Asymptotic behavior of solutions to the multidimensional semidiscrete diffusion equation

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Abstract. We study the asymptotic behavior of solutions to the multidimensional diffusion (heat) equation with continuous time and discrete space. We focus on initial-value problems with bounded initial data, and provide sufficient conditions for the existence of pointwise and uniform limits of solutions.

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1 Introduction

In the present paper, we are concerned with the n-dimensional diffusion (heat) equation with continuous time and discrete space, i.e., with the equation

$$\frac{\partial u}{\partial t}(x,t) = a\left(\sum_{i=1}^{n} u(x+e_i,t) - 2nu(x,t) + \sum_{i=1}^{n} u(x-e_i,t)\right), \quad x \in \mathbb{Z}^n, \quad t \geq 0, \qquad (1.1)$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$, the constant $a > 0$ is the diffusion strength, and the terms inside the parentheses represent the n-dimensional discrete Laplace operator. The study of Eq. (1.1) is meaningful not only from the viewpoint of numerical mathematics, but the equation is of independent interest; for example, it describes the continuous-time symmetric random walk on $\mathbb{Z}^n$, with a being the intensity of transitions between two neighboring lattice points in $\mathbb{Z}^n$. In this case, the value $u(x,t)$ is the probability that the random walk visits point $x \in \mathbb{Z}^n$ at time $t \geq 0$.

We impose the initial condition

$$u(x,0) = c_x, \quad x \in \mathbb{Z}^n, \qquad (1.2)$$

where $\{c_x\}_{x \in \mathbb{Z}^n}$ is a collection of real numbers such that $|c_x| \leq M$ for a certain $M \geq 0$ and all $x \in \mathbb{Z}^n$, i.e., $\{c_x\}_{x \in \mathbb{Z}^n} \in \ell^\infty(\mathbb{Z}^n)$. We refer to $\{c_x\}_{x \in \mathbb{Z}^n}$ as a bounded array of real numbers, and

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occasionally write $c_{x_1,\dots,x_n}$ instead of c_x if we need to refer to the components of x. According to [13, Section 5.2], the problem (1.1)–(1.2) has a unique bounded solution, which can be expressed in terms of the modified Bessel functions as follows (see also [9, Section 6] for closely related results):

$$u(x,t) = e^{-2ant} \sum_{k \in \mathbb{Z}^n} c_k I_{x_1-k_1}(2at) \cdots I_{x_n-k_n}(2at), \quad x \in \mathbb{Z}^n, \quad t \geq 0. \tag{1.3}$$

Note that the series is absolutely convergent, because the modified Bessel functions are non-negative, and if we replace c_k with $|c_k|$, the series represents the solution of the problem with initial values $\{|c_k|\}_{k \in \mathbb{Z}^n}$.

Our goal is to investigate the asymptotic behavior of solutions, focusing on the pointwise limits $\lim_{t \to \infty} u(x,t)$ and the question whether they are uniform with respect to $x \in \mathbb{Z}^n$.

The asymptotic behavior of the classical diffusion equation with continuous time and space was studied in numerous papers, see e.g. [2, 7, 11, 15] and the references therein. The one-dimensional semidiscrete case, i.e., the equation

$$\frac{\partial u}{\partial t}(x,t) = a\left(u(x+1,t) - 2u(x,t) + u(x-1,t)\right), \quad x \in \mathbb{Z}, \quad t \geq 0,$$

was treated in [12], where it was shown that the solution converges to the average of the initial values, provided that the average exists. Generalization of the results from [12] to the multidimensional case is not completely straightforward, and we believe it will be of interest to the readers, also in view of the recent popularity of semidiscrete evolution equations (including those with fractional derivatives), see e.g. [1, 3–6, 8, 14], and the references therein.

2   Main results

Recall that I_k, $k \in \mathbb{Z}$, denotes the modified Bessel function of the first kind of order k. Throughout the paper, we use only a few basic properties of modified Bessel functions, all of which can be found e.g. in the online handbook [10]. Thus, the exposition is accessible also to readers with no prior knowledge of Bessel functions.

Our first goal is to transform the formula (1.3) into an alternative formula, which shows the dependence of the solution on sums (or averages) of initial values. The following statement corresponds to Lemma 2.1 from [12], where it was derived using summation by parts.

Lemma 2.1. *Let $\{c_k\}_{k \in \mathbb{Z}}$ be an arbitrary real sequence. Then for each $N \in \mathbb{N}$ and $t \geq 0$, we have*

$$\sum_{k=-N+1}^{N} c_k I_k(t) = \sum_{k=0}^{N-1} (I_k(t) - I_{k+1}(t)) \sum_{l=-k}^{k} c_l + I_N(t) \sum_{k=-N+1}^{N} c_k.$$

We need the multidimensional version of Lemma 2.1, which reads as follows.

Lemma 2.2. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k \in \mathbb{Z}^n}$ be an array of real numbers. Then for each $N \in \mathbb{N}$ and $t \geq 0$, we have*

$$\sum_{k_1,\dots,k_n=-N+1}^{N} c_{k_1,\dots,k_n} I_{k_1}(t) \cdots I_{k_n}(t) = \sum_{k_1,\dots,k_n=0}^{N-1} \prod_{j=1}^{n}(I_{k_j}(t) - I_{k_j+1}(t)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\dots,l_n}$$

$$+ I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\dots,k_n=0}^{N-1} \sum_{k_1,\dots,k_j=-N+1}^{N} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n}(I_{k_i}(t) - I_{k_i+1}(t)) \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} c_{k_1,\dots,k_j,l_{j+1},\dots,l_n}.$$

Proof. We use induction with respect to n. For $n = 1$, the statement reduces to Lemma 2.1. Suppose next that the statement holds for $n \in \mathbb{N}$ and let us show that it holds for $n+1$. Using Lemma 2.1, we get

$$
\sum_{k_1,\ldots,k_{n+1}=-N+1}^{N} c_{k_1,\ldots,k_{n+1}} I_{k_1}(t) \cdots I_{k_{n+1}}(t) = \sum_{k_1,\ldots,k_n=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) \sum_{k_{n+1}=-N+1}^{N} I_{k_{n+1}}(t) c_{k_1,\ldots,k_{n+1}}
$$

$$
= \sum_{k_1,\ldots,k_n=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) \left(\sum_{k_{n+1}=0}^{N-1} (I_{k_{n+1}}(t) - I_{k_{n+1}+1}(t)) \sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{k_1,\ldots,k_n,l_{n+1}} + I_N(t) \sum_{k_{n+1}=-N+1}^{N} c_{k_1,\ldots,k_{n+1}} \right)
$$

$$
= \sum_{k_{n+1}=0}^{N} \sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} (I_{k_{n+1}}(t) - I_{k_{n+1}+1}(t)) \sum_{k_1,\ldots,k_n=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) c_{k_1,\ldots,k_n,l_{n+1}}
$$

$$
+ I_N(t) \sum_{k_1,\ldots,k_{n+1}=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) c_{k_1,\ldots,k_{n+1}}.
$$

Using the induction hypothesis to rewrite the inner sum in the first term on the right-hand side, we get

$$
\sum_{k_{n+1}=0}^{N} \sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} (I_{k_{n+1}}(t) - I_{k_{n+1}+1}(t)) \left(\sum_{k_1,\ldots,k_n=0}^{N-1} \prod_{j=1}^{n} (I_{k_j}(t) - I_{k_j+1}(t)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\ldots,l_{n+1}} \right.
$$

$$
\left. + I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\ldots,k_n=0}^{N-1} \sum_{k_1,\ldots,k_j=-N+1}^{N} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} c_{k_1,\ldots,k_j,l_{j+1},\ldots,l_{n+1}} \right)
$$

$$
+ I_N(t) \sum_{k_1,\ldots,k_{n+1}=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) c_{k_1,\ldots,k_{n+1}}.
$$

Expanding the product inside the first term and performing some elementary manipulations, we get:

$$
\sum_{k_1,\ldots,k_{n+1}=0}^{N-1} \prod_{j=1}^{n+1} (I_{k_j}(t) - I_{k_j+1}(t)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{l_1,\ldots,l_{n+1}}
$$

$$
+ I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\ldots,k_{n+1}=0}^{N-1} \sum_{k_1,\ldots,k_j=-N+1}^{N} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n+1} (I_{k_i}(t) - I_{k_i+1}(t)) \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{k_1,\ldots,k_j,l_{j+1},\ldots,l_{n+1}}
$$

$$
+ I_N(t) \sum_{k_1,\ldots,k_{n+1}=-N+1}^{N} I_{k_1}(t) \cdots I_{k_n}(t) c_{k_1,\ldots,k_{n+1}}.
$$

This completes the proof, because the third term can be incorporated into the second term as the summand corresponding to $j = n+1$. $\qquad\square$

Proposition 2.3. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k \in \mathbb{Z}^n}$ be a bounded array of real numbers. Then the unique bounded solution of the problem (1.1)–(1.2) is given by the formula*

$$
u(x,t) = e^{-2ant} \sum_{k_1,\ldots,k_n=0}^{\infty} \left(\prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at)) \right) \left(\sum_{l_1=x_1-k_1}^{x_1+k_1} \cdots \sum_{l_n=x_n-k_n}^{x_n+k_n} c_{l_1,\ldots,l_n} \right)
$$

for all $x \in \mathbb{Z}^n$, $t \geq 0$.

Proof. It suffices to prove the statement for $x = 0$, since for a nonzero $x \in \mathbb{Z}^n$, one can consider the shifted solution satisfying shifted initial conditions (cf. the proof of Lemma 2.2 in [12]).

Using formula (1.3) and the fact that $I_{-k}(t) = I_k(t)$ for all $k \in \mathbb{Z}$ and $t \geq 0$, we have

$$u(0,t) = e^{-2ant} \sum_{k \in \mathbb{Z}^n} c_k I_{k_1}(2at) \cdots I_{k_n}(2at) = e^{-2ant} \lim_{N \to \infty} \sum_{k_1,\ldots,k_n = -N+1}^{N} c_{k_1,\ldots,k_n} I_{k_1}(2at) \cdots I_{k_n}(2at).$$

The sum can be rewritten using the formula from Lemma 2.2. But let us first observe that the second term on the right-hand of that formula tends to zero as $N \to \infty$. To see this, we perform some estimates. Let $M \geq 0$ be such that $|c_{k_1,\ldots,k_n}| \leq M$ for all $k \in \mathbb{Z}^n$. Using the fact that the modified Bessel functions are nonnegative and nonincreasing with respect to the order, we get

$$\left| \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) \right| \leq I_0(t)^{n-1}.$$

For $k_{j+1}, \ldots, k_n \in \{0, \ldots, N-1\}$, this implies that

$$\left| \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) c_{k_1,\ldots,k_j,l_{j+1},\ldots,l_n} \right| \leq (2N-1)^{n-j} I_0(t)^{n-1} M.$$

Consequently,

$$\left| I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\ldots,k_n=0}^{N-1} \sum_{k_1,\ldots,k_j=-N+1}^{N} \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) c_{k_1,\ldots,k_j,l_{j+1},\ldots,l_n} \right|$$
$$\leq I_N(t) n N^{n-j} (2N)^j (2N-1)^{n-j} I_0(t)^{n-1} M \leq I_N(t) n N^n (2N)^n I_0(t)^{n-1} M,$$

which tends to zero as $N \to \infty$, because $I_N(t) \sim \frac{1}{\sqrt{2\pi N}} \left(\frac{et}{2N} \right)^N$ for $N \to \infty$ (see formula 10.41.1 in [10]). Returning to the beginning of the proof and applying Lemma 2.2, we now see that

$$u(0,t) = e^{-2ant} \lim_{N \to \infty} \left(\sum_{k_1,\ldots,k_n=0}^{N-1} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\ldots,l_n} \right)$$
$$= e^{-2ant} \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\ldots,l_n},$$

and the statement for $x = 0$ is proved. $\square$

We need two more auxiliary lemmas to be able to prove our main result.

Lemma 2.4. *For every $n \in \mathbb{N}$ and $t \geq 0$, we have*

$$e^{-nt} \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j + 1)] = 1.$$

Proof. According to Proposition 2.3, the formula on the left-hand side corresponds to the unique bounded solution of the initial-value problem (1.1)–(1.2) with $a = 1/2$ and $c_x = 1$ for all $x \in \mathbb{Z}$. But this problem admits the constant solution $u(x,t) = 1$ for all $x \in \mathbb{Z}$ and $t \geq 0$, and therefore the equality is proved. $\square$

Lemma 2.5. *Let $n, k_0 \in \mathbb{N}$, $l \in \{1, \dots, n\}$. If $i_1, \dots, i_l, j_1, \dots, j_{n-l} \in \mathbb{N}$ are distinct integers such that $\{i_1, \dots, i_l, j_1, \dots, j_{n-l}\} = \{1, \dots, n\}$, then*

$$\lim_{t \to \infty} \sum_{k_{i_1}, \dots, k_{i_l} = 0}^{k_0 - 1} \sum_{k_{j_1}, \dots, k_{j_{n-l}} = k_0}^{\infty} e^{-nt} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j + 1)] = 0.$$

Proof. We have

$$0 \leq \sum_{k_{i_1}, \dots, k_{i_l} = 0}^{k_0 - 1} \sum_{k_{j_1}, \dots, k_{j_{n-l}} = k_0}^{\infty} e^{-nt} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j + 1)]$$

$$\leq \sum_{k_{i_1}, \dots, k_{i_l} = 0}^{k_0 - 1} e^{-lt} \prod_{m \in \{i_1, \dots, i_l\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m + 1)] \sum_{k_{j_1}, \dots, k_{j_{n-l}} = 0}^{\infty} e^{-(n-l)t} \prod_{m \in \{j_1, \dots, j_{n-l}\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m + 1)]$$

$$= \sum_{k_{i_1}, \dots, k_{i_l} = 0}^{k_0 - 1} e^{-lt} \prod_{m \in \{i_1, \dots, i_l\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m + 1)],$$

where the last equality follows from Lemma 2.4 (with n replaced by $n - l$). Because $I_k(t) \sim \frac{e^t}{\sqrt{2\pi t}}$ for $t \to \infty$ (see formula 10.30.4 in [10]), we get $\lim_{t \to \infty} e^{-t} I_k(t) = 0$ for each $k \in \mathbb{Z}$. Thus, $\lim_{t \to \infty} e^{-t}(I_k(t) - I_{k+1}(t))(2k + 1) = 0$ for each fixed $k \in \mathbb{N}_0$, which completes the proof. $\square$

Here is the main result dealing with the asymptotic behavior of solutions to the problem (1.1)–(1.2).

Theorem 2.6. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k \in \mathbb{Z}^n}$ be a bounded array of real numbers. Denote*

$$A_{k_1, \dots, k_n}(x) = \frac{1}{\prod_{j=1}^{n}(2k_j + 1)} \sum_{l_1 = x_1 - k_1}^{x_1 + k_1} \cdots \sum_{l_n = x_n - k_n}^{x_n + k_n} c_{l_1, \dots, l_n}, \quad x \in \mathbb{Z}^n, \quad k_1, \dots, k_n \in \mathbb{N}_0. \quad (2.1)$$

Then the unique bounded solution of the problem (1.1)–(1.2) has the following properties:

1. *For every $x \in \mathbb{Z}^n$,*

$$\liminf_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x) \leq \liminf_{t \to \infty} u(x, t) \leq \limsup_{t \to \infty} u(x, t) \leq \limsup_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x).$$

2. *If $x \in \mathbb{Z}^n$ and $\lim_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x) = d$, then $\lim_{t \to \infty} u(x, t) = d$.*

3. *If $\lim_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x) = d$ uniformly for all $x \in \mathbb{Z}^n$, then $\lim_{t \to \infty} u(x, t) = d$ uniformly with respect to $x \in \mathbb{Z}^n$.*

Proof. Fix an arbitrary $x \in \mathbb{Z}^n$ and denote

$$\underline{A} = \liminf_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x), \quad \overline{A} = \limsup_{k_1, \dots, k_n \to \infty} A_{k_1, \dots, k_n}(x).$$

Using Proposition 2.3, we get

$$u(x, t) = e^{-2ant} \sum_{k_1, \dots, k_n = 0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j + 1)] A_{k_1, \dots, k_n}(x). \quad (2.2)$$

Let $M > 0$ be such that $|c_l| \leq M$ for all $l \in \mathbb{Z}^n$. Then $|A_{k_1,\dots,k_n}(x)| \leq M$ for all $k_1,\dots,k_n \in \mathbb{N}_0$. Given an $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that for all $k_1,\dots,k_n \geq k_0$, we have $\underline{A} - \varepsilon < A_{k_1,\dots,k_n}(x) < \overline{A} + \varepsilon$.

From Lemma 2.4, we know that for each $t \geq 0$,

$$1 = e^{-2ant} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)].$$

We split the sum in two parts, one containing all terms with $k_1,\dots,k_n \geq k_0$, and the second one containing all remaining terms, i.e., those where $l \in \{1,\dots,n\}$ indices, say $k_{i_1},\dots,k_{i_l}$, are smaller than k_0:

$$1 = e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)]$$

$$+ e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)].$$

By Lemma 2.5, the second term tends to zero as $t \to \infty$. Thus, there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, we have

$$0 < e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] < \varepsilon,$$

$$1 - \varepsilon < e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] < 1.$$

We now use these estimates together with $|A_{k_1,\dots,k_n}(x)| \leq M$ and (2.2) to obtain

$$u(x,t) = e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] A_{k_1,\dots,k_n}(x)$$

$$+ e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] A_{k_1,\dots,k_n}(x)$$

$$< (\overline{A} + \varepsilon) e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] + M\varepsilon.$$

If $\overline{A} + \varepsilon$ is nonnegative, the first term on the right-hand side is majorized by $\overline{A} + \varepsilon$. Otherwise, if $\overline{A} + \varepsilon$ is nonpositive, the term is majorized by $(\overline{A} + \varepsilon)(1 - \varepsilon) = \overline{A} + \varepsilon - \varepsilon\overline{A} - \varepsilon^2$. In any case, we get the estimate

$$u(x,t) < \max(\overline{A} + \varepsilon, \overline{A} + \varepsilon - \varepsilon\overline{A} - \varepsilon^2) + \varepsilon M = \overline{A} + \varepsilon M + \varepsilon + \varepsilon \max(0, -\overline{A} - \varepsilon), \quad t \geq t_0.$$

This proves that $\limsup_{t\to\infty} u(x,t) \leq \overline{A}$. Similarly, we have

$$u(x,t) > (\underline{A} - \varepsilon) e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1) - M\varepsilon.$$

If $\underline{A} - \varepsilon$ is nonnegative, the first term on the right-hand side is minorized by $(\underline{A} - \varepsilon)(1 - \varepsilon) = \underline{A} - \varepsilon - \varepsilon\underline{A} + \varepsilon^2$. Otherwise, if $\underline{A} - \varepsilon$ is nonpositive, the term is minorized by $\underline{A} - \varepsilon$. In any case, we get the estimate

$$u(x, t) > -\varepsilon M + \min(\underline{A} - \varepsilon - \varepsilon\underline{A} + \varepsilon^2, \underline{A} - \varepsilon) = \underline{A} - \varepsilon M - \varepsilon + \varepsilon\min(-\underline{A} + \varepsilon, 0), \quad t \geq t_0.$$

This proves that $\liminf_{t \to \infty} u(x, t) \geq \underline{A}$.

The second statement of the theorem follows from the first one.

If $\lim_{k_1, \ldots, k_n \to \infty} A_{k_1, \ldots, k_n}(x) = d$ uniformly for all $x \in \mathbb{Z}$, then the previous estimates are independent of x, which proves the third statement. $\qquad\square$

The second part of Theorem 2.6 says that $u(x, t)$ tends to the limit of averages of initial conditions over hyperrectangles centered at x, provided that the limit exists. The next result implies that in fact, one can consider hyperrectangles centered at an arbitrary point. The reason is that if we take two sufficiently large hyperrectangles, then their intersection is large, while their symmetric difference is small. We use the notation introduced in (2.1).

Proposition 2.7. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k \in \mathbb{Z}^n}$ be a bounded array of real numbers. For every $x \in \mathbb{Z}^n$, we have $\lim_{k_1, \ldots, k_n \to \infty} A_{k_1, \ldots, k_n}(x) = \lim_{k_1, \ldots, k_n \to \infty} A_{k_1, \ldots, k_n}(0)$ whenever at least one of the limits exists.*

Proof. For each $k = (k_1, \ldots, k_n) \in (\mathbb{N}_0)^n$, let

$$S_k = \{x_1 - k_1, \ldots, x_1 + k_1\} \times \cdots \times \{x_n - k_n, \ldots, x_n + k_n\},$$
$$R_k = \{-k_1, \ldots, k_1\} \times \cdots \times \{-k_n, \ldots, k_n\}.$$

We need to show that

$$\lim_{k_1, \ldots, k_n \to \infty} \frac{1}{\prod_{j=1}^n (2k_j + 1)} \left(\sum_{(l_1, \ldots, l_n) \in S_k} c_{l_1, \ldots, l_n} - \sum_{(l_1, \ldots, l_n) \in R_k} c_{l_1, \ldots, l_n} \right) = 0.$$

Let $M > 0$ be such that $|c_l| \leq M$ for all $l \in \mathbb{Z}^n$. Then

$$\frac{1}{\prod_{j=1}^n (2k_j + 1)} \left| \sum_{(l_1, \ldots, l_n) \in S_k} c_{l_1, \ldots, l_n} - \sum_{(l_1, \ldots, l_n) \in R_k} c_{l_1, \ldots, l_n} \right| \leq \frac{M \cdot |R_k \Delta S_k|}{\prod_{j=1}^n (2k_j + 1)},$$

where $R_k \Delta S_k = (R_k \setminus S_k) \cup (S_k \setminus R_k)$ is the symmetric difference of the two hyperrectangles. Since both have the same dimensions, it follows from symmetry that $|R_k \Delta S_k| = 2|R_k \setminus S_k| = 2(|R_k| - |R_k \cap S_k|)$. The intersection $R_k \cap S_k$ is again a hyperrectangle. For each $j \in \{1, \ldots, n\}$, consider its orthogonal projection on the j-th coordinate axis. If $x_j \geq 0$ and k_j is sufficiently large, then the projection is $\{x_j - k_j, \ldots, k_j\}$. If $x_j \leq 0$ and k_j is sufficiently large, then the projection is $\{-k_j, \ldots, x_j + k_j\}$. In both cases, the projection contains $2k_j + 1 - |x_j|$ points. Thus, for sufficiently large $k_1, \ldots, k_n \in \mathbb{N}$, we have $|R_k \cap S_k| = \prod_{j=1}^n (2k_j + 1 - |x_j|)$, and

$$\frac{|R_k \Delta S_k|}{\prod_{j=1}^n (2k_j + 1)} = 2 \frac{\prod_{j=1}^n (2k_j + 1) - \prod_{j=1}^n (2k_j + 1 - |x_j|)}{\prod_{j=1}^n (2k_j + 1)}.$$

In the numerator of the last fraction, note that $\prod_{j=1}^n (2k_j + 1 - |x_j|)$ equals $\prod_{j=1}^n (2k_j + 1)$ plus $2^n - 1$ additional terms, each of which is a constant multiple of at most $n - 1$ terms of the form $2k_j + 1$. Hence, the whole fraction tends to zero when $k_1, \ldots, k_n \to \infty$, and the proof is complete. $\qquad\square$

Proposition 2.8. *If $\{c_l\}_{l \in \mathbb{Z}^n}$ is an array of real numbers such that*

$$\lim_{\max(|l_1|,\ldots,|l_n|) \to \infty} c_{l_1,\ldots,l_n} = d \in \mathbb{R},$$

then the unique bounded solution of the problem (1.1)–(1.2) *satisfies* $\lim_{t \to \infty} u(x,t) = d$ *uniformly with respect to x. In particular, if $\{c_l\}_{l \in \mathbb{Z}^n} \in \ell^p(\mathbb{Z}^n)$ for a certain $p \in [1, \infty)$, then $\lim_{t \to \infty} u(x,t) = 0$ uniformly with respect to x.*

Proof. It follows from the assumption that there is an $M \geq 0$ such that $|c_l| \leq M$ for all $l \in \mathbb{Z}^n$. Given an $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such if $\max(|l_1|,\ldots,|l_n|) > k_0$, then $|c_{l_1,\ldots,l_n} - d| < \varepsilon$.

For each $x \in \mathbb{Z}^n$ and all $k_1,\ldots,k_n \in \mathbb{N}_0$, consider the average $A_{k_1,\ldots,k_n}(x)$ given by (2.1). In the n-fold sum, there are at most $(2k_0 + 1)^n$ terms with $\max(|l_1|,\ldots,|l_n|) \leq k_0$; their values lie between $-M$ and M. The values of the remaining $\prod_{j=1}^n (2k_j + 1) - (2k_0 + 1)^n$ terms lie between $d - \varepsilon$ and $d + \varepsilon$. Thus, if at least one of $k_1,\ldots,k_n$ is sufficiently large, then $A_{k_1,\ldots,k_n}(x)$ will lie between $d - 2\varepsilon$ and $d + 2\varepsilon$. This shows that $\lim_{k_1,\ldots,k_n \to \infty} A_{k_1,\ldots,k_n}(x) = d$. The convergence is uniform with respect to x, because the previous estimate does not depend on x. The third part of Theorem 2.6 implies that $\lim_{t \to \infty} u(x,t) = d$ uniformly with respect to x. □

References

[1] L. Abadias, J. Gonález-Camus, P. J. Miana, J. C. Pozo, Large time behaviour for the heat equation on $\mathbb{Z}$, moments and decay rates, *J. Math. Anal. Appl.* **500**(2021), No. 125137. https://doi.org/10.1016/j.jmaa.2021.125137; MR4232677; Zbl 1464.39008

[2] M.-S. Chang, D.-H. Tsai, On the oscillation behavior of solutions to the heat equation on $\mathbb{R}^n$, *J. Differential Equations* **268**(2020), 2040–2062. https://doi.org/10.1016/j.jde.2019.09.021; MR4046182; Zbl 1431.35052

[3] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, J. L. Varona, Harmonic analysis associated with a discrete Laplacian, *J. Anal. Math.* **132**(2017), 109–131. https://doi.org/10.1007/s11854-017-0015-6; MR3666807; Zbl 1476.39003

[4] Ó. Ciaurri, C. Lizama, L. Roncal, J. L. Varona, On a connection between the discrete fractional Laplacian and superdiffusion, *Appl. Math. Lett.* **49**(2015), 119–125. https://doi.org/10.1016/j.aml.2015.05.007; MR3361705; Zbl 1381.35215

[5] J. González-Camus, C. Lizama, P. J. Miana, Fundamental solutions for semidiscrete evolution equations via Banach algebras, *Adv. Differ. Equ.* **2021**, No. 35. https://doi.org/10.1186/s13662-020-03206-7; MR4197351

[6] J. González-Camus, V. Keyantuo, C. Lizama, M. Warma, Fundamental solutions for discrete dynamical systems involving the fractional Laplacian, *Math. Meth. Appl. Sci.* **42**(2019), 4688–4711. https://doi.org/10.1002/mma.5685; MR3992934; Zbl 1423.35398

[7] S. Kamin, On stabilisation of solutions of the Cauchy problem for parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **76**(1976/77), 43–53. https://doi.org/10.1017/S0308210500019478; MR0440200; Zbl 0346.35014

[8] V. Keyantuo, C. Lizama, M. Warma, Lattice dynamical systems associated with a fractional Laplacian, *Numer. Funct. Anal. Optim.* **40**(2019), 1315–1343. https://doi.org/10.1080/01630563.2019.1602542; MR3949124; Zbl 1417.49031

[9] C. Lizama, L. Roncal, Hölder–Lebesgue regularity and almost periodicity for semidiscrete equations with a fractional Laplacian, *Discrete Contin. Dyn. Syst.* **38**(2018), 1365–1403. https://doi.org/10.3934/dcds.2018056; MR3808998; Zbl 1397.34034

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, eds., *NIST handbook of mathematical functions*, Cambridge University Press, New York, 2010. Online version at http://dlmf.nist.gov/. MR2723248; Zbl 1198.00002

[11] V. D. Repnikov, S. D. Eidel'man, A new proof of the theorem on the stabilization of the solution of the Cauchy problem for the heat equation, *Math. USSR Sb.* **2**(1967), 135–139. https://doi.org/10.1070/SM1967v002n01ABEH002328; MR0212424; Zbl 0162.15201

[12] A. Slavík, Asymptotic behavior of solutions to the semidiscrete diffusion equation, *Appl. Math. Lett.* **106**(2020), No. 106392. https://doi.org/10.1016/j.aml.2020.106392; MR4090371; Zbl 1439.35502

[13] A. Slavík, P. Stehlík, Explicit solutions to dynamic diffusion-type equations and their time integrals, *Appl. Math. Comput.* **234**(2014), 486–505. https://doi.org/10.1016/j.amc.2014.01.176; MR3190559; Zbl 1305.39010

[14] A. Slavík, P. Stehlík, Dynamic diffusion-type equations on discrete-space domains, *J. Math. Anal. Appl.* **427**(2015), 525–545. https://doi.org/10.1016/j.jmaa.2015.02.056; MR3318214; Zbl 1338.35449

[15] D.-H. Tsai, C.-H. Nien, On the oscillation behavior of solutions to the one-dimensional heat equation, *Discrete Contin. Dyn. Syst.* **39**(2019), 4073–4089. https://doi.org/10.3934/dcds.2019164; MR3960497; Zbl 1415.35144