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On global attractivity of a higher order difference equation and its applications

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Abstract. Consider the following higher order difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots$$

where a, b and c are constants with $0 < a < 1, 0 \le b < 1, 0 \le c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$, and k is a positive integer. Our aim in this paper is to study the global attractivity of positive solutions of this equation and its applications to some population models.

Keywords: higher order difference equation, positive equilibrium, global attractivity, population models.

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1 Introduction

Consider the following higher order difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots, \tag{1.1}$$

where a, b and c are constants with $0 < a < 1, 0 \le b < 1, 0 \le c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. Our aim in this paper is to study the global attractivity of positive solutions of Eq. (1.1) and its applications to some population models.

In a recent paper [1], the authors study the dynamics of the following difference system

$$\begin{cases} x(n+1) = (1-\epsilon)f(x(n)) + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon f(x(n)), \\ x(0) \ge 0,\ y(0) \ge 0,\ x(0) + y(0) > 0, \end{cases} \qquad n = 0, 1, \dots, \tag{1.2}$$

where $0 < \epsilon < 1$ is a positive constant. Sys. (1.2) is a population model proposed by Newman et al. [14] which assumes symmetric dispersal between active population $x(n)$ and refuge

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population $y(n)$. The function f describes density-dependent reproduction of the active population. Newman et al. explore the effects of coupling an otherwise chaotic population to a refuge. Specifically, the logistic function $f(x) = \lambda x(1-x)$ and the exponential function $f(x) = \lambda x \exp(-x)$ are studied in [14]. Using numerical simulations, it is concluded that the presence of a passive refuge can greatly stabilize a population that would otherwise exhibit chaotic dynamics [14]. While in [1], Chow et al. assume that the growth function f is monotonically increasing to rule out the chaotic behavior explored in [14]. In particular, the authors study the following two cases: the growth rate is of Beverton–Holt type, that is, $f(x) = \frac{\lambda x}{1+kx}$, and the population is also subject to Allee effects, that is, $f(x) = \frac{\lambda x^2}{(1+kx)(m+x)}$, where λ is the density-independent growth rate, k relates to the population's carrying capacity, and m is the reciprocal of the searching efficiency of an individual when the population is subject to Allee effects. Various properties of solutions of Sys. (1.2) are studied in [1]. Some other results on Beverton–Holt and related equations can be found, e.g., in [15] and [16].

Motivated by these studies, in the present paper we are interested in obtaining an explicit sufficient condition to guarantee the global stability of positive solutions of Sys. (1.2) no matter if f is monotonic or not.

Noting that (1.2) can be converted into a second order scalar difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f(x(n)) + (2\epsilon - 1)f(x(n-1)), \tag{1.3}$$

we are led to the more general equation (1.1). When $b = 0$, Eq. (1.1) reduces to the form

$$x(n+1) = ax(n) + cf(x(n-k)), \tag{1.4}$$

which includes several discrete models derived from mathematical biology. The global attractivity and global stability of positive solutions of Eq. (1.4) and some related forms has been studied by many authors; see, for example, [2–13, 17] and the references cited therein.

In the present paper, we are interested in positive solutions of Eq. (1.1). Clearly, if we let

$$x(-k),\ x(-k+1), \dots, x(0) \tag{1.5}$$

be $k+1$ given nonnegative constants with $x(0) > 0$, then Eq.(1.1) has a unique positive solution $\{x(n)\}$ with the initial values (1.5). In the following, we assume that f has a unique positive fixed point $\bar{x}$. It is not difficult to see that $\bar{x}$ is the unique positive equilibrium of (1.1). In the next section, we establish a sufficient condition for $\bar{x}$ to be a global attractor of all positive solutions of Eq. (1.1). Then, in Section 3, we show that our result may be applied to Sys. (1.2) and some other difference equations derived from mathematical biology.

In the following discussion, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n} s(i) = 1$ and $\sum_{i=m}^{n} s(i) = 0$ whenever $\{s(n)\}$ is a real sequence and $m > n$.

2 Main result

In this section we establish a sufficient condition for the global attractivity of positive solutions of Eq. (1.1). The following lemma is needed.

Lemma 2.1. *Assume that f satisfies the negative feedback condition*

$$(x - \bar{x})(f(x) - x) < 0, \qquad x > 0,\ x \neq \bar{x}. \tag{2.1}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) is bounded and persistent.

Proof. First we show that $\{x(n)\}$ is bounded. Otherwise, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$x(n_i) = \max\{x(n) : -k \le n \le n_i\}, \qquad i = 1, 2, \ldots, \quad \text{and} \quad \lim_{i\to\infty} x(n_i) = \infty. \tag{2.2}$$

Then it follows from Eq. (1.1) that

$$b(f(x(n_i - 1)) - x(n_i)) + c(f(x(n_i - 1 - k)) - x(n_i)) = a(x(n_i) - x(n_i - 1)) \ge 0,$$

which, together with (2.2), implies there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) \ge x(n_{i_j}) \ge x(n_{i_j} - 1), \qquad j = 1, 2, \ldots \tag{2.3}$$

or

$$f(x(n_{i_j} - 1 - k)) \ge x(n_{i_j}) \ge x(n_{i_j} - 1 - k), \qquad j = 1, 2, \ldots \tag{2.4}$$

Assume that (2.3) holds. Then by noting (2.1) we see that $x(n_{i_j} - 1) \le \bar{x}$. Since $f(x)$ is bounded on the closed and finite interval $[0, \bar{x}]$, $\{f(x(n_{i_j} - 1))\}$ is bounded. Clearly, this contradicts (2.2) and (2.3). Hence, (2.3) can not hold. Similarly, we can show that (2.4) can not hold either. Hence, $\{x(n)\}$ must be bounded.

Next we show that $\{x(n)\}$ is persistent. Otherwise, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$x(n_i) = \min\{x(n) : -k \le n \le n_i\}, \qquad i = 1, 2, \ldots, \quad \text{and} \quad \lim_{i\to\infty} x(n_i) = 0. \tag{2.5}$$

Then it follows from Eq. (1.1) that

$$b(f(x(n_i - 1)) - x(n_i)) + c(f(x(n_i - 1 - k)) - x(n_i)) = a(x(n_i) - x(n_i - 1)) \le 0,$$

which, together with (2.5), implies there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) \le x(n_{i_j}) \le x(n_{i_j} - 1), \qquad j = 1, 2, \ldots \tag{2.6}$$

or

$$f(x(n_{i_j} - 1) - k)) \le x(n_{i_j}) \le x(n_{i_j} - 1 - k), \qquad j = 1, 2, \ldots \tag{2.7}$$

Assume that (2.6) holds. Then by noting (2.1) we see that $x(n_{i_j} - 1) \ge \bar{x}$. Since we have shown that $\{x(n)\}$ is bounded, $\{x(n_{i_j} - 1)\}$ is bounded. Hence there is a subsequence $\{x(n_{i_{jm}} - 1)\}$ of $\{x(n_{i_j} - 1)\}$ and a constant δ such that

$$\lim_{m\to\infty} x(n_{i_{jm}} - 1) = \delta,$$

where $\delta \ge \bar{x}$. However, from (2.6) we see that

$$f(x(n_{i_{jm}} - 1)) \le x(n_{i_{jm}}),$$

which implies that $f(\delta) \le 0$. This contradicts the assumption $f(x) > 0$ for $x > 0$. Hence, (2.6) can not hold. Similarly, we can show that (2.7) can not hold either. Hence, there is no such case (2.5) and so $\{x(n)\}$ is persistent. The proof is complete. □

The following theorem is our main result which establishes a sufficient condition for the global attractivity of positive solutions of Eq. (1.1).

Theorem 2.2. *Assume that $ax + bf(x)$ is increasing, $f(x)$ satisfies the negative feedback condition (2.1), and $f(x)$ is L-Lipschitz with*

$$\frac{c(1-a^{k+1})}{c+a^k b} L < 1. \tag{2.8}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) converges to $\bar{x}$ as $n \to \infty$.

Proof. First assume that $\{x(n)\}$ does not oscillate about $\bar{x}$. Then there are two cases : $x(n) - \bar{x}$ is eventually positive or $x(n) - \bar{x}$ is eventually negative. For the case that $x(n) - \bar{x}$ is eventually positive, let

$$\limsup_{n\to\infty} x(n) = l.$$

From Lemma 2.1, we know that $\{x(n)\}$ is bounded. Hence, $\bar{x} \leq l < \infty$. We now show that $l = \bar{x}$. First assume that $\{x(n)\}$ is eventually decreasing. Then $\lim_{n\to\infty} x(n) = l$. If $l > \bar{x}$, it follows from Eq. (1.1) that

$$\begin{aligned} a(x(n) - x(n-1)) &= (a-1)x(n) + bf(x(n-1)) + cf(x(n-1-k)) \\ &\to (a-1)l + bf(l) + cf(l) \quad \text{as } n \to \infty. \end{aligned} \tag{2.9}$$

Noting that $a + b + c = 1$, and $f(l) < l$ since $l > \bar{x}$, we see that

$$(a-1)l + bf(l) + cf(l) = (a-1)(l - f(l)) < 0.$$

Hence, there is a positive integer N such that

$$a(x(n) - x(n-1)) \leq \frac{1}{2}(a-1)(l - f(l)), \qquad n \geq N. \tag{2.10}$$

Summing (2.10) from $N+1$ to n, we find that

$$a(x(n) - x(N)) \leq \frac{1}{2}(a-1)(l-f(l))(n-N) \to -\infty \quad \text{as } n \to \infty,$$

which is a contradiction. Hence, we must have $l = \bar{x}$.

Next, consider the case that $\{x(n)\}$ is not eventually decreasing. Then, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$\lim_{i\to\infty} x(n_i) = l \quad \text{and} \quad x(n_i) > x(n_i - 1), \qquad i = 1, 2, \ldots \tag{2.11}$$

Hence it follows from Eq. (1.1) that

$$b(f(x(n_i-1)) - x(n_i)) + c(f(x(n_i-1-k)) - x(n_i)) = a(x(n_i) - x(n_i-1)) > 0, \quad i = 1, 2, \ldots$$

and so there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) > x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.12}$$

or

$$f(x(n_{i_j} - 1 - k)) > x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.13}$$

If (2.12) holds, then by noting $x(n_{i_j}-1) \geq \bar{x}$, we have $f(x(n_{i_j}-1)) \leq x(n_{i_j}-1)$ and so it follows that $x(n_{i_j}-1) > x(n_{i_j})$, which contradicts (2.11). Hence, (2.12) can not hold and we must have (2.13). Then, by noting $x(n_{i_j}-1-k) \geq \bar{x}$, we have $f(x(n_{i_j}-1-k)) \leq x(n_{i_j}-1-k)$ which yields $x(n_{i_j}-1-k) \geq x(n_{i_j})$. Hence

$$\lim_{j\to\infty} x(n_{i_j}-1-k) = l.$$

Then, by taking limit on both sides of (2.13), we see that $f(l) \geq l$ which yields $l \leq \bar{x}$. Hence, $l = \bar{x}$.

In the above, we have shown that $\{x(n)\}$ converges to $\bar{x}$ when $x(n) - \bar{x}$ is eventually positive. Next we show that $\{x(n)\}$ converges to $\bar{x}$ also when $x(n) - \bar{x}$ is eventually negative. To this end, let

$$\lim\inf_{n\to\infty} x(n) = r.$$

From Lemma 2.1, we know that $\{x(n)\}$ is persistent. Hence, $0 < r \leq \bar{x}$. We will show that $r = \bar{x}$. First assume that $\{x(n)\}$ is eventually increasing. Then $\lim_{n\to\infty} x(n) = r$. If $r < \bar{x}$, then it follows from Eq. (1.1) that

$$\begin{aligned} a(x(n) - x(n-1)) &= (a-1)x(n) + bf(x(n-1)) + cf(x(n-1-k)) \\ &\to (a-1)r + bf(r) + cf(r) \text{ as } n \to \infty. \end{aligned} \tag{2.14}$$

Noting that $a+b+c=1$, and $f(r) > r$ since $r < \bar{x}$, we see that

$$(a-1)r + bf(r) + cf(r) = (a-1)(r - f(r)) > 0.$$

Hence, there is a positive integer N such that

$$a(x(n) - x(n-1)) \geq \frac{1}{2}(a-1)(r-f(r)), \qquad n \geq N. \tag{2.15}$$

Summing (2.15) from $N+1$ to n, we find that

$$a(x(n) - x(N)) \geq \frac{1}{2}(a-1)(r-f(r))(n-N) \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction. Hence, we must have $r = \bar{x}$.

Next, consider the case that $\{x(n)\}$ is not eventually increasing. Then, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$\lim_{i\to\infty} x(n_i) = r \text{ and } x(n_i) < x(n_i - 1), \qquad i = 1, 2, \ldots \tag{2.16}$$

Hence, it follows from Eq. (1.1) that

$$b(f(x(n_i-1)) - x(n_i)) + c(f(x(n_i-k)) - x(n_i)) = a(x(n_i) - x(n_i-1)) < 0, \qquad i = 1, 2, \ldots$$

and so there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j}-1)) < x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.17}$$

or

$$f(x(n_{i_j}-1-k)) < x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.18}$$

If (2.17) holds, then by noting $x(n_{i_j}-1)<\bar{x}$, we have $f(x(n_{i_j}-1))\geq x(n_{i_j}-1)$ and so it follows that $x(n_{i_j}-1)<x(n_{i_j})$, which contradicts (2.16). Hence, (2.17) can not hold and we must have (2.18). Then, noting that $x(n_{i_j}-1-k)<\bar{x}$, we have $f(x(n_{i_j}-1-k))\geq x(n_{i_j}-1-k)$ which yields $x(n_{i_j}-1-k)\leq x(n_{i_j})$. Hence

$$\lim_{j\to\infty} x(n_{i_j}-1-k)=r.$$

Then by taking limit on both sides of (2.18), we see that $f(r)\leq r$ which yields $r\geq\bar{x}$. Hence, $r=\bar{x}$.

In the above, we have shown that every nonoscillatory (about $\bar{x}$) positive solution of Eq. (1.1) converges to $\bar{x}$ as $n\to\infty$. Next, consider the case that $\{x(n)\}$ is a positive solution of Eq. (1.1) and oscillates about $\bar{x}$, that is, $x(n)-\bar{x}$ is not of eventually constant sign. We show that $x(n)$ converges to $\bar{x}$ also as $x\to\infty$. To this end, let $y(n)=x(n)-\bar{x}$. Then $\{y(n)\}$ satisfies the equation

$$y(n+1)=ay(n)+b(f(y(n)+\bar{x})-\bar{x})+c(f(y(n-k)+\bar{x})-\bar{x}) \tag{2.19}$$

and $\{y(n)\}$ oscillates about zero. Let $y(i)<y(j)$ be two consecutive members of the solution $\{y(n)\}$ such that

$$y(i)\leq 0,\quad y(j+1)\leq 0\quad \text{and}\quad y(n)>0\quad \text{for } i+1\leq n\leq j. \tag{2.20}$$

Let

$$y(t)=\max\{y(i+1),y(i+2),\dots,y(j)\}. \tag{2.21}$$

We claim that

$$t-(i+1)\leq k. \tag{2.22}$$

Otherwise $t-(i+1)>k$. Then

$$y(t)\geq y(t-1)>0\quad \text{and}\quad y(t)\geq y(t-1-k)>0.$$

By noting $y(t-1)+\bar{x}>\bar{x}$, $y(t-1-k)+\bar{x}>\bar{x}$ and $f(x)<x$ for $x>\bar{x}$, we see that

$$\begin{aligned} &b(f(y(t-1)+\bar{x})-y(t)-\bar{x})+c(f(y(t-1-k)+\bar{x})-y(t)-\bar{x})\\ &\quad< b((y(t-1)+\bar{x})-y(t)-\bar{x})+c((y(t-1-k)+\bar{x})-y(t)-\bar{x})\\ &\quad= b(y(t-1)-y(t))+c(y(t-1-k)-y(t))\\ &\quad\leq 0. \end{aligned} \tag{2.23}$$

However, on the other hand, (2.19) yields

$$b(f(y(t-1)+\bar{x})-y(t)-\bar{x})+c(f(y(t-1-k)+\bar{x})-y(t)-\bar{x})=a(y(t)-y(t-1))>0,$$

which contradicts (2.23). Hence, (2.22) must hold.

Now, observe that (2.19) yields

$$\frac{y(n+1)}{a^{n+1}}-\frac{y(n)}{a^n}=\frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}]+\frac{c}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}] \tag{2.24}$$

Summing (2.24) from i to $t-1$, we see that

$$\frac{y(t)}{a^t} - \frac{y(i)}{a^i} = \sum_{n=i}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}]$$

and so it follows that

$$\begin{aligned} y(t) &= a^t \left(\frac{y(i)}{a^i} + \sum_{n=i}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x})] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x})] \right) \\ &= a^t \left(\frac{1}{a^{i+1}}(ay(i)+bf(y(i)+\bar{x})-b\bar{x}) + \sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}] \right. \\ &\quad \left. + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}] \right). \end{aligned} \tag{2.25}$$

Noting that $ax + bf(x)$ is increasing, $y(i) \le 0$, and $f(\bar{x}) = \bar{x}$, we see that

$$a(y(i)+\bar{x}) + bf(y(i)+\bar{x}) \le a\bar{x} + bf(\bar{x}),$$

which yields

$$ay(i) + bf(y(i)+\bar{x}) - b\bar{x} \le 0.$$

In addition, by noting (2.21) and the fact that $0 < y(n) \le y(t)$ for $n = i+1, i+2, \dots, t-1$, we see that

$$f(y(n)+\bar{x}) < y(n)+\bar{x} \le y(t)+\bar{x}, \qquad n = i+1, i+2, \dots, t-1.$$

Hence, it follows from (2.25) that

$$\begin{aligned} y(t) &\le a^t \left(\sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}}[y(t)+\bar{x}-\bar{x}] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}] \right) \\ &= a^t \left(y(t) \sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}} + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})] \right) \\ &= \frac{(1-a^{t-i-1})b}{1-a} y(t) + a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})] \\ &\le \frac{(1-a^k)b}{1-a} y(t) + a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})], \end{aligned}$$

which yields

$$\left(1 - \frac{(1-a^k)b}{1-a}\right) y(t) \le a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})]. \tag{2.26}$$

Then, by noting

$$1 - \frac{(1-a^k)b}{1-a} = \frac{c+a^k b}{1-a},$$

it follows from (2.26) that

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})]. \tag{2.27}$$

Since $\{x(n)\}$ is bounded, there is a positive constant M such that $|y(n)| = |x(n) - \bar{x}| \leq M, n = 0, 1, \ldots$ Then by noting the Lipschitz property of f, we see that

$$|f(y(n-k) + \bar{x}) - f(\bar{x})| \leq L|y(n-k)| \leq LM, \qquad n \geq k.$$

Hence, (2.27) yields

$$y(t) \leq \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} LM = \frac{1-a}{c+a^k b} \left(\frac{1-a^{t-i}}{1-a} \right) cLM \leq \frac{1-a}{c+a^k b} \left(\frac{1-a^{k+1}}{1-a} \right) cLM,$$

that is,

$$y(t) \leq \frac{c(1-a^{k+1})}{c+a^k b} LM.$$

It follows that

$$y(n) \leq \frac{c(1-a^{k+1})}{c+a^k b} LM \quad \text{for } i \leq n \leq j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.21), we see that there is a positive integer N_1' such that

$$y(n) \leq \frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \geq N_1'.$$

Then, by a similar argument, it can be shown that there is a positive integer N_1'' such that

$$y(n) \geq -\frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \geq N_1''.$$

Hence, there is a positive integer N_1 such that

$$|y(n)| \leq \frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \geq N_1. \tag{2.28}$$

Now, by noting (2.28) and the Lipschitz property of $f(x)$ again, we see that

$$|f(y(n-k) + \bar{x}) - f(\bar{x})| \leq L|y(n-k)| \leq \frac{c(1-a^{k+1})}{c+a^k b} L^2 M, \qquad n \geq N_1 + k.$$

Let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ with $N_1 + k \leq i < j$ such that (2.20) holds. Let $y(t)$ be defined by (2.21). By a similar argument, we may show that (2.22) holds and

$$y(t) \leq \left(\frac{c(1-a^{k+1})}{c+a^k b} L \right)^2 M.$$

Then it follows that

$$y(t) \leq \left(\frac{c(1-a^{k+1})}{c+a^k b} L \right)^2 M, \qquad i \leq n \leq j$$

and so again by noting $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.20), there is a positive integer $N_2' \geq N_1 + k$ such tat

$$y(n) \leq \left(\frac{c(1-a^{k+1})}{c+a^k b} L \right)^2 M, \qquad n \geq N_2'.$$

Similarly, it can be shown that there is a positive integer $N_2'' \geq N_1 + K$ such that

$$y(n) \geq -\left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^2 M, \qquad n \geq N_2''.$$

Hence, there is a positive integer $N_2 \geq N_1 + k$ such that

$$|y(n)| \leq \left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^2 M, \qquad n \geq N_2.$$

Finally, by induction, we find that for any positive integer m, there is a positive integer N_m with $N_m \to \infty$ as $m \to \infty$ such that

$$|y_n| \leq \left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^m M, \qquad n \geq N_m.$$

Then, by noting the hypotheses $\frac{c(1-a^{k+1})}{c+a^k b}L < 1$, we see that $y(n) \to 0$ as $n \to \infty$, and so it follows that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. $\square$

3 Applications

In this section, we apply our result obtained in the last section to some difference equations derived from mathematical biology.

Consider the system (1.2) which has been mentioned in Section 1. By a simple calculation, Sys. (1.2) can be converted into the second order difference equation (1.3). Let us assume that $f \in C[[0,\infty),[0,\infty)]$ with $f(x) > 0$ for $x > 0$, and f has a unique positive fixed point $\bar{x}$. It is not difficult to see that $(\bar{x},\bar{x})$ is the unique positive equilibrium of Sys. (1.2). By Theorem 2.2, we may have the following result.

Theorem 3.1. *Assume that $1/2 \leq \epsilon < 1$. Suppose also that $x + f(x)$ is increasing, f satisfies the negative feedback condition*

$$(x-\bar{x})(f(x)-x) < 0, \qquad x > 0,\ x \neq \bar{x} \tag{3.1}$$

and f is L-Lipschitz with

$$(2-\epsilon)(2-1/\epsilon)L < 1. \tag{3.2}$$

Then every positive solution $(x(n), y(n))$ of Sys. (1.2) tends to its positive equilibrium $(\bar{x},\bar{x})$ as $n \to \infty$. Furthermore, if f is differentiable and

$$(1-\epsilon)|1+f'(\bar{x})| < 1 + (1-2\epsilon)f'(\bar{x}) < 2, \tag{3.3}$$

then $(\bar{x},\bar{x})$ is globally asymptotically stable.

Proof. Eq. (1.3) is in the form of (1.1) with $a = b = 1-\epsilon, c = 2\epsilon - 1$ and $k = 1$. By the hypotheses, $ax + bf(x) = (1-\epsilon)(x+f(x))$ is increasing. In addition,

$$\frac{c(1-a^{k+1})}{c+a^k b}L = (2-\epsilon)(2-1/\epsilon)L < 1.$$

Hence, by Theorem 2.2 every positive solution $\{x(n)\}$ of Eq. (1.3) converges to $\bar{x}$ as $n \to \infty$. Then from (1.2) we see that

$$\epsilon y(n) = x(n+1) - (1-\epsilon)f(x(n)) \to \bar{x} - (1-\epsilon)f(\bar{x}) \quad \text{as } n \to \infty,$$

which yields

$$y(n) \to \bar{x} \quad \text{as } n \to \infty.$$

Hence it follows that every positive solution $(x(n), y(n))$ of Sys. (1.2) converges to $(\bar{x}, \bar{x})$ as $n \to \infty$.

Clearly, to show that $(\bar{x}, \bar{x})$ is globally asymptotically stable when (3.3) holds, it suffices to show that $(\bar{x}, \bar{x})$ is stable. Note that Sys. (1.2) can be converted into the scalar equation (1.3) and the linearized equation of Eq. (1.3) about $\bar{x}$ is

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f'(\bar{x})x(n) + (2\epsilon - 1)f'(\bar{x})x(n-1),$$

that is,

$$x(n+1) + (\epsilon - 1)(1 + f'(\bar{x}))x(n) + (1 - 2\epsilon)f'(\bar{x})x(n-1) = 0. \tag{3.4}$$

It is well-known (see, for example [9]) that for the linear equation

$$z(n+1) + \alpha z(n) + \beta z(n-1) = 0,$$

where α and β are constants, a necessary and sufficient condition for the asymptotic stability is

$$|\alpha| < 1 + \beta < 2. \tag{3.5}$$

Hence, when (3.3) holds, the zero solution of Eq. (3.4) is asymptotically stable. Then by linearized stability theory, the positive equilibrium $\bar{x}$ of Eq. (1.3) is asymptotically stable and so it follows that the positive equilibrium $(\bar{x}, \bar{x})$ of Sys. (1.2) is asymptotically stable. This, together with the global attractivity, we have shown above implies that $(\bar{x}, \bar{x})$ is globally asymptotically stable. The proof is complete. □

Sys. (1.2) is a population model proposed by Newman et al. [14] which assumes symmetrical dispersal between an active population $x(n)$ and a refuge population $y(n)$. The exponential function $f(x) = \lambda x e^{-x}$ is a function studied in [14]. Using numerical simulations, it is concluded that the presence of a passive refuge can greatly stabilize a population that would otherwise exhibit chaotic dynamics. Now, by applying Theorem 3.1, we may get an explicit sufficient condition for the global asymptotic stability of Sys. (1.2) when $f(x) = \lambda x e^{-x}$.

When $\lambda > 1$, $\bar{x} = \ln \lambda$ is the unique positive fixed point of f. Clearly, f satisfies the negative feedback condition (3.1). Noting that

$$(x + f(x))' = 1 + (1-x)\lambda e^{-x} \quad \text{and} \quad (x + f(x))'' = \lambda(x-2)e^{-x},$$

we see that when $\lambda \le e^2$,

$$(x + f(x))' \ge (x + f(x))'|_{x=2} = 1 - \lambda e^{-2} \ge 0,$$

and so $x + f(x)$ is increasing. In addition, noting that

$$f'(x) = \lambda(1-x)e^{-x} \quad \text{and} \quad f''(x) = \lambda(x-2)e^{-x},$$

we see that

$$f'(0) = \lambda \quad \text{and} \quad f'(2) = -\lambda e^{-2},$$

and it follows that

$$|f'(x)| \le \lambda, \qquad x \ge 0.$$

Hence, f is L-Lipschitz with $L = \lambda$. Now observe that

$$f'(\bar{x}) = \lambda(1 - \ln\lambda)e^{-\ln\lambda} = 1 - \ln\lambda,$$

$$1 + (1-2\epsilon)f'(\bar{x}) = 1 + (1-2\epsilon)(1 - \ln\lambda) < 1 + (2\epsilon - 1)(\ln e^2 - 1) = 2\epsilon < 2, \qquad \lambda \le e^2$$

and

$$\begin{aligned} 1 + (1-2\epsilon)f'(\bar{x}) &= 1 + (1-2\epsilon)(1 - \ln\lambda) > 1 + (1-2\epsilon) \\ &= 2(1-\epsilon) > (1-\epsilon)(2 - \ln\lambda) = (1-\lambda)|1 + f'(\bar{x})|. \end{aligned}$$

Hence, by Theorem 3.1, we have the following conclusion: if $1 < \lambda \le e^2$ and

$$(2-\epsilon)(2 - 1/\epsilon)\lambda < 1,$$

then Sys. (1.2) has a unique positive equilibrium $(\ln\lambda, \ln\lambda)$ that is globally asymptotically stable.

In the above, we showed that our result can be applied to the case that f is not monotonic. Clearly, our result can be applied to the case that f is monotonic also. The following is an example in which the function f is decreasing. Consider the equation

$$x(n+1) = ax(n) + b\frac{q}{1+x^p} + c\frac{q}{1+x^p(n-k)}, \tag{3.6}$$

where a, b and c are the same as those assumed in Eq. (1.1), and $p \ge 1$, $q \ge 0$ are constants. If $b = 0$, (3.6) reduces to the equation of the form

$$x(n+1) = ax(n) + \frac{B}{1+x^p(n-k)}, \tag{3.7}$$

where $B = cq$. Eq. (3.7) is a discrete analogue of a model that has been used to study blood cells production [10]. The global attractivity of positive solutions of this equation has been studied by numerous authors, see for example, [2–4,7–9] and the references cited therein. Let

$$f(x) = \frac{q}{1+x^p}.$$

Then f is decreasing and has a unique positive fixed point $\bar{x}$. Clearly, f satisfies the feedback condition (3.1). Now observe that

$$f'(x) = \frac{-qpx^{p-1}}{(1+x^p)^2}$$

and

$$f''(x) = \frac{-qpx^{p-2}((p-1) - (p+1)x^p)}{(1+x^p)^3}.$$

We find that $f'(x)$ has a minimum at $x^* = \left(\frac{p-1}{p+1}\right)^{1/p}$ and

$$f'(x^*) = -\frac{q}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p}.$$

Hence, f is L-Lipschitz with

$$L = \frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p}.$$

In addition, if

$$\frac{bq}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < a, \tag{3.8}$$

then

$$(ax + bf(x))' = a + bf'(x) \geq a + bf'(x^*) = a - \frac{bq}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} > 0$$

and so the function $ax + bf(x)$ is increasing, if

$$\frac{c(1-a^{k+1})}{c+a^k b}\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.9}$$

then (2.8) is satisfied. Therefore, by Theorem 2.2, if (3.8) and (3.9) hold, then every positive solution of Eq. (3.6) tends to its positive equilibrium $\bar{x}$.

When $f(x) = \frac{q}{1+x^p}$, Sys. (1.2) can be converted into the second order difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)\frac{q}{1+x^p(n)} + (2\epsilon-1)\frac{q}{1+x^p(n-1)}, \tag{3.10}$$

which is in the form of Eq. (1.1) with $a = b = 1-\epsilon,\ c = 2\epsilon - 1$ and $k = 1$. In this case, since

$$\frac{c(1-a^{k+1})}{c+a^k b} = (2-\epsilon)\left(2 - \frac{1}{\epsilon}\right),$$

we see that (3.8) and (3.9) reduce to

$$\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.11}$$

and

$$(2-\epsilon)(2-\frac{1}{\epsilon})\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.12}$$

respectively. Then by noting $(2-\epsilon)(2-\frac{1}{\epsilon}) < 1$, we see that (3.11) yields (3.12).

From the above discussion and (3.12), we know that

$$-f'(\bar{x}) \leq -f'(x^*) = \frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1.$$

Then it follows that

$$1 + (1-2\epsilon)f'(\bar{x}) = 1 + (2\epsilon-1)(-f'(\bar{x})) \leq 1 + (2\epsilon - 1) = 2\epsilon < 2$$

and

$$(1-\epsilon)|1+f'(\bar{x})| \leq (1-\epsilon)(1-f'(\bar{x})) \leq 2(1-\epsilon) \leq 1 \leq 1 + (2\epsilon-1)(-f'(\bar{x})).$$

Hence, by Theorem 3.1, we find that if (3.11) holds, then Sys. (1.2) with $f(x) = \frac{q}{1+x^p}$ has a unique positive equilibrium $(\bar{x}, \bar{x})$ which is globally asymptotically stable.

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