \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, and this proves Claim 1.

Claim 2. $v_\lambda - \underline{v} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. The proof is essentially equal to the one of Claim 1. Indeed, we set

$$c_1 = \|a g^{-1-\gamma}(\underline{v})\|_\infty(\gamma + 2g^2(\|\overline{v}\|_\infty)),$$

and from (4.4) and mean value theorem one has

$$\begin{aligned} -\Delta(v_\lambda - \underline{v}) &\geq a(x)((g(\theta_1(x)))^{-\gamma} g'(\theta_1(x)))'(v_\lambda - \underline{v}) + \lambda(g(v_\lambda))^p g'(v_\lambda) \\ &\geq -c_1(v_\lambda - \underline{v}) \end{aligned}$$

in Ω, because $\underline{v}(x) \leq \theta_1(x) \leq v_\lambda(x)$, and since $v_\lambda - \underline{v} \neq 0$, we can apply Theorem 3 of [10] to deduce the existence of constants $c_3, c_4 > 0$ such that

$$v_\lambda - \underline{v} \geq c_3 d(x) \geq c_4 \phi_1(x) \quad \text{in } \Omega.$$

As a consequence we obtain

$$\frac{\partial (v_\lambda - \underline{v})}{\partial \nu} \leq c_4 \frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

which jointly with $v_\lambda - \underline{v} > 0$ in Ω means that $v_\lambda - \underline{v} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, and this proves Claim 2.

Claim 3. There exists a ball $B = B_\epsilon(v_\lambda)$ in the $C_0^1(\overline{\Omega})$ topology satisfying

$$B \subset [\underline{v}, \overline{v}] := \left\{ v \in C_0^1(\overline{\Omega}) : \underline{v} \leq v \leq \overline{v} \text{ in } \Omega \right\}.$$

From Claims 1 and 2 there exists $\epsilon > 0$ such that the balls $B_1 = B_\epsilon(\overline{v} - v_\lambda), B_2 = B_\epsilon(v_\lambda - \underline{v}) \subset \mathrm{int}(C_0^1(\overline{\Omega})_+)$. We define $B = B_\epsilon(v_\lambda)$. Let $v \in B$. Notice that

$$\overline{v} - B_1 = B_\epsilon(v_\lambda) \quad \text{and} \quad \underline{v} + B_2 = B_\epsilon(v_\lambda),$$

and as a consequence there exist $z \in B_1, w \in B_2$ with

$$\underline{v} + w = v = \overline{v} - z,$$

which implies $\underline{v} < v < \overline{v}$ in Ω, that is, $v \in [\underline{v}, \overline{v}]$. Hence $B \subset [\underline{v}, \overline{v}]$.

We can finally complete the proof of the lemma. Let B as in Claim 3 and consider $v \in B$. Then,

$$\begin{aligned}
J_\lambda(v) - I_\lambda(v) &= -\frac{\lambda}{p+1}\int_\Omega g^{p+1}(v) + \lambda \int_\Omega \tilde{G}(x,v) \\
&= -\frac{\lambda}{p+1}\int_\Omega g^{p+1}(v) + \lambda \int_\Omega \int_0^{\underline{v}(x)} \tilde{g}(x,t)dtdx + \lambda \int_\Omega \int_{\underline{v}(x)}^{v(x)} \tilde{g}(x,t)dtdx \\
&= -\frac{\lambda}{p+1}\int_\Omega g^{p+1}(v) + \lambda \int_\Omega \int_0^{\underline{v}(x)} g^p(\underline{v}(x))g'(\underline{v}(x))dtdx \\
&\quad + \lambda \int_\Omega \int_{\underline{v}(x)}^{v(x)} g^p(t)g'(t)dtdx \\
&= \lambda \int_\Omega g^p(\underline{v}(x))g'(\underline{v}(x))\underline{v}(x)dx - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(\underline{v}(x))dx =: c
\end{aligned}$$

where c is a constant.

By virtue of the above equality, we obtain that v_λ is a $C_0^1(\overline{\Omega})$-local minimizer of J_λ. This finishes the proof. □

Remark 4.3. Since $\underline{v}, \overline{v} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, it follows that $[\underline{v}, \overline{v}] \subset \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and then, by Remark 2.9, $J_\lambda(v), I_\lambda(v) \in \mathbb{R}$ for all $v \in [\underline{v}, \overline{v}]$. Furthermore, arguing as in Lemma 2.5 we infer

$$\left\{ v \in H_0^1(\Omega) : \underline{v} \leq v \leq \overline{v} \text{ in } \Omega \right\} \subset D.$$

Corollary 4.4. *Let $B = B_\epsilon(0) + v_\lambda$ be as in the proof of Lemma 4.2. Then for all $v \in B_\epsilon(0)$ we have*

$$J_\lambda(v_\lambda + v^+) - J_\lambda(v_\lambda) \geq 0,$$

holds.

Proof. As we have seen in the proof of Lemma 4.2,

$$\underline{v} < v_\lambda + v < \overline{v} \quad \text{in } \Omega, \tag{4.5}$$

for all $v \in B_\epsilon(0)$. We claim that

$$\underline{v} < v_\lambda + v^+ < \overline{v} \quad \text{in } \Omega,$$

for all $v \in B_\epsilon(0)$. Indeed, by using (4.5) one has

$$\underline{v} < v_\lambda + v = v_\lambda + v^+ - v^- \leq v_\lambda + v^+ \quad \text{in } \Omega.$$

Now, let us show that $v_\lambda + v^+ < \overline{v}$ in Ω. Arguing by contradiction, suppose that there exists $x \in \Omega$ such that $v_\lambda(x) + v^+(x) \geq \overline{v}(x)$. Then, from $v_\lambda(x) < \overline{v}(x)$ we infer that $v(x) > 0$, and therefore $v^-(x) = 0$. Thus, the inequality (4.5) implies

$$\overline{v}(x) \leq v_\lambda(x) + v^+(x) = v_\lambda(x) + v^+(x) - v^-(x) = v_\lambda(x) + v(x) < \overline{v}(x),$$

a contradiction.

Finally, we can argue as in the proof of Lemma 4.2 to get

$$J_\lambda(v_\lambda + v^+) - I_\lambda(v_\lambda + v^+) = c,$$

where c is a constant, and since $v_\lambda + v^+ \in H_0^1(\Omega)$, by Theorem 2.8, we deduce that

$$J_\lambda(v_\lambda + v^+) - J_\lambda(v_\lambda) = I_\lambda(v_\lambda + v^+) - I_\lambda(v_\lambda) \geq 0,$$

proving the corollary. □

For fixed $\lambda \in (0, \lambda_*)$, we look for a second solution in the form $z = w + v$, where $v \gneqq 0$ and $w = v_\lambda$ is the solution found in the preceding lemma. A straight calculation shows that v satisfies

$$\begin{aligned} -\Delta v = {} & a(x)((g(w+v))^{-\gamma} g'(w+v) - (g(w))^{-\gamma} g'(w)) \\ & + \lambda((g(w+v))^p g'(w+v) - (g(w))^p g'(w)). \end{aligned} \tag{4.6}$$

Denote by $g_\lambda(x,t)$ the right hand side of the preceding equation (with $g_\lambda(x,t) = 0$ for $t \leq 0$) and set

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G_\lambda(x,v), \tag{4.7}$$

where

$$G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds = \begin{cases} 0 & \text{if } t \leq 0, \\ H_1(x,t) + H_2(x,t) + H_3(x,t) & \text{if } t \geq 0, \end{cases}$$

and

$$\begin{aligned} H_1(x,t) &= a(x)(G(w+t) - G(w)), \\ H_2(x,t) &= \frac{\lambda}{p+1}(g^{p+1}(w+t) - g^{p+1}(w)), \\ H_3(x,t) &= -a(x)g^{-\gamma}(w)g'(w)t - \lambda(g(w))^p g'(w)t, \end{aligned}$$

for $t \geq 0$.

We observe that by $(H)_\infty$, Lemma 2.3 $(3), (6), (11), (12)$ and (4.1) one has

$$|g_\lambda(x,t)| \leq c_1 + 2^{(p-3)/4}\lambda c_2 |t|^{(p-1)/2}, \tag{4.8}$$

where $c_1, c_2 > 0$ are constants which depends of $\|ag^{-\gamma}(w)g'(w)\|_\infty$, $\|w\|_\infty$ and p. From this it follows that $\mathcal{J}_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$.

We shall use the Mountain Pass Theorem to prove the existence of a nontrivial solution to (4.6). In order to do this, we need some preliminary lemmas.

Lemma 4.5. *$v = 0$ is a local minimum of $\mathcal{J}_\lambda$ in $H_0^1(\Omega)$.*

Proof. We write $v = v^+ - v^-$. Using the fact that w is a solution of (Q_λ) and $G(x,t) = 0$ for $t \leq 0$ we get

$$\begin{aligned}
\mathcal{J}_\lambda(v) &= \frac{1}{2}\|v^+\|^2 + \frac{1}{2}\|v^-\|^2 - \int_\Omega G_\lambda(x, v^+) + \frac{1}{2}\|w + v^+\|^2 - \frac{1}{2}\|w + v^+\|^2 \\
&= \frac{1}{2}\|v^-\|^2 - \int_\Omega \nabla w \nabla v^+ + \int_\Omega a(x)(g(w))^{-\gamma} g'(w) v^+ + \lambda \int_\Omega (g(w))^p g'(w) v^+ \\
&\quad + \frac{1}{2}\|w + v^+\|^2 - \int_\Omega a(x) G(w + v^+) - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(w + v^+) \\
&\quad - \frac{1}{2}\|w\|^2 + \int_\Omega a(x)G(w) + \frac{\lambda}{p+1}\int_\Omega g^{p+1}(w) \\
&= \frac{1}{2}\|v^-\|^2 + J_\lambda(w + v^+) - J_\lambda(w).
\end{aligned}$$

This and Corollary 4.4 imply that $\mathcal{J}_\lambda(v) \geq 0$ for all $v \in B_\epsilon(0)$, where $B_\epsilon(0)$ is as in Corollary 4.4. This proves that $v = 0$ is a local minimum in the $C_0^1(\overline{\Omega})$ topology. Therefore, in view of (4.8), Theorem 1 in [10] applies and $v = 0$ is a local minimum of $\mathcal{J}_\lambda$ in the $H_0^1(\Omega)$ topology. This finishes the proof. $\square$

Lemma 4.6. *If $v, w \in L^\infty(\Omega) \cap D$ are positive functions, then*

$$\lim_{t\to\infty} \int_\Omega \frac{a(x)G(v + tw)}{t^{(p+1)/2}} = 0$$

and

$$\int_\Omega g^{p+1}(v + tw) \geq t^{(p+1)/2} \int_\Omega g^{p+1}\Big(\frac{v}{t} + w\Big),$$

for all $t > 1$.

Proof. First we prove the limit. We divide the proof into three cases.

Case 1. $\gamma < 1$. In this case, by Lemma 2.3 (5) one has

$$0 < \frac{a(x)G(v + tw)}{t^{(p+1)/2}} = \frac{a(x)g^{1-\gamma}(v + tw)}{(1-\gamma)t^{(p+1)/2}} \leq \frac{a(x)(\frac{v}{t} + w)^{1-\gamma}}{(1-\gamma)t^{((p+1)/2)+\gamma-1}} \leq \frac{a(x)(v + w)^{1-\gamma}}{1-\gamma},$$

for all $t \geq 1$. Then taking the limit as $t \to \infty$ we get

$$\frac{a(x)G(v + tw)}{t^{(p+1)/2}} \to 0,$$

and from the Lebesgue dominated convergence theorem we find

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

This proves the case 1.

Case 2. $\gamma=1$. By Lemma 2.3 (3), (5)

$$\frac{a(x)\ln(g(v))}{t^{(p+1)/2}}\le\frac{a(x)G(v+tw)}{t^{(p+1)/2}}=\frac{a(x)\ln(g(v+tw))}{t^{(p+1)/2}}\le\frac{a(x)\left(\frac{v}{t}+w\right)}{t^{((p+1)/2)-1}}\le a(x)(v+w)$$

for all $t\ge1$, and thus

$$\left|\frac{a(x)G(v+tw)}{t^{(p+1)/2}}\right|\le\max\left\{|a(x)\ln(g(v))|,a(x)(v+w)\right\}.$$

Again, by the Lebesgue dominated convergence theorem we have

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

Case 3. $\gamma>1$. By Lemma 2.3 (3), (10) one has

$$0<\left|\frac{a(x)G(v+tw)}{t^{(p+1)/2}}\right|=\frac{a(x)g^{1-\gamma}(v+tw)}{|1-\gamma|t^{(p+1)/2}}\le\frac{a(x)g^{1-\gamma}(v)}{|1-\gamma|t^{(p+1)/2}}\le\frac{a(x)g^{1-\gamma}(v)}{|1-\gamma|},$$

for all $t\ge1$. By the Lebesgue dominated convergence theorem one finds

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

We now fix $t>1$. Then, from Lemma 2.3 (13) we have

$$g^{p+1}(v+tw)=\left[g^2\left(t\left(\frac{v}{t}+w\right)\right)\right]^{(p+1)/2}\ge\left[tg^2\left(\frac{v}{t}+w\right)\right]^{(p+1)/2}=t^{(p+1)/2}g^{p+1}\left(\frac{v}{t}+w\right),$$

and this implies that

$$\int_\Omega g^{p+1}(v+tw)\ge t^{(p+1)/2}\int_\Omega g^{p+1}\left(\frac{v}{t}+w\right),$$

for all $t>1$. The lemma is proved. □

Lemma 4.7. *Let $2<\theta<p+1$. Then, for all $t\ge0$,*

a) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c-\frac{a(x)}{1-\gamma}t^{1-\gamma}$ *for some constant $c\in\mathbb{R}$, if $0<\gamma<1$;*

b) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c-a(x)t+a(x)\ln(g(w))$ *for some constant $c\in\mathbb{R}$, if $\gamma=1$;*

c) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c+\frac{a(x)}{1-\gamma}g^{1-\gamma}(w)$ *for some constant $c\in\mathbb{R}$, if $\gamma>1$.*

Proof. For convenience of notation we write

$$\begin{aligned} h_1(x,t) &= a(x)(g(w+t))^{-\gamma}g'(w+t), \\ h_2(x,t) &= \lambda(g(w+t))^p g'(w+t), \\ h_3(x,t) &= -a(x)(g(w))^{-\gamma}g'(w) - \lambda(g(w))^p g'(w), \end{aligned}$$

for $t \geq 0$. Thus,

$$\begin{aligned} -G_\lambda(x,t) + \frac{\theta}{p+1}g_\lambda(x,t)t = & -H_1(x,t) + \frac{\theta}{p+1}h_1(x,t)t \\ & -H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t \\ & -H_3(x,t) + \frac{\theta}{p+1}h_3(x,t)t \end{aligned}$$

for $t \geq 0$.
a) In this case, from Lemma 2.3 (5) we have

$$\begin{aligned} -H_1(x,t) + \frac{\theta}{p+1}h_1(x,t)t \geq -H_1(x,t) &\geq -\frac{a(x)}{1-\gamma}g^{1-\gamma}(w+t) \\ &\geq -\frac{a(x)}{1-\gamma}(w+t)^{1-\gamma} \\ &\geq -\frac{a(x)}{1-\gamma}(w^{1-\gamma}+t^{1-\gamma}) \\ &\geq -\frac{\|aw^{1-\gamma}\|_\infty}{1-\gamma} - \frac{a(x)}{1-\gamma}t^{1-\gamma} \end{aligned}$$

and since $p+1 > \theta$,

$$-H_3(x,t) + \frac{\theta}{p+1}h_3(x,t)t = \left(1 - \frac{\theta}{p+1}\right)(a(x)(g(w))^{-\gamma}g'(w) + \lambda(g(w))^p g'(w))t \geq 0. \tag{4.9}$$

Let us observe that this inequality is valid for all $\gamma > 0$.
Now, let us estimate $-H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t$. From Lemma 2.3 (4) one has

$$\begin{aligned} -H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t &= \frac{\lambda}{p+1}(-g^{p+1}(w+t) + g^{p+1}(w)) \\ &\quad + \frac{\theta\lambda}{p+1}(g(w+t))^p g'(w+t)t \\ &\geq \frac{\lambda}{p+1}\left[-g^{p+1}(w+t) + \frac{\theta}{2}\frac{g^{p+1}(w+t)t}{w+t} + g^{p+1}(w)\right] \\ &\geq \frac{\lambda}{p+1}\left[g^{p+1}(w+t)\left(-1 + \frac{\theta}{2}\frac{t}{\|w\|_\infty + t}\right) + g^{p+1}(w)\right], \end{aligned}$$

and therefore

$$-H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t > 0,$$

for all $t > \bar{t} := (2\|w\|_\infty)/(\theta - 2)$.

Moreover, for $0 \le t \le \bar{t}$, by using Lemma 2.3 (5) we get

$$-H_2(x,t) + \frac{\theta}{p+1} h_2(x,t)t \ge -\frac{\lambda}{p+1} g^{p+1}(w+t) \\ \ge -\frac{\lambda}{p+1}(w+t)^{p+1} \ge -\frac{\lambda}{p+1}(\|w\|_\infty + \bar{t})^{p+1}.$$

By setting $c_1 = -\frac{\lambda}{p+1}(\|w\|_\infty + \bar{t})^{p+1}$, we have proved that

$$-H_2(x,t) + \frac{\theta}{p+1} h_2(x,t)t \ge c_1, \quad \text{for all } t \ge 0. \tag{4.10}$$

Let us observe that this inequality is valid independent of $\gamma > 0$.

In view of the above inequalities we deduce that

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \ge c - \frac{a(x)}{1-\gamma} t^{1-\gamma} \quad \text{for all } t \ge 0,$$

where $c = -\frac{\|aw^{1-\gamma}\|_\infty}{1-\gamma} + c_1$.

b) When $\gamma = 1$, by Lemma 2.3 (5), one has the inequality

$$-H_1(x,t) + \frac{\theta}{p+1} h_1(x,t)t \ge -H_1(x,t) = -a(x)\ln(g(w+t)) + a(x)\ln(g(w)) \\ \ge -a(x)(w+t) + a(x)\ln(g(w)) \\ \ge -\|aw\|_\infty - a(x)t + a(x)\ln(g(w)),$$

which combined with (4.9) and (4.10) yield

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \ge c - a(x)t + a(x)\ln(g(w))$$

for some constant $c \in \mathbb{R}$.

c) Indeed, the inequality

$$-H_1(x,t) + \frac{\theta}{p+1} h_1(x,t)t \ge -H_1(x,t) = -\frac{a(x)}{1-\gamma} g^{1-\gamma}(w+t) + \frac{a(x)}{1-\gamma} g^{1-\gamma}(w) \\ \ge \frac{a(x)}{1-\gamma} g^{1-\gamma}(w),$$

combined with (4.9) and (4.10) yield

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \ge c + \frac{a(x)}{1-\gamma} g^{1-\gamma}(w)$$

for some constant $c \in \mathbb{R}$. This concludes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.5 $u_0 = 0$ is a local minimizer of $\mathcal{J}_\lambda$ with respect to the topology of $H_0^1(\Omega)$. In the case where u_0 is not a strict local minimizer of $\mathcal{J}_\lambda$, we deduce the existence of further critical points of $\mathcal{J}_\lambda$, and then we are done. In this way, we may assume that

$$u_0 = 0 \text{ is a strict local minimizer of } \mathcal{J}_\lambda. \tag{4.11}$$

For all $t > 1$ we have

$$\mathcal{J}_\lambda(t\phi_1) = J_\lambda(w + t\phi_1) - J_\lambda(w)$$

and by Lemma 4.6 it follows that

$$\mathcal{J}_\lambda(t\phi_1) \le \frac{1}{2}\|w + t\phi_1\|^2 - \int_\Omega a(x)G(w + t\phi_1) - \frac{t^{(p+1)/2}\lambda}{p+1}\int_\Omega g^{p+1}\left(\frac{w}{t} + \phi_1\right) - J_\lambda(w)$$

and using again Lemma 4.6 and the Lebesgue dominated convergence theorem we yield $\lim_{t\to\infty}\mathcal{J}(t\phi_1) = -\infty$. From this and (4.11), we conclude that $\mathcal{J}_\lambda$ has the mountain pass geometry (see [5, Theorem 2.1]). It remains to prove the Palais–Smale condition. Let $4 < 2\theta < p+1$. Let $v_n \in H_0^1(\Omega)$ be such that $\mathcal{J}_\lambda(v_n) \to c$ ($c \in \mathbb{R}$) and $\mathcal{J}'_\lambda(v_n) \to 0$. From the former, respectively the latter multiplied by $\theta v_n/(p+1)$, we get

$$\frac{1}{2}\|v_n\|^2 - \int_\Omega G_\lambda(x, v_n) = c + o(1),$$

$$o(1)\|v_n\| \ge |\frac{\theta}{p+1}\|v_n\|^2 - \frac{\theta}{p+1}\int_\Omega g_\lambda(x, v_n)v_n| \ge \frac{-\theta}{p+1}\|v_n\|^2 + \frac{\theta}{p+1}\int_\Omega g_\lambda(x, v_n)v_n,$$

and therefore (remember that $G_\lambda(x,t) = g_\lambda(x,t)t = 0$ for $t \le 0$),

$$c + o(1) + o(1)\|v_n\| \ge \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + \int_\Omega (-G_\lambda(x, v_n^+) + \frac{\theta}{p+1}g_\lambda(x, v_n^+)v_n^+).$$

From this and Lemma 4.7 we deduce that

$$c + o(1) + o(1)\|v_n\| \ge \begin{cases} \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| - \int_\Omega \frac{a(x)}{1-\gamma}(v_n^+)^{1-\gamma} & \text{if } \gamma < 1, \\ \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| - \int_\Omega a(x)v_n^+ + \int_\Omega a(x)\ln(g(w)) & \text{if } \gamma = 1, \\ \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| + \int_\Omega \frac{a(x)}{1-\gamma}g^{1-\gamma}(w) & \text{if } \gamma > 1. \end{cases}$$

Thus, in any case, by the Sobolev embedding theorem we have that the sequence $\{v_n\}$ is bounded in $H_0^1(\Omega)$ and a standard argument shows that, up to a subsequence, there exists $v \in H_0^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$. Therefore, the Palais–Smale condition has been verified.

Finally, an application of the mountain pass theorem yields a nontrivial critical point v of $\mathcal{J}_\lambda$ (see [5, heorem 2.1]) and by elliptic regularity $v \in C_0^1(\overline{\Omega})$. Moreover, since $g_\lambda(x,t) = 0$ for $t \le 0$ one has $-\|v^-\|^2 = 0$, which implies that $v \gneqq 0$ and $z = w + v \in C_0^1(\overline{\Omega})$ is a second solution of (Q_λ). This finishes the proof of Theorem 1.2. □

We end this section with the following proposition.

Proposition 4.8. *Suppose that $(H)_\infty$ and $3 < p < 22^* - 1$ hold. If $0 < \gamma < 1$, then $\lambda_* \in \mathcal{L}$.*

Proof. In order to prove the proposition one uses the following properties:

- if v_λ is the solution obtained in Theorem 1.1, then $J_\lambda(v_\lambda) < c$ for some constant $c > 0$ independent of $\lambda \in (0, \lambda_*)$. Indeed, as we have seen in the proof of Lemma 4.2,

$$J_\lambda(v_\lambda) = I_\lambda(v_\lambda) + \lambda \int_\Omega g^p(\underline{v}(x))g'(\underline{v}(x))\underline{v}(x)dx - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(\underline{v}(x))dx,$$

which, jointly with Remark 2.9 b), gives

$$J_\lambda(v_\lambda) \le \lambda_* \int_\Omega g^p(\underline{v}(x))g'(\underline{v}(x))\underline{v}(x)dx =: c.$$

- $\frac{-g^{p+1}(t)}{p+1} + \frac{\theta}{p+1}g^p(t)g'(t)t \ge 0$ for all $t > 0$ and $4 < 2\theta < p+1$. Indeed, from Lemma 2.3 (4) we get

$$\frac{-g^{p+1}(t)}{p+1} + \frac{\theta}{p+1}g^p(t)g'(t)t \ge \frac{g^{p+1}(t)}{p+1}\left(-1+\frac{\theta}{2}\right) > 0, \tag{4.12}$$

for all $t > 0$.

Now, let $\lambda_n \in (0, \lambda_*)$ be an increasing sequence such that $\lambda_n \to \lambda_*$ as $n \to \infty$ and let $v_n := v_{\lambda_n}$ be a solution of (Q_λ) obtained in Theorem 1.1 for $\lambda = \lambda_n$. Then

$$J_{\lambda_n}(v_n) = \frac{1}{2}\|v_n\|^2 - \int_\Omega a(x)G(v_n) - \frac{\lambda_n}{p+1}\int_\Omega g^{p+1}(v_n) < c,$$

for some constant $c > 0$ independent of λ_n and

$$\|v_n\|^2 - \int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)v_n - \lambda_n \int_\Omega (g(v_n))^p g'(v_n)v_n = 0.$$

Thus, by using (4.12), one deduces

$$\left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 - \frac{1}{1-\gamma}\int_\Omega a(x)g^{1-\gamma}(v_n) + \frac{\theta}{p+1}\int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)v_n < c,$$

whence, by Lemma 2.3 (3),

$$\left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 < \frac{1}{1-\gamma}\int_\Omega a(x)g^{1-\gamma}(v_n) + c \le \frac{\|a\|_\infty}{1-\gamma}\int_\Omega v_n^{1-\gamma} + c.$$

From the previous relation it is easy to see that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exists $v^* \in H_0^1(\Omega)$ such that, up to a subsequence, we have as $n \to \infty$

$$v_n \rightharpoonup v^* \text{ in } H_0^1(\Omega),$$
$$v_n \to v^* \text{ a.e. in } \Omega.$$

Remember that $v_n \ge \underline{v} = v_0$ in Ω and thus, by Lemma 2.3 (9), (11),

$$|a(x)(g(v_n))^{-\gamma}g'(v_n)\psi| \le |a(x)(g(v_0))^{-\gamma}g'(v_0)\psi| \quad \text{in } \Omega.$$

Because v_n is a solution of (Q_{λ_n}), we have

$$\int_\Omega \nabla v_n \nabla \psi = \int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)\psi + \lambda_n \int_\Omega (g(v_n))^p g'(v_n)\psi,$$

for all $\psi \in H_0^1(\Omega)$. Passing to the limit in the previous equality and using Lebesgue's theorem, we deduce that v^* is a weak solution of (Q_{λ_*}). Finally, we can adapt the arguments in the proof of Theorem 1 c) in [1] to obtain $v^* \in C_0^1(\overline{\Omega})$. This ends the proof of the proposition. □

Proposition 4.8 suggests that $\lambda_* \in \mathcal{L}$ for arbitrary $\gamma > 0$. However, for $\gamma > 1$ and $\lambda \in (0, \lambda_*)$ one has $J_\lambda(v) > 0$ for any solution v of (Q_λ), and thus the proof of Proposition 4.8 cannot be applied to deduce that $\lambda_* \in \mathcal{L}$.

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