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Multi-bump solutions for the magnetic Schrödinger–Poisson system with critical growth

Chao Ji[1], **Yongde Zhang**[1] and **Vicențiu D. Rădulescu** ✉ [2,3,4]

[1]School of Mathematics, East China University of Science and Technology, Shanghai, 200237, China
[2]Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland
[3]Department of Mathematics, University of Craiova, 200585 Craiova, Romania
[4]China–Romania Research Centre in Applied Mathematics

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Abstract. In this paper, we are concerned with the following magnetic Schrödinger–Poisson system

$$\begin{cases} -(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, f is a subcritical nonlinearity, the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function verifying some conditions, the magnetic potential $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$. Assuming that the zero set of $V(x)$ has several isolated connected components $\Omega_1, \dots, \Omega_k$ such that the interior of Ω_j is non-empty and $\partial\Omega_j$ is smooth, where $j \in \{1, \dots, k\}$, then for $\lambda > 0$ large enough, we use the variational methods to show that the above system has at least $2^k - 1$ multi-bump solutions.

Keywords: Schrödinger–Poisson system, multi-bump solutions, magnetic field, critical growth, variational methods.

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1 Introduction

In the past few decades, there is a vast literature concerning the nonlinear Schrödinger–Poisson system

$$\begin{cases} -i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi + \phi(x)\psi - |\psi|^{p-1}\psi, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \psi^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a nonnegative continuous function with $\inf_{x\in\mathbb{R}^3} V(x) > 0$, $1 < p < 5$ and $\psi : \mathbb{R}^3 \to \mathbb{C}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ are two unknown functions. In fact, the first equation in the above system describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. And $\phi(x)$ satisfies the second equation (Poisson

✉ Corresponding author. Email: radulescu@inf.ucv.ro

equation) in the system, because the potential $\phi(x)$ is determined by the charge of wave function itself. Therefore, system (1.1) can be regarded as the coupling of the Schrödinger equation and Poisson equation.

If one looks for stationary solutions $\psi(x,t) := e^{-it}u(x)$ of system (1.1), the system can be reduced by

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

In [4], Azzollini and Pomponio considered system (1.2). More precisely, if V is a positive constant, they proved the existence of a ground state solution (u, ϕ) for $2 < p < 5$. If V is a nonconstant potential that is measurable and (possibly) not bounded from below, they obtained a similar existence result for $3 < p < 5$. Existence and nonexistence results were also proved when the nonlinearity exhibits a critical growth.

In a celebrated paper [13], by using the variational methods, Ding and Tanaka established multiplicity of multi-bump solutions for a semilinear elliptic equation with deepening potential well. Recently, in [2], Alves and Yang considered system (1.2) which having a general nonlinear term f and assumed the potential $V(x)$ has the form $V(x) = \lambda a(x) + 1$, where λ is a positive parameter and $a : \mathbb{R}^3 \to \mathbb{R}^3$ is a nonnegative continuous function. In the interesting paper, the authors proved the existence of positive multi-bump solutions for the system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u + \phi(x)u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

For more results on the Schrödinger–Poisson system, we refer the reader to [3,5,7,10,11,18,19,23–26,28,31–34,36,38,40,41] and the references therein.

In recent years, the magnetic nonlinear Schrödinger equation has also received considerable attention

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A(x)\right)^2 \psi + U(x)\psi - f(|\psi|^2)\psi, \quad \text{in } \mathbb{R}^N \times \mathbb{R},$$

where i is the imaginary unit, $\hbar$ is the Planck constant, and $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential. When one looks for standing wave solutions $\psi(x,t) := e^{-iEt/\hbar}u(x)$, with $E \in \mathbb{R}$, of the above equation, the problem can be reduced by

$$\left(\frac{\hbar}{i}\nabla - A(x)\right)^2 u + V(x)u = f\left(|u|^2\right)u, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

From a physical point of view, the existence of such solutions and the study of their shape in the semiclassical limit, namely, as $\hbar \to 0^+$ is of the greatest importance, since the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant $\hbar$ to zero.

As far as we know, the first result involving the magnetic field was obtained by Esteban and Lions [15]. In [15], for $\hbar > 0$ fixed and special classes of magnetic fields, the authors found the existence of standing waves to problem (1.3) by solving an appropriate minimization problem for the corresponding energy functional in the cases of $N = 2$ and 3. Afterwards, in [27], Kurata assumed a technical condition relating $V(x)$ and $A(x)$. Under these assumptions, he proved that the associated functional satisfies the Palais–Smale condition at any level and further obtained a least energy solution of the problem for any $\epsilon > 0$. Also, Alves *et al.*

[1] studied the multiple solutions by combining a local assumption on V, the penalization techniques of del Pino and Felmer [12] and the Ljusternic–Schnirelmann theory.

Recently, Tang [35] considered multi-bump solutions of the following nonlinear magnetic Schrödinger equation with critical frequency

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + E)u = f\left(|u|^2\right) u, \quad \text{in } \mathbb{R}^2,$$

where $\lambda > 0$, $E \in \mathbb{R}$ is a constant, $\inf_{x\in\mathbb{R}^N} V(x) = E$ and f satisfies subcritical growth. Later, by using the variational methods, Ji and Rădulescu [22] established the existence and multiplicity of multi-bump solutions for the following nonlinear magnetic Schrödinger equation

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + Z(x))u = f\left(|u|^2\right) u, \quad \text{in } \mathbb{R}^2,$$

where $\lambda > 0$, $f(t)$ is a continuous function with exponential critical growth, the magnetic potential $A : \mathbb{R}^2 \to \mathbb{R}^2$ is in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ and the potentials $V, Z : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions verifying some conditions. Recently, Ma and Ji [30] studied the existence and multiplicity of multi-bump solutions for the magnetic Schrödinger–Poisson system with subcritical growth. It is natural to consider multiplicity of multi-bump solutions for the magnetic Schrödinger–Poisson system with *critical* growth. To the best of our knowledge, this problem has not ever been studied. For more results related to the nonlinear partial differential equations with magnetic field, we refer to [6,8,9,14,17,20,21,39,42] and references therein.

Inspired by the previous works of [22,30,35], the aim of this paper is to study existence of multi-bump solutions for the magnetic Schrödinger–Poisson system with critical growth

$$\begin{cases} -(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a parameter, the magnetic potential A is in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, f has subcritical growth and the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Due to the appearance of magnetic field $A(x)$, problem (1.4) can not be changed into a pure real-valued problem, hence we should deal with a complex-valued directly. Also, since the electrostatic potential $\phi(x)$ depends on the wave function, $\phi(x)u$ is nonlocal which will make some estimates more difficult and complicated. Moreover, since the problem we deal with has critical growth, we need more refined estimates to overcome the lack of compactness.

Now we present the general assumptions on the potentials in this paper:

(A) $A : \mathbb{R}^3 \to \mathbb{R}^3$ be in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$;

(V_1) $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ and $V(x) \geq 0$, for all $x \in \mathbb{R}^3$;

(V_2) $\Omega = \operatorname{int} V^{-1}(0)$ is a nonempty bounded open subset with smooth boundary and $\overline{\Omega} = V^{-1}(0)$ where $\operatorname{int} V^{-1}(0)$ denotes the set of the interior points of $V^{-1}(0)$, Ω consists of k components:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for all $i \neq j$.

Furthermore, the nonlinearity f is a continuous function satisfying the following conditions:

(f_1) $f(t) = 0, \forall\, t \leq 0$, and $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$;

(f_2) There exists $q, \iota \in (4,6)$ and $\varsigma > 0$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0, \text{ and } f(t) \geq \varsigma t^{(\iota-2)/2} \text{ for any } t > 0;$$

(f_3) There exists $\theta \in (4,6)$ such that

$$0 < \frac{\theta}{2} F(t) \leq t f(t), \text{ for any } t > 0$$

where $F(t) = \int_0^t f(s)ds$;

(f_4) $f(t)$ is an increasing function in $t > 0$.

The main result of this paper to be proved is the theorem below:

Theorem 1.1. *Assume that $(A), (V_1) - (V_2)$ and $(f_1) - (f_4)$ hold. Then, for any non-empty subset Γ of $\{1, 2, \ldots, k\}$, there exist constants $\alpha^* > 0$ and $\lambda^* = \lambda^*(\alpha^*)$ such that, for all $\alpha \geq \alpha^*$ and $\lambda \geq \lambda^*$, problem (1.4) has a nontrivial solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda\geq\lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges in $H_A^1(\mathbb{R}^3, \mathbb{C})$ to a function u, which satisfies $u = 0$ for $x \notin \Omega_\Gamma = \cup_{j\in\Gamma}\Omega_j$, and the restriction $u\,|_{\Omega_j}$ is a least energy solution of*

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \left(\frac{1}{4\pi}\int_{\Omega_j} \frac{|u(y)|^2}{|x-y|} dy\right) u = f(|u|^2)u + |u|^4 u, \ x \in \Omega_j, \\ u \in H_A^{0,1}(\Omega_j), \end{cases}$$

where $j \in \Gamma$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exist $\alpha_* > 0$ and $\lambda_* = \lambda_*(\alpha_*)$ such that, for all $\alpha \geq \alpha_*$ and $\lambda \geq \lambda_*$, problem (1.4) has at least $2^k - 1$ nontrivial solutions.*

The paper is organized as follows. In Section 2, we shall introduce the variational setting and give some necessary preliminaries. In Section 3, we study an modified problem, and prove the Palais–Smale condition for the modified problem and study the behavior of $(PS)_\infty$ sequence. Moreover, we establish L^∞ estimate of the solution of the modified problem. In Section 4, by adapting the deformation flow method, we show that the existence of a special critical point and prove the main theorem.

2 Preliminaries

In this section, we shall present the variational framework for problem (1.4) and some useful preliminary lemmas.

For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u = (\nabla + iA)\, u,$$

and

$$H_A^1(\mathbb{R}^3, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R})\right\}.$$

The space $H_A^1(\mathbb{R}^3, \mathbb{C})$ is an Hilbert space under the scalar product

$$\langle u, v\rangle = \operatorname{Re}\int_{\mathbb{R}^3} \left(\nabla_A u \overline{\nabla_A v} + u\bar{v}\right) dx, \qquad \forall u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover, the norm induced by the product $\langle \cdot,\cdot \rangle$ is $\|u\|_A = \left(\int_{\mathbb{R}^3} |\nabla_A u|^2 + |u|^2\, dx\right)^{\frac{1}{2}}$.

By (A), on $H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$, we have the important diamagnetic inequality (see [29], Theorem 7.21) which is frequently used in this paper:

$$|\nabla_A u(x)| \geq |\nabla |u(x)||\,. \tag{2.1}$$

Let

$$E_\lambda = \left\{ u \in H^1_A\left(\mathbb{R}^3,\mathbb{C}\right) : \int_{\mathbb{R}^3} \lambda V(x)\, |u|^2\, dx < \infty \right\},$$

with the norm

$$\|u\|^2_\lambda = \int_{\mathbb{R}^3} \left(|\nabla_A u|^2 + (\lambda V(x) + 1)\, |u|^2 \right) dx.$$

For $\lambda \geq 0$, a direct computation gives that $(E_\lambda, \|\cdot\|_\lambda)$ is an Hilbert space and $E_\lambda \subset H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$.

Also, for an open set $K \subset \mathbb{R}^3$,

$$\begin{aligned}
H^1_A(K,\mathbb{C}) &:= \left\{ u \in L^2(K,\mathbb{C}) : |\nabla_A u| \in L^2(K,\mathbb{R}) \right\}, \\
\|u\|_{H^1_A(K,\mathbb{C})} &= \left(\int_K \left(|\nabla_A u|^2 + |u|^2 \right) dx \right)^{\frac{1}{2}}, \\
E_\lambda(K,\mathbb{C}) &:= \left\{ u \in H^1_A(K,\mathbb{C}) : \int_K \lambda V(x)\, |u|^2\, dx < \infty \right\}, \\
\|u\|^2_{\lambda,K} &= \int_K \left(|\nabla_A u|^2 + (\lambda V(x) + 1)\, |u|^2 \right) dx.
\end{aligned}$$

Let $H^{0,1}_A(K,\mathbb{C})$ be the Hilbert space obtained as the closure of $C^\infty_0(K,\mathbb{C})$ under the norm $\|u\|_{H^1_A(K,\mathbb{C})}$.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$, then $|u| \in H^1\left(\mathbb{R}^3,\mathbb{R}\right)$ and $\|u\| \leq \|u\|_A$. Therefore, the embedding $H^1_A\left(\mathbb{R}^3,\mathbb{C}\right) \hookrightarrow L^r\left(\mathbb{R}^3,\mathbb{C}\right)$ is continuous for $2 \leq r \leq 6$ and the embedding $H^1_A\left(\mathbb{R}^3,\mathbb{C}\right) \hookrightarrow L^r_{loc}\left(\mathbb{R}^3,\mathbb{C}\right)$ is compact for $1 \leq r < 6$.

By the continuous embedding $H^1\left(\mathbb{R}^3,\mathbb{R}\right) \hookrightarrow L^r\left(\mathbb{R}^3,\mathbb{R}\right)$ for $2 \leq r \leq 6$, we have

$$H^1\left(\mathbb{R}^3,\mathbb{R}\right) \hookrightarrow L^{\frac{12}{5}}\left(\mathbb{R}^3,\mathbb{R}\right).$$

For any $u \in H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$, we obtain that $|u| \in H^1\left(\mathbb{R}^3,\mathbb{R}\right)$, and the linear functional $\mathcal{L}_{|u|} : D^{1,2}\left(\mathbb{R}^3,\mathbb{R}\right) \to \mathbb{R}$ given by

$$\mathcal{L}_{|u|}(v) = \int_{\mathbb{R}^3} |u|^2 v dx$$

is well defined and continuous in view of the Hölder inequality and (2.2). Indeed, we can see that

$$\left|\mathcal{L}_{|u|}(v)\right| \leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |v|^6 dx \right)^{\frac{1}{6}} \leq C\|u\|^2_A \|v\|_{D^{1,2}}. \tag{2.2}$$

Then, given $u \in H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$, $|u| \in H^1\left(\mathbb{R}^3,\mathbb{R}\right)$, by the Lax–Milgram Theorem, there exists an unique $\phi = \phi_{|u|} \in D^{1,2}\left(\mathbb{R}^3,\mathbb{R}\right)$ such that

$$-\Delta\phi = u^2.$$

Moreover, $\phi_{|u|}$ can be expressed as

$$\phi_{|u|}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy.$$

Next, we provide the following properties about $\phi_{|u|}$ in the following lemma whose proof is similar to one in [11,32,41], so we omit it.

Lemma 2.1. *For any $u \in H_A^1\left(\mathbb{R}^3,\mathbb{C}\right)$, we have*

(i) there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} \left|\nabla\phi_{|u|}\right|^2 dx = \int_{\mathbb{R}^3} \phi_{|u|}\left|u\right|^2 dx \le C\left\|u\right\|_A^4, \qquad \forall u \in H_A^1\left(\mathbb{R}^3,\mathbb{C}\right);$$

(ii) $\phi_{|u|} \ge 0$, $\forall u \in H_A^1\left(\mathbb{R}^3,\mathbb{C}\right)$;

(iii) $\phi_{|tu|} = t^2\phi_{|u|}$, $\forall t \in \mathbb{R}$ and $u \in H_A^1\left(\mathbb{R}^3,\mathbb{C}\right)$;

(iv) if $u_n \rightharpoonup u$ in $H_A^1\left(\mathbb{R}^3,\mathbb{C}\right)$, then $\phi_{|u_n|} \rightharpoonup \phi_{|u|}$ in $D^{1,2}\left(\mathbb{R}^3,\mathbb{R}\right)$ and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{|u_n|}|u_n|^2dx \ge \int_{\mathbb{R}^3}\phi_{|u|}|u|^2dx;$$

(v) if $u_n \to u$ in $H_A^1\left(\mathbb{R}^3,\mathbb{C}\right)$, then $\phi_{|u_n|} \to \phi_{|u|}$ in $D^{1,2}\left(\mathbb{R}^3,\mathbb{R}\right)$. Hence,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{|u_n|}|u_n|^2dx = \int_{\mathbb{R}^3}\phi_{|u|}|u|^2dx.$$

Now, we define the energy functional I_λ associated with problem (1.4) given by

$$\begin{aligned} I_\lambda\left(u\right) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla_A u|^2 + \left(\lambda V(x)+1\right)|u|^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{|u|}\left(x\right)|u|^2dx \\ &-\frac{\alpha}{2}\int_{\mathbb{R}^3}F\left(|u|^2\right)dx - \frac{1}{6}\int_{\mathbb{R}^3}|u_n|^6dx, \end{aligned}$$

it is standard to prove that $I_\lambda\left(u\right) \in C^1\left(E_\lambda,\mathbb{R}\right)$, and for any $\varphi \in E_\lambda$, we have

$$\begin{aligned} \left\langle I_\lambda{}'\left(u\right),\varphi\right\rangle = &\operatorname{Re}\int_{\mathbb{R}^3}\left(\nabla_A u\overline{\nabla_A\varphi} + \left(\lambda V\left(x\right)+1\right)u\overline{\varphi}\right)dx + \operatorname{Re}\int_{\mathbb{R}^3}\phi_{|u|}\left(x\right)u\overline{\varphi}dx \\ &-\operatorname{Re}\int_{\mathbb{R}^3}\alpha f\left(|u|^2\right)u\overline{\varphi}dx - \operatorname{Re}\int_{\mathbb{R}^3}|u|^4u\overline{\varphi}dx. \end{aligned}$$

Definition 2.2. A pair $(u,\phi) \in E_\lambda \times D^{1,2}\left(\mathbb{R}^3,\mathbb{R}\right)$ is said to be a weak solution of problem (1.4), if $I_\lambda'\left(u\right)\varphi = 0$, $\forall\varphi \in E_\lambda$, where $\phi_{|u|} = \phi$.

By (V_3), we can derive that for any open set $K \subset \mathbb{R}^3$,

$$M_0\|u\|_{2,K}^2 \le \int_K\left(|\nabla_A u|^2 + (\lambda V(x)+1)|u|^2\right)dx,$$

for all $u \in E_\lambda\left(K\right)$, and $\lambda > 0$, where $\|u\|_{2,K}^2 = \int_K|u|^2dx$. So, from this relation, we have the following result:

Lemma 2.3. *There exist $\delta_0, \nu_0 > 0$ with $\delta_0 \approx 1$ and $\nu_0 \approx 0$ such that for any open set $K \subset \mathbb{R}^3$,*

$$\delta_0\|u\|_{\lambda,K}^2 \le \|u\|_{\lambda,K}^2 - \nu_0\|u\|_{2,K}^2, \quad \textit{for all } u \in E_\lambda(K,\mathbb{C}),\ \textit{and } \lambda > 0.$$

3 A modified problem

Since $\mathbb{R}^3$ is unbounded and nonlinear term has the critical growth, we know that the Sobolev embeddings are not compact, as so I_λ can not verify the Palais–Smale condition. In order to overcome this difficulty, we adapt the argument of the penalization method introduced by del Pino and Felmer [12] and Ding and Tanaka [13], and consider a modified problem satisfying the Palais–Smale condition.

Let $\nu_0 > 0$ be a constant given in Lemma 2.3, $\kappa > \frac{\theta}{\theta-2}$ and $a > 0$ verifying $\alpha f(a) + a^2 = \frac{\nu_0}{\kappa}$ and $\widetilde{f}, \widetilde{F} : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}(t) = \begin{cases} \alpha f(t) + t^2, & t \le a, \\ \frac{\nu_0}{\kappa}, & t \ge a, \end{cases}$$

thus

$$\widetilde{f}(t) \le \alpha f(t) + t^2, \qquad t \ge 0. \tag{3.1}$$

Also,

$$\widetilde{F}(t) = \int_0^t \widetilde{f}(s)\, ds.$$

Now, since the potential well $\Omega = \operatorname{int} V^{-1}(0)$ can be decomposed into k connected components $\Omega_1, \dots, \Omega_k$ with $\operatorname{dist}(\Omega_i, \Omega_j) > 0$, $i \ne j$, then for each $j \in \{1, 2, \dots, k\}$, we fix a smooth bounded domain Ω_j' such that

(i) $\overline{\Omega_j} \subset \Omega_j'$;

(ii) $\overline{\Omega_i'} \cap \overline{\Omega_j'} = \emptyset$ for all $i \ne j$.

Next, we fix a non-empty subset $\Gamma \subset \{1, \dots, k\}$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \qquad \Omega_\Gamma' = \bigcup_{j\in\Gamma} \Omega_j',$$

$$\chi_\Gamma(x) := \begin{cases} 1 & \text{for } x \in \Omega_\Gamma', \\ 0 & \text{for } x \notin \Omega_\Gamma'. \end{cases}$$

Using the above notations, we set the functions

$$\begin{aligned} g(x,t) &= \chi_\Gamma(x)(\alpha f(t) + t^2) + (1 - \chi_\Gamma(x))\, \tilde{f}(t), \\ G(x,t) &= \int_0^t g(x,s) ds = \chi_\Gamma(x) \alpha F(t) + (1 - \chi_\Gamma(x))\, \tilde{F}(t). \end{aligned} \tag{3.2}$$

In view of (f_1)–(f_4), we have that g is a Carathéodory function satisfying the following properties:

(g_1) $g(x,t) = 0$ for each $t \le 0$;

(g_2) $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

(g_3) $g(x,t) \le \alpha f(t) + t^2$ for all $t \ge 0$ and any $x \in \mathbb{R}^3$;

(g_4) $0 < \theta G(x,t) \le 2g(x,t)t$ for each $x \in \Omega_\Gamma'$ and $t > 0$;

(g_5) $0 < G(x,t) \leq g(x,t)t \leq \nu_0 t/\kappa$, for each $x \in \mathbb{R}^3 \setminus \Omega'_\Gamma$, $t > 0$;

(g_6) for each $x \in \Omega'_\Gamma$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0, +\infty)$ and for each $x \in \mathbb{R}^3 \setminus \Omega'_\Gamma$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0, a)$.

Moreover, we have the modified problem

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi_{|u|} u = g\left(x, |u|^2\right) u, \qquad x \in \mathbb{R}^3, \tag{3.3}$$

and the energy functional $\Phi_\lambda(u) : E_\lambda\left(\mathbb{R}^3, \mathbb{C}\right) \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{|u|}|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G\left(x, |u|^2\right) dx.$$

We want to get some nontrivial solutions of (3.3) are ones of the original problem (1.4), more precisely, if u_λ is a nontrivial solution of (3.3) verifying $|u_\lambda(x)|^2 \leq a$ in $\mathbb{R}^3 \setminus \Omega'_\Gamma$, then it is a nontrivial solution to (1.4).

Next, we prove that the energy functional $\Phi_\lambda(u)$ satisfies the (PS) condition.

Lemma 3.1. *All $(PS)_c$ sequences for Φ_λ are bounded in E_λ.*

Proof. Let (u_n) be a $(PS)_c$ sequence for Φ_λ. Thus, we have

$$\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi'_\lambda(u_n)\, u_n = c + o_n(1) + o_n(1)\, \|u_n\|_\lambda .$$

On the other hand, by (g_4), (g_5), $\kappa > \frac{\theta}{\theta - 2}$, and Lemma 2.3, we derive

$$\begin{aligned}
\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi'_\lambda(u_n)\, u_n &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\lambda + \left(\frac{1}{4} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} \phi_{|u_n|}(x)\, |u_n|^2 dx \\
&\quad + \int_{\mathbb{R}^3} \left(\frac{1}{\theta} g\left(x, |u_n|^2\right)|u_n|^2 - \frac{1}{2} G\left(x, |u_n|^2\right)\right) dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\lambda + \frac{2-\theta}{2\theta}\int_{\mathbb{R}^3 \setminus \Omega'_\Gamma} \tilde{F}(|u_n|^2) dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2_\lambda + \frac{(\theta - 2)\nu_0}{2\theta\kappa}\int_{\mathbb{R}^3 \setminus \Omega'_\Gamma} |u_n|^2 dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)(1 - \frac{1}{\kappa})\, \|u_n\|^2_\lambda .
\end{aligned}$$

So,

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)(1 - \frac{1}{\kappa})\, \|u_n\|^2_\lambda \leq c + o_n(1) + o_n(1)\, \|u_n\|_\lambda .$$

This shows that (u_n) is bounded in E_λ. □

For each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the functional $I_j \colon H^{0,1}_A(\Omega_j, \mathbb{C}) \to \mathbb{R}$ given by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx + \frac{1}{4}\int_{\Omega_j} \phi_{|u|}|u|^2 dx - \frac{\alpha}{2}\int_{\Omega_j} F(|u|^2) dx - \frac{1}{6}\int_{\Omega_j} |u|^6 dx,$$

and

$$c_j = \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)),$$

where

$$\Lambda_j = \left\{ \gamma \in C([0,1], H_A^{0,1}(\Omega_j, \mathbb{C})) : \gamma(0) = 0, I_j(\gamma(1)) < 0 \right\}.$$

It is well-known that the critical points of are the weak solutions of the problem

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \phi_{|u|} u = \alpha f\left(|u|^2\right) u + |u|^4 u, & \text{in } \Omega_j, \\ u = 0, & \text{on } \partial\Omega_j. \end{cases} \tag{3.4}$$

Moreover, we have the following important result.

Lemma 3.2. *There exists $\alpha^* > 0$ such that, for all $\alpha \geq \alpha^*$, we have*

$$c_j \in \left(0, \frac{1}{3(k+1)} S^{3/2}\right), \quad \textit{for all } j \in \{1, \cdots, k\} \textit{ and all } \alpha \in [\alpha^*, +\infty).$$

Proof. We choose a function $\varphi_j \in H_A^{0,1}(\Omega_j, \mathbb{C})) \setminus \{0\}$ for each $j \in \{1, \cdots, k\}$. There exists $t_{\alpha,j} \in (0, +\infty)$ such that

$$c_j \leq I_j(t_{\alpha,j}\varphi_j) = \max_{t \geq 0} I_j(t\varphi_j)$$

and hence, by (f_4), one has

$$\begin{aligned} t_{\alpha,j}^2 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2 \right) dx + t_{\alpha,j}^4 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \\ = \alpha \int_{\mathbb{R}^3} f\left(|t_{\alpha,j}\varphi_j|^2 \right) |t_{\alpha,j}\varphi_j|^2 dx + t_{\alpha,j}^6 \int_{\mathbb{R}^3} |\varphi_j|^6 dx \\ \geq \alpha \int_{\mathbb{R}^3} f\left(|t_{\alpha,j}\varphi_j|^2 \right) |t_{\alpha,j}\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx. \end{aligned} \tag{3.5}$$

If $|t_{\alpha,j}| \leq 1$, by (3.5), we have

$$t_{\alpha,j}^2 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2 \right) dx + t_{\alpha,j}^2 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx.$$

The above inequality implies that

$$t_{\alpha,j} \leq \left[\frac{\int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2 \right) dx + \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx}{\alpha \varsigma \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx} \right]^{1/(\iota - 2)}.$$

If $|t_{\alpha,j}| \geq 1$, by (3.5), one has

$$t_{\alpha,j}^4 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2 \right) dx + t_{\alpha,j}^4 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx.$$

The above inequality implies that

$$t_{\alpha,j} \leq \left[\frac{\int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2 \right) dx + \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx}{\alpha \varsigma \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx} \right]^{1/(\iota - 4)}.$$

Using the above limits, we have $t_{\alpha,j} \to 0$ as $\alpha \to +\infty$. This fact yields that $I_j(t_{\alpha,j}\varphi_j) \to 0$ as $\alpha \to +\infty$. Thus, there exists $\alpha^* > 0$ such that

$$c_j \in \left(0, \frac{1}{3(k+1)} S^{3/2}\right), \quad \text{for all } j \in \{1, \cdots, k\}. \qquad \square$$

Remark 3.3. In particular, the above lemma implies for $\alpha > 0$ large that

$$\sum_{j=1}^{k} c_j \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right). \tag{3.6}$$

Proposition 3.4. *For any $\lambda > 0$, the functional Φ_λ satisfies the $(PS)_c$ condition at the level $c < \frac{1}{3}S^{\frac{3}{2}}$.*

Proof. Let $(u_n) \subset E_\lambda$ be a $(PS)_c$ sequence for $\Phi_\lambda$at the level $c < \frac{1}{3}S^{\frac{3}{2}}$, that is

$$\Phi_\lambda(u_n) \to c < \frac{1}{3}S^{\frac{3}{2}} \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0.$$

From Lemma 3.1, we know that the sequence (u_n) is bounded in E_λ. Thus, there exists $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ, up to a subsequence if necessary. Then it is standard to check that for any $C_0^\infty\left(\mathbb{R}^3, \mathbb{C}\right) \subset E_\lambda$,

$$\text{Re} \int_{\mathbb{R}^3} \nabla_A u_n \overline{\nabla_A \varphi} dx \to \text{Re} \int_{\mathbb{R}^3} \nabla_A u \overline{\nabla_A \varphi} dx,$$

$$\text{Re} \int_{\mathbb{R}^3} (\lambda V(x) + 1) u_n \overline{\varphi} dx \to \text{Re} \int_{\mathbb{R}^3} (\lambda V(x) + 1) u \overline{\varphi} dx,$$

and

$$\text{Re} \int_{\mathbb{R}^3} g\left(x, |u_n|^2\right) u_n \overline{\varphi} dx \to \text{Re} \int_{\mathbb{R}^3} g\left(x, |u|^2\right) u \overline{\varphi} dx. \tag{3.7}$$

Form (3.7), the density of $C_0^\infty\left(\mathbb{R}^3, \mathbb{C}\right)$ in E_λ, and $\Phi'_\lambda(u_n) \to 0$, we can obtain that the weak limit u is a critical point of Φ_λ and so

$$\|u\|_\lambda^2 + \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx = \int_{\mathbb{R}^3} g(x, |u|^2) |u|^2 dx. \tag{3.8}$$

On the other hand, we know that $< \Phi'_\lambda\left(u_n\right), u_n >= o_n(1)$ which implies that

$$\|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \phi_{|u_n|} |u_n|^2 dx = \int_{\mathbb{R}^3} g(x, |u_n|^2) |u_n|^2 dx + o_n(1). \tag{3.9}$$

Step 1: We show that for any given $\zeta > 0$, there exists $R > 0$ large enough such that $\Omega'_\Gamma \subset B_{R/2}(0)$ and

$$\limsup_n \int_{B_R^c(0)} (|\nabla_A u_n|^2 + (\lambda V(x) + 1)|u_n|^2) dx \le \zeta. \tag{3.10}$$

Now, we take $R > 0$ large such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfying

$$\eta_R = 0 \quad x \in B_{\frac{R}{2}}(0), \quad \eta_R = 1 \quad x \in B_R^c(0), \quad 0 \le \eta_R \le 1, \quad \text{and} \quad |\nabla \eta_R| \le \frac{C}{R},$$

where $C > 0$ is a constant independent of R.

By a direct computation, we have

$$\begin{aligned} o_n(1) = \langle \Phi'_\lambda\left(u_n\right), u_n \eta_R \rangle = & \int_{\mathbb{R}^3} \left(|\nabla_A u_n|^2 + (\lambda V(x) + 1)\left|u_n\right|^2\right) \eta_R dx \\ & + \int_{\mathbb{R}^3} \phi_{|u_n|}(x) \left|u_n\right|^2 \eta_R dx + \text{Re}\left(\int_{\mathbb{R}^3} \overline{u_n} \nabla_A u_n \nabla \eta_R dx\right) \\ & - \int_{\mathbb{R}^3} \tilde{f}\left(|u_n|^2\right) |u_n|^2 \eta_R dx. \end{aligned}$$

Notice that

$$|\mathrm{Re}\,(\overline{u_n}\nabla_A u_n)| = |\mathrm{Re}\,((\nabla u_n + iAu_n)\,\overline{u_n})| = |\mathrm{Re}\,(\overline{u_n}\nabla u_n)| = |u_n|\,|\nabla|u_n||.$$

Using the Hölder inequality and the above equality, we derive

$$\limsup_{n\to\infty}\left|\mathrm{Re}\left(\int_{\mathbb{R}^3}\overline{u_n}\nabla_A u_n \nabla\eta_R dx\right)\right| \le \frac{C}{R}.$$

So, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla_A u_n|^2 + (\lambda V(x)+1)\,|u_n|^2\right)\eta_R dx\\ &\le \int_{\mathbb{R}^3}\tilde{f}\left(|u_n|^2\right)|u_n|^2\eta_R dx + \frac{C}{R} + o_n(1)\\ &\le \frac{\nu_0}{\kappa}\int_{\mathbb{R}^3}|u_n|^2\eta_R dx + \frac{C}{R} + o_n(1),\end{aligned}$$

which implies that for any $\zeta > 0$, choosing a $R > 0$ larger if necessary, we have

$$\limsup_{n\to\infty}\int_{B_R^c(0)}\left(|\nabla_A u_n|^2 + (\lambda V(x)+1)\,|u_n|^2\right)dx \le \zeta.$$

Step 2: We show that

$$\lim_n \int_{\mathbb{R}^3}\phi_{|u_n|}|u_n|^2 dx = \int_{\mathbb{R}^3}\phi_{|u|}|u|^2 dx. \tag{3.11}$$

By (3.10) and the Sobolev embedding, for any $\zeta > 0$, there exists $R > 0$ such that for n large enough and $q \in [2,6)$

$$\begin{aligned}\|u_n - u\|_{L^q(\mathbb{R}^3)} &= \|u_n - u\|_{L^q(B_R(0))} + \|u_n - u\|_{L^q(B_R^c(0))}\\ &\le \|u_n - u\|_{L^q(B_R(0))} + \|u_n\|_{L^q(B_R^c(0))} + \|u\|_{L^q(B_R^c(0))}\\ &\le C\zeta,\end{aligned}$$

which implies

$$u_n \to u \quad \text{in } L^q(\mathbb{R}^3,\mathbb{C}), \qquad \forall q \in [2,6).$$

Since $||u_n| - |u|| \le |u_n - u||$ and $\frac{12}{5} \in (2,6)$, one has

$$|u_n| \to |u| \quad \text{in } L^{12/5}(\mathbb{R}^3,\mathbb{R}). \tag{3.12}$$

Let

$$\mathbb{D}(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy,$$

we have

$$\begin{aligned}|\mathbb{D}(u_n) - \mathbb{D}(u)| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2\,|u_n(y)|^2}{|x-y|}dxdy - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy\right|\\ &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(|u_n(x)|^2 - |u(x)|^2)(|u_n(y)|^2 + |u(y)|^2)}{|x-y|}dxdy\right|\\ &\le \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\,|u_n(x)|^2 - |u(x)|^2\,|(|u_n(y)|^2 + |u(y)|^2)}{|x-y|}dxdy\right|\\ &\le C\sqrt{\mathbb{D}(||u_n|^2 - |u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2 + |u|^2|^{1/2})}\end{aligned}$$

Then, by the Hardy–Littlewood–Sobolev inequality, the Hölder inequality and (3.12), it follows that

$$
\begin{aligned}
|\mathbb{D}(u_n) - \mathbb{D}(u)|^2 &= C\| ||u_n|^2 - |u|^2|^{1/2} \|^4_{L^{12/5}(\mathbb{R}^3)} \| ||u_n|^2 + |u|^2|^{1/2} \|^4_{L^{12/5}(\mathbb{R}^3)} \\
&\leq C\| ||u_n|^2 - |u|^2|^{1/2} \|^4_{L^{12/5}(\mathbb{R}^3)} \to 0.
\end{aligned}
$$

Step 3:

$$
\lim_n \int_{\mathbb{R}^3} g(x, |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} g(x, |u|^2)|u|^2 dx. \tag{3.13}
$$

By (g_3), (f_1) and (f_2), (3.10), for n large enough,

$$
\begin{aligned}
\int_{B^c_R(0)} \left| g(x, |u_n|^2)\, |u_n|^2 \right| dx &\leq C_1 \int_{B^c_R(0)} (|u_n|^2 + |u_n|^q + |u_n|^6) dx \\
&\leq C_2(\zeta + \zeta^{\frac{q}{2}} + \zeta^3)
\end{aligned} \tag{3.14}
$$

On the other hand, choosing $R > 0$ large if necessary, we may assume that

$$
\int_{B^c_R(0)} \left| g(x, |u|^2)\, |u|^2 \right| dx \leq \zeta.
$$

Hence, from the last inequality and (3.14), we have that

$$
\lim_n \int_{B^c_R(0)} g(x, |u_n|^2)\, |u_n|^2\, dx = \int_{B^c_R(0)} g(x, |u|^2)\, |u|^2\, dx. \tag{3.15}
$$

By the definition of g, one has

$$
g(x, |u_n|^2)\, |u_n|^2 \leq \alpha f(|u_n|^2)\, |u_n|^2 + a^3 + \frac{\nu_0}{\kappa} |u_n|^2, \quad \text{for any } x \in \mathbb{R}^3 \setminus \Omega'_\Gamma.
$$

Since the set $B_R(0) \bigcap (\mathbb{R}^3 \setminus \Omega'_\Gamma)$ is bounded, we can use the above estimates, (f_1), (f_2) and Lebesgue dominated convergence theorem to obtain that

$$
\lim_n \int_{B_R(0) \bigcap (\mathbb{R}^3 \setminus \Omega'_\Gamma)} g(x, |u_n|^2)\, |u_n|^2\, dx = \int_{B_R(0) \bigcap (\mathbb{R}^3 \setminus \Omega'_\Gamma)} g(x, |u|^2)\, |u|^2\, dx. \tag{3.16}
$$

We show now

$$
\lim_n \int_{\Omega'_\Gamma} |u_n|^6 dx = \int_{\Omega'_\Gamma} |u|^6 dx. \tag{3.17}
$$

If (3.17) holds, by (g_3), (f_1), (f_2) and Lebesgue dominated convergence theorem, we have

$$
\lim_n \int_{B_R(0) \cap \Omega'_\Gamma} g(x, |u_n|^2)\, |u_n|^2\, dx = \int_{B_R(0) \cap \Omega'_\Gamma} g(x, |u|^2)\, |u|^2\, dx. \tag{3.18}
$$

Hence, by (3.16) and (3.18), $\lim_n \int_{\mathbb{R}^3} g(x, |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} g(x, |u|^2)|u|^2 dx$. Using (3.10) and the diamagnetic inequality (2.1), the sequence $(|u_n|)$ is tight in, we may assume that

$$
|\nabla |u_n||^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^6 \rightharpoonup \nu \tag{3.19}
$$

in the sense of measures. By the concentration-compactness principle in [37], we can find an at most countable index I, sequences $(x_i) \subset \mathbb{R}^3, (\mu_i), (\nu_i) \subset (0, \infty)$ such that

$$
\begin{aligned}
&\mu \geq |\nabla |u||^2 dx + \sum_{i \in I} \mu_i \delta_{x_i}, \\
&\nu = |u|^6 + \sum_{i \in I} \nu_i \delta_{x_i} \quad \text{and} \quad S\nu_i^{1/3} \leq \mu_i
\end{aligned} \tag{3.20}
$$

for any $i \in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $(x_i)_{i\in I} \cap \Omega'_\Gamma = \emptyset$. Assume, by contradiction, that $x_i \in \Omega'_\Gamma$ for some $i \in I$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$ where $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ is such that $\psi = 1$ in B_1, $\psi = 0$ in $\mathbb{R}^3 \setminus B_2$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3,\mathbb{R})} \leq 2$. We suppose that $\rho > 0$ is such that $\mathrm{supp}(\psi_\rho) \subset \Omega'_\Gamma$. Since $(\psi_\rho u_n)$ is bounded in E_λ, we can see that $\Phi'_\lambda(u_n)[\psi_\rho u_n] = o_n(1)$, that is

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla_A u_n|^2 \psi_\rho dx + \mathrm{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n \nabla\psi_\rho dx + \int_{\mathbb{R}^3}(\lambda V(x)+1)|u_n|^2\psi_\rho dx \\
&\quad = \int_{\mathbb{R}^3} g(x,|u_n|^2)|u_n|^2\psi_\rho dx + o_n(1) \\
&\quad = \alpha\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx + o_n(1).
\end{aligned}$$

Using the diamagnetic inequality (2.1) again, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla|u_n||^2 \psi_\rho dx + \mathrm{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n \nabla\psi_\rho dx \\
&\qquad \leq \alpha\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx + o_n(1).
\end{aligned} \tag{3.21}$$

Due to the fact that f has the subcritical growth and ψ_ρ has the compact support, we have that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} f(|u|^2)|u|^2\psi_\rho dx = 0. \tag{3.22}$$

Now, we show that

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\left|\mathrm{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n\nabla\psi_\rho dx\right| = 0. \tag{3.23}$$

Because of the boundedness of (u_n) in E_λ, using the Hölder inequality, the strong convergence of $(|u_n|)$ in $L^2_{loc}(\mathbb{R}^3,\mathbb{R})$, $|u| \in L^6(\mathbb{R}^3,\mathbb{R})$, $|\nabla\psi_\rho| \leq C\rho^{-1}$ and $|B_{2\rho}(x_i)| \sim \rho^3$, we have that

$$\begin{aligned}
0 &\leq \lim_{\rho\to 0}\limsup_{n\to\infty}\left|\mathrm{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n\nabla\psi_\rho dx\right| \\
&\leq \lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\overline{u_n}\nabla\psi_\rho||\nabla_A u_n| dx \\
&\leq \lim_{\rho\to 0}\lim_{n\to\infty}\left(\int_{B_{2\rho}(x_i)}|\overline{u_n}\nabla\psi_\rho|^2 dx\right)^{1/2}\|u_n\|_\lambda \\
&\leq C\lim_{\rho\to 0}\left(\int_{B_{2\rho}(x_i)}|u|^2 dx\right)^{1/2} = 0
\end{aligned}$$

which shows that (3.23) holds.

Then, taking into account (3.19), (3.21), (3.22) and (3.23), we can conclude that $\nu_i \geq \mu_i$ for all $i \in I$. Together with the inequality $S\nu_i^{1/3} \leq \mu_i$ in (3.20), we have

$$\nu_i \geq S^{\frac{3}{2}}. \tag{3.24}$$

Now, from (f_3), (g_4) and (g_5), we have

$$
\begin{aligned}
c &= \Phi_\lambda(u_n) - \frac{1}{4} < \Phi'_\lambda(u_n), u_n > + o_n(1) \\
&= \frac{1}{4}\|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \Big(\frac{1}{4} g(x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(x,|u_n|^2)\Big)dx + o_n(1) \\
&\geq \frac{1}{4}\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3\setminus\Omega'_\Gamma} \Big(\frac{1}{4} g(x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(x,|u_n|^2)\Big)dx \\
&\quad + \frac{1}{12}\int_{\Omega'_\Gamma} |u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\Big(\int_{\Omega'_\Gamma} \psi_\rho |\nabla|u_n||^2 dx + \int_{\mathbb{R}^3\setminus\Omega'_\Gamma} (\lambda V(x)+1)|u_n|^2\Big) - \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} G(x,|u_n|^2)dx \\
&\quad + \frac{1}{12}\int_{\Omega'_\Gamma} |u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon} \psi_\rho |\nabla|u_n||^2 dx + (\frac{1}{4} - \frac{1}{4\kappa})\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} (\lambda V(x)+1)|u_n|^2 dx + \frac{1}{12}\int_{\Lambda_\varepsilon} \psi_\rho |u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Omega'_\Gamma} \psi_\rho |\nabla|u_n||^2 dx + \frac{1}{12}\int_{\Omega'_\Gamma} \psi_\rho |u_n|^6 dx + o_n(1).
\end{aligned}
$$

From the above arguments, (3.20) and (3.24), we have

$$
\begin{aligned}
c &\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}} \psi_\rho(x_i)\mu_i + \frac{1}{12}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}} \psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}} \psi_\rho(x_i) S\nu_i^{1/3} + \frac{1}{12}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}} \psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4}S^{\frac{3}{2}} + \frac{1}{12}S^{\frac{3}{2}} = \frac{1}{3}S^{\frac{3}{2}}
\end{aligned}
$$

which gives a contradiction. This means that (3.17) holds.

From (3.8), (3), (3.12) and (3.13), we may obtain that $\|u_n\|_\lambda^2 \to \|u\|_\lambda^2$ which means that $u_n \to u$ in E_λ. □

Next we study the behavior of a $(PS)_\infty$ sequence, that is, a sequence $(u_n) \subset H_A^1(\mathbb{R}^3, \mathbb{C})$ satisfying

$$
\begin{aligned}
& u_n \in E_{\lambda_n} \quad \text{and} \quad \lambda_n \to \infty, \\
& \Phi_{\lambda_n}(u_n) \to c, \\
& \left\|\Phi'_{\lambda_n}(u_n)\right\|_{E^*_{\lambda_n}} \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Proposition 3.5. *Let $(u_n) \subset H_A^1(\mathbb{R}^3, \mathbb{C})$ be a $(PS)_\infty$ sequence with $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$. Then, up to a subsequence, there exists $u \in H_A^1(\mathbb{R}^3, \mathbb{C})$ such that $u_n \rightharpoonup u$ in $H_A^1(\mathbb{R}^3, \mathbb{C})$. Moreover,*

(i) $u = 0$ in $\mathbb{R}^3 \setminus \Omega_\Gamma$, and for all $j \in \Gamma$, $u|_{\Omega_j}$ is a solution for

$$
\begin{cases}
-(\nabla + iA(x))^2 u + u + \phi_{|u|}u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \Omega_j, \\
u = 0, & \text{on } \partial\Omega_j;
\end{cases}
\tag{3.25}
$$

(ii) $u_n \to u$ in E_{λ_n}. Hence

$$
u_n \to u \quad \text{in } H_A^1(\mathbb{R}^3, \mathbb{C}); \tag{3.26}
$$

(iii) $\lambda_n \int_{\mathbb{R}^3} V(x)\,|u_n|^2\,dx \to 0.$

(iv) $\|u_n\|^2_{\lambda_n,\Omega'_j} \to \int_{\Omega_j}(|\nabla_A u|^2 + |u|^2)dx,$ *for* $j \in \Gamma$*;*

(v) $\|u_n\|^2_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Gamma} \to 0$*;*

(vi) $\Phi_{\lambda_n}(u_n) \to \frac{1}{2}\int_{\Omega_\Gamma}\left(|\nabla_A u|^2 + |u|^2\right)dx + \frac{1}{4}\int_{\Omega_\Gamma}\phi_{|u|}|u|^2dx - \alpha\int_{\Omega_\Gamma}F(|u|^2)dx - \frac{1}{6}\int_{\Omega_\Gamma}|u|^6dx.$

Proof. As in Lemma 3.1, we know that (u_n) is bounded in $H^1_A(\mathbb{R}^3,\mathbb{C})$. Thus we may assume that for some $u \in H^1_A(\mathbb{R}^3,\mathbb{C})$, up to a subsequence, if necessary

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1_A\left(\mathbb{R}^3,\mathbb{C}\right), \\ u_n &\to u \quad \text{in } L^r_{loc}\left(\mathbb{R}^3,\mathbb{C}\right), \quad \forall r \geq 1, \\ |u_n| &\to |u| \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

(i) We fix the set $C_m = \left\{x \in \mathbb{R}^3; V(x) \geq \frac{1}{m}\right\}$, for each $m \in \mathbb{N}$. Then, we have

$$\begin{aligned} \int_{C_m}|u_n|^2\,dx &\leq \frac{m}{\lambda_n}\int_{\mathbb{R}^3}\lambda_n V(x)\,|u_n|^2\,dx \\ &\leq \frac{2m}{\lambda_n}\int_{\mathbb{R}^3}\left(|\nabla_A u_n|^2 + (\lambda_n V(x)+1)\,|u_n|^2\right)dx \\ &= \frac{2m}{\lambda_n}\,\|u_n\|^2_{\lambda_n}. \end{aligned}$$

By the Fatou's lemma, we derive

$$\int_{C_m}|u|^2\,dx = 0.$$

So, $u = 0$ in $\cup_{m=1}^{+\infty}C_m = \mathbb{R}^3 \setminus \Omega$, from which we can assert that $u|_{\Omega_j} \in H^{0,1}_A\left(\Omega_j,\mathbb{C}\right)$ for any $j \in \{1,2,\ldots,k\}$.

From (f_1), (f_2), for any $\zeta > 0$, there exists $C_\zeta > 0$ such that

$$|f(t)| \leq \zeta\,|t| + C_\zeta\,|t|^{\frac{q-2}{2}}.$$

So, we derive

$$\left|\mathrm{Re}\int_{\mathbb{R}^3}g\left(x,|u_n|^2\right)u_n\bar{v}dx\right| \leq \zeta\alpha\int_{\mathbb{R}^3}|u_n|^3\,|\bar{v}|dx + C_\zeta\alpha\int_{\mathbb{R}^3}|u_n|^{q-1}\,|\bar{v}|dx + \int_{\mathbb{R}^3}|u_n|^5\,|\bar{v}|dx.$$

Therefore,

$$\mathrm{Re}\int_{\mathbb{R}^3}g\left(x,|u_n|^2\right)u_n\bar{v}dx \to \mathrm{Re}\int_{\mathbb{R}^3}g\left(x,|u|^2\right)u\bar{v}dx.$$

Since for each $v \in C_0^\infty\left(\Omega_j,\mathbb{C}\right)$, $\Phi'_{\lambda_n}(u_n)\,v \to 0$ as $n \to \infty$, from the above information and the argument explored in Proposition 3.4, we have

$$\mathrm{Re}\left(\int_{\Omega_j}\left(\nabla_A u\overline{\nabla_A v} + u\bar{v}\right)dx + \int_{\Omega_j}\phi_{|u|}u\bar{v}dx - \int_{\Omega_j}g\left(x,|u|^2\right)u\bar{v}dx\right) = 0,$$

which implies that $u|_{\Omega_j}$ is a solution of problem (3.25) for each $j \in \Gamma$.

On the other hand, if $j \in \{1,2,\ldots,k\} \setminus \Gamma$, setting $v = u|_{\Omega_j}$,

$$\int_{\Omega_j}\left(|\nabla_A u|^2 + |u|^2\right)dx + \int_{\Omega_j}\phi_{|u|}\,|u|^2\,dx - \int_{\Omega_j}\widetilde{f}\left(|u|^2\right)|u|^2\,dx = 0.$$

By Lemma 2.3 and the definition of $\widetilde{f}$, we have

$$
\begin{aligned}
0 \leq \delta_0 \|u\|^2_{\lambda,\Omega_j} &\leq \|u\|^2_{\lambda,\Omega_j} - \frac{\nu_0}{k}\|u\|^2_{2,\Omega_j} \\
&\leq \int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx - \int_{\Omega_j} \tilde{f}\left(|u|^2\right)|u|^2 dx \leq 0.
\end{aligned}
$$

Thus $u|_{\Omega_j} = 0$ for $j \in \{1, 2, \ldots, k\} \setminus \Gamma$. This proves that $u = 0$ in $\mathbb{R}^3 \setminus \Omega_\Gamma$.

(ii) From the similar arguments in the proof of Proposition 3.4,

$$
\begin{aligned}
\int_{\mathbb{R}^3} g\left(x, |u_n|^2\right)|u_n|^2 dx &\to \int_{\mathbb{R}^3} g\left(x, |u|^2\right)|u|^2 dx \\
&= \alpha \int_{\Omega_\Gamma} f\left(|u|^2\right)|u|^2 dx + \int_{\Omega_\Gamma} |u|^6 dx \quad \text{as } n \to +\infty.
\end{aligned}
$$

By (i), we have

$$
\begin{aligned}
o_n(1) &= \Phi'_{\lambda_n}(u_n)(u_n) \\
&= \|u_n\|^2_{\lambda_n} + \int_{\mathbb{R}^3} \phi_{|u_n|}(x)|u_n|^2 dx - \int_{\mathbb{R}^3} g\left(x, |u_n|^2\right)|u_n|^2 dx \\
&= \|u_n\|^2_{\lambda_n} - \|u\|^2_{\lambda_n} + o_n(1),
\end{aligned}
$$

which implies $u_n \to u$ in E_{λ_n}. Hence $u_n \to u$ in $H^1_A(\mathbb{R}^3, \mathbb{C})$.

(iii) By (ii),

$$
\begin{aligned}
\lambda_n \int_{\mathbb{R}^3} V(x)|u_n|^2 dx &= \lambda_n \int_{\mathbb{R}^3} V(x)|u_n - u|^2 dx \\
&\leq C\|u_n - u\|^2_{\lambda_n} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

(iv) Let $j \in \Gamma$. By (ii),

$$
|u_n - u|^2_{2,\Omega'_j} \to 0, \qquad |\nabla_A u_n - \nabla_A u|^2_{2,\Omega'_j} \to 0,
$$

therefore,

$$
\int_{\Omega'_\Gamma} \left(|\nabla_A u_n|^2 - |\nabla_A u|^2\right) dx \to 0 \quad \text{and} \quad \int_{\Omega'_\Gamma} \left(|u_n|^2 - |u|^2\right) dx \to 0.
$$

Also, by (iii),

$$
\int_{\Omega'_\Gamma} \lambda_n V(x)|u_n|^2 dx \to 0.
$$

Thus,

$$
\|u_n\|^2_{\lambda_n,\Omega'_\Gamma} \to \int_{\Omega_\Gamma} \left(|\nabla_A u|^2 + u^2\right) dx.
$$

(v) By (ii), it is easy to obtain that

$$
\|u_n\|^2_{\lambda,\mathbb{R}^3\setminus\Omega_\Gamma} \to 0.
$$

(vi) Since

$$
\begin{aligned}
\Phi_{\lambda_n}(u_n) = &\sum_{j\in\Gamma} \left[\frac{1}{2}\int_{\Omega'_j} \left(|\nabla_A u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2\right) dx + \frac{1}{4}\int_{\Omega'_j} \phi_{|u_n|}|u_n|^2 dx\right] \\
&+ \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} \left(|\nabla_A u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} \phi_{|u_n|}|u_n|^2 dx \\
&- \int_{\mathbb{R}^3} G(x, u_n)\, dx,
\end{aligned}
$$

by (i)–(v), we can derive

$$\Phi_{\lambda_n}(u_n) \to \frac{1}{2}\int_{\Omega_\Gamma}\left(|\nabla_A u|^2+|u|^2\right)dx+\frac{1}{4}\int_{\Omega_\Gamma}\phi_{|u|}u^2dx-\alpha\int_{\Omega_\Gamma}F(|u|^2)dx-\frac{1}{6}\int_{\Omega_\Gamma}|u|^6dx. \quad \square$$

Now, we study L^∞ estimate of the solution of problem (3.3).

Proposition 3.6. *Let (u_λ) be a family of nontrivial solutions of (3.3). Then, there exists $\lambda^*>0$ such that*

$$\|u_\lambda\|^2_{L^\infty(\mathbb{R}^3\setminus\Omega'_\Gamma)} \le a, \quad \forall \lambda \ge \lambda^*.$$

In particular, u_λ is a solution of the original problem (1.4) for any $\lambda \ge \lambda^*$.

Proof. We give notation $B_r(x)=\{y\in\mathbb{R}^3 : |x-y|<r\}$. Since $u_\lambda \in E_\lambda$ is a critical point of $\Phi_\lambda(u)$, that is, u_λ satisfies the following equation

$$-\Delta_A u_\lambda + (\lambda V(x)+1)u_\lambda + \phi_{|u_\lambda|}u_\lambda = g\left(x,|u_\lambda|^2\right)u_\lambda, \quad x\in\mathbb{R}^3.$$

By the Kato's inequality

$$\Delta|u_\lambda| \ge \mathrm{Re}\left(\frac{\overline{u_\lambda}}{|u_\lambda|}(\nabla+iA(x))^2u_\lambda(x)\right),$$

there holds

$$\Delta|u_\lambda(x)|-(\lambda V(x)+1)|u_\lambda(x)|-\phi_{|u_\lambda|}|u_\lambda(x)|-g(x,|u_\lambda|^2)|u_\lambda(x)|\ge 0, \qquad x\in\mathbb{R}^3,$$

since $|u_\lambda|\ge 0$, $\phi_{|u_\lambda|}\ge 0$ and $(\lambda V(x)+1)\ge M_0>0$ if $\lambda\ge 1$, we have

$$\Delta|u_\lambda(x)|-g\left(x,|u_\lambda|^2\right)|u_\lambda(x)|\ge 0, \qquad x\in\mathbb{R}^3.$$

We use the subsolution estimate (see [16], Theorem 8.17) and obtain that there exists a constant $C(r)$ such that for $1<q<2$

$$\sup_{y\in B_r(x)}|u_\lambda(y)|\le C(r)\Big(\int_{B_{2r}(x)}|u_\lambda|^q dy\Big)^{1/q}.$$

By Proposition 3.5, for any sequence $\lambda_n\to\infty$, we can extract a subsequence λ_{n_i} such that

$$u_{\lambda_{n_i}} \to u \in H_A^{0,1}(\Omega_\Gamma,\mathbb{C}) \quad \text{strongly in } H_A^1(\mathbb{R}^N,\mathbb{C}).$$

In particular,

$$u_{\lambda_{n_i}} \to 0 \quad \text{in } L^2(\mathbb{R}^N\setminus\overline{\Omega_\Gamma},\mathbb{C}).$$

Since $\lambda_n\to\infty$ is arbitrary, we have

$$u_\lambda \to 0 \quad \text{in } L^2(\mathbb{R}^N\setminus\overline{\Omega_\Gamma},\mathbb{C}) \quad \text{as } \lambda\to\infty.$$

Thus, choosing $r\in(0,\mathrm{dist}(\Omega_\Gamma,\mathbb{R}^N\setminus\Omega'_\Gamma))$, we have uniformly in $x\in\mathbb{R}^N\setminus\Omega'_\Gamma$ that

$$\begin{aligned}|u_\lambda(y)| &\le C(r)\|u_\lambda\|_{L^q(B_{2r}(x))}\\ &\le C(r)|B_{2r}(x)|^{\frac{2-q}{2q}}\|u_\lambda\|_{L^2(\mathbb{R}^N\setminus\overline{\Omega_\Gamma})}\\ &\to 0.\end{aligned}$$

This finishes the proof. $\square$

4 Existence of multi-bump solutions

In this section, we start to prove the existence of multi-bump solutions. First of all, for each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the functional $I_j : H_A^{0,1}(\Omega_j, \mathbb{C}) \to \mathbb{R}$ given by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx + \frac{1}{4}\int_{\Omega_j} \phi_{|u|}|u|^2 dx - \frac{\alpha}{2}\int_{\Omega_j} F(|u|^2) dx - \frac{1}{6}\int_{\Omega_j} |u|^6 dx,$$

and

$$c_j = \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)),$$

where

$$\Lambda_j = \left\{\gamma \in C([0,1], H_A^{0,1}(\Omega_j, \mathbb{C})) : \gamma(0) = 0, I_j(\gamma(1)) < 0\right\}.$$

For each $j \in \Gamma$, we denote by $\Phi_{\lambda,j} : H_A^1\left(\Omega'_j, \mathbb{C}\right) \to \mathbb{R}$ the functional given by

$$\begin{aligned}\Phi_{\lambda,j}(u) = &\frac{1}{2}\int_{\Omega'_j} \left(|\nabla_A u|^2 + (\lambda V(x)+1)|u|^2\right) dx \\ &+ \frac{1}{4}\int_{\Omega'_j}\left(\frac{1}{4\pi}\int_{\Omega'_j}\frac{|\widetilde{u}|^2}{|x-y|}dy\right)u^2 dx - \frac{\alpha}{2}\int_{\Omega'_j} F(|u|^2)dx - \frac{1}{6}\int_{\Omega'_j}|u|^6 dx,\end{aligned}$$

and the above functional is associated to the following problem

$$\begin{cases} -\Delta_A u + (\lambda V(x)+1)u + \left(\frac{1}{4\pi}\int_{\Omega'_j}\frac{|\widetilde{u}|^2}{|x-y|}dy\right)u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \Omega'_j, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega'_j, \end{cases}$$

where

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{in } \Omega'_j, \\ 0, & \text{in } \mathbb{R}^3 \setminus \Omega'_j. \end{cases}$$

In what follows, we denote by $c_{\lambda,j}$ the minimax level of the above functional given by

$$c_{\lambda,j} = \inf_{\gamma \in \Lambda_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)),$$

where

$$\Lambda_{\lambda,j} = \left\{\gamma \in C\left([0,1], H_A^1\left(\Omega'_j, \mathbb{C}\right)\right) : \gamma(0) = 0, \Phi_{\lambda,j}(\gamma(1)) < 0\right\}.$$

Repeating the same method used in the previous section, we are able to prove that there exist $\omega_j \in H_A^{0,1}(\Omega_j, \mathbb{C})$ and $\omega_{\lambda,j} \in H_A^1\left(\Omega'_j, \mathbb{C}\right)$ such that

$$I_j(\omega_j) = c_j \quad \text{and} \quad I'_j(\omega_j) = 0,$$

and

$$\Phi_{\lambda,j}(\omega_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \Phi'_{\lambda,j}(\omega_{\lambda,j}) = 0.$$

Furthermore, we have the following important lemma.

Lemma 4.1. *The following statements hold:*

(i) $0 < c_{\lambda,j} \leq c_j$, *for* $\lambda \geq 1$ *and* $j \in \Gamma$.

(ii) c_j *(*$c_{\lambda,j}$ *respectively) is a least energy level for* $I_j(u)$ *(*$\Phi_{\lambda,j}(u)$ *respectively), that is*

$$c_j = \inf\left\{ I_j(u) : u \in H_A^{0,1}(\Omega_j, \mathbb{C}) \setminus \{0\}, I_j'(u)u = 0 \right\},$$

and

$$c_{\lambda,j} = \inf\left\{ \Phi_{\lambda,j}(u) : u \in H_A^1\left(\Omega_j', \mathbb{C}\right) \setminus \{0\}, \Phi_{\lambda,j}'(u)u = 0 \right\}.$$

(iii) $c_{\lambda,j} \to c_j$, *as* $\lambda \to \infty$ *for any* $j \in \Gamma$.

Proof. (i) From (f_3), we have $c_j > 0$ and $c_{\lambda,j} > 0$ for any $j \in \Gamma$ and $\lambda \geq 1$. For any $u \in H_A^{0,1}(\Omega_j, \mathbb{C})$, we may extend u to $\widehat{u} \in H_A^1\left(\Omega_j', \mathbb{C}\right)$ by

$$\widehat{u}(x) = \begin{cases} u(x), & \text{in } \Omega_j, \\ 0, & \text{in } \Omega_j' \setminus \overline{\Omega}_j. \end{cases}$$

Using the fact that $H_A^{0,1}(\Omega_j, \mathbb{C}) \subset H_A^1\left(\Omega_j', \mathbb{C}\right)$, we have

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma \in \Lambda_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &\leq \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &= \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)) = c_j. \end{aligned}$$

(ii) By the monotonicity of the term $f(t)$ with respect to t for $t > 0$, we are able to prove this.

(iii) Using Proposition 3.5, for sequences (λ_n) with $\lambda_n \to \infty$, as $n \to \infty$, there exists $\omega \in H_A^{0,1}(\Omega_j, \mathbb{C})$ is a solution of (3.25) such that

$$\omega_{\lambda_n,j} \to \omega \quad \text{in } H_A^1(\Omega_j', \mathbb{C}),$$

and

$$\Phi_{\lambda_n,j}(\omega_{\lambda_n,j}) \to I_j(\omega).$$

By the definition of c_j, we have

$$\limsup_{\lambda \to \infty} c_{\lambda,j} = \limsup_{\lambda \to \infty} \Phi_{\lambda,j}\left(\omega_{\lambda,j}\right) \geq I_j(\omega) \geq c_j.$$

Together with (i), we get the asserted result. $\square$

In what follows, we fix $R > 1$ verifying

$$\left| I_j\left(\frac{1}{R}\omega_j\right) \right| < \frac{1}{2}c_j, \qquad \forall j \in \Gamma, \tag{4.1}$$

and

$$\left| I_j\left(R\omega_j\right) - c_j \right| \geq 1, \qquad \forall j \in \Gamma. \tag{4.2}$$

By the definition of c_j, we are able to obtain

$$\max_{s_j \in [1/R^2,1]} I_j\left(s_j R\omega_j\right) = c_j, \qquad \forall j \in \Gamma.$$

Then, for $\Gamma = \{1,2,\ldots,l\}$ $(l \leq k)$, we define

$$\gamma_0(\boldsymbol{s})(x) = \textstyle\sum_{j=1}^{l} s_j R\omega_j(x) \quad \forall \boldsymbol{s} = (s_1, s_2, \ldots, s_l) \in \left[1/R^2, l\right]^l,$$

$$\Lambda_* = \left\{\gamma \in C\left(\left[1/R^2,1\right]^l, E_\lambda \setminus \{0\}\right) : \gamma = \gamma_0 \text{ on } \partial\left(\left[1/R^2,1\right]^l\right)\right\},$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma\in\Lambda_*} \max_{\boldsymbol{s}\in[1/R^2,1]^l} \Phi_\lambda\left(\gamma\left(\boldsymbol{s}\right)\right).$$

Next, let us denote by $c_\Gamma = \sum_{j=1}^{l} c_j$ and $c_{\lambda,\Gamma} = \sum_{j=1}^{l} c_{\lambda,j}$. Moreover, from Remark 3.3, we know that $c_\Gamma \in (0, \frac{1}{3}S^{\frac{3}{2}})$. To prove an important relation among $b_{\lambda,\Gamma}$, c_Λ and $c_{\lambda,\Gamma}$, we need to the following lemma.

Lemma 4.2. *For any $\gamma \in \Lambda_*$, there exists $(t_1, t_2, \ldots, t_l) \in \left[1/R^2, 1\right]^l$ such that*

$$\Phi'_{\lambda,j}\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right)\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right) = 0 \quad \textit{for all } j \in \{1,2,\ldots,l\}.$$

Proof. Given $\gamma \in \Lambda_*$, consider $\widetilde{\gamma} : \left[1/R^2, 1\right]^l \to \mathbb{C}^l$ defined by

$$\widetilde{\gamma}\left(s_1, s_2, \ldots, s_l\right) = \left(\Phi'_{\lambda,1}(\gamma)(\gamma), \Phi'_{\lambda,2}(\gamma)(\gamma), \ldots, \Phi'_{\lambda,l}(\gamma)(\gamma)\right),$$

where

$$\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right)\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right) \qquad \text{for all } j \in \Gamma.$$

By (f_4) and $I'_j(\omega_j) = 0$, we have

$$I'_j\left(R\omega_j\right)\left(R\omega_j\right) < 0 \quad \text{and} \quad I'_j\left(\frac{1}{R}\omega_j\right)\left(\frac{1}{R}\omega_j\right) > 0.$$

For $\boldsymbol{s} \in \partial(\left[1/R^2,1\right]^l)$, it holds $\gamma(\boldsymbol{s}) = \gamma_0(\boldsymbol{s})$, and

$$\Phi'_{\lambda,j}\left(\gamma_0\left(\boldsymbol{s}\right)\right)\left(\gamma_0\left(\boldsymbol{s}\right)\right) = 0 \Rightarrow s_j \notin \left\{1/R^2, 1\right\}, \qquad \forall j \in \Gamma.$$

Thus,

$$(0,0,\ldots,0) \notin \widetilde{\gamma}\left(\partial\left(\left[1/R^2,1\right]^l\right)\right).$$

Since

$$\deg\left(\widetilde{\gamma}, \left(1/R^2,1\right)^l, (0,\ldots,0)\right) = \deg\left(\widetilde{\gamma}_0, \left(1/R^2,1\right)^l, (0,\ldots,0)\right)$$

and, for $\boldsymbol{s} \in \left(1/R^2,1\right)^l$,

$$\widetilde{\gamma}_0(\boldsymbol{s}) = 0 \Longleftrightarrow \boldsymbol{s} = \left(\frac{1}{R}, \ldots, \frac{1}{R}\right),$$

we have

$$\deg\left(\widetilde{\gamma}, \left(1/R^2,1\right)^l, (0,\ldots,0)\right) \neq 0.$$

This shows what was stated. $\square$

Proposition 4.3. *The following facts hold:*

(i) $c_{\lambda,\Gamma} \leq b_{\lambda,\Gamma} \leq c_\Gamma, \forall \lambda \geq 1$;

(ii) $b_{\lambda,\Gamma} \to c_\Gamma$, *as* $\lambda \to \infty$;

(iii) $\Phi_\lambda(\gamma(\boldsymbol{s})) < c_\Gamma, \forall \lambda \geq 1, \gamma \in \Lambda_*$ *and* $\boldsymbol{s} = (s_1, \ldots, t_l) \in \partial([1/R^2, 1]^l)$;

(iv) $b_{\lambda,\Gamma}$ *is a critical point of* Φ_λ *for large* λ.

Proof. (i) Since $\gamma_0 \in \Lambda_*$,

$$\begin{aligned} b_{\lambda,\Gamma} &\leq \max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]} \Phi_\lambda\left(\gamma_0\left(s_1, s_2, \ldots, s_l\right)\right) \\ &= \max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]} \sum_{j=1}^{l} I_j\left(s_j R\omega_j\right) \\ &= \sum_{j=1}^{l} c_j = c_\Gamma. \end{aligned}$$

Fixing $(t_1, t_2, \ldots, t_l) \in \left[1/R^2, 1\right]^l$ given in Lemma 4.2 and recalling that $c_{\lambda,j}$ can be characterized by

$$c_{\lambda,j} = \inf\left\{\Phi_{\lambda,j}(u) : u \in H_A^1\left(\Omega_j', \mathbb{C}\right) \setminus \{0\}, \Phi_{\lambda,j}'(u)u = 0\right\},$$

it follows that

$$\Phi_{\lambda,j}\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right) \geq c_{\lambda,j} \qquad \forall j \in \Gamma.$$

Since $\forall u \in H_A^1\left(\mathbb{R}^3 \setminus \Omega_\Gamma', \mathbb{C}\right)$, $\Phi_{\lambda,\mathbb{R}^2\setminus\Omega_\Gamma'}(u) \geq 0$, we have

$$\Phi_\lambda\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right) \geq \sum_{j=1}^{l} \Phi_{\lambda,j}\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right).$$

Hence

$$\max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]^l} \Phi_\lambda\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right) \geq \Phi_\lambda\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right) \geq \sum_{j=1}^{l} c_{\lambda,j}$$

showing that

$$b_{\lambda,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j} = c_{\lambda,\Gamma}.$$

(ii) Since $c_{\lambda,j} \to c_j$, as $\lambda \to \infty$, by the previous item,

$$b_{\lambda,\Gamma} \to c_\Gamma, \quad as\ \lambda \to \infty.$$

(iii) For $\boldsymbol{s} \in \partial([1/R^2, 1]^l)$, it holds $\gamma(\boldsymbol{s}) = \gamma_0(\boldsymbol{s})$. Hence,

$$\Phi_\lambda\left(\gamma_0\left(s_1, s_2, \ldots, s_l\right)\right) = \sum_{j=1}^{l} I_j\left(s_j R\omega_j\right).$$

From (4.1) and (4.2), we have

$$\Phi_\lambda\left(\gamma_0\left(s_1, s_2, \ldots, s_l\right)\right) \leq c_\Gamma - \epsilon.$$

for some $\epsilon > 0$, so (iii) holds.
(iv) By (ii), we can choose λ large enough such that $b_{\lambda,\Gamma}, c_\Gamma \in (0, \frac{1}{3}S^{\frac{3}{2}})$. From Proposition 3.4 and (3.6), we know that any $(PS)_{b_{\lambda,\Gamma}}$ sequence of Φ_λ has a convergence subsequence in E_λ. Moreover, from the deformation lemma, we can conclude that $b_{\lambda,\Gamma}$ is a critical level of Φ_λ for λ large. □

To prove Theorem 1.1, we need to find a nontrivial solution u_λ for the large λ which approaches a least energy solution in each Ω_j $(j \in \Gamma)$ and to 0 elsewhere as $\lambda \to \infty$. Therefore, we shall show two propositions which imply together with the estimates made in the previous section that Theorem 1.1 holds.

Henceforth, let

$$\Phi_\lambda^{c_\Gamma} = \{u \in E_\lambda : \Phi_\lambda(u) \leq c_\Gamma\}.$$

For small $\mu > 0$, we denote by

$$A_\mu^\lambda = \left\{u \in E_\lambda : \|u\|_{\lambda,\mathbb{R}^3\setminus\Omega_j'} \leq \mu, \left|\Phi_{\lambda,j}(u) - c_j\right| \leq \mu, \forall j \in \Gamma\right\},$$

and observe that $\omega = \sum_{j=1}^{l} \omega_j \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, showing that $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma} \neq \emptyset$. Fixing

$$0 < \mu < \frac{1}{3}\min\left\{c_j, j \in \Gamma\right\}. \tag{4.3}$$

We obtain the following uniform estimate of $\left\|\Phi_\lambda'(u)\right\|_\lambda$ on the annulus $\left(A_{2\mu}^\lambda \setminus A_\mu^\lambda\right) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 4.4. *Let $\mu > 0$ satisfying (4.3). Then there exist $\sigma_0 > 0$ and $\lambda^* \geq 1$ independent of λ such that*

$$\left\|\Phi_\lambda'(u)\right\|_\lambda \geq \sigma_0 \quad \text{for } \lambda \geq \lambda^* \quad \text{for all } u \in \left(A_{2\mu}^\lambda \setminus A_\mu^\lambda\right) \cap \Phi_\lambda^{c_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and $u_n \in \left(A_{2\mu}^{\lambda_n} \setminus A_\mu^{\lambda_n}\right) \cap \Phi_{\lambda_n}^{c_\Gamma}$ such that $\left\|\Phi_{\lambda_n}'(u)\right\|_{\lambda_n} \to 0$.

Since $u_n \in A_{2\mu}^{\lambda_n}$, we can obtain that $\left\{\|u_n\|_{\lambda_n}\right\}$ is a bounded in $E_{\lambda_n}\left(\mathbb{R}^3, \mathbb{C}\right)$ and $H_A^1\left(\mathbb{R}^3, \mathbb{C}\right)$, and $\{\Phi_{\lambda_n}(u_n)\}$ is also bounded. Thus, passing a subsequence if necessary, we may assume that

$$\Phi_{\lambda_n}(u_n) \to c \in (-\infty, c_\Gamma].$$

From Proposition 3.5, there exists $u \in H_A^{0,1}(\Omega_\Gamma, \mathbb{C})$ such that u is a solution of (3.25),

$$\begin{aligned}
&u_n \to u \quad \text{in } H_A^1(\mathbb{R}^3, \mathbb{C}),\\
&\lim_{n\to\infty} \Phi_{\lambda_n}(u_n) = \sum_{j=1}^{l} I_j(u) \leq c_\Gamma,\\
&\|u_n\|_{\lambda_n,\Omega_j'}^2 \to \int_{\Omega_j}\left(|\nabla_A u|^2 + |u|^2\right)dx, \qquad \forall j \in \Gamma,\\
&\lambda_n \int_{\mathbb{R}^3} V(x)\,|u_n|^2\,dx \to 0,\\
&\|u_n\|_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Gamma}^2 \to 0.
\end{aligned}$$

Since $c_\Gamma = \sum_{j=1}^{l} c_j$ and c_j is the least energy level for I_j, we have two possibilities:
(i) $I_j(u|_{\Omega_j}) = c_j$ $\forall j \in \Gamma$;
(ii) $I_{j_0}(u|_{\Omega_{j_0}}) = 0$, that is $u|_{\Omega_{j_0}} \equiv 0$ for some $j_0 \in \Gamma$.

If (i) occurs, we have

$$\frac{1}{2}\int_{\Omega_j}\left(|\nabla_A u|^2+|u|^2\right)dx+\frac{1}{4}\int_{\Omega_j}\phi_{|u|}\,|u|^2\,dx-\frac{\alpha}{2}\int_{\Omega_j}F(|u|^2)dx-\frac{1}{6}\int_{\Omega_j}|u|^6dx=c_j,\qquad \forall j\in\Gamma.$$

Thus, $\left|\Phi_{\lambda,j}(u)-c_j\right|\le\mu$, $\forall j\in\Gamma$, that is, $u_n\in A_\mu^{\lambda_n}$ for large n, which is a contradiction to the assumption $u_n\in A_{2\mu}^{\lambda_n}\setminus A_\mu^{\lambda_n}$.

If (ii) occurs, we have

$$\left|\Phi_{\lambda_n,j_0}(u_n)-c_{j_0}\right|\to c_{j_0}\ge 3\mu,$$

which is a contradiction with the fact that $u_n\in A_{2\mu}^{\lambda_n}\setminus A_\mu^{\lambda_n}$. Thus neither (i) nor (ii) can hold, and the proof is completed. □

Proposition 4.5. *Let $\mu>0$ satisfying (4.3) and $\lambda^*\ge 0$ be a constant given in Proposition 4.4. Then, for any $\lambda\ge\lambda^*$, there exists a nontrivial solution u_λ of (3.3) satisfying $u_\lambda\in A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}$.*

Proof. Arguing by contradiction, we assume that there are no critical points in $A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}$. Since Φ_λ verifies the (PS) condition in the level $(0,\frac{1}{3}S^{\frac{3}{2}})$, there exists a constant $d_\lambda>0$ such that

$$\left\|\Phi_\lambda'(u)\right\|\ge d_\lambda\quad\text{for all}\quad u\in A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}.$$

From Proposition 4.4, we have

$$\left\|\Phi_\lambda'(u)\right\|\ge\sigma_0\quad\text{for all } u\in\left(A_{2\mu}^\lambda\setminus A_\mu^\lambda\right)\cap\Phi_\lambda^{c_\Gamma},$$

where $\sigma_0>0$ does not depend on λ. In what follows, $\Psi:E_\lambda\to\mathbb{R}$ is a continuous functional verifying

$$\begin{array}{ll}\Psi(u)=1 & \text{for } u\in A_{3\mu/2}^\lambda,\\ \Psi(u)=0 & \text{for } u\notin A_{2\mu}^\lambda,\\ 0\le\Psi(u)\le 1 & \text{for } u\in E_\lambda\left(\mathbb{R}^3,\mathbb{C}\right).\end{array}$$

We consider $H:\Phi_\lambda^{c_\Gamma}\to E_\lambda$ given by

$$H(u)=\begin{cases}-\Psi(u)\frac{\Phi_\lambda'(u)}{\left\|\Phi_\lambda'(u)\right\|_\lambda}, & u\in A_{2\mu}^\lambda,\\ 0, & u\notin A_{2\mu}^\lambda.\end{cases}$$

Hence, we have the inequality

$$\|H(u)\|_\lambda\le 1\quad\forall\lambda\ge\Lambda_*\quad\text{and}\quad u\in\Phi_\lambda^{c_\Gamma}.$$

Considering the deformation flow $\eta:[0,\infty)\times\Phi_\lambda^{c_\Gamma}\to\Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt}=H(\eta)\quad\text{and}\quad\eta(0,u)=u\in\Phi_\lambda^{c_\Gamma}.$$

Thus η has the following properties

$$\frac{d}{dt}\Phi_\lambda(\eta(t,u))=-\Psi(\eta(t,u))\left\|\Phi_\lambda'(\eta(t,u))\right\|_\lambda\le 0,\tag{4.4}$$

$$\eta(t,u)=u\quad\text{for all } t\ge 0 \text{ and } u\in\Phi_\lambda^{c_\Gamma}\setminus A_{2\mu}^\lambda,\tag{4.5}$$

$$\left\| \frac{d\eta}{dt} \right\|_\lambda \leq 1 \quad \text{for all } t, u. \tag{4.6}$$

Now let $\gamma_0(\boldsymbol{s}) \in \Lambda_*$ and we consider $\eta(t, \gamma_0(\boldsymbol{s}))$ for large t. If μ satisfies (4.3), we have that

$$\gamma_0(\boldsymbol{s}) \notin \mathcal{A}^\lambda_{2\mu}, \qquad \forall \boldsymbol{s} \in \partial\left([1/R^2, 1]^l\right).$$

Since

$$\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) < c_\Gamma, \qquad \forall \boldsymbol{s} \in \partial\left([1/R^2, 1]^l\right),$$

from (4.5), it follows that

$$\eta\left(t, \gamma_0(\boldsymbol{s})\right) = \gamma_0(\boldsymbol{s}), \quad \forall \boldsymbol{s} \in \partial\left([1/R^2, 1]^l\right).$$

So, $\eta\left(t, \gamma_0(\boldsymbol{s})\right) \in \Lambda_*$, for each $t \geq 0$.

On the other hand, $\operatorname{supp}\gamma_0(\boldsymbol{s})(x) \subset \overline{\Omega}_\Gamma$ for all $\boldsymbol{s} \in \partial([1/R^2, 1]^l)$, then $\Phi_\lambda(\gamma_0(\boldsymbol{s}))$ does not depend on $\lambda \geq 0$. Moreover,

$$\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) \leq c_\Gamma, \quad \forall \boldsymbol{s} \in [1/R^2, 1]^l$$

and $\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) = c_\Gamma$ if and only if $s_j = \frac{1}{R}, \forall j \in \Gamma$.

Therefore, we have that

$$m_0 = \max\left\{\Phi_\lambda(u) : u \in \gamma_0([1/R^2, 1]^l) \setminus A^\lambda_\mu\right\}$$

is independent of λ and $m_0 \leq c_\Gamma$. From (4.6), it is easy to see that for any $t > 0$,

$$\left\|\eta\left(0, \gamma_0\left(s_1, s_2, \ldots, s_l\right)\right) - \eta\left(t, \gamma_0\left(s_1, s_2, \ldots, s_l\right)\right)\right\|_\lambda \leq t.$$

Since $\Phi_{\lambda,j}(u) \in C^1(E_\lambda, \mathbb{R})$ for all $j = 1, 2, \ldots, l$, and the assumptions $(f_1) - (f_4)$, it is easy to see that for large number $T > 0$, there exists a positive number $\rho_0 > 0$ which is independent of λ such that for all $j = 1, 2, \ldots, l$ and $t \in [0, T]$,

$$\left\|\Phi'_{\lambda,j}(\eta(t, \gamma_0(s_1, s_2, \ldots, s_l)))\right\|_\lambda \leq \rho_0. \tag{4.7}$$

We claim that for large T,

$$\max_{\boldsymbol{s} \in [1/R^2, 1]^l} \Phi_\lambda\left(\eta\left(T, \gamma_0(\boldsymbol{s})\right)\right) \leq \max\left\{m_0, c_\Gamma - \frac{1}{2}\tau_0\mu\right\},$$

where $\tau_0 = \max\{\sigma_0, \frac{\sigma_0}{\rho_0}\}$.

In fact, if $\gamma_0(\boldsymbol{s}) \notin A^\lambda_\mu$, from (4.4),

$$\Phi_\lambda(\eta(t, \boldsymbol{s})) \leq \Phi_\lambda(\boldsymbol{s}) \leq m_0, \qquad \forall t \geq 0.$$

If $\gamma_0(\boldsymbol{s}) \in A^\lambda_\mu$, we set

$$\widetilde{\eta}(t) = \eta(t, \boldsymbol{s}), \qquad \widetilde{d_\lambda} = \min\left\{d_\lambda, \sigma_0\right\} \quad \text{and} \quad T = \frac{\sigma_0 \mu}{2\widetilde{d_\lambda}}.$$

Next we differentiate two cases:

(1) $\widetilde{\eta}(t) \in A^\lambda_{3\mu/2}$ for all $t \in [0, T]$.

(2) $\widetilde{\eta}\,(t_0) \in \partial A^{\lambda}_{3\mu/2}$ for some $t_0 \in [0, T]$.

If (1) holds, we have $\Psi(\widetilde{\eta}(t)) \equiv 1$ and $\left\|\Phi'_\lambda(\widetilde{\eta}(t))\right\|_\lambda \geq \widetilde{d_\lambda}$ for all $t \in [0, T]$. Hence, from (4.4), we get

$$\begin{aligned}
\Phi_\lambda(\widetilde{\eta}(T)) &= \Phi_\lambda\left(\gamma_0(s)\right) + \int_0^T \frac{d}{ds}\Phi_\lambda(\widetilde{\eta}(s))ds \\
&= \Phi_\lambda\left(\gamma_0(s)\right) - \int_0^T \Psi(\widetilde{\eta}(s))\left\|\Phi'_\lambda(\widetilde{\eta}(s))\right\|_\lambda ds \\
&\leq c_\Gamma - \int_0^T \widetilde{d}_\lambda ds \\
&= c_\Gamma - \widetilde{d}_\lambda T \\
&= c_\Gamma - \frac{1}{2}\sigma_0\mu \\
&\leq c_\Gamma - \frac{1}{2}\tau_0\mu.
\end{aligned}$$

If (2) holds, there exists $0 \leq t_1 \leq t_2 \leq T$ such that

$$\widetilde{\eta}\,(t_1) \in \partial A^{\lambda}_{u}, \tag{4.8}$$

$$\widetilde{\eta}\,(t_2) \in \partial A^{\lambda}_{3\mu/2}, \tag{4.9}$$

$$\widetilde{\eta}(t) \in A^{\lambda}_{3\mu/2} \setminus A^{\lambda}_{u}, \quad \text{for all } t \in [t_1, t_2]\,.$$

It follows from (4.9)

$$\|\widetilde{\eta}\,(t_2)\|_{\lambda,\mathbb{R}^3\setminus\Omega'_\Gamma} = \frac{3\mu}{2},$$

or

$$\left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}\,(t_2)\right) - c_{j_0}\right| = \frac{3\mu}{2},$$

for some $j_0 \in \Gamma$.

Now we consider the later case, the former case can be obtained in a similar way. By (4.8),

$$\left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}\,(t_1)\right) - c_{j_0}\right| \leq \mu,$$

thus, we obtain

$$\left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}(t_2)\right) - \Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}(t_1)\right)\right| \geq \left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}\,(t_2)\right) - c_{j_0}\right| - \left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}\,(t_1)\right) - c_{j_0}\right| \geq \frac{1}{2}\mu.$$

On the other hand, by the mean value theorem, there exists $t_3 \in (t_1, t_2)$ such that

$$\left|\Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}(t_2)\right) - \Phi_{\lambda,\Omega'_{j_0}}\left(\widetilde{\eta}(t_1)\right)\right| = \left|\Phi'_{\lambda,\Omega'_{j_0}} \cdot \frac{d\widetilde{\eta}}{dt}(t_3)\right|(t_2 - t_1).$$

Moreover, from (4.6) and (4.7), we have

$$t_2 - t_1 \geq \frac{\mu}{2\rho_0}.$$

Hence, we obtain

$$\begin{aligned}\Phi_\lambda(\widetilde{\eta}(T)) &= \Phi_\lambda\left(\gamma_0(\mathbf{s})\right) + \int_0^T \frac{d}{ds}\Phi_\lambda(\widetilde{\eta}(s))ds \\ &= \Phi_\lambda\left(\gamma_0(\mathbf{s})\right) - \int_0^T \Psi(\widetilde{\eta}(s))\left\|\Phi'_\lambda(\widetilde{\eta}(s))\right\|_\lambda ds \\ &\leq c_\Gamma - \int_{t_1}^{t_2} \Psi(\widetilde{\eta}(s))\left\|\Phi'_\lambda(\widetilde{\eta}(s))\right\|_\lambda ds \\ &= c_\Gamma - \sigma_0(t_2 - t_1) \\ &\leq c_\Gamma - \frac{1}{2}\tau_0\mu,\end{aligned}$$

and so (4.7) is proved. Now we recall that $\widetilde{\eta}(T) = \eta(T, \gamma_0(\mathbf{0})) \in \Lambda_*$, thus

$$b_{\lambda,\Gamma} \leq \Phi_\lambda(\widetilde{\eta}(T)) \leq \max\left\{m_0, c_\Gamma - \frac{1}{2}\tau_0\mu\right\},$$

which contradicts the fact that $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$. □

Proof of Theorem 1.1. From Proposition 4.5, there exists a nontrivial solutions u_λ to problem (3.3) such that $u_\lambda \in A^\lambda_\mu \cap \Phi^{c_\Gamma}_\lambda$, for all $\lambda \geq \lambda^*$. So, using the proof of Proposition 3.6, we can derive that

$$\|u_\lambda\|^2_{L^\infty(\mathbb{R}^3\setminus\Omega'_\Gamma)} \leq a, \qquad \forall\lambda \geq \lambda^*,$$

which shows that u_λ is a nontrivial solution to the original problem (1.4).

Moreover, for any given sequence (λ_n) with $\lambda_n \to +\infty$, up to a subsequence if necessary, it is easy to show that (u_{λ_n}) is a $(PS)_\infty$ sequence. Hence, by Proposition 3.5, we obtain

$$u_{\lambda_n} \to u \quad \text{in } H^1_A\left(\mathbb{R}^3, \mathbb{C}\right) \text{ with } u \in H^{0,1}_A\left(\Omega_\Gamma, \mathbb{C}\right), \quad u \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega_\Gamma,$$

and the restriction $u\,|_{\Omega_j}$ is a least energy solution of

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \left(\frac{1}{4\pi}\int_{\Omega_j} \frac{|u(y)|^2}{|x-y|}dy\right) u = \alpha f(|u|^2)u + |u|^4 u, \ x \in \Omega_j, \\ u \in H^{0,1}_A\left(\Omega_j\right), \end{cases}$$

where $j \in \Gamma$. We complete the proof of Theorem 1.1. □

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