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A double phase equation with convection

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Abstract. We consider a double phase problem with a gradient dependent reaction (convection). Using the theory of nonlinear operators of monotone type, we show the existence of a nontrivial, positive, bounded solution.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following double phase Dirichlet problem with gradient dependent reaction (convection)

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) + E(z)|Du(z)|^{q-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, 1 < q < p. \end{cases} \tag{1.1}$$

Here Δ_p^a denotes the weighted p-Laplace differential operator defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

Problem (1.1) has two interesting features. The first is that in the weighted operator, the weight $a \in L^\infty(\Omega)$ is not bounded away from zero. This means that the integrand

$$\theta(z, x) = a(z)x^p + x^q \quad \forall z \in \Omega,\ \forall x \geq 0,$$

which is associated with the energy functional of the differential operator exhibits unbalanced growth, that is,

$$x^q \leq \theta(z, x) \leq c_1[x^p + x^q] \quad \text{for all } z \in \Omega, \text{ all } x \geq 0, \text{ some } c_1 > 0.$$

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Such functionals were first examined by Marcellini [11] and Zhikov [19] in the context of problems of the calculus of variations and of nonlinear elasticity theory. More recently Marcellini and co-workers and Mingione and co-workers, produced important local regularity results for such problems. We refer to the papers of Marcellini [12] and Baroni–Colombo–Mingione [1] and the references therein. We also mention the recent informative survey paper of Mingione–Rădulescu [13]. A global regularity theory (that is, regularity up to the boundary), remains so far elusive and this makes double phase problems more difficult to deal with. The second distinguishing feature of problem (1.1), is that the reaction (right hand side) of (1.1) is gradient dependent. This makes the problem nonvariational and this eliminates the use of minimax theorems from the critical point theory. For this reason, our approach is based on the theory of nonlinear operators of monotone type. Variational double phase problems have been studied recently using a variety of methods. We mention the works of Colasuonno–Squassina [2], Gasiński–Winkert [5], Ge–Lv–Lu [7], Liu–Dai [9], Liu–Papageorgiou [10], Papageorgiou–Rădulescu-Repovš [15], Papageorgiou–Vetro–Vetro [16]. On the other hand the study of double phase problems with convection, is lagging behind. There are only the works of Gasiński–Winkert [6] and Zeng-Bai-Gasiński–Winkert [18].

Finally we should mention the very recent work of Repovš–Vetro [17], who studied parametric, variational (that is, no convection term is presented) Dirichlet problems, driven by a weighted (p, q)-Laplacian. However the weights in [17] are bounded away from zero and so the differential operator in [17] exhibits balanced growth. This facilitates the analysis since for such problems there is a global regularity theory available.

2 Mathematical background-hypotheses

The unbalanced growth of the integrand corresponding to the differential operator, leads to a functional framework based on Musielak–Orlicz spaces. We introduce the following conditions on the weight $a(\cdot)$, the coefficient $E(\cdot)$ and the exponents p, q, r. In what follows by $C^{0,1}(\overline{\Omega})$ we denote the space of locally Lipschitz functions from $\overline{\Omega}$ into $\mathbb{R}$.

H_0: $a \in C^{0,1}(\overline{\Omega}), a \neq 0, a(z) \geq 0$ for all $z \in \overline{\Omega}$, $E \in L^\infty(\Omega)$, $E(z) \neq 0, E(z) \geq 0$ for a.a. $z \in \Omega$, $1 < q < p < N, \frac{p}{q} < 1 + \frac{1}{N}$.

Remark 2.1. The relation $\frac{p}{q} < 1 + \frac{1}{N}$ is standard in Dirichlet double phase problems and implies $p < q* = \frac{Nq}{N-q}$. So the relation $p \leq r < q*$ makes sense and we have useful embeddings of the relevant Musielak–Orlicz–Sobolev spaces.

Let $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ $(\mathbb{R}_+ = [0, +\infty))$ be the integrand $\theta(z, x) = a(z)x^p + x^q$. Evidently $\theta(\cdot, \cdot)$ is continuous and uniformly convex in $x \in \mathbb{R}_+$. Let $M(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable function$\}$. As usual we identify two such functions which differ only on a Lebesgue-null set. The Musielak–Orlicz space $L^\theta(\Omega)$ is defined by

$$L^\theta(\Omega) = \{u \in M(\Omega) : \rho_\theta(u) < \infty\},$$

with $\rho_\theta(\cdot)$ being the modular function defined by

$$\rho_\theta(u) = \int_\Omega [a(z)|u|^p + |u|^q] dz.$$

We equip $L^\theta(\Omega)$ with the so called "Luxemburg norm" defined by

$$\|u\|_\theta = \inf\left[\lambda > 0 : \rho_\theta\left(\frac{u}{\lambda}\right) \le 1\right].$$

Then $L^\theta(\Omega)$ becomes a Banach space which is also separable and reflexive (in fact uniformly convex). Using $L^\theta(\Omega)$ we can define the corresponding Musielak–Orlicz–Sobolev space $W^{1,\theta}(\Omega)$ by

$$W^{1,\theta}(\Omega) = \{u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega)\}.$$

Here Du denotes the weak gradient of $u(\cdot)$. We equip $W^{1,\theta}(\Omega)$ with the following norm

$$\|u\|_{1,\theta} = \|u\|_\theta + \|Du\|_\theta \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Here $\|Du\|_\theta = \||Du|\|_\theta$. Also, we define

$$W_0^{1,\theta}(\overline{\Omega}) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\theta}}.$$

For this space the Poincaré inequality holds and so on $W_0^{1,\theta}(\Omega)$ we consider the equivalent norm

$$\|u\| = \|Du\|_\theta \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Both spaces are separable and reflexive (in fact uniformly convex).

Given $u \in W_0^{1,\theta}(\Omega)$, we define

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}.$$

We know that $u^+ \in W_0^{1,\theta}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

We have the following useful embeddings.

Proposition 2.2. *If hypotheses H_0 hold, then*

(a) $L^\theta(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *continuously and densely for all* $1 \le r \le q$;

(b) $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ *continuously (resp. compactly) and densely for all* $1 \le r \le q^*$ *(resp.* $1 \le r < q^*$*);*

(c) $L^p(\Omega) \hookrightarrow L^\theta(\Omega)$ *continuously and densely.*

Also there is a close relation between the norm $\|\cdot\|_\theta$ and the modular function $\rho_\theta(\cdot)$.

Proposition 2.3. *If hypotheses H_0 hold, then*

(a) $\|u\|_\theta = \lambda \Leftrightarrow \rho_\theta(\frac{u}{\lambda}) = 1$;

(b) $\|u\|_\theta < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_\theta(u) < 1$ *(resp.* $= 1, > 1$*);*

(c) $\|u\|_\theta < 1 \Rightarrow \|u\|_\theta^p \le \rho_\theta(u) \le \|u\|_\theta^q$;

(d) $\|u\|_\theta > 1 \Rightarrow \|u\|_\theta^q \le \rho_\theta(u) \le \|u\|_\theta^p$;

(e) $\|u\|_\theta \to 0$ *(resp.* $\to +\infty$*)* $\Leftrightarrow \rho_\theta(u) \to 0$ *(resp.* $\to +\infty$*).*

We consider the nonlinear operators A_p^a, $A_q : W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)^*$ defined by

$$\langle A_p^a(u), h\rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz,$$
$$\langle A_q(u), h\rangle = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,\theta}(\Omega).$$

We set $V = A_p^a + A_q : W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)^*$. This operator has the following properties.

Proposition 2.4. *If hypotheses H_0 hold, then $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ that is, "$u_n \xrightarrow{w} u$ in $W_0^{1,\theta}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$ imply that $u_n \to u$ in $W_0^{1,\theta}(\Omega)$."*

For details on Musielak–Orlicz spaces, we refer to the book of Harjulehto–Hästo [8].

By $\widehat{\lambda}_1(q)$ we denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\widehat{\lambda}_1(q) > 0$, is simple, isolated and has the following variational characterization

$$\widehat{\lambda}_1(q) = \inf\left[\frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega), u \neq 0\right]. \tag{2.1}$$

The infimum in (2.1) is realized on the corresponding one-dimensional eigenspace. So, we see that the elements of this eigenspace have fixed sign. By $\widehat{u}_1(q)$ we denote the L^q-normalized (that is, $\|\widehat{u}(q)\|_q = 1$), positive eigenfunction corresponding to $\widehat{\lambda}_1(q)$. We know that $\widehat{u}_1(q) \in C^1(\Omega)$ and $\widehat{u}_1(q)(z) > 0$ for all $z \in \Omega$. For details we refer to Gasiński–Papageorgiou [4].

The hypotheses on the perturbation $f(z, x)$ are the following:

(H_1): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for *a.a.* $z \in \Omega$, $x \to f(z, x)$ is continuous) such that $f(z, 0) = 0$ for *a.a.* $z \in \Omega$ and

(i) $|f(z, x)| \leq \widehat{a}(z)[1 + x^{r-1}]$ for *a.a.* $z \in \Omega$, all $x \geq 0$, with $\widehat{a} \in L^\infty(\Omega)$, $p \leq r < q^*$;

(ii) there exists $M > 1$ such that $f(z, x) \leq 0$ for *a.a.* $z \in \Omega$, all $x \geq M$;

(iii) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1(q) \leq \eta(z) \quad \text{for a.a. } z \in \Omega, \quad \eta \not\equiv \widehat{\lambda}_1(q),$$
$$\eta(z) \leq \liminf_{x\to 0^+} \frac{f(z, x)}{x^{q-1}} \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 2.5. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z, x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

On account of hypotheses H_1(i), (iii), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$f(z, x) \geq [\eta(z) - \varepsilon]x^{q-1} - c_\varepsilon x^{r-1} \quad \text{for a.a. } z \in \Omega \text{ all } x \geq 0. \tag{2.2}$$

3 An auxiliary problem

The unilateral growth restriction (2.2) on $f(z,\cdot)$, leads to the following auxiliary double phase problem:

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = [\eta(z) - \varepsilon] u(z)^{q-1} - c_\varepsilon u(z)^{r-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0. \end{cases} \tag{3.1}$$

Proposition 3.1. *If hypotheses H_0 hold, then for all $\varepsilon > 0$ small problem (2.2) has a unique solution $\underline{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$, $\underline{u}(z) > 0$ for a.a. $z \in \Omega$.*

Proof. Consider the C^1-functional $\varphi_\varepsilon : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\varepsilon(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{c_\varepsilon}{r}\|u^+\|_r^r - \frac{1}{q}\int_\Omega [\eta(z) - \varepsilon](u^+)^q dz.$$

Here $\rho_a(Du) = \int_\Omega a(z)|Du|^p dz$. Since $q < p \leq r$, we see that $\varphi_\varepsilon(\cdot)$ is coercive. Also since $\rho_a(\cdot)$ is continuous, convex, exploiting the compact embedding of $W_0^{1,\theta}(\Omega)$ into $L^r(\Omega)$ (see Proposition 2.2), we infer that $\varphi_\varepsilon(\cdot)$ is sequentially weakly lower semi-continuous. So, by the Weierstrass–Tonelli theorem, we can find $\underline{u} \in W_0^{1,\theta}(\Omega)$ such that

$$\varphi_\varepsilon(\underline{u}) = \inf[\varphi_\varepsilon(u) : u \in W_0^{1,\theta}(\Omega)]. \tag{3.2}$$

Let $\widehat{\lambda}_1 = \widehat{\lambda}_1(q)$, $\widehat{u}_1 = \widehat{u}_1(q)$, and $t \in (0,1)$. We have

$$\varphi_\varepsilon(t\widehat{u}_1) = \frac{t^p}{p}\rho_a(D\widehat{u}_1) + \frac{t^q}{q}\left[\int_\Omega (\widehat{\lambda}_1 - \eta(z))\widehat{u}_1^q dz + \varepsilon\right] + \frac{t^r c_\varepsilon}{r}\|\widehat{u}_1\|_r^r.$$

Since $\widehat{u}_1(z) > 0$ for all $z \in \Omega$, hypotheses H_1(iii) implies that

$$\mu_0 = \int_\Omega [\eta(z) - \widehat{\lambda}_1]\widehat{u}_1^q dz > 0.$$

So, choosing $\varepsilon \in (0, \mu_0)$ and since $p \leq r$ and $t \in (0,1)$, we have

$$\varphi_\varepsilon(t\widehat{u}_1) \leq c_1 t^p - c_2 t^q \text{ for some } c_1, c_2 > 0.$$

Recall that $q < p$. So, choosing $t \in (0,1)$ even smaller if necessary, we see that

$$\begin{aligned} &\varphi_\varepsilon(t\widehat{u}_1) < 0, \\ \Rightarrow\ &\varphi_\varepsilon(\underline{u}) < 0 = \varphi_\varepsilon(0) \quad (\text{see}(3.2)), \\ \Rightarrow\ &\underline{u} \neq 0. \end{aligned}$$

From (3.2) we have $\varphi'_\varepsilon(\underline{u}) = 0$,

$$\Rightarrow \langle V(u), h\rangle = \int_\Omega [(\eta(z) - \varepsilon)\underline{u}^{q-1} - c_\varepsilon \underline{u}^{r-1}] h dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega). \tag{3.3}$$

Choosing $h = -\underline{u}^- \in W_0^{1,\theta}(\Omega)$ in (3.3), we obtain

$$\rho_\theta(D\underline{u}^-) = 0 \quad \Rightarrow \quad \underline{u} \geq 0,\ \underline{u} \neq 0.$$

Therefore $\underline{u}$ is a weak solution of (3.1). From Theorem 3.1 of Gasiński–Winkert [5], we have that

$$\underline{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

Moreover, Proposition 2.4 of Papageorgiou–Vetro–Vetro [16] implies that

$$\underline{u}(z) > 0 \quad \text{for a.a. } z \in \Omega.$$

Next we show that this positive solution of (3.1) is unique. So, suppose that $\underline{v} \in W_0^{1,\theta}(\Omega)$ is another positive solution of (3.1). Again we show that

$$\underline{v} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \quad \underline{v}(z) > 0 \quad \text{for a.a. } z \in \Omega.$$

Let $\underline{u}_\delta = \underline{u} + \delta$, $\underline{v}_\delta = \underline{v} + \delta$, $\delta > 0$. If $L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : u(z) \geq 0 \quad \text{for a.a. } z \in \Omega\}$ (the positive (order) cone of the ordered Banach space $L^\infty(\Omega)$), then $\underline{u}_\delta, \underline{v}_\delta \in \operatorname{int} L^\infty(\Omega)_+$. Hence using Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [14], we have

$$\frac{\underline{u}_\delta}{\underline{v}_\delta} \in L^\infty(\Omega), \qquad \frac{\underline{v}_\delta}{\underline{u}_\delta} \in L^\infty(\Omega). \tag{3.4}$$

We consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_a(Du^{\frac{1}{q}}) + \frac{1}{q}\|Du^{\frac{1}{q}}\|_q^q & \text{if } u \geq 0,\ u^{\frac{1}{q}} \in W^{1,\theta}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

The convexity of $\rho_a(\cdot)$ implies that $j(\cdot)$ is convex (see Diaz–Saá [3]). On account of (3.4), if $h = \underline{u}_\delta^q - \underline{v}_\delta^q \in W^{1,\theta}(\Omega)$ and $|t| < 1$ is small, we have

$$\underline{u}_\delta^q + th \in \operatorname{dom} j, \quad \underline{v}_\delta^q + th \in \operatorname{dom} j,$$

where $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). Then using the convexity of $j(\cdot)$, we see that $j(\cdot)$ is Gateaux differentiable at $\underline{u}_\delta^q$ and at $\underline{v}_\delta^q$ in the direction h. Moreover, using the chain rule and the nonlinear Green's identity (see [14, p. 34]), we have

$$\begin{aligned} j'(\underline{u}_\delta^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \underline{u}_\delta - \Delta_q \underline{u}_\delta}{\underline{u}_\delta^{q-1}} h dz \\ &= \frac{1}{q}\int_\Omega \frac{[\eta(z) - \varepsilon]\underline{u}_\delta^{q-1} - c_\varepsilon \underline{u}_\delta^{r-1}}{\underline{u}_\delta^{q-1}} h dz, \quad \text{(see (3.1))}. \end{aligned}$$

and

$$\begin{aligned} j'(\underline{v}_\delta^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \underline{v}_\delta - \Delta_q \underline{v}_\delta}{\underline{v}_\delta^{q-1}} h dz \\ &= \frac{1}{q}\int_\Omega \frac{[\eta(z) - \varepsilon]\underline{v}_\delta^{q-1} - c_\varepsilon \underline{v}_\delta^{r-1}}{\underline{v}_\delta^{q-1}} h dz, \quad \text{(see (3.1))}. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$, Hence

$$0 \leq \int_\Omega [\eta(z) - \varepsilon]\left[\frac{\underline{u}^{q-1}}{\underline{u}_\delta^{q-1}} - \frac{\underline{v}^{q-1}}{\underline{v}_\delta^{q-1}}\right](\underline{u}_\delta^q - \underline{v}_\delta^q)dz + \int_\Omega c_\varepsilon \left[\frac{\underline{v}^{r-1}}{\underline{v}_\delta^{q-1}} - \frac{\underline{u}^{r-1}}{\underline{u}_\delta^{q-1}}\right](\underline{u}_\delta^q - \underline{v}_\delta^q)dz. \tag{3.5}$$

Note that for $\delta \in (0, 1]$, we have

$$\left|\frac{\underline{u}^{q-1}}{\underline{u}_\delta^{q-1}} - \frac{\underline{v}^{q-1}}{\underline{v}_\delta^{q-1}}\right| |\underline{u}_\delta^q - \underline{v}_\delta^q| \leq 2^q \left[\|\underline{u}\|_\infty^q + \|\underline{v}\|_\infty^q + 2\right],$$

$$\left[\frac{\underline{u}^{q-1}}{\underline{u}_\delta^{q-1}} - \frac{\underline{v}^{q-1}}{\underline{v}_\delta^{q-1}}\right](\underline{u}_\delta^q - \underline{v}_\delta^q) \to 0 \quad \text{for a.a. } z \in \Omega, \text{ as } \delta \to 0^+.$$

So, invoking the dominated convergence theorem, we obtain

$$\int_\Omega [\eta(z) - \varepsilon]\left[\frac{\underline{u}^{q-1}}{\underline{u}_\delta^{q-1}} - \frac{\underline{v}^{q-1}}{\underline{v}_\delta^{q-1}}\right](\underline{u}_\delta^q - \underline{v}_\delta^q)dz \to 0 \quad \text{as } \delta \to 0^+. \tag{3.6}$$

Also, for $\delta \in (0,1]$ we have

$$\left|\frac{\underline{v}^{r-1}}{\underline{v}_\delta^{q-1}} - \frac{\underline{u}^{r-1}}{\underline{u}_\delta^{q-1}}\right| |\underline{u}_\delta^q - \underline{v}_\delta^{q-1}| \le 2^{q-1}[\|\underline{v}\|_\infty^{r-q} + \|\underline{u}\|_\infty^{r-q}][\|\underline{u}\|_\infty^q + \|\underline{v}\|_\infty^q + 2],$$

$$\left|\frac{\underline{v}^{r-1}}{\underline{v}_\delta^{q-1}} - \frac{\underline{u}^{r-1}}{\underline{u}_\delta^{q-1}}\right|(\underline{u}_\delta^q - \underline{v}_\delta^q) \to (\underline{v}^{r-q} + \underline{u}^{r-q})(\underline{u}^q - \underline{v}^q) \quad \text{for a.a. } z \in \Omega, \text{ as } \delta \to 0^+.$$

Then once again the dominated convergence theorem gives

$$\int_\Omega c_\varepsilon \left[\frac{\underline{v}^{r-1}}{\underline{v}_\delta^{q-1}} - \frac{\underline{u}^{r-1}}{\underline{u}_\delta^{q-1}}\right](\underline{u}_\delta^q - \underline{v}_\delta^q)dz \to \int_\Omega c_\varepsilon \left[\underline{v}^{r-q} - \underline{u}^{r-q}\right](\underline{u}^q - \underline{v}^q)dz \quad \text{as } \delta \to 0^+. \tag{3.7}$$

We return to (3.5), pass to the limit as $\delta \to 0^+$ and use (3.6) and (3.7). We obtain

$$\begin{aligned} 0 &\le \int_\Omega c_\varepsilon \left[\underline{v}^{r-q} - \underline{u}^{r-q}\right](\underline{u}^q - \underline{v}^q)dz \le 0, \\ \Rightarrow \underline{u} &= \underline{v}. \end{aligned}$$

This proves the uniqueness of the positive solution of (3.1). □

In the next section, we will use this solution $\underline{u} \in W_0^{1,\theta}(\Omega) \bigcap L^\infty(\Omega)$ of (3.1), to produce a nontrivial positive solution for problem (1.1).

4 Positive solution

Let $M > 1$ be as in hypothesis H_1(ii). Choose $\bar{u} \ge M > 1$ big so that $\|\underline{u}\|_\infty < \bar{u}$. We have $\underline{u} < \bar{u}$. Then on account of hypothesis H_1(iii), we have

$$f(z, \bar{u}) \le 0 \quad \text{a.a. } z \in \Omega.$$

We introduce the truncation map $\tau : L^q(\Omega) \to L^q(\Omega)$ defined by

$$\tau(u)(z) = \begin{cases} \underline{u}(z) & \text{if } u(z) < \underline{u}(z); \\ u(z) & \text{if } \underline{u}(z) \le u(z) \le \bar{u}; \\ \bar{u} & \text{if } \bar{u} < u(z). \end{cases} \tag{4.1}$$

Evidently $\tau(\cdot)$ is continuous and $\tau(u) \in W_0^{1,\theta}(\Omega)$ if $u \in W_0^{1,\theta}(\Omega)$.

Let $N_f(\tau(u))(\cdot) = f(\cdot, \tau(u)(\cdot))$ (the Nemitsky map corresponding to f). We define

$$N_\tau(u)(\cdot) = N_f(\tau(u))(\cdot) + E(\cdot)|D\tau(u)|^{q-1} \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

We consider the map $K : W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)^*$ defined by

$$K(u) = V(u) - N_\tau(u) \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Proposition 4.1. *If hypotheses H_0, H_1 hold, then $K(\cdot)$ is pseudomonotone.*

Proof. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ such that

$$\left\{ \begin{array}{l} u_n \xrightarrow{w} u \text{ in } W_0^{1,\theta}(\Omega),\ K(u_n) \xrightarrow{w} u^* \text{ in } W_0^{1,\theta}(\Omega)^*, \\ \limsup_{n\to\infty} \langle K(u_n), u_n - u\rangle \le 0. \end{array} \right\} \tag{4.2}$$

From (4.2) and since $W_0^{1,\theta}(\Omega) \hookrightarrow L^q(\Omega)$ compactly (see Proposition 2.2), we have $u_n \to u$ in $L^q(\Omega)$. This implies that $\tau(u_n) \to \tau(u)$ in $L^q(\Omega)$. Then by Krasnoselskii's theorem (see Gasiński–Papageorgiou [4], p. 407), we have

$$N_f(\tau(u_n)) \to N_f(\tau(u)) \quad \text{in } L^{q'}(\Omega) \qquad \left(\frac{1}{q} + \frac{1}{q'} = 1\right). \tag{4.3}$$

Moreover, we have

$$\{D\tau(u_n)\}_{n\in\mathbb{N}} \subseteq L^\theta(\Omega, \mathbb{R}^N) \hookrightarrow L^q(\Omega, \mathbb{R}^N) \quad \text{is bounded (see Proposition 2.2).}$$

Therefore

$$\langle E(\cdot)|D\tau(u_n)|^{q-1}, u_n - u\rangle = \int_\Omega E(z)|D\tau(u_n)|^{q-1}(u_n - u)dz \to 0. \tag{4.4}$$

From (4.2), (4.3) and (4.4), it follows that

$$\limsup_{n\to\infty} \langle V(u_n), u_n - u\rangle \le 0 \quad \Rightarrow \quad u_n \to u \text{ in } W_0^{1,\theta}(\Omega) \text{ (see Proposition 2.4).}$$

Then we have

$$V(u_n) \to V(u) \text{ in } W_0^{1,\theta}(\Omega)^*,$$
$$N_f(\tau(u_n)) \to N_f(\tau(u)) \text{ in } L^{q'}(\Omega) \hookrightarrow W_0^{1,\theta}(\Omega)^* \quad \text{(see Gasiński–Papageorgiou [4], p. 141),}$$
$$E(\cdot)|D\tau(u_n)|^{q-1} \to E(\cdot)|D\tau(u)|^{q-1} \quad \text{in } L^{q'}(\Omega) \hookrightarrow W_0^{1,\theta}(\Omega)^*.$$

So, finally we have

$$u^* = V(u) - N_\tau(u) = K(u) \quad \text{(see (4.2)),}$$
$$\langle K(u_n), u_n\rangle \to \langle K(u), u\rangle.$$

This means that $K(\cdot)$ is generalized pseudomonotone and by Proposition 3.2.49, p. 333, of Gasiński–Papageorgiou [4], we conclude that $K(\cdot)$ is pseudomonotone. $\square$

Proposition 4.2. *If hypotheses H_0, H_1 hold, then the map $K(\cdot)$ is strongly coercive (see [14], p. 130).*

Proof. For every $u \in W_0^{1,\theta}(\Omega)$ with $\|u\| > 1$, we have

$$\begin{aligned} \langle K(u), u\rangle &= \rho_\theta(Du) - \int_\Omega f(z, \tau(u))u dz - \int_\Omega E(z)|D\tau(u)|^{q-1} u dz \\ &\ge c_3\|u\|^q - c_4\|u\|^{q-1} \text{ for some } c_3, c_4 > 0 \quad \text{(see Proposition 2.3 and (4.1))} \\ &\Rightarrow K(\cdot) \text{ is strongly coercive.} \end{aligned}$$

$\square$

Now we are ready for the existence theorem.

Theorem 4.3. *If hypotheses H_0 and H_1 hold, then problem (1.1) has a positive solution $u_0 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$ with $u_0(z) > 0$ for a.a. $z \in \Omega$.*

Proof. Propositions 4.1 and 4.2 together with Theorem 3.2.52, p. 336, of Gasiński–Papageorgiou [4], imply that $K(\cdot)$ is surjective. So we can find $u_0 \in W_0^{1,\theta}(\Omega)$ such that

$$K(u_0) = 0.$$

Then we have

$$\begin{aligned}
\langle V(u_0), (\underline{u} - u_0)^+ \rangle &\geq \int_\Omega f(z, \underline{u})(\underline{u} - u_0)^+ dz \text{ (see (4.1) and recall } E \geq 0) \\
&\geq \int_\Omega \left([\eta(z) - \varepsilon]\underline{u}^{q-1} - c_\varepsilon \underline{u}^{r-1} \right) (\underline{u} - u_0)^+ dz \quad \text{(see (2.2))} \\
&= \langle V(\underline{u}), (\underline{u} - u_0)^+ \rangle \quad \text{(see Proposition 4)} \\
&\Rightarrow \underline{u} \leq u_0 \quad \text{(see Proposition 3).}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle V(u_0), (u_0 - \bar{u})^+ \rangle &= \int_\Omega f(z, \bar{u})(u_0 - \bar{u})^+ dz \quad \text{(see (4.1) and note } D\bar{u} = 0) \\
&\leq 0 = \langle V(\bar{u}), (u_0 - \bar{u})^+ \rangle \quad \text{(see } H_1(ii)) \\
&\Rightarrow u_0 \leq \bar{u} \quad \text{(see Proposition 3).}
\end{aligned}$$

So we have proved

$$\begin{aligned}
u_0 \in [\underline{u}, \bar{u}] = \{u \in W_0^{1,\theta}(\Omega) : \underline{u}(z) \leq u(z) \leq \bar{u} \quad \text{for a.a. } z \in \Omega\}, \\
\Rightarrow u_0 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{is a positive solution of (1.1).}
\end{aligned}$$

Moreover, we have

$$0 < \underline{u}(z) \leq u_0(z) \quad \text{for a.a. } z \in \Omega.$$

□

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