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Random invariant manifolds and foliations for slow-fast PDEs with strong multiplicative noise

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Abstract. This article is devoted to the dynamical behaviors of a class of slow-fast PDEs perturbed by strong multiplicative noise. We will accomplish the existence of random invariant manifolds and foliations, and show exponential tracking property of them. Moreover, the asymptotic approximation for both objects will be presented.

Keywords: slow-fast systems; random invariant manifolds; random invariant foliations.
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1 Introduction

Various kinds of mathematical models arising from physics, engineering and biology not only involve random effects such as uncertain parameters, stochastic perturbation, but also relate to multiple disparate time or spatial scales [2,16,21]. Many important physical models, such as Burger's equation, Ginzburg–Laudau equation, Swift–Hohenberg equation are highly referred in this field. In order to investigate a variety of equations in the context of random influences, by combining probability theory, functional analysis and the theory of partial differential equations, mathematicians gradually developed and perfected a systematic framework of stochastic partial differential equations (SPDEs) in recent decades [13,30]. In terms of SPDEs evolving on multi-scales, there are many methods used to analyze the dynamical behaviours of SPDEs, such as averaging method [9,10], amplitude equations [4,5] and the theory of invariant manifolds [17,29].

Among these methods, the theory of invariant manifolds is considered as a practicable tool, which can provide a geometric structure of complex systems [1,35,37]. For deterministic systems, the pioneering results were obtained by Hadamard [22], Lyapunov [23] and Perron [25]. Duan et al. [14,15] extended this theory to random dynamic systems and show the existence of random invariant manifolds for SPDEs with simple multiplicative noise. Equations with more general multiplicative noise were studied by Caraballo et al. [7] and Mohammed

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et al. [24]. Also other dynamical properties of SPDEs have been already addressed in the literature, just to list a few but far from being complete: random invariant foliations [26,32,34], asymptotic dynamical behaviors [19,33,34,36], geometric shape [6,11,18,20], etc.

Applying the property that the random invariant manifolds contribute to the reduction of SPDEs, mathematicians can eliminate the fast variable of slow-fast systems to reduce the original system to a lower dimensional system. At earlier stage of the research, Schmalfuß and Schneider [31] studied a class of slow-fast systems with noise in the finite dimensional case by Hadamard method, and obtained that inertial manifolds tend to slow manifolds if the scaling parameter ε tends to 0. Fu et al. [17] applied Lyapunov–Perron method to a class of stochastic evolution equations with slow and fast components, and proved that slow manifolds asymptotically approximate to critical manifolds. Qiao et al. [28,29] obtained a reduced system of a class of SDEs under slow-fast Gaussian noisy fluctuations on the random invariant manifolds, and showed the delicate error between the filter of the original system and that of the reduced system. The slow invariant foliation, another interesting object in this field, was originally studied by Chen et al. [12]. They constructed random invariant foliations for a class of slow-fast stochastic evolutionary systems, and presented the approximation of slow foliations. Recently, slow-fast systems with non-Gaussian noise have gained substantial attention from researchers. For details, please see [27,38,39], etc.

In this paper, we investigate a class of slow-fast PDEs driven by strong multiplicative noise:

$$\mathrm{d}X^\varepsilon = \left[\frac{A}{\varepsilon}X^\varepsilon + \frac{f(X^\varepsilon, Y^\varepsilon)}{\varepsilon}\right]\mathrm{d}t + \frac{X^\varepsilon}{\sqrt{\varepsilon}} \circ \mathrm{d}W, \quad \text{in} \quad H_1, \tag{1.1}$$

$$\mathrm{d}Y^\varepsilon = [BY^\varepsilon + g(X^\varepsilon, Y^\varepsilon)]\,\mathrm{d}t + \frac{Y^\varepsilon}{\sqrt{\varepsilon}} \circ \mathrm{d}W, \quad \text{in} \quad H_2, \tag{1.2}$$

where H_1 and H_2 are separable Hilbert spaces, ε is small parameter ($0 < \varepsilon \ll 1$), $W(t)$ is a two-sided Wiener process taking value in $\mathbb{R}$, $\circ$ means Stratonovich stochastic differential, and A, B, f, g will be introduced later. Briefly, the main goal of this paper is to construct the random invariant manifolds and foliations for (1.1)–(1.2) and to derive corresponding approximations for both. Compared with [17,28,29,31], the system we study is forced by multiplicative noise rather than additive noise. To the best of our knowledge, this is the first research to consider the slow manifolds and slow foliations for slow-fast SPDEs with multiplicative noise.

This paper is organized as follows. In next section, we present some assumptions and recall some basic concepts in random dynamical systems. In Section 3, the existence of random invariant manifolds of (1.1)–(1.2) is established. Moreover, we show the orbit starting from random invariant manifold can exponentially approach to the other orbits in forward time, and prove that invariant manifolds can converge to slow ones as ε tends to 0. Section 4 is aimed at the theory of random invariant foliations including existence, exponential tracking property in backward time, and asymptotic foliations.

2 Preliminaries

The section is devoted to presenting some conditions that we need later, and reviewing some background materials in random dynamic systems.

2.1 Notations and assumptions

Let H_1 and H_2 be separable Hilbert spaces in (1.1) and (1.2). Denote their norms by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Set $H := H_1 \times H_2$ with norm $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$. A, B, f, g in (1.1)–(1.2) satisfy the following conditions.

Assumption 1. Suppose that linear operator A generates a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on H_1 fulfilling

$$\|e^{At}x\|_1 \leq e^{-\gamma_1 t}\|x\|_1, \quad \text{for} \quad x \in H_1,\ t \geq 0,$$

and linear operator B generates a C_0-group $\{e^{Bt}\}_{t\in\mathbb{R}}$ on H_2 fulfilling

$$\|e^{Bt}y\|_2 \leq e^{\gamma_2 t}\|y\|_2, \quad \text{for} \quad y \in H_2,\ t \leq 0,$$

where $\gamma_1 > 0, \gamma_2 \geq 0$.

Assumption 2. Suppose that nonlinear terms

$$f : H_1 \times H_2 \to H_1,$$
$$g : H_1 \times H_2 \to H_2,$$

satisfy $f(0,0) = 0$ and $g(0,0) = 0$, and there exists a constant $K > 0$ such that

$$\|f(x_1,y_1) - f(x_2,y_2)\|_1 \leq K(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2),$$
$$\|g(x_1,y_1) - g(x_2,y_2)\|_2 \leq K(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2),$$

for all $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$.

Assumption 3. $f(x,y)$ and $g(x,y)$ are C^1 functions, and all the first order partial derivatives of them are uniformly bounded.

Assumption 4. The Lipschitz constant K and decay rate γ_1 of A satisfy

$$K < \gamma_1. \tag{2.1}$$

Assumption 5. The Lipschitz constant K, decay rate γ_1 of A and decay rate γ_2 of B satisfy

$$K < \frac{\gamma_1\gamma_2}{2\gamma_1 + \gamma_2}. \tag{2.2}$$

Remark 2.1. (1) We remark that our main theorems hold when H_1 and H_2 are real or complex separable Hilbert spaces. For simplicity, we ignore it.

(2) Assumption 3 and Assumption 4 will be imposed in Section 3 and Section 4, respectively. We would like to point out that condition (2.2) is sufficient for condition (2.1), which implies that the condition used for the study of the random invariant manifolds is weaker than that used for the study of the random invariant foliations.

(3) Moreover, we remark that there are other conditions, which can also play the same role as condition (2.2). For the details, please see Remark 4.3 and Remark 4.4 in [12].

2.2 Random dynamical systems

Referring to the literature [1,12,14,15,26], we introduce some concepts of random dynamical systems.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and a flow θ of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ be defined by $\theta : \mathbb{R} \times \Omega \to \Omega$ such that

$$\begin{aligned}
&\theta_0 = \mathrm{id}_\Omega,\\
&\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}, \quad \forall t_1, t_2 \in \mathbb{R},\\
&\text{the flow is } (\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})\text{-measurable},\\
&\theta_t \mathbb{P} = \mathbb{P}, \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is called a metric dynamical system.

Definition 2.3. A random dynamical system on the topological space X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R} \times \Omega \times X \to X, \qquad (t, \omega, x) \to \varphi(t, \omega, x),$$

such that

$$\begin{aligned}
&\varphi \text{ is } (\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))\text{-measurable},\\
&\varphi(0, \omega) = \mathrm{id}_X, \quad \forall \omega \in \Omega,\\
&\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot)), \quad \forall s, t \in \mathbb{R}, \ \omega \in \Omega,\\
&\varphi(t, \omega, x) \text{ is continuous with respect to } t, \text{ for fixed } \omega \in \Omega, x \in X.
\end{aligned}$$

In what follows, we consider $\varphi(t, \omega, \cdot)$ as a random dynamical system on a complete separable metric space (H, d_H) over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Definition 2.4. A family of nonempty closed sets $M = \{M(\omega)\}$ contained in (H, d_H) is called a random set if

$$\omega \to \inf_{y\in M(\omega)} d_H(x, y)$$

is a random variable for $x \in H$.

Definition 2.5. A random set $M = \{M(\omega)\}$ is called a positively invariant set contained in (H, d_H) if

$$\varphi(t, \omega, M(\omega)) \subset M(\theta_t\omega), \quad t \geq 0, \quad \omega \in \Omega.$$

Furthermore, if, for every $\omega \in \Omega$, we can represent M by a graph of a Lipschitz mapping

$$\phi(\omega, \cdot) : H_2 \to H_1,$$

i.e.,

$$M(\omega) = \{(\phi(\omega, y), y) | y \in H_2\},$$

then $M(\omega)$ is called a Lipschitz continuous invariant manifold.

Definition 2.6. (i) Fixing $x \in H$, we call $\mathcal{W}_{\alpha s}(x, \omega)$ is an α-stable fiber passing through x with $\alpha \in \mathbb{R}^-$, if $\|\varphi(t, \omega, x) - \varphi(t, \omega, \bar{x})\|_H = \mathcal{O}(e^{\alpha t}), \forall \omega \in \Omega$ as $t \to +\infty$ for all $\bar{x} \in \mathcal{W}_{\alpha s}(x, \omega)$.

(ii) Fixing $x \in H$, we call $\mathcal{W}_{\beta u}(x,\omega)$ is a β-unstable fiber passing through x with $\beta \in \mathbb{R}^+$, if $\|\varphi(t,\omega,x) - \varphi(t,\omega,\bar{x})\|_H = \mathcal{O}(e^{\beta t}), \forall \omega \in \Omega$ as $t \to -\infty$ for all $\bar{x} \in \mathcal{W}_{\beta u}(x,\omega)$.

(iii) $\mathcal{W}_{\alpha s}(\omega) := \cup_{x\in H}\mathcal{W}_{\alpha s}(x,\omega)$ is called stable foliation.

(iv) $\mathcal{W}_{\beta u}(\omega) := \cup_{x\in H}\mathcal{W}_{\beta u}(x,\omega)$ is called unstable foliation.

(v) A foliation $\mathcal{W}_{\beta u}(\omega)$ is invariant with respect to random dynamical system φ if each fiber of it satisfies that

$$\varphi(t,\omega,\mathcal{W}_{\beta u}(x,\omega)) \subset \mathcal{W}_{\beta u}(\varphi(t,\omega,x),\theta_t\omega).$$

2.3 Transformation from SPDEs to RPDEs

The motivation of this subsection is to transform SPDEs (1.1)–(1.2) into random partial differential equations (RPDEs), and show the relationship between them. For our applications, we introduce the metric dynamical system induced by Wiener process. Let $W(t)$ be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R},\mathbb{R})$ which is the collection of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$. Set $\bar{\Omega} := C_0(\mathbb{R},\mathbb{R})$. This set is equipped with a compact-open topology (please see the Appendix in [1]). Let $\bar{\mathcal{F}}$ be its Borel σ-field and $\bar{\mathbb{P}}$ be the Wiener measure. Set

$$\theta_t\omega(\cdot) := \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

Note that $\bar{\mathbb{P}}$ is ergodic with respect to θ_t. Then $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}},\{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

In order ot obtain RPDEs, we need the following preparation. Consider the linear stochastic differential equation:

$$\mathrm{d}z^\varepsilon = -\frac{z^\varepsilon}{\varepsilon}\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\mathrm{d}W. \tag{2.3}$$

The solution of (2.3) is called an Ornstein–Uhlenbeck process. Following Lemma 2.1 in [14], we present the properties of $z^\varepsilon(t)$ as follows.

Lemma 2.7.

(1) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ of full measure with sublinear growth:

$$\lim_{t\to\pm\infty}\frac{|\omega(t)|}{|t|} = 0, \omega \in \Omega.$$

(2) For $\omega \in \Omega$ the random variable

$$z^\varepsilon(\omega) = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\omega(\tau)\mathrm{d}\tau$$

exists and generates a unique stationary solution of (2.3) given by

$$\Omega\times\mathbb{R} \ni (\omega,t) \to z^\varepsilon(\theta_t\omega) = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\theta_t\omega(\tau)\mathrm{d}\tau = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\omega(\tau+t)\mathrm{d}\tau + \varepsilon^{-\frac{1}{2}}\omega(t).$$

The mapping $t \to z^\varepsilon(\theta_t\omega)$ is continuous.

(3) In particular, we have

$$\lim_{t\to\pm\infty}\frac{|z^\varepsilon(\theta_t\omega)|}{|t|} = 0 \ \textit{for } \omega \in \Omega.$$

(4) In addition,

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z^\varepsilon(\theta_\tau\omega)\mathrm{d}\tau = 0 \ \textit{for}\ \omega\in\Omega.$$

In the followings of this paper, we consider (1.1)–(1.2) on the new metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$, where Ω is given in Lemma 2.7, $\mathcal{F}:=\{\Omega\cap\mathcal{A},\mathcal{A}\in\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))\}$, and $\mathbb{P}$ is the restriction of the Wiener measure $\bar{\mathbb{P}}$ to $\mathcal{F}$. We proceed to show the solution of (1.1)–(1.2) can generate a random dynamical system over the metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$. Letting $\bar{X}^\varepsilon = e^{-z^\varepsilon(\theta_t\omega)}X^\varepsilon$, $\bar{Y}^\varepsilon = e^{-z^\varepsilon(\theta_t\omega)}Y^\varepsilon$, we obtain RPDEs:

$$\mathrm{d}\bar{X}^\varepsilon = \left[\frac{A}{\varepsilon}\bar{X}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\bar{X}^\varepsilon}{\varepsilon} + \frac{F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega)}{\varepsilon}\right]\mathrm{d}t, \tag{2.4}$$

$$\mathrm{d}\bar{Y}^\varepsilon = [B\bar{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\bar{Y}^\varepsilon}{\varepsilon} + G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega)]\mathrm{d}t, \tag{2.5}$$

where

$$F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega) = e^{-z^\varepsilon(\theta_t\omega)}f(e^{z^\varepsilon(\theta_t\omega)}\bar{X}^\varepsilon, e^{z^\varepsilon(\theta_t\omega)}\bar{Y}^\varepsilon),$$
$$G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega) = e^{-z^\varepsilon(\theta_t\omega)}g(e^{z^\varepsilon(\theta_t\omega)}\bar{X}^\varepsilon, e^{z^\varepsilon(\theta_t\omega)}\bar{Y}^\varepsilon).$$

Since F and G are also Lipschitz functions with the same Lipschitz constant K for $\omega\in\Omega$, there exists a unique solution $Z^\varepsilon(t)=(X^\varepsilon(t),Y^\varepsilon(t))$ of (2.4)–(2.5) for $\omega\in\Omega$. Hence, the mapping

$$(t,\omega,Z^\varepsilon(0))\to Z^\varepsilon(t,\omega,Z^\varepsilon(0))$$

is $(\mathbb{R}\otimes\mathcal{F}\otimes\mathcal{B}(H),\mathcal{B}(H))$-measurable and generates a random dynamical system. We introduce the transform

$$T(\omega,x) = xe^{-z^\varepsilon(\omega)} \tag{2.6}$$

and its inverse transform

$$T^{-1}(\omega,x) = xe^{z^\varepsilon(\omega)} \tag{2.7}$$

for $x\in H$ and $\omega\in\Omega$.

Lemma 2.8. *Suppose that $u^\varepsilon(t,\omega,x)$ is the random dynamical system generated by (2.4)–(2.5). Then*

$$(t,\omega,x)\to T^{-1}(\theta_t\omega,u^\varepsilon(t,\omega,T(\omega,x))) =: \hat{u}^\varepsilon(t,\omega,x)$$

is a random dynamical system. For any $x\in H$ this process $(t,\omega)\to\hat{u}^\varepsilon(t,\omega,x)$ is a solution to (1.1)–(1.2).

Proof. Note that $T(\omega,\cdot)$ is a homeomorphism for any $\omega\in\Omega$, $T(\cdot,x)$, $T^{-1}(\cdot,x)$ are measurable for any $x\in H$, and $u^\varepsilon(t,\omega,x)$ is a random dynamical system. Hence, $\hat{u}^\varepsilon(t,\omega,x)$ is a random dynamical system. For $x\in H$, applying Itô's formula to $T(\theta_t\omega,\hat{u}^\varepsilon(t,\omega,T^{-1}(\omega,x)))$, we can obtain a solution of (2.4)–(2.5). Because $T(\theta_t\omega,x)$ and $u(t,\omega,x)$ are well defined for any $\omega\in\Omega$, and T^{-1} is the inverse of T, the converse is also true, which implies $\hat{u}^\varepsilon(t,\omega,x)$ is a solution of (1.1)–(1.2). □

Based on the above lemma, we can investigate (1.1)–(1.2) via (2.4)–(2.5). Then we are concerned with the random partial differential equations (RPDEs) (2.4)–(2.5) in the remainder of this paper.

3 Random invariant manifolds and slow manifolds

In this section, we use Lyapunov–Perron's method to prove the existence of random invariant manifolds for (2.4)–(2.5), and state that any orbit can be exponentially attracted by random invariant manifolds. Moreover, we show slow manifolds can approach to random invariant manifolds as the parameter ε tends to 0.

3.1 Random invariant manifolds

Let us give some notations. For $\alpha \in \mathbb{R}$, a real-valued stochastic process $p(t,\omega)$ and $i = 1,2$, define Banach Space

$$C^{i,-}_{\alpha,p} := \Big\{\phi : (-\infty,0] \to H_i | \phi \ \text{ is continuous and } \sup_{t\in(-\infty,0]} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i < \infty\Big\}$$

with the norm $\|\phi\|_{C^{i,-}_{\alpha,p}} = \sup\limits_{t\in(-\infty,0]} e^{-\alpha t-\int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i$, and

$$C^{i,+}_{\alpha,p} := \Big\{\phi : [0,+\infty) \to H_i | \phi \ \text{ is continuous and } \sup_{t\in[0,+\infty)} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i < \infty\Big\}$$

with the norm $\|\phi\|_{C^{i,+}_{\alpha,p}} = \sup_{t\in[0,+\infty)} e^{-\alpha t-\int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i$. Furthermore, define product Banach space $C^{\pm}_{\alpha,p} := C^{1,\pm}_{\alpha,p} \times C^{2,\pm}_{\alpha,p}$ with the norm $\|z\|_{C^{\pm}_{\alpha,p}} = \|x\|_{C^{1,\pm}_{\alpha,p}} + \|y\|_{C^{2,\pm}_{\alpha,p}}$, $z = (x,y) \in C^{\pm}_{\alpha,p}$.

Let μ be a positive number satisfying $\gamma_1 - \mu > K$. Let $\bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)$ be the solution of (2.4)-(2.5) with the initial value $\bar{Z}_0 \in H$. Set $M^\varepsilon(\omega) = \big\{\bar{Z}_0 \in H | \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}\big\}$. More precisely, $M^\varepsilon(\omega)$ is the set containing all initial data such that corresponding solutions belong to $C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$.

Following the idea from [15, 17], we will show $M^\varepsilon(\omega)$ can be represented by a Lipschitz function with Lyapunov–Perron's method.

Lemma 3.1. *Suppose that Assumptions 1, 2, 4 hold. $\bar{Z}_0 = (\bar{X}_0, \bar{Y}_0) \in M^\varepsilon(\omega)$ if and only if there exists a function $\bar{Z}^\varepsilon(\cdot) = (\bar{X}^\varepsilon(\cdot), \bar{Y}^\varepsilon(\cdot)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ with $\bar{Z}^\varepsilon(0) = \bar{Z}_0$ and satisfies*

$$\bar{X}^\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s, \tag{3.1}$$

$$\bar{Y}^\varepsilon(t) = e^{Bt+\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s. \tag{3.2}$$

Proof. The proof can be completed by that of Theorem 3.1 in [15], so it is omitted here. □

Theorem 3.2 (Existence of invariant manifold)**.** *Under Assumptions 1, 2, 4, for sufficiently small $\varepsilon > 0$, there exists a Lipschitz invariant manifold $M^\varepsilon(\omega)$ for (2.4)–(2.5) represented as a graph*

$$M^\varepsilon(\omega) = \{(H^\varepsilon(\omega, Y_0), Y_0) | Y_0 \in H_2\}, \tag{3.3}$$

where $H^\varepsilon(\cdot,\cdot) : \Omega \times H_2 \to H_1$ is a Lipschitz continuous mapping with Lipschitz constant $\mathrm{Lip}\, H^\varepsilon(\omega,\cdot)$ satisfying that

$$\mathrm{Lip}\, H^\varepsilon(\omega,\cdot) \le \frac{K}{(\gamma_1-\mu)\Big[1 - K\Big(\frac{1}{\gamma_1-\mu} + \frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\Big]}, \quad \omega \in \Omega. \tag{3.4}$$

Moreover, $H^\varepsilon(\omega,0) = 0$.

Proof. The proof consists of four steps.

Step1. Claim that for $\varepsilon > 0$ sufficiently small, (3.1)–(3.2) will have a unique solution $\bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0) = (\bar{X}^\varepsilon(\cdot,\omega,\bar{Z}_0),\bar{Y}^\varepsilon(\cdot,\omega,\bar{Z}_0)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. We will use Banach's Fixed Point Theorem to achieve the claim.

Define two operators $J_1^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ and $J_2^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ satisfying

$$J_1^\varepsilon(\bar{Z}^\varepsilon(t)) = \frac{1}{\varepsilon}\int_{-\infty}^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{3.5}$$

$$J_2^\varepsilon(\bar{Z}^\varepsilon(t)) = e^{Bt+\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{3.6}$$

for $t \le 0$. Define $J^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ by means of $J^\varepsilon(\bar{Z}^\varepsilon(\cdot)) = (J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot)), J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot)))$. Firstly, let us show that J^ε maps $C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ into itself. Since F, G keep the Lipschitz condition, we have

$$\begin{aligned}
\|J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} &= \frac{1}{\varepsilon}\sup_{t\in(-\infty,0]}\Big\{\int_{-\infty}^t e^{\frac{A(t-s)+\mu t+\int_s^0 z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s\Big\} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\Big\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}\mathrm{d}s\Big\}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&= \frac{K}{\gamma_1-\mu}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}
\end{aligned}$$

and

$$\begin{aligned}
\|J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} &= \sup_{t\in(-\infty,0]}\Big\{e^{Bt+\frac{\mu}{\varepsilon}t}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\mu t+\int_s^0 z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s\Big\} \\
&\le K\sup_{t\in(-\infty,0]}\Big\{\int_t^0 e^{(\gamma_2+\frac{\mu}{\varepsilon})(t-s)}\mathrm{d}s\Big\}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|\bar{Y}_0\|_2 \\
&= \frac{\varepsilon K}{\mu+\varepsilon\gamma_2}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|\bar{Y}_0\|_2.
\end{aligned}$$

Thus, $\|J^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \rho(K,\mu,\gamma_1,\gamma_2,\varepsilon) + \|\bar{Y}_0\|_2$, where $\rho(K,\mu,\gamma_1,\gamma_2,\varepsilon) = \frac{K}{\gamma_1-\mu} + \frac{\varepsilon K}{\mu+\varepsilon\gamma_2}$.

Next, we verify that J^ε is a contractive mapping. Let $Z^\varepsilon(\cdot)$ and $\bar{Z}^\varepsilon(\cdot) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. Then

$$\begin{aligned}
&\|J_1^\varepsilon(Z^\varepsilon(\cdot)) - J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z}{\varepsilon}}} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\Big\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}(\|X^\varepsilon-\bar{X}^\varepsilon\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|Y^\varepsilon-\bar{Y}^\varepsilon\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}})\mathrm{d}s\Big\} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\Big\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}\mathrm{d}s\Big\}\|Z^\varepsilon-\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&= \frac{K}{\gamma_1-\mu}\|Z^\varepsilon-\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}
\end{aligned}$$

and

$$\begin{aligned}
&\|J_2^\varepsilon(Z^\varepsilon(\cdot)) - J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&\quad\le K \sup_{t\in(-\infty,0]} \Big\{ \int_0^t e^{(-\gamma_2+\frac{\mu}{\varepsilon})(t-s)} (\|X^\varepsilon - \bar{X}^\varepsilon\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|Y^\varepsilon - \bar{Y}^\varepsilon\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}) \mathrm{d}s \Big\} \\
&\quad\le K \sup_{t\in(-\infty,0]} \Big\{ \int_t^0 e^{(\gamma_2+\frac{\mu}{\varepsilon})(t-s)} \mathrm{d}s \Big\} \|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&\quad= \frac{\varepsilon K}{\mu+\varepsilon\gamma_2} \|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}$$

Therefore, $\|J^\varepsilon(Z^\varepsilon(\cdot)) - J^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \rho(K,\mu,\gamma_1,\gamma_2,\varepsilon)\|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}$. Since $\gamma_1 - \mu > K$, we have $\rho < 1$ if $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 = \frac{(\gamma_1-\mu-K)\mu}{K(\gamma_1-\mu)-(\gamma_1-\mu-K)\gamma_2}$.

Consequently, we use Banach's Fixed Point Theorem to obtain the existence of the unique solution $\bar{Z}^\varepsilon(\cdot) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ for (2.4)–(2.5), and the standard a-priori estimate:

$$\|\bar{Z}^\varepsilon(\cdot,\omega,Z_0) - \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0)\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \frac{1}{1-\rho(K,\mu,\gamma_1,\gamma_2,\varepsilon)} \|Y_0 - \bar{Y}_0\|_2, \tag{3.7}$$

for all $\omega \in \Omega$, $Y_0, \bar{Y}_0 \in H_2$.

Step 2. Construct the random invariant manifold $M^\varepsilon(\omega)$.

Define

$$H^\varepsilon(\omega, Y_0) := \frac{1}{\varepsilon} \int_{-\infty}^0 e^{\frac{-As - \int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon(s,\omega,Y_0), \bar{Y}^\varepsilon(s,\omega,Y_0), \theta_s\omega)\mathrm{d}s.$$

Then, the Lipschitz constant of $H^\varepsilon(\omega, Y_0)$ is given by

$$\begin{aligned}
\|H^\varepsilon(\omega,Y_0) - H^\varepsilon(\omega,\bar{Y}_0)\|_1 &\le \frac{K}{\gamma_1-\mu} \|\bar{Z}^\varepsilon(\cdot,\omega,Z_0) - \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0)\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&\le \frac{K}{(\gamma_1-\mu)\Big[1 - K\Big(\frac{1}{\gamma_1-\mu} + \frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\Big]} \|Y_0 - \bar{Y}_0\|_2
\end{aligned}$$

for all $\omega \in \Omega$, $Y_0, \bar{Y}_0 \in H_2$. Lemma 3.1 yields that $M^\varepsilon(\omega) = \{(H^\varepsilon(\omega,Y_0), Y_0) | Y_0 \in H_2\}$.

Step 3. We need to prove $M^\varepsilon(\omega)$ is a random set. To this end, we show that

$$\omega \to \inf_{z'\in H} \|(x,y) - (H^\varepsilon(\omega,\mathcal{P}z'),\mathcal{P}z')\| \tag{3.8}$$

is measurable, where $\mathcal{P}$ is the projection from H to H_2. Let H_c be a countable dense set of the separable space H. The continuity of $H^\varepsilon(\omega,\cdot)$ yields that the right-hand side of (3.8) is equivalent to

$$\omega \to \inf_{z'\in H_c} \|(x,y) - (H^\varepsilon(\omega,\mathcal{P}z'),\mathcal{P}z')\|. \tag{3.9}$$

Since $\omega \to H^\varepsilon(\omega,\mathcal{P}z')$ is measurable, we obtain that measurability of any expression under the infimum of (3.9). Then the fact that M^ε is a random set follows from Theorem III.9 in [8].

Step 4. We are going to show that M^ε is positively invariant which means that for every $\bar{Z}_0 = (\bar{X}_0, \bar{Y}_0) \in M^\varepsilon(\omega)$, $\bar{Z}(s,\omega,\bar{Z}_0) \in M^\varepsilon(\theta_s\omega)$ for all $s \ge 0$. For every fixed $s \ge 0$, we claim

that $\bar{Z}(t+s,\omega,\bar{Z}_0)$ is the solution of

$$\mathrm{d}\bar{X}^\varepsilon = \left[\frac{A}{\varepsilon}\bar{X}^\varepsilon + \frac{z^\varepsilon(\theta_t(\theta_s\omega))\bar{X}^\varepsilon}{\varepsilon} + \frac{F(\bar{X},\bar{Y},\theta_t(\theta_s\omega))}{\varepsilon}\right]\mathrm{d}t,$$
$$\mathrm{d}\bar{Y}^\varepsilon = [B\bar{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t(\theta_s\omega))\bar{Y}^\varepsilon}{\varepsilon} + G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t(\theta_s\omega))]\mathrm{d}t$$

with initial value $\bar{Z}(0) = \bar{Z}(s,\omega,\bar{Z}_0)$. Then, $\bar{Z}(t+s,\omega,\bar{Z}_0) = \bar{Z}(t,\theta_s\omega,\bar{Z}(s,\omega,\bar{Z}_0))$. Since $\bar{Z}(\cdot,\omega,\bar{Z}_0) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$, we get $\bar{Z}(\cdot,\theta_s\omega,\bar{Z}(s,\omega,\bar{Z}_0)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. Thus $\bar{Z}(s,\omega,\bar{Z}_0) \in M^\varepsilon(\theta_s\omega)$. The proof is completed. □

Next, we want to show that the Lipschitz invariant manifolds for (2.4)–(2.5) given in (3.3) has exponential tracking property

Theorem 3.3 (Exponential tracking property in forward time). *Under Assumptions 1, 2, 4, for sufficiently small $\varepsilon > 0$, there exists a positive constant C and a random process $D(t,\omega)$ such that for any $\check{Z}_0 = (\check{X}_0,\check{Y}_0) \in H$, there exists $\bar{Z}_0 = (\bar{X}_0,\bar{Y}_0) \in M^\varepsilon(\omega)$ satisfying that*

$$\|\bar{Z}^\varepsilon(t,\omega,\check{Z}_0) - \bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)\| \le D(t,\omega)e^{-Ct}\|\check{Z}_0 - \bar{Z}_0\|, \quad t \ge 0.$$

Proof. Suppose that $\check{Z}^\varepsilon(t) = (\check{X}^\varepsilon(t),\check{Y}^\varepsilon(t))$ and $\bar{Z}^\varepsilon(t) = (\bar{X}^\varepsilon(t),\bar{Y}^\varepsilon(t))$ are two solutions of (2.4)–(2.5) with initial data $\check{Z}^\varepsilon(0) = \check{Z}_0$ and $\bar{Z}^\varepsilon(0) = \bar{Z}_0$, respectively. Then $\widetilde{Z}^\varepsilon(t) = \check{Z}^\varepsilon(t) - \bar{Z}^\varepsilon(t) = (\widetilde{X}^\varepsilon(t),\widetilde{Y}^\varepsilon(t))$ satisfies that the following system:

$$\mathrm{d}\widetilde{X}^\varepsilon = \left[\frac{A}{\varepsilon}\widetilde{X}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\widetilde{X}^\varepsilon}{\varepsilon} + \frac{\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega)}{\varepsilon}\right]\mathrm{d}t, \tag{3.10}$$
$$\mathrm{d}\widetilde{Y}^\varepsilon = [B\widetilde{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\widetilde{Y}^\varepsilon}{\varepsilon} + \widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega)]\mathrm{d}t, \tag{3.11}$$

where

$$\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega) = F(\widetilde{X}^\varepsilon + \bar{X}^\varepsilon,\widetilde{Y}^\varepsilon + \bar{Y}^\varepsilon,\theta_t\omega) - F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega), \tag{3.12}$$
$$\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega) = G(\widetilde{X}^\varepsilon + \bar{X}^\varepsilon,\widetilde{Y}^\varepsilon + \bar{Y}^\varepsilon,\theta_t\omega) - G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega). \tag{3.13}$$

Acoording to the variation of constants formula, we state that $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot),\widetilde{Y}^\varepsilon(\cdot))$ with initial value $\widetilde{Z}^\varepsilon(0) = \check{Z}_0 - \bar{Z}_0 = (\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0))$ is the solution in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ of (3.10)–(3.11) if and only if

$$\widetilde{X}^\varepsilon(t) = e^{\frac{At+\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{X}^\varepsilon(0) + \frac{1}{\varepsilon}\int_0^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{3.14}$$
$$\widetilde{Y}^\varepsilon(t) = \int_{+\infty}^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s. \tag{3.15}$$

Now, let us show that there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot),\widetilde{Y}^\varepsilon(\cdot))$ in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ with initial value $(\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0))$ such that

$$(\bar{X}_0,\bar{Y}_0) = (\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0)) + (\check{X}_0,\check{Y}_0) \in M^\varepsilon(\omega). \tag{3.16}$$

It follows from Theorem 3.2 that

$$\begin{aligned}\widetilde{X}^\varepsilon(0) &= -\check{X}_0 + H^\varepsilon(\omega,\widetilde{Y}^\varepsilon(0) + \check{Y}_0)\\ &= -\check{X}_0 + \frac{1}{\varepsilon}\int_{-\infty}^0 e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}F(\bar{X}^\varepsilon(s,\widetilde{Y}^\varepsilon(0)+\check{Y}_0),\bar{Y}^\varepsilon(s,\widetilde{Y}^\varepsilon(0)+\check{Y}_0),\theta_s\omega)\mathrm{d}s.\end{aligned}$$

Let $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot), \widetilde{Y}^\varepsilon(\cdot)) \in C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$. For $t \geq 0$, define two operators $C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^{1,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ and $C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^{2,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ by means of

$$\begin{aligned}
\mathcal{J}_1^\varepsilon(\widetilde{Z}^\varepsilon(t)) &= e^{\frac{At + \int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} \widetilde{X}^\varepsilon(0) + \frac{1}{\varepsilon}\int_0^t e^{\frac{A(t-s) + \int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\widetilde{X}^\varepsilon, \widetilde{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s, \\
\mathcal{J}_2^\varepsilon(\widetilde{Z}^\varepsilon(t)) &= \int_{+\infty}^t e^{B(t-s) + \frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\widetilde{X}^\varepsilon, \widetilde{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s,
\end{aligned}$$

where $\widetilde{X}^\varepsilon(0)$ is from (3.16). Furthermore, define $\mathcal{J}^\varepsilon : C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ as

$$\mathcal{J}^\varepsilon(\widetilde{Z}^\varepsilon(\cdot)) = (\mathcal{J}_1^\varepsilon(\widetilde{Z}^\varepsilon(\cdot)), \mathcal{J}_2^\varepsilon(\widetilde{Z}^\varepsilon(\cdot))).$$

As the proof of Theorem 3.2, we apply Banach's Fixed Point Theorem to (3.14)–(3.15). Obviously, $\mathcal{J}^\varepsilon$ is self-map. It remains to show that $\mathcal{J}^\varepsilon$ is contractive. Note

$$\begin{aligned}
&\|e^{\frac{A\cdot + \int_0^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}(\widetilde{X}_1^\varepsilon(0) - \widetilde{X}_2^\varepsilon(0))\|_{C^{1,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \\
&\leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \|\widetilde{Y}_1^\varepsilon(0) - \widetilde{Y}_2^\varepsilon(0))\|_2 \\
&\leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \| \int_{+\infty}^0 e^{-Bs + \frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} \big(\widetilde{G}(\widetilde{X}_1^\varepsilon, \widetilde{Y}_1^\varepsilon, \theta_s\omega) - \widetilde{G}(\widetilde{X}_2^\varepsilon, \widetilde{Y}_2^\varepsilon, \theta_s\omega)\big)\mathrm{d}s\|_2 \\
&\leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \cdot K \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \int_0^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})s}\mathrm{d}s.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\mathcal{J}_1^\varepsilon(\widetilde{Z}_1^\varepsilon - \widetilde{Z}_2^\varepsilon)\|_{C^{1,+}_{\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} &\leq \operatorname{Lip} H^\varepsilon \cdot K \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0}\left\{ e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \int_0^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})s}\mathrm{d}s \right\} \\
&\quad + \frac{K}{\varepsilon}\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0}\left\{ \int_0^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}\mathrm{d}s \right\} \\
&\leq \Big(\frac{\operatorname{Lip} H^\varepsilon \cdot \varepsilon K}{\mu + \varepsilon\gamma_2} + \frac{K}{\gamma_1 - \mu}\Big)\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}},
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{J}_2^\varepsilon(\widetilde{Z}_1^\varepsilon - \widetilde{Z}_2^\varepsilon)\|_{C^{2,+}_{\frac{\mu}{\varepsilon}, \frac{z}{\varepsilon}}} &\leq K\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0}\left\{ \int_t^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})(t-s)}\mathrm{d}s \right\} \\
&\leq \frac{\varepsilon K}{\mu + \varepsilon\gamma_2}\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}$$

By (3.4), we further obtain

$$\|\mathcal{J}^\varepsilon(\widetilde{Z}_1^\varepsilon(\cdot)) - \mathcal{J}^\varepsilon(\widetilde{Z}_2^\varepsilon(\cdot))\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \leq \beta(K, \gamma_1, \gamma_2, \mu)\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}},$$

where

$$\beta(K, \gamma_1, \gamma_2, \mu) = \frac{\varepsilon K^2}{(\gamma_1 - \mu)(\mu + \varepsilon\gamma_2)[1 - K(\frac{1}{\gamma_1 - \mu} + \frac{\varepsilon}{\mu + \varepsilon\gamma_2})]} + K(\frac{1}{\gamma_1 - \mu} + \frac{\varepsilon}{\mu + \varepsilon\gamma_2}).$$

Note $\beta(K,\gamma_1,\gamma_2,\mu)<1$ if ε is sufficiently small. Then there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot)$ for (3.14)–(3.15) in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$.

Furthermore,

$$\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}\le\|\widetilde{Z}^\varepsilon(0)\|+K\Big(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}},$$

which implies that

$$\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}\le\frac{1}{1-K\left(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\right)}\|\widetilde{Z}^\varepsilon(0)\|.$$

Thus, we obtain

$$\|\bar{Z}^\varepsilon(t,\omega,\check{Z}_0)-\bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)\|\le\frac{e^{\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}}{1-K\left(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\right)}e^{-\frac{\mu}{\varepsilon}t}\|\check{Z}_0-\bar{Z}_0\|,\quad t\ge0.$$

The proof is completed. □

3.2 Slow manifolds

In this subsection, we are going to present the approximation of $M^\varepsilon(\omega)$ in slow time-scale $T=\frac{t}{\varepsilon}$. Scaling $t=\varepsilon T$ in (2.4)–(2.5), we have

$$\mathrm{d}\bar{X}^\varepsilon=[A\bar{X}^\varepsilon+z^\varepsilon(\theta_{\varepsilon T}\omega)\bar{X}^\varepsilon+\bar{F}(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_{\varepsilon T}\omega)]\mathrm{d}T,\tag{3.17}$$
$$\mathrm{d}\bar{Y}^\varepsilon=[\varepsilon B\bar{Y}^\varepsilon+z^\varepsilon(\theta_{\varepsilon T}\omega)\hat{Y}^\varepsilon+\varepsilon\bar{G}(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_{\varepsilon T}\omega)]\mathrm{d}T.\tag{3.18}$$

Let $\eta(\theta_T\omega)$ be the stationary solution of

$$\mathrm{d}\eta=-\eta\mathrm{d}T+\mathrm{d}\widetilde{W}(T),\tag{3.19}$$

where $\widetilde{W}(T)$ and $\varepsilon^{-\frac{1}{2}}W(\varepsilon T)$ are identical in distribution. Replacing $z^\varepsilon(\theta_{\varepsilon T}\omega)$ by $\eta(\theta_T\omega)$ in (3.17)–(3.18), we have

$$\mathrm{d}\widehat{X}^\varepsilon=[A\widehat{X}^\varepsilon+\eta(\theta_T\omega)\widehat{X}^\varepsilon+\widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)]\mathrm{d}T,\tag{3.20}$$
$$\mathrm{d}\widehat{Y}^\varepsilon=[\varepsilon B\widehat{Y}^\varepsilon+\eta(\theta_T\omega)\widehat{Y}^\varepsilon+\varepsilon\widehat{G}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)]\mathrm{d}T,\tag{3.21}$$

where

$$\widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)=e^{-\eta(\theta_T\omega)}f(e^{\eta(\theta_T\omega)}\widehat{X}^\varepsilon,e^{\eta(\theta_T\omega)}\widehat{Y}^\varepsilon),$$
$$\widehat{G}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)=e^{-\eta(\theta_T\omega)}g(e^{\eta(\theta_T\omega)}\widehat{X}^\varepsilon,e^{\eta(\theta_T\omega)}\widehat{Y}^\varepsilon).$$

Since $z^\varepsilon(\theta_{\varepsilon T}\omega)$ is the same as $\eta(\theta_T\omega)$ in distribution (please see Lemma 3.2 in [31]), the distribution of the solution (3.17)–(3.18) coincides with that of (3.20)–(3.21). Using the similar procedure as the proof of Theorem 3.2, we obtain that (3.20)–(3.21) has a random invariant manifold $\widehat{M}^\varepsilon(\omega)$ represented as

$$\widehat{M}^\varepsilon(\omega)=\{(\widehat{H}^\varepsilon(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in H_2\}$$

with

$$\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)=\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_s\omega)\mathrm{d}s,$$

where

$$\begin{aligned}\widehat{X}^{\varepsilon}(T)&=\int_{-\infty}^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_s\omega)\mathrm{d}s,\\ \widehat{Y}^{\varepsilon}(T)&=e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0+\varepsilon\int_0^T e^{\varepsilon B(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{G}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_s\omega)\mathrm{d}s.\end{aligned}$$

Then, for fixed $\widehat{Y}_0\in H_2$, $H^{\varepsilon}(\omega,\widehat{Y}_0)=\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)$ in distribution. In fact,

$$\begin{aligned}H^{\varepsilon}(\omega,\widehat{Y}_0)&=\frac{1}{\varepsilon}\int_{-\infty}^{0}e^{\frac{-As-\int_0^s z^{\varepsilon}(\theta_r\omega)\mathrm{d}r}{\varepsilon}}F(\bar{X}(s,\omega,\widehat{Y}_0),\bar{Y}(s,\omega,\widehat{Y}_0),\theta_s\omega)\mathrm{d}s\\ &=\int_{-\infty}^{0}e^{-As-\int_0^s z^{\varepsilon}(\theta_{\varepsilon r}\omega)\mathrm{d}r}F(\bar{X}(\varepsilon s,\omega,\widehat{Y}_0),\bar{Y}(\varepsilon s,\omega,\widehat{Y}_0),\theta_{\varepsilon s}\omega)\mathrm{d}s\\ &\overset{d}{=}\int_{-\infty}^{0}e^{-As-\int_0^s\eta^{\varepsilon}(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}(s,\omega,\widehat{Y}_0),\widehat{Y}(s,\omega,\widehat{Y}_0),\theta_s\omega)\mathrm{d}s\\ &=\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0),\end{aligned}\tag{3.22}$$

where $\overset{d}{=}$ denotes the equivalence in distribution.

We proceed to want to explore the approximation form of the invariant manifold $\widehat{M}^{\varepsilon}(\omega)$ as $\varepsilon\to 0$. To achieve it, we observe (3.20)–(3.21) when $\varepsilon=0$.

Consider

$$\mathrm{d}\widehat{X}^0=[A\widehat{X}^0+\eta(\theta_T\omega)\widehat{X}^0+\widehat{F}(\widehat{X}^0,\widehat{Y}^0,\theta_T\omega)]\mathrm{d}T,\tag{3.23}$$

$$\mathrm{d}\widehat{Y}^0=\eta(\theta_T\omega)\widehat{Y}^0\mathrm{d}T.\tag{3.24}$$

We comment that (3.23)–(3.24) is the critical system of (3.20)–(3.21). It is clear that there exists a random invariant manifold $\widehat{M}^0(\omega)$ for (3.23)–(3.24) which is

$$\widehat{M}^0(\omega)=\{(\widehat{H}^0(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in H_2\},\tag{3.25}$$

where

$$\widehat{H}^0(\omega,\widehat{Y}_0)=\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^0,e^{\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0,\theta_s\omega)\mathrm{d}s.\tag{3.26}$$

Furthermore, using the idea coming from Theorem 5.1 in [17], we state that $\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)=\widehat{H}^0(\omega,\widehat{Y}_0)+\mathcal{O}(\varepsilon)$, for $\omega\in\Omega$.

Theorem 3.4. *Under Assumptions 1, 2, 4, for sufficiently small $\varepsilon>0$, we have*

$$\|\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)-\widehat{H}^0(\omega,\widehat{Y}_0)\|_1=\mathcal{O}(\varepsilon),$$

for all $\widehat{Y}_0\in\mathcal{D}(B)$, $\omega\in\Omega$.*

Proof. For $T\le 0$ and $\widehat{Y}_0\in\mathcal{D}(B)$,

$$\begin{aligned}\|e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0-e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0\|_2&\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|\int_{\varepsilon T}^{0}e^{Bs}B\widehat{Y}_0\mathrm{d}s\|_2\\ &\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}.\end{aligned}$$

*$\mathcal{D}(B)$ means the domain of operator B

Then,

$$
\begin{aligned}
&\|\widehat{Y}^{\varepsilon}(T)-\widehat{Y}^{0}(T)\|_2\\
&\quad\le \|e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0-e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0\|_2\\
&\qquad+\|\varepsilon\int_0^T e^{\varepsilon B(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{G}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_s\omega)\mathrm{d}s\|_2\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}\\
&\qquad+\varepsilon K\int_T^0 e^{\varepsilon\gamma_2(T-s)-\mu s+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\big(\|\widehat{X}^{\varepsilon}\|_{C^{1,-}_{-\mu,\eta}}+\|\widehat{Y}^{\varepsilon}\|_{C^{2,-}_{-\mu,\eta}}\big)\mathrm{d}s\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}+\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\rho}\int_T^0 e^{\varepsilon\gamma_2(T-s)-\mu s+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\mathrm{d}s,\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}+e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\rho}\frac{e^{-\mu T}-e^{\varepsilon\gamma_2 T}}{\mu+\varepsilon\gamma_2},
\end{aligned}
\tag{3.27}
$$

where we use the estimation (3.7) in the third inequality. By (3.27), we obtain

$$
\begin{aligned}
&\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\\
&\quad\le K\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\sup_{T\le 0}\Big\{\int_{-\infty}^T e^{-(\gamma_1-\mu)(T-s)}\mathrm{d}s\Big\}\\
&\qquad+K\sup_{T\le 0}\Big\{\int_{-\infty}^T e^{-\gamma_1(T-s)+\mu T+\int_s^0\eta(\theta_r\omega)\mathrm{d}r}\|\widehat{Y}^{\varepsilon}-\widehat{Y}^{0}\|_2\mathrm{d}s\Big\}\\
&\quad\le\frac{K}{\gamma_1-\mu}\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}+\mathcal{R},
\end{aligned}
\tag{3.28}
$$

where

$$
\begin{aligned}
\mathcal{R}=&\sup_{T\le 0}\Bigg\{Ke^{\mu T}\Big(\frac{\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2}-\frac{\|B\widehat{Y}_0\|_2e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2}-\frac{\varepsilon K\|\widehat{Y}_0\|_2e^{\gamma_2\varepsilon T}}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\Big)\\
&\qquad+\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1-\mu)}\Bigg\}\\
:=&\sup_{T\le 0}\Sigma(T).
\end{aligned}
$$

Note that there exists $T_{sup}<0$ such that $\frac{\mathrm{d}\Sigma(T)}{\mathrm{d}T}|_{T=T_{sup}}=0$, which implies that $\mathcal{R}=\Sigma(T_{sup})=\mathcal{O}(\varepsilon)$. Then we have

$$
\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\le\frac{(\gamma_1-\mu)\mathcal{R}}{\gamma_1-\mu-K}.
$$

Therefore,

$$\begin{aligned}
&\|\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)-\widehat{H}^{0}(\omega,\widehat{Y}_0)\|_1\\
&\quad\le K\int_{-\infty}^{0}e^{(\gamma_1-\mu)s}\|\widehat{X}^{\varepsilon}-\widehat{X}^{0}\|_{C^{1,-}_{-\mu,\eta}}\mathrm{d}s+K\int_{-\infty}^{0}e^{\gamma_1 s-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\|\check{Y}^{\varepsilon}-\check{Y}^{0}\|_2\mathrm{d}s\\
&\quad\le\frac{K\mathcal{R}}{\gamma_1-\mu-K}+\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1-\mu)}+\frac{K\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2}\\
&\qquad-\frac{K\|B\widehat{Y}_0\|_2}{(\gamma_1+\gamma_2\varepsilon)\gamma_2}-\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\\
&\quad=\mathcal{O}(\varepsilon).
\end{aligned}$$

□

We now show the better approximation of slow manifolds. According to Assumption 3, we know $\widehat{F}(x,y)$ has the partial derivatives. Let

$$\begin{aligned}
\widehat{X}^{\varepsilon}(T)&=\widehat{X}^{0}(T)+\varepsilon\widehat{X}^{1}(T)+\varepsilon^2\widehat{X}^{2}(T)+\cdots,\\
\widehat{Y}^{\varepsilon}(T)&=\widehat{Y}^{0}(T)+\varepsilon\widehat{Y}^{1}(T)+\varepsilon^2\widehat{Y}^{2}(T)+\cdots.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\widehat{F}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_T\omega)&=\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)+\varepsilon\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{X}^{1}\\
&\quad+\varepsilon\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{Y}^{1}+\mathcal{O}(\varepsilon^2),\\
\varepsilon\widehat{G}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_T\omega)&=\varepsilon\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)+\mathcal{O}(\varepsilon^2),
\end{aligned}$$

where $\widehat{F}_x$ and $\widehat{F}_y$ are the partial derivatives of $\widehat{F}(x,y)$ with respect to x and y, respectively. Equating the same degree of ε, we have

$$\mathrm{d}\widehat{X}^{0}=[A\widehat{X}^{0}+\eta(\theta_T\omega)\widehat{X}^{0}+\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)]\mathrm{d}T, \tag{3.29}$$

$$\mathrm{d}\widehat{Y}^{0}=\eta(\theta_T\omega)\widehat{Y}^{0}\mathrm{d}T, \tag{3.30}$$

and

$$\mathrm{d}\widehat{X}^{1}=[A\widehat{X}^{1}+\eta(\theta_T\omega)\widehat{X}^{1}+\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{X}^{1}+\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{Y}^{1}]\mathrm{d}T, \tag{3.31}$$

$$\mathrm{d}\widehat{Y}^{1}=[B\widehat{Y}^{0}+\eta(\theta_T\omega)\widehat{Y}^{1}+\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)]\mathrm{d}T. \tag{3.32}$$

We note (3.29)–(3.30) is the same as (3.23)–(3.24). Hence, (3.29)–(3.30) has a random invariant manifold $\widehat{M}^{0}(\omega)$ given in (3.25).

Let us consider

$$\widehat{X}^{1}(T)=\int_{-\infty}^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\widehat{X}^{1}+\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\widehat{Y}^{1}]\mathrm{d}s, \tag{3.33}$$

$$\widehat{Y}^{1}(T)=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_1+\int_0^{T}e^{\int_s^T\eta(\theta_r\omega)\mathrm{d}r}[B\widehat{Y}^{0}+\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)]\mathrm{d}s, \tag{3.34}$$

where

$$\begin{aligned}
\widehat{X}^{0}(T)&=e^{AT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{H}^{0}(Y_0,\omega)+\int_0^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\mathrm{d}T,\\
\widehat{Y}^{0}(T)&=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0,
\end{aligned}$$

with $\widehat{H}^0(Y_0,\omega)$ given in (3.26).

We state there exists a unique solution for (3.33)–(3.34) in $C^+_{-\frac{\mu}{\varepsilon},\frac{z\varepsilon}{\varepsilon}}$ without proof. Then, it is easy to obtain that (3.31)–(3.32) has the random invariant manifold represented as

$$M^1(\omega)=\{(\check{H}^1(\omega,\widehat{Y}_1),\widehat{Y}_1)|\widehat{Y}_1\in H_2\},$$

where

$$\check{H}^1(\omega,\widehat{Y}_1)=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\widehat{X}^1+\widehat{F}_y(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\widehat{Y}^1]\mathrm{d}s.$$

Set

$$\begin{aligned}\widehat{H}^1(\omega,\widehat{Y}_0)&=\check{H}^1(\omega,0)\\&=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\Big\{\widehat{F}_x(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\widehat{X}^1+\widehat{F}_y(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\\&\quad\times\Big[\int_0^s e^{\int_\tau^s\eta(\theta_r\omega)\mathrm{d}r}\Big(B\widehat{Y}_0+\widehat{G}(\widehat{X}^0,\widehat{Y}^0,\theta_\tau\omega)\Big)\mathrm{d}\tau\Big]\Big\}\mathrm{d}s.\end{aligned}\tag{3.35}$$

Then, we can formally show the first order approximation of $H^\varepsilon(\widehat{Y}_0,\omega)$ as follow:

$$\begin{aligned}&H^\varepsilon(\omega,\widehat{Y}_0)\\&\quad\overset{d}{=}\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)\\&\quad=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}(s,\omega,Y_0),\widehat{Y}(s,\omega,Y_0),\theta_s\omega)\mathrm{d}s\\&\quad=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\mathrm{d}s\\&\qquad+\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\widehat{X}^1+\widehat{F}_y(\widehat{X}^0,\widehat{Y}^0,\theta_s\omega)\widehat{Y}^1]\mathrm{d}s+\mathcal{O}(\varepsilon^2)\\&\quad=\widehat{H}^0(\omega,\widehat{Y}_0)+\varepsilon\widehat{H}^1(\omega,\widehat{Y}_0)+\mathcal{O}(\varepsilon^2).\end{aligned}\tag{3.36}$$

Note that $H^\varepsilon(\omega,\widehat{Y}_0)$ coincides with $\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)$ in distribution. We have, in fact, proved the following theorem.

Theorem 3.5 (First order approximation of slow manifold). *Under Assumptions 1-4, for sufficiently small $\varepsilon>0$, we obtain the approximation of the random invariant manifold for (2.4)–(2.5) as*

$$\begin{aligned}M^\varepsilon(\omega)&=\{(H^\varepsilon(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\}\\&\overset{d}{=}\{(\widehat{H}^\varepsilon(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\}\\&=\{(\widehat{H}^0(\omega,\widehat{Y}_0)+\varepsilon\widehat{H}^1(\omega,\widehat{Y}_0)+\mathcal{O}(\varepsilon^2),\widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\},\end{aligned}$$

where the second equality holds in distribution, that means for fixed $\widehat{Y}_0\in\mathcal{D}(B)$, $H^\varepsilon(\omega,\widehat{Y}_0)$ and $\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)$ are identical in distribution, while the third equality holds for all $\omega\in\Omega$, $\widehat{H}^0(\omega,\widehat{Y}_0)$ is the critical manifold as (3.26), and $\widehat{H}^1(\omega,\widehat{Y}_0)$ is the first order manifold as (3.35).

We now go back to investigate the approximation of the random invariant manifold for SPDEs (1.1)–(1.2). Recall the transforms T and T^{-1} defined in (2.6) and (2.7). Let $\widetilde{M}^\varepsilon(\omega) := T^{-1}(\omega, M^\varepsilon(\omega))$, $\widetilde{Z}^\varepsilon(t,\omega,\cdot)$ be the solution of SPDEs (1.1)–(1.2) and $\bar{Z}^\varepsilon(t,\omega,\cdot)$ be the solution of RPDEs (2.4)–(2.5). By Lemma 2.8, we have

$$\begin{aligned}\widetilde{Z}^\varepsilon(t,\omega,\widetilde{M}^\varepsilon(\omega)) &= T^{-1}(\theta_t\omega, \bar{Z}^\varepsilon(t,\omega,T(\omega,\widetilde{M}^\varepsilon(\omega))))\\ &= T^{-1}(\theta_t\omega, \bar{Z}^\varepsilon(t,\omega,M^\varepsilon(\omega)))\\ &\subset T^{-1}(\theta_t\omega, M^\varepsilon(\theta_t\omega))\\ &= \widetilde{M}^\varepsilon(\theta_t\omega),\end{aligned} \tag{3.37}$$

which implies that $\widetilde{M}^\varepsilon(\omega)$ is an invariant set. Moreover, we notice that

$$\begin{aligned}\widetilde{M}^\varepsilon(\omega) &= T^{-1}(\omega, M^\varepsilon(\omega))\\ &= \{(e^{z^\varepsilon(\omega)}H^\varepsilon(\omega,\widehat{Y}_0), e^{z^\varepsilon(\omega)}\widehat{Y}_0)|\widehat{Y}_0\in H_2\}\\ &= \{(e^{z^\varepsilon(\omega)}H^\varepsilon(\omega,e^{-z^\varepsilon(\omega)}\widehat{Y}_0), \widehat{Y}_0)|\widehat{Y}_0\in H_2\}\\ &= \{(\widetilde{H}^\varepsilon(\omega,\widehat{Y}_0), \widehat{Y}_0)|\widehat{Y}_0\in H_2\},\end{aligned} \tag{3.38}$$

where $\widetilde{H}^\varepsilon(\omega,\widehat{Y}_0) = e^{z^\varepsilon(\omega)}H^\varepsilon(\omega,e^{-z^\varepsilon(\omega)}\widehat{Y}_0)$. Then, $\widetilde{M}^\varepsilon(\omega)$ can be represented by a graph of a Lipschitz function $\widetilde{H}^\varepsilon(\omega,\cdot)$. Therefore, $\widetilde{M}^\varepsilon(\omega)$ is the random invariant manifold for (1.1)–(1.2). With the help of Theorem 3.5, we show the approximation of the random invariant manifold for SPDEs (1.1)–(1.2) as follows.

Theorem 3.6. *Under Assumptions 1-4, for sufficiently small $\varepsilon > 0$, we obtain the approximation of the random invariant manifold for* (1.1)–(1.2) *as*

$$\begin{aligned}\widetilde{M}^\varepsilon(\omega) &= \{(\widetilde{H}^\varepsilon(\omega,\widehat{Y}_0), \widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\}\\ &= \{(e^{z^\varepsilon(\omega)}H^\varepsilon(\omega,e^{-z^\varepsilon(\omega)}\widehat{Y}_0), \widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\}\\ &\overset{d}{=} \{(e^{\eta(\omega)}\widehat{H}^\varepsilon(\omega,e^{-\eta(\omega)}\widehat{Y}_0), \widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\}\\ &= \{(e^{\eta(\omega)}\widehat{H}^0(\omega,e^{-\eta(\omega)}\widehat{Y}_0) + \varepsilon e^{\eta(\omega)}\widehat{H}^1(\omega,e^{-\eta(\omega)}\widehat{Y}_0) + \mathcal{O}(\varepsilon^2), \widehat{Y}_0)|\widehat{Y}_0\in\mathcal{D}(B)\},\end{aligned}$$

where the third equality holds in distribution while the fourth equality holds for all $\omega\in\Omega$, $\eta(\theta_T\omega)$ is the stationary solution of (3.19)*, $e^{\eta(\omega)}\widehat{H}^0(\omega,e^{-\eta(\omega)}\widehat{Y}_0)$ is the critical manifold, and $e^{\eta(\omega)}\widehat{H}^1(\omega,e^{-\eta(\omega)}\widehat{Y}_0)$ is the first order manifold.*

4 Random invariant foliations and slow foliations

In the section, we are going to show there also exist random invariant foliations for RPDEs (2.4)–(2.5), and any two orbits start from the same fiber can approach to each other as exponential rate in backward time. Then, we prove that random invariant foliations converge to slow foliations as the parameter ε tends to 0.

4.1 Random invariant foliations

For $(\bar{X}_0,\bar{Y}_0),(\check{X}_0,\check{Y}_0)\in H$, let $\widetilde{Y}^\varepsilon(0) = \check{Y}_0 - \bar{Y}_0$. Set $\widetilde{Z}^\varepsilon(t,\omega) = \bar{Z}^\varepsilon(t,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0)) = (\widetilde{X}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon(0)), \widetilde{Y}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon(0)))$. Introduce a set:

$$\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega) := \{(\check{X}_0,\check{Y}_0)\in H|\bar{Z}^\varepsilon(t,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0))\in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\}. \tag{4.1}$$

In the followings, we will prove $\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big)$ is a fiber of the random invariant foliations for (2.4)–(2.5).

Lemma 4.1. *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon>0$, we have the following results:*

(1) $(\check{X}_0,\check{Y}_0)\in\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big)$ if and only if $\widetilde{Z}^\varepsilon(\cdot)\in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$ satisfies

$$\widetilde{X}^\varepsilon(t)=\frac{1}{\varepsilon}\int_{-\infty}^{t}e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.2}$$

$$\widetilde{Y}^\varepsilon(t)=e^{Bt+\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{Y}^\varepsilon(0)+\int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.3}$$

where nonlinear functions $\widetilde{F}$ and $\widetilde{G}$ are defined in (3.12)–(3.13).

(2) There exists a unique solution $\widetilde{Z}^\varepsilon(\cdot)\in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$ for (4.2)–(4.3).

(3) Let $\widetilde{Y}^\varepsilon_1(0)$ and $\widetilde{Y}^\varepsilon_2(0)\in H_2$. Then

$$\begin{aligned}&\|\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon_1(0))-\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon_2(0))\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\&\quad\le\frac{1}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\widetilde{Y}^\varepsilon_1(0)-\widetilde{Y}^\varepsilon_2(0)\|_2,\end{aligned}$$

where

$$\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)=\frac{2K}{\gamma_2}+\frac{2K}{2\gamma_1+\varepsilon\gamma_2}. \tag{4.4}$$

Proof. The proof of (i) follows from the variation of constants formula. With the help of Banach's Fixed Point Theorem, we can prove (2). Using the same techniques as in the proof of Theorem 3.3, we can obtain (3). □

For $\zeta\in H_2$, we define

$$\begin{aligned}l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega):=\bar{X}_0+\frac{1}{\varepsilon}\int_{-\infty}^{0}e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r^\varepsilon\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}\Big(&\widetilde{X}^\varepsilon(s,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{Y}_0),\\&\widetilde{Y}^\varepsilon(s,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{Y}_0),\theta_s\omega\Big)\mathrm{d}s.\end{aligned} \tag{4.5}$$

By (4.5), we can show the existence of random invariant foliations for (2.4)–(2.5) as follows.

Theorem 4.2 (Existence of random invariant foliations)**.** *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon>0$, we have the following results:*

(1) (2.4)–(2.5) has a random invariant foliation, whose each fiber is represented as

$$\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big)=\big\{\big(l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega),\zeta\big)|\zeta\in H_2\big\}, \tag{4.6}$$

where $l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega)$ is defined in (4.5).

(2) l^ε is a Lipschitz mapping with respect to ζ, whose Lipschitz constant $\mathrm{Lip}\,l^\varepsilon(\cdot,\omega)$ satisfies that

$$\mathrm{Lip}\,l^\varepsilon(\cdot,\omega)\le\frac{2K}{(\varepsilon\gamma_2+2\gamma_1)[1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)]},$$

with $\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)$ given in (4.4).

Proof. (1) According to (4.2)–(4.3), we obtain

$$\begin{aligned}\check{X}_0-\bar{X}_0&=\frac{1}{\varepsilon}\int_{-\infty}^{0}e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X},\widetilde{Y},\theta_s\omega)\mathrm{d}s\\&=\frac{1}{\varepsilon}\int_{-\infty}^{0}e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X}(s,\omega,(\bar{X}_0,\bar{Y}_0);(\check{Y}_0-\bar{Y}_0)),\\&\qquad\widetilde{Y}(s,\omega,(\bar{X}_0,\bar{Y}_0);(\check{Y}_0-\bar{Y}_0)),\theta_s\omega)\mathrm{d}s.\end{aligned}$$

Then, replacing $\check{Y}_0$ with ζ, we have (4.6) by (4.1), (4.5) and Lemma 4.1.

We proceed to verify that each fiber is invariant. Let $(\check{X}_0,\check{Y}_0)\in\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega)$. Since $\bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0,\check{Y}_0))-\bar{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0))\in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$, we have $\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\check{X}_0,\check{Y}_0))-\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\bar{X}_0,\bar{Y}_0))\in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$. Then, the cocycle property leads to

$$\begin{aligned}\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\check{X}_0,\check{Y}_0))&=\bar{Z}^\varepsilon(\cdot,\theta_\tau\omega,\bar{Z}^\varepsilon(\tau,\omega,(\check{X}_0,\check{Y}_0))),\\\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\bar{X}_0,\bar{Y}_0))&=\bar{Z}^\varepsilon(\cdot,\theta_\tau\omega,\bar{Z}^\varepsilon(\tau,\omega,(\bar{X}_0,\bar{Y}_0))),\end{aligned}$$

which can yield that $\bar{Z}^\varepsilon(\tau,\omega,(\check{X}_0,\check{Y}_0))\in\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}(\bar{Z}^\varepsilon(\tau,\omega,(\bar{X}_0,\bar{Y}_0)),\theta_\tau\omega)$.

(2) Let ζ and $\bar{\zeta}\in H_2$. Then

$$\begin{aligned}&\|l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0))-l^\varepsilon(\bar{\zeta},(\bar{X}_0,\bar{Y}_0))\|_1\\&\quad\le\|\widetilde{X}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{X}_0)-\widetilde{X}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\bar{\zeta}-\bar{X}_0)\|_{C^{1,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\&\quad\le\frac{2K}{\varepsilon\gamma_2+2\gamma_1}\|\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{X}_0)-\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\bar{\zeta}-\bar{X}_0)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\&\quad\le\frac{2K}{(\varepsilon\gamma_2+2\gamma_1)[1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)]}\|\zeta-\bar{\zeta}\|_2,\end{aligned}$$

where the second inequality is from direct calculation and the last one is from Lemma 4.1. □

Theorem 4.3 (Exponential tracking property in backward time)**.** *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon>0$, any two points $(\check{X}_0^1,\check{Y}_0^1)$ and $(\check{X}_0^2,\check{Y}_0^2)$ in a same fiber $\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega)$, we have*

$$\|\bar{Z}^\varepsilon(t,\omega,(\check{X}_0^1,\check{Y}_0^1))-\bar{Z}^\varepsilon(t,\omega,(\check{X}_0^2,\check{Y}_0^2))\|\le\frac{e^{\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}+\frac{\gamma_2 t}{2}}}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\check{Y}_0^1-\check{Y}_0^2\|_2\tag{4.7}$$

with $t\le 0$.

Proof. Let

$$\begin{aligned}\widetilde{Z}_1^\varepsilon(t)&=\bar{Z}^\varepsilon(t,\omega,(\check{X}_0^1,\check{Y}_0^1))-\bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0)),\\\widetilde{Z}_2^\varepsilon(t)&=\bar{Z}^\varepsilon(t,\omega,(\check{X}_0^2,\check{Y}_0^2))-\bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0)).\end{aligned}$$

Applying Lemma 4.1, we know that

$$\begin{aligned}
&\|\bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0^1,\check{Y}_0^1))-\bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0^2,\check{Y}_0^2))\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\quad=\|\widetilde{Z}_1^\varepsilon(\cdot)-\widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\quad\le\frac{1}{\varepsilon}\|\int_{-\infty}^{\cdot}e^{\frac{A(\cdot-s)+\int_s^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}[\widetilde{F}(\widetilde{X}_1^\varepsilon,\widetilde{Y}_1^\varepsilon,\theta_s\omega)-\widetilde{F}(\widetilde{X}_2^\varepsilon,\widetilde{Y}_2^\varepsilon,\theta_s\omega)]\mathrm{d}s\|_{C^{1,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\qquad+\|e^{B\cdot+\frac{\int_0^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}(\check{Y}_0^1-\check{Y}_0^2)\|_{C^{2,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\qquad+\|\int_0^{\cdot}e^{B(\cdot-s)+\frac{\int_s^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}[\widetilde{G}(\widetilde{X}_1^\varepsilon,\widetilde{Y}_1^\varepsilon,\theta_s\omega)-\widetilde{G}(\widetilde{X}_2^\varepsilon,\widetilde{Y}_2^\varepsilon,\theta_s\omega)]\mathrm{d}s\|_{C^{2,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\quad\le\|\check{Y}_0^1-\check{Y}_0^2\|_2+\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)\|\widetilde{Z}_1^\varepsilon(\cdot)-\widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}$$

For sufficiently small $\varepsilon>0$, we have

$$\|\widetilde{Z}_1^\varepsilon(\cdot)-\widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\le\frac{1}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\check{Y}_0^1-\check{Y}_0^2\|_2,$$

which implies (4.7). □

4.2 Slow foliations

The motivation of this subsection is to investigate the approximation of the random invariant foliations for RPDEs (2.4)–(2.5) in slow time-scale $T=\frac{t}{\varepsilon}$. As the arguments in Subsection 3.2, we will study the approximation of the random invariant foliations for RPDEs (2.4)–(2.5) via (3.20)–(3.21).

Let $\widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0))$ and $\widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$ be the solutions of (3.20)–(3.21) with initial data $(\check{X}_0,\check{Y}_0)$ and $(\widehat{X}_0,\widehat{Y}_0)$, respectively. Set $\widetilde{Z}^\varepsilon(T,\omega)=\widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0))-\widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$, $\widetilde{Y}^\varepsilon(0)=\check{Y}_0-\widehat{Y}_0$. According to the variation of constants formula, we state $\widetilde{Z}^\varepsilon(\cdot,\omega)=(\widetilde{X}^\varepsilon(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^\varepsilon(0)),\widetilde{Y}^\varepsilon(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^\varepsilon(0)))\in C^-_{\frac{\gamma_2}{2},\eta}$ if and only if $\widetilde{Z}^\varepsilon(T,\omega)$ satisfies

$$\widetilde{X}^\varepsilon(T)=\int_{-\infty}^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s,\tag{4.8}$$

$$\widetilde{Y}^\varepsilon(T)=e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widetilde{Y}^\varepsilon(0)+\varepsilon\int_0^{T}e^{\varepsilon B(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\mathring{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s,\tag{4.9}$$

where

$$\begin{aligned}
\mathring{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)&=\widehat{F}(\widetilde{X}^\varepsilon+\widehat{X}_2^\varepsilon,\widetilde{Y}^\varepsilon+\widehat{Y}_2^\varepsilon,\theta_s\omega)-\widehat{F}(\widehat{X}_2^\varepsilon,\widehat{Y}_2^\varepsilon,\theta_s\omega),\\
\mathring{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)&=\widehat{G}(\widetilde{X}^\varepsilon+\widehat{X}_2^\varepsilon,\widetilde{Y}^\varepsilon+\widehat{Y}_2^\varepsilon,\theta_s\omega)-\widehat{G}(\widehat{X}_2^\varepsilon,\widehat{Y}_2^\varepsilon,\theta_s\omega).
\end{aligned}$$

By Banach's Fixed Point Theorem, we can prove there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot)\in C^-_{\frac{\gamma_2}{2},\eta}$ of (4.8)–(4.9). Then, by the similar arguments as in Theorem 4.2, we can prove that there exists a random invariant foliation for (3.20)–(3.21), each fiber of which is represented as

$$\begin{aligned}
\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\eta}((\widehat{X}_0,\widehat{Y}_0),\omega)&=\{(\check{X}_0,\check{Y}_0)\in H|\widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0))-\widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0))\in C^-_{\frac{\gamma_2}{2},\eta}\}\\
&=\{(\widehat{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in H_2\},
\end{aligned}\tag{4.10}$$

where

$$\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega):=\widehat{X}_0+\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}\Big(\widetilde{X}^{\varepsilon}(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\\ \widetilde{Y}^{\varepsilon}(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\theta_s\omega\Big)\mathrm{d}s.$$

Using the similar discussion as (3.22), we derive

$$\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)\overset{d}{=}l^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega). \tag{4.11}$$

Based on (4.11), we turn to the study of the approximation of $\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)$. Taking into account the critical system (3.23)–(3.24). Let $\widehat{Z}_1^0(T,\omega,(\check{X}_0,\check{Y}_0))$ and $\widehat{Z}_2^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$ be the solutions of (3.23)–(3.24) with initial data $(\check{X}_0,\check{Y}_0)$ and $(\widehat{X}_0,\widehat{Y}_0)$, respectively. Set $\widetilde{Z}^0(T,\omega)=\widehat{Z}_1^0(T,\omega,(\check{X}_0,\check{Y}_0))-\widehat{Z}_2^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$, $\widetilde{Y}^0(0)=\check{Y}_0-\widehat{Y}_0$.

Clearly, $\widetilde{Z}^0(\cdot,\omega)=(\widetilde{X}^0(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^0(0)),\widetilde{Y}^0(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^0(0)))\in C^{-}_{\frac{\gamma_2}{2},\eta}$ if and only if $\widetilde{Z}^0(T,\omega)$ satisfies

$$\widetilde{X}^0(T)=\int_{-\infty}^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}(\widetilde{X}^0,\widetilde{Y}^0,\theta_s\omega)\mathrm{d}s, \tag{4.12}$$

$$\widetilde{Y}^0(T)=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widetilde{Y}^0(0), \tag{4.13}$$

where

$$\mathring{F}(\widetilde{X}^0,\widetilde{Y}^0,\theta_T\omega)=\widehat{F}(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)-\widehat{F}(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega).$$

Furthermore, we claim that (3.23)–(3.24) has a random invariant foliation, whose each fiber is represented as

$$\mathcal{W}^0_{\frac{\gamma_2}{2},\eta}((\widehat{X}_0,\widehat{Y}_0),\omega)=\{(\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in H_2\}, \tag{4.14}$$

where

$$\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega):=\widehat{X}_0+\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}\Big(\widetilde{X}^0(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\\ \widetilde{Y}^0(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\theta_s\omega\Big)\mathrm{d}s. \tag{4.15}$$

Inspired by the technique from Theorem 5.2 in [12], we are going to prove that the random invariant foliation for (3.20)–(3.21) converges to that for (3.23)–(3.24) as $\varepsilon\to 0$.

Theorem 4.4. *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon>0$, we have*

$$\hat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)=\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\mathcal{O}(\varepsilon),$$

for all $\hat{X}_0\in H_1$, $\widehat{Y}_0,\zeta\in\mathcal{D}(B)$, $\omega\in\Omega$.

Proof. Due to the representations of $\widehat{l}^{\varepsilon}$ and $\widehat{l}^0$, we obtain

$$\|\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)-\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)\|_1\\ =\|\widetilde{X}^{\varepsilon}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))-\widetilde{X}^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))\|_1|_{T=0}.$$

Hence, if we can estimate the error between $\widetilde{X}^{\varepsilon}(T)$ and $\widetilde{X}^{0}(T)$, the proof will be done. For $T \leq 0$, we have

$$\begin{aligned}
&\|\widetilde{X}^{\varepsilon}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0)) - \widetilde{X}^{0}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))\|_1 \\
&\quad= \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\mathring{F}(\widetilde{X}^{\varepsilon}(s),\widetilde{Y}^{\varepsilon}(s),\theta_s\omega) - \mathring{F}(\widetilde{X}^{0}(s),\widetilde{Y}^{0}(s),\theta_s\omega)]\mathrm{d}s \\
&\quad= \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}(\widetilde{X}^{\varepsilon}(s)+\widehat{X}_2^{\varepsilon}(s),\widetilde{Y}^{\varepsilon}(s)+\widehat{Y}_2^{\varepsilon}(s),\theta_s\omega) \\
&\qquad- \widehat{F}(\widetilde{X}^{0}(s)+\widehat{X}_2^{0}(s),\widetilde{Y}^{0}(s)+\widehat{Y}_2^{0}(s),\theta_s\omega) + \widehat{F}(\widehat{X}_2^{0}(s),\widehat{Y}_2^{0}(s),\theta_s\omega) \\
&\qquad- \widehat{F}(\widehat{X}_2^{\varepsilon}(s),\widehat{Y}_2^{\varepsilon}(s),\theta_s\omega)]\mathrm{d}s \\
&\quad\leq K\int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\|\widetilde{X}^{\varepsilon}(s)-\widetilde{X}^{0}(s)\|_1 + \|\widetilde{Y}^{\varepsilon}(s)-\widetilde{Y}^{0}(s)\|_2 \\
&\qquad+ 2\|\widehat{X}_2^{\varepsilon}(s)-\widehat{X}_2^{0}(s)\|_1 + 2\|\widehat{Y}_2^{\varepsilon}(s)-\widehat{Y}_2^{0}(s)\|_2]\mathrm{d}s.
\end{aligned}$$

In order to show the bounds of $\|\widetilde{X}^{\varepsilon}(T)-\widetilde{X}^{0}(T)\|_1$, we need the estimates of $\|\widetilde{Y}^{\varepsilon}(T)-\widetilde{Y}^{0}(T)\|_2$, $\|\widehat{X}_2^{\varepsilon}(T)-\widehat{X}_2^{0}(T)\|_1$ and $\|\widehat{Y}_2^{\varepsilon}(T)-\widehat{Y}_2^{0}(T)\|_2$, respectively. Choose $\bar{\mu}$ satisfying $\bar{\mu} \in (0, \frac{2\gamma_1^2}{\gamma_2+2\gamma_1}]$, which implies $K < \gamma_1 - \bar{\mu}$. Similar to the deduction in (3.27), we derive

$$\|\widehat{Y}_2^{\varepsilon}(T)-\widehat{Y}_2^{0}(T)\|_2 \leq e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2} + e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\widehat{\rho}}\frac{e^{-\bar{\mu}T}-e^{\varepsilon\gamma_2 T}}{\bar{\mu}+\varepsilon\gamma_2},$$

and

$$\|\widetilde{Y}^{\varepsilon}(T)-\widetilde{Y}^{0}(T)\|_2 \leq e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\|B(\zeta-\widehat{Y}_0)\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2} + e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\zeta-\widehat{Y}_0\|_2}{1-\widehat{\rho}}\frac{e^{-\bar{\mu}T}-e^{\varepsilon\gamma_2 T}}{\bar{\mu}+\varepsilon\gamma_2},$$

where

$$\widehat{\rho}(K,\bar{\mu},\gamma_1,\gamma_2,\varepsilon) = \frac{K}{\gamma_1-\bar{\mu}} + \frac{\varepsilon K}{\bar{\mu}+\varepsilon\gamma_2}.$$

Analogous argument as (3.28) yields

$$\|\widehat{X}_2^{\varepsilon}(\cdot)-\widehat{X}_2^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} \leq \frac{(\gamma_1-\bar{\mu})\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K},$$

where

$$\begin{aligned}
\widehat{\mathcal{R}} &= \sup_{T\leq 0}\left\{Ke^{\bar{\mu}T}\left(\frac{\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2} - \frac{\|B\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2} - \frac{\varepsilon K\|\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\right)\right. \\
&\qquad\left. + \frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1-\bar{\mu})}\right\} \\
&= \mathcal{O}(\varepsilon).
\end{aligned}$$

Furthermore, combining the above estimates, we obtain that

$$\begin{aligned}
\|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} &\leq K\int_{-\infty}^{T} e^{(-\gamma_1+\bar{\mu})(T-s)}(2\|\widehat{X}_2^{\varepsilon}(\cdot)-\widehat{X}_2^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} + \|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}})\mathrm{d}s \\
&\quad + K\int_{-\infty}^{T} e^{-\gamma_1(T-s)+\bar{\mu}T+\int_s^0 \eta(\theta_r\omega)\mathrm{d}r}(2\|\widehat{Y}_2^{\varepsilon}-\widehat{Y}_2^{0}\|_2 + \|\widetilde{Y}^{\varepsilon}-\widetilde{Y}^{0}\|_2)\mathrm{d}s \\
&\leq \frac{K}{\gamma_1-\bar{\mu}}\|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} + \frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K} + 2\widehat{\mathcal{R}} + \widetilde{\mathcal{R}},
\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathcal{R}} = \sup_{T\leq 0}\Big\{ Ke^{\bar{\mu}T}\big(&\frac{\|B(\zeta-\widehat{Y}_0)\|_2}{\gamma_1\gamma_2} - \frac{\|B(\zeta-\widehat{Y}_0)\|_2 e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2} \\ &- \frac{\varepsilon K\|\zeta-\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\big) + \frac{\varepsilon K^2\|\zeta-\widehat{Y}_0\|_2}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1-\bar{\mu})}\Big\} \\ &= \mathcal{O}(\varepsilon).\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}\|\widetilde{X}^\varepsilon(\cdot)-\widetilde{X}^0(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} &\leq \frac{\frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K}+2\widehat{\mathcal{R}}+\widetilde{\mathcal{R}}}{1-\frac{K}{\gamma_1-\bar{\mu}}} \\ &= \mathcal{O}(\varepsilon),\end{aligned}$$

which implies

$$\|\widetilde{X}^\varepsilon(T)-\widetilde{X}^0(T)\|_1 \leq \frac{\frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K}+2\widehat{\mathcal{R}}+\widetilde{\mathcal{R}}}{1-\frac{K}{\gamma_1-\bar{\mu}}} e^{-\bar{\mu}T+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}.$$

Our proof is completed. □

In the following, we study the first order approximation of the random invariant foliation for (3.20)–(3.21). Let $\widehat{Z}^1(T,\omega,(\check{X}_0^1,\check{Y}_0^1))$ and $\widehat{Z}^1(T,\omega,(\widehat{X}_0^1,\widehat{Y}_0^1))$ be the solutions of (3.31)–(3.32) with initial data $(\check{X}_0^1,\check{Y}_0^1)$ and $(\widehat{X}_0^1,\widehat{Y}_0^1)$, respectively. Set $\widetilde{Z}^1(T,\omega)=\widehat{Z}_1^1(T,\omega,(\check{X}_0^1,\check{Y}_0^1))-\widehat{Z}_2^1(T,\omega,(\widehat{X}_0^1,\widehat{Y}_0^1))$.

According to Assumption 3, we know $\widehat{F}(x,y)$ has the partial derivatives. Expanding $\widetilde{Z}^\varepsilon(T,\omega)=(\widetilde{X}^\varepsilon(T,\omega),\widetilde{Y}^\varepsilon(T,\omega))$ with respect to ε, we have $\widetilde{Z}^\varepsilon(T,\omega)=\widetilde{Z}^0(T,\omega)+\varepsilon\widetilde{Z}^1(T,\omega)+\mathcal{O}(\varepsilon^2)$, where $\widetilde{Z}^0(T,\omega)$ satisfies

$$\begin{aligned}\mathrm{d}\widetilde{X}^0 &= [A\widetilde{X}^0+\eta(\theta_T\omega)\widetilde{X}^0+\widehat{F}(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)-\widehat{F}(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega)]\mathrm{d}T,\\ \mathrm{d}\widetilde{Y}^0 &= \eta(\theta_T\omega)\widetilde{Y}^0\mathrm{d}T,\end{aligned}$$

and $\widetilde{Z}^1(T,\omega)$ satisfies

$$\begin{aligned}\mathrm{d}\widetilde{X}^1 = [&A\widetilde{X}^1+\eta(\theta_T\omega)\widetilde{X}^1+\widehat{F}_x(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)(\widetilde{X}^1+\widehat{X}_2^1)\\ &+\widehat{F}_y(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)(\widetilde{Y}^1+\widehat{Y}_2^1)-\widehat{F}_x(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega)\widehat{X}_2^1\\ &-\widehat{F}_y(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega)\widehat{Y}_2^1]\mathrm{d}T,\\ \mathrm{d}\widetilde{Y}^1 = [&B\widetilde{Y}^0+\eta(\theta_T\omega)\widetilde{Y}^1+\widehat{G}(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)-\widehat{G}(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega)]\mathrm{d}T.\end{aligned}$$

The critical foliation for (3.23)–(3.24) has been presented in (4.14). We proceed to investigate the first order of the random invariant foliations for (3.23)–(3.24). Consider

$$\begin{aligned}\widetilde{X}^1(T) = \int_{-\infty}^T e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}&[\widehat{F}_x(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_s\omega)(\widetilde{X}^1+\widehat{X}_2^1)\\ &+\widehat{F}_y(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_s\omega)(\widetilde{Y}^1+\widehat{Y}_2^1)-\widehat{F}_x(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_s\omega)\widehat{X}_2^1\\ &-\widehat{F}_y(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_s\omega)\widehat{Y}_2^1]\mathrm{d}s, \qquad (4.16)\\ \widetilde{Y}^1(T) = \int_0^T e^{\int_s^T\eta(\theta_r\omega)\mathrm{d}r}&[B\widetilde{Y}^0+\widehat{G}(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_s\omega)\\ &-\widehat{G}(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_s\omega)]\mathrm{d}s, \qquad (4.17)\end{aligned}$$

where $\widetilde{X}^0(T)$ and $\widetilde{Y}^0(T)$ are from (4.12)–(4.13). By Banach's Fixed Point Theorem, we know that there exists a unique solution $(\widetilde{X}^1(\cdot), \widetilde{Y}^1(\cdot))$ of (4.16)–(4.17) in $C^-_{\frac{\lambda_2}{2},\eta}$. Similar to the arguments as in Theorem 4.2, we can obtain the first order approximation of the random invariant foliation for (3.20)–(3.21). Fixing $\widehat{X}_0 \in H_1$, $\widehat{Y}_0 \in \mathcal{D}(B)$, define

$$\begin{aligned}\widehat{l}^1(\check{Y}_0,(\widehat{X}_0,\widehat{Y}_0),\omega) := \int_{-\infty}^0 e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[&\widehat{F}_x(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_s\omega)(\widetilde{X}^1+\widehat{X}_2^1)\\ &+\widehat{F}_y(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_s\omega)(\widetilde{Y}^1+\widehat{Y}_2^1)-\widehat{F}_x(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_s\omega)\widehat{X}_2^1\\ &-\widehat{F}_y(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_s\omega)\widehat{Y}_2^1]\mathrm{d}s. \end{aligned} \tag{4.18}$$

Similar to (3.36), we formally obtain

$$\widehat{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)=\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\widehat{l}^1(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\mathcal{O}(\varepsilon^2).$$

Ultimately, we have the following theorem.

Theorem 4.5 (First order approximation of slow foliation)**.** *Under Assumptions 1, 2, 3, 5, for sufficiently small $\varepsilon>0$, we obtain the approximation fo the random invariant foliation for* as

$$\begin{aligned}\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0,\widehat{Y}_0),\omega)&=\{(l^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in\mathcal{D}(B)\}\\ &\overset{d}{=}\{(\widehat{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in\mathcal{D}(B)\}\\ &=\{(\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\varepsilon\widehat{l}^1(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\mathcal{O}(\varepsilon^2),\zeta)|\zeta\in\mathcal{D}(B)\},\end{aligned}$$

where $\widehat{X}_0\in H_1$, $\widehat{Y}_0\in\mathcal{D}(B)$, the second equality holds in distribution while the third equality holds for all $\omega\in\Omega$, $\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)$ is the critical foliation as (4.15), and $\widehat{l}^1(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)$ is the first order foliation as (4.18).

We are going to study the the approximation fo the random invariant foliation for SPDEs (1.1)–(1.2) in the followings. Recall the transforms T and T^{-1} defined in (2.6) and (2.7). Let $(\widehat{X}_0,\widehat{Y}_0)\in H$. By Lemma 2.8 and the similar arguments as in (3.19)–(3.37), we can obtain a random invariant foliation for SPDEs (1.1)–(1.2), each fiber of which is represented as

$$\begin{aligned}\widetilde{\mathcal{W}}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0,\widehat{Y}_0),\omega)&=T^{-1}\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((T\widehat{X}_0,T\widehat{Y}_0),\omega)\\ &=\{(e^{z^\varepsilon(\omega)}l^\varepsilon(\zeta,(e^{-z^\varepsilon(\omega)}\widehat{X}_0,e^{-z^\varepsilon(\omega)}\widehat{Y}_0),\omega),e^{z^\varepsilon(\omega)}\zeta)|\zeta\in H_2\}\\ &=\{(\widetilde{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in H_2\},\end{aligned}$$

where l^ε is defined in (4.5), and $\widetilde{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)=e^{z^\varepsilon(\omega)}l^\varepsilon(e^{-z^\varepsilon(\omega)}\zeta,(e^{-z^\varepsilon(\omega)}\widehat{X}_0,e^{-z^\varepsilon(\omega)}\widehat{Y}_0),\omega)$. By Theorem 4.5, we can obtain the first order approximation of the random invariant foliation for SPDEs (1.1)–(1.2) as follows.

Theorem 4.6. *Under Assumptions 1, 2, 3, 5, for sufficiently small $\varepsilon>0$, we obtain the approximation of the random invariant foliation for* (1.1)–(1.2) *as*

$$\begin{aligned}&\widetilde{\mathcal{W}}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0,\widehat{Y}_0),\omega)\\ &\quad=\{(\widetilde{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in\mathcal{D}(B)\}\\ &\quad=\{(e^{z^\varepsilon(\omega)}l^\varepsilon(\zeta,(e^{-z^\varepsilon(\omega)}\widehat{X}_0,e^{-z^\varepsilon(\omega)}\widehat{Y}_0),\omega),e^{z^\varepsilon(\omega)}\zeta)|\zeta\in\mathcal{D}(B)\}\\ &\quad\overset{d}{=}\{(e^{\eta(\omega)}\widehat{l}^\varepsilon(\zeta,(e^{-\eta(\omega)}\widehat{X}_0,e^{-\eta(\omega)}\widehat{Y}_0),\omega),e^{\eta(\omega)}\zeta)|\zeta\in\mathcal{D}(B)\}\\ &\quad=\{(e^{\eta(\omega)}\widehat{l}^0(\zeta,(e^{-\eta(\omega)}\widehat{X}_0,e^{-\eta(\omega)}\widehat{Y}_0),\omega)+\varepsilon e^{\eta(\omega)}\widehat{l}^1(\zeta,(e^{-\eta(\omega)}\widehat{X}_0,e^{-\eta(\omega)}\widehat{Y}_0),\omega)\\ &\qquad+\mathcal{O}(\varepsilon^2),e^{\eta(\omega)}\zeta)|\zeta\in\mathcal{D}(B)\},\end{aligned}$$

where $\widehat{X}_0 \in H_1$, $\widehat{Y}_0 \in \mathcal{D}(B)$, *the third equality holds in distribution while the fourth equality holds for all* $\omega \in \Omega$, $e^{\eta(\omega)}\widehat{l}^0(\zeta, (e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0), \omega)$ *is the critical foliation, and* $e^{\eta(\omega)}\widehat{l}^1(\zeta, (e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0), \omega)$ *is the first order foliation.*

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