Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **34**, 1–9; https://doi.org/10.14232/ejqtde.2022.1.34 www.math.u-szeged.hu/ejqtde/

Iterative solution of elliptic equations

Philip Korman[✉][1] and **Dieter S. Schmidt**[2]

[1]University of Cincinnati, Cincinnati OH 45221-0025, USA
[2]University of Cincinnati, Cincinnati OH 45221-0030, USA

Received 10 February 2022, appeared 23 July 2022
Communicated by Gabriele Bonanno

Abstract. We reduce solution of the Dirichlet problem ($x \in D \subset R^m$)

$$\Delta u(x) + a(x)u(x) = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

to iterative solution of a simpler problem

$$\Delta u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,,$$

for which one can use either Fourier series or Green's function method. The method is suitable for numerical computations, particularly when one uses Newton's method for semilinear problems

$$\Delta u + g(x,u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,,$$

in dimensions $m \geq 3$.

Keywords: iterative method, Lyapunov–Schmidt reduction.

2020 Mathematics Subject Classification: 35J25, 65N80.

1 Introduction

If Green's function $G(x,y)$ is available for a domain $D \subset R^m$, it is easy to solve numerically the Dirichlet problem for Laplace's equation

$$-\Delta u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,. \tag{1.1}$$

The solution is $u(x) = \int_D G(x,y)f(y)\,dy$. *Mathematica* software can compute such integrals quickly and accurately even in dimensions $m > 2$, say for $m = 5$. When solving semilinear problems

$$\Delta u + g(x,u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

one usually uses Newton's method

$$\Delta u_{p+1} + g(x,u_p) + g_u(x,u_p)\left(u_{p+1} - u_p\right) = 0 \quad \text{in } D, \qquad u_{p+1} = 0 \quad \text{on } \partial D\,,$$

[✉]Corresponding author. Email: kormanp@ucmail.uc.edu

which requires repeated solution of the linear problems

$$\Delta u + a(x)u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,, \tag{1.2}$$

with given functions $a(x)$ and $f(x)$. It is very unlikely to have eigenfunctions (or Green's function) available for the problem (1.2). *Question: can one reduce solving (1.2) to iterative solution of (1.1)?* It turns out that the answer is affirmative for any bounded $a(x)$. We show that either the iterations

$$-\Delta u_{n+1} = a(x)u_n - f(x) \quad \text{in } D, \qquad u_{n+1} = 0 \quad \text{on } \partial D \tag{1.3}$$

converge to the solution of (1.2), or else there is a modified iterative process that converges to the solution of (1.2). Eigenfunctions of the Laplacian, or Green's functions, are available for some domains. For other domains their computation is a one time effort, while solving nonlinear problems requires repeated solutions of the problem (1.2), particularly in connection to curve following.

Turning to the description of the method, let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of $-\Delta$ with zero boundary conditions on D (λ_1 is simple, while some other eigenvalues may be repeated), and $\varphi_1 > 0, \varphi_2, \varphi_3, \ldots$ be the corresponding eigenfunctions of $-\Delta$, forming an orthonormal set in $L^2(D)$, so that $\int_D \varphi_k^2\,dx = 1$. Represent $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$, with $f_k = \int_D f(x)\varphi_k(x)\,dx$. Recall that $\|f\|^2_{L^2(D)} = \int_D f^2(x)\,dx = \sum_{k=1}^{\infty} f_k^2$ (Parseval's identity), see e.g., W. Craig [1] or P. Korman [2]. The solution of (1.1) is

$$u(x) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k} \varphi_k(x) \equiv (-\Delta)^{-1}\left(f(x)\right),$$

where $(-\Delta)^{-1}$ is the common notation for the solution operator of (1.1). By Parseval's identity

$$\|(-\Delta)^{-1} f\|^2_{L^2(D)} = \sum_{k=1}^{\infty} \frac{f_k^2}{\lambda_k^2} \le \frac{1}{\lambda_1^2} \sum_{k=1}^{\infty} f_k^2 = \frac{1}{\lambda_1^2} \|f\|^2_{L^2(D)}\,. \tag{1.4}$$

In case $f_1 = 0$, or $f \perp \varphi_1$ in L^2, the same argument shows that

$$\|(-\Delta)^{-1} f\|_{L^2} \le \frac{1}{\lambda_2} \|f\|_{L^2}\,, \tag{1.5}$$

and if $f_1 = f_2 = \cdots = f_j = 0$, then

$$\|(-\Delta)^{-1} f\|_{L^2} \le \frac{1}{\lambda_{j+1}} \|f\|_{L^2}\,. \tag{1.6}$$

Proposition 1.1. *Assume that $a(x) \in C(\bar{D})$ satisfies*

$$\max_{\bar{D}} |a(x)| < \lambda_1\,. \tag{1.7}$$

Then the iterates given by (1.3) converge in $L^2(D)$, to a solution $u(x) \in H^2(D)$ of (1.2), for any $f(x) \in L^2(D)$.

Proof. Write (1.3) in the form

$$u_{n+1} = (-\Delta)^{-1}\left[a(x)u_n - f(x)\right].$$

Subtracting a similar formula for u_n, and then using (1.4), we obtain

$$u_{n+1} - u_n = (-\Delta)^{-1} \left[a(x) \left(u_n - u_{n-1} \right) \right],$$
$$\|u_{n+1} - u_n\|_{L^2} \leq \frac{1}{\lambda_1} \|a(x) \left(u_n - u_{n-1} \right)\|_{L^2} \leq \theta \|u_n - u_{n-1}\|_{L^2},$$

where $\theta \equiv \frac{\max_{\bar{D}} |a(x)|}{\lambda_1} < 1$, which implies that $\{u_n(x)\}$ is a Cauchy sequence in $L^2(D)$, and the proof follows in view of completeness of $L^2(D)$. □

If the condition (1.7) is violated then the iterations (1.3) diverge in $L^2(D)$, in general, as the following example shows.

Example 1.2. Let $a(x) = a$, a constant, with $\lambda_1 < a < \lambda_2$. For the iterations

$$-\Delta u_{n+1} = a u_n - f(x) \quad \text{in } D, \qquad u_{n+1} = 0 \quad \text{on } \partial D, \tag{1.8}$$

write $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k$, and $u_n = \sum_{k=1}^{\infty} u_n^k \varphi_k$, to obtain

$$\lambda_k u_{n+1}^k = a u_n^k - f_k.$$

Denoting $\delta = \frac{a}{\lambda_1} > 1$, obtain for the $k = 1$ component

$$u_{n+1}^1 - u_n^1 = \delta \left(u_n^1 - u_{n-1}^1 \right),$$

so that the iterations (1.8) diverge (because the first component diverges).

Now suppose that the condition (1.7) does not hold, but we have

$$\max_{\bar{D}} |a(x)| < \lambda_2 \tag{1.9}$$

instead. Decompose

$$u(x) = \xi_1 \varphi_1(x) + U(x), \tag{1.10}$$

with $\int_D U(x)\varphi_1(x)\,dx = 0$, i.e., $u(x)$ is the sum of the first harmonic of $u(x)$, and the projection of $u(x)$ on $\varphi_1^\perp$, the orthogonal complement of φ_1 in $L^2(D)$. Now the iterates given by (1.3) diverge, in general, but we shall show that both ξ_1 and the U part can be obtained by using two converging iteration processes. (Unless $a(x)$ is a constant, the harmonics do not decouple, making the problem nontrivial.) Then we extend the method for any $a(x)$ bounded on $\bar{D}$.

2 The case $\max_{\bar{D}} |a(x)| < \lambda_2$

Let P denote the projection operator on $\varphi_1^\perp$ in $L^2(D)$ ($Pv = v - \left(\int_D v\varphi_1\,dx\right)\varphi_1$). Then one can write $U(x) = Pu(x)$ in the decomposition (1.10). Similarly, decompose $f(x) = \mu_1\varphi_1 + e(x)$, with $e(x) = Pf(x)$. Applying the operator P to the equation (1.2) gives

$$\Delta U + P\left[a(x)\left(\xi_1\varphi_1(x) + U(x)\right)\right] = e(x) \quad \text{in } D, \qquad U = 0 \quad \text{on } \partial D. \tag{2.1}$$

Projection of (1.2) onto φ_1 gives

$$\int_D \left(\Delta u + a(x)u\right)\varphi_1\,dx = \int_D f(x)\varphi_1\,dx = \mu_1. \tag{2.2}$$

Clearly, $u(x) = \xi_1 \varphi_1(x) + U(x)$ is a solution of (1.2) if and only if (2.1) and (2.2) hold. The decomposition (2.1), (2.2) is similar to the Lyapunov–Schmidt reduction, see e.g., L. Nirenberg [5].

We now modify the problem (2.1): find $V(x) \in \varphi_1^\perp \cap H^2(D)$ solving

$$\Delta V + P\left[a(x)V(x)\right] = e(x) \quad \text{in } D, \qquad V = 0 \quad \text{on } \partial D\,. \tag{2.3}$$

Proposition 2.1. *Assume that the condition* (1.9) *holds. Then the problem* (2.3) *can be solved by the converging iterations* $V_n(x) \in \varphi_1^\perp \cap H^2(D)$

$$-\Delta V_{n+1} = P\left[a(x)V_n(x)\right] - e(x) \quad \textit{in } D, \qquad V_{n+1} = 0 \quad \textit{on } \partial D\,, \tag{2.4}$$

beginning with $V_0 = 0$.

Proof. The iterates belong to $\varphi_1^\perp$, since the right hand sides of (2.4) do. Subtracting the equations for two consecutive iterates, and then using (1.5) and $\|Pv\|_{L^2} \le \|v\|_{L^2}$, we obtain from (2.4):

$$V_{n+1} - V_n = (-\Delta)^{-1} P\left[a(x)\left(V_n - V_{n-1}\right)\right],$$
$$\|V_{n+1} - V_n\|_{L^2} \le \frac{1}{\lambda_2}\|a(x)\left(V_n - V_{n-1}\right)\|_{L^2} \le \theta \|V_n - V_{n-1}\|_{L^2}\,,$$

where $\theta \equiv \frac{\max_D |a(x)|}{\lambda_2} < 1$ by (1.9), and the proof follows. ☐

The difference $W(x) = U(x) - V(x)$ satisfies

$$\Delta W + P\left[a(x)W(x)\right] = -\xi_1 P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D\,.$$

It follows that $W = \xi_1 \bar{W}$, where $\bar{W}$ is the unique solution of

$$\Delta W + P\left[a(x)W(x)\right] = -P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D\,, \tag{2.5}$$

which in view of Proposition 2.1 is the limit of the iterations

$$-\Delta W_{n+1} = P\left[a(x)W_n(x)\right] + P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W_{n+1} = 0 \quad \text{on } \partial D\,, \tag{2.6}$$

starting with $W_0 = 0$.

We conclude that $U = V + \xi_1 \bar{W}$, so that $u = \xi_1 \varphi_1 + U = \xi_1 \varphi_1 + V + \xi_1 \bar{W}$, and it remains to determine the value of ξ_1. Substitute this $u(x)$ into (1.2)

$$-\lambda_1 \xi_1 \varphi_1 + \Delta V + \xi_1 \Delta \bar{W} + a(x)\left(\xi_1 \varphi_1 + V + \xi_1 \bar{W}\right) = f(x)\,.$$

Multiplication by φ_1 and integration over D gives a linear equation for ξ_1, with the solution (observe that both ΔV and $\Delta \bar{W}$ are in $\varphi_1^\perp$)

$$\bar{\xi}_1 = \frac{\int_D f \varphi_1 \, dx - \int_D a(x) V \varphi_1 \, dx}{-\lambda_1 + \int_D a(x) \varphi_1^2 \, dx + \int_D a(x) \bar{W} \varphi_1 \, dx}\,. \tag{2.7}$$

Then the solution of (1.2) is

$$u(x) = \bar{\xi}_1 \varphi_1 + V + \bar{\xi}_1 \bar{W}\,. \tag{2.8}$$

Remark 2.2. In case $\max_D a(x) > \lambda_1$, it is possible to have resonance, when the problem

$$\Delta u + a(x)u = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

has a nontrivial solution. In such a case the denominator in (2.7) is zero, and the problem (1.2) is not solvable for general $f(x)$.

Example 2.3. As a feasibility check we solved the problem

$$u''(x) + \left(2 + \frac{1}{3}x\right) u(x) = x^2 \quad \text{for } 0 < x < \pi, \qquad u(0) = u(\pi) = 0\,. \tag{2.9}$$

Here $\lambda_1 = 1$, $\lambda_2 = 4$, so that $a(x) = 2 + \frac{1}{3}x$ satisfies $\lambda_1 < a(x) < \lambda_2$ on $(0, \pi)$. Calculate $\varphi_1(x) = \sqrt{\frac{2}{\pi}}\sin x$, $e(x) = f(x) - \left(\int_0^\pi f(x)\varphi_1(x)\,dx\right)\varphi_1(x)$, with $f(x) = x^2$. We achieved good accuracy performing twelve iterations for both (2.4) and (2.6). The graph of the solution of (2.9) was identical to the one produced by *Mathematica's* NDSolve command.

3 The general $a(x)$

We now prove directly that the formulas (2.3), (2.5), (2.7), (2.8) give the solution of (1.2), and then generalize for any bounded $a(x)$.

Theorem 3.1. *Assume that the condition* (1.9) *holds. Then the formulas* (2.3), (2.5), (2.7), (2.8) *give the solution of* (1.2).

Proof. We will show that $u(x) = \bar{\xi}_1\varphi_1(x) + U(x)$, with $U(x) = V + \bar{\xi}_1\bar{W}$ satisfies (2.1) and (2.2) (where V and $\bar{W}$ are the unique solutions (2.3) and (2.5) respectively, and $\bar{\xi}$ is determined by (2.7)). Indeed,

$$\begin{aligned}
\Delta U + P\left[a(x)\left(\bar{\xi}_1\varphi_1(x) + U(x)\right)\right] &= \Delta V + \bar{\xi}_1\Delta\bar{W} + P\left[a(x)\left(\bar{\xi}_1\varphi_1(x) + V + \bar{\xi}_1\bar{W}\right)\right] \\
&= \Delta V + P\left[a(x)V\right] + \bar{\xi}_1\left\{\Delta\bar{W} + P\left[a(x)\bar{W}\right] + P\left[a(x)\varphi_1\right]\right\} \\
&= e(x)\,,
\end{aligned}$$

verifying (2.1). Using (2.7) we obtain

$$\begin{aligned}
&\int_D (\Delta u + a(x)u)\,\varphi_1\,dx = -\lambda_1\bar{\xi}_1 + \int_D a(x)\left[\bar{\xi}_1\varphi_1 + V + \bar{\xi}_1\bar{W}\right]\varphi_1\,dx \\
&\quad = \bar{\xi}_1\left[-\lambda_1 + \int_D a(x)\varphi_1^2\,dx + \int_D \bar{a}(x)W\varphi_1\,dx\right] + \int_D a(x)V\varphi_1\,dx = \int_D f\varphi_1\,dx\,,
\end{aligned}$$

justifying (2.2). □

Turning to any $a(x) \in C(\bar{D})$, we can find the first index j so that

$$\max_{\bar{D}} |a(x)| < \lambda_{j+1}\,. \tag{3.1}$$

Decompose

$$u(x) = \sum_{i=1}^{j} \xi_i\varphi_i(x) + U(x)\,, \tag{3.2}$$

with $\int_D U(x)\varphi_i(x)\,dx = 0$ for all $i = 1, \ldots, j$. Let P denote the projection operator on the orthogonal complement of the first j eigenfunctions, i.e., the projection on $\text{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_j\}^\perp$ in $L^2(D)$. Decompose $f(x) = \sum_{i=1}^j \mu_i\varphi_i + e(x)$, with $e(x) = Pf(x)$. Applying P to the equation (1.2) gives

$$\Delta U + P\left[a(x)\left(\sum_{i=1}^{j}\xi_i\varphi_i(x) + U(x)\right)\right] = e(x) \quad \text{in } D, \qquad U = 0 \quad \text{on } \partial D. \tag{3.3}$$

We now modify the problem (3.3): find $V(x) \in \text{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_j\}^\perp$ solving

$$\Delta V + P\left[a(x)V(x)\right] = e(x) \quad \text{in } D, \qquad V = 0 \quad \text{on } \partial D. \tag{3.4}$$

The following proposition is proved the same way as Proposition 2.1.

Proposition 3.2. *Under the condition (3.1) the problem (3.4) can be solved by the converging iterations* $V_n(x) \in \text{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_j\}^\perp$

$$-\Delta V_{n+1} = P\left[a(x)V_n(x)\right] - e(x) \quad \text{in } D, \qquad V_{n+1} = 0 \quad \text{on } \partial D, \tag{3.5}$$

beginning with $V_0 = 0$.

The difference $W(x) = U(x) - V(x) \in \text{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_j\}^\perp$ satisfies

$$\Delta W + P\left[a(x)W(x)\right] = -\sum_{i=1}^{j}\xi_i P\left[a(x)\varphi_i\right] \quad \text{in } D, \quad W = 0 \quad \text{on } \partial D.$$

By linearity $W = \sum_{i=1}^j \xi_i \bar{W}_i$, where $\bar{W}_i$ is the unique solution of

$$\Delta W + P\left[a(x)W(x)\right] = -P\left[a(x)\varphi_i\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D, \tag{3.6}$$

which in view of Proposition 3.2 is the limit of the iterations

$$-\Delta W_{n+1} = P\left[a(x)W_n(x)\right] + P\left[a(x)\varphi_i\right] \quad \text{in } D, \qquad W_{n+1} = 0 \quad \text{on } \partial D, \tag{3.7}$$

starting with $W_0 = 0$. It follows that

$$u = \sum_{i=1}^{j}\xi_i\varphi_i + U = \sum_{i=1}^{j}\xi_i\varphi_i + V + \sum_{i=1}^{j}\xi_i\bar{W}_i, \tag{3.8}$$

and it remains to determine the values of ξ_i. Substitute this $u(x)$ into (1.2)

$$-\sum_{i=1}^{j}\lambda_i\xi_i\varphi_i + \Delta V + \sum_{i=1}^{j}\xi_i\Delta\bar{W}_i + a(x)\left(\sum_{i=1}^{j}\xi_i\varphi_i + V + \sum_{i=1}^{j}\xi_i\bar{W}_i\right) = f(x).$$

Multiplication by φ_k and integration over D gives a $j \times j$ system of linear equations for ξ_i's ($k = 1, 2, \ldots, j$)

$$-\lambda_k\xi_k + \sum_{i=1}^{j}\xi_i\left[\int_D a(x)\left(\varphi_i + \bar{W}_i\right)\varphi_k\,dx\right] = \int_D \left(f(x) - a(x)V\right)\varphi_k\,dx. \tag{3.9}$$

This system has a unique solution, provided that (1.2) is solvable. Using the solution of (3.9) in (3.8) provides the solution of (1.2).

So that in case the condition (3.1) holds, the algorithm for solving (1.2) is as follows.

1. Solve the problem (3.4) by using the iterates (3.5).
2. Solve j problems (3.6) by using the iterates (3.7) for each problem.
3. Solve the $j \times j$ linear algebraic system (3.9) to find $\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_j$.
4. The solution is $u(x) = \sum_{i=1}^{j} \bar{\xi}_i \varphi_i + V + \sum_{i=1}^{j} \bar{\xi}_i \bar{W}_i$.

4 Semilinear Poisson equation in higher dimensions

Elliptic PDE's, like the problem (1.2), are rarely solved numerically in dimensions $m > 2$. Using finite differences in the dimension $m = 4$ with 20 subdivision points along each axis (which is not many), requires solving a system of $20^4 = 160000$ linear equations. Richard Bellman coined a phrase "the curse of dimensionality" to describe the computational challenges in higher dimensions. Since then there has been a tremendous advance in computer power and software (e.g., parallel computations). In particular, a system of $20^4 = 160000$ linear equations nowadays is not considered to be very large. However the accuracy will be low with only 20 subdivision points along each axis, so that challenges remain. Another problem in higher dimensions is representation of solutions. Once a solution in dimension $m = 4$ is computed, should the result be presented as a graph in 5 dimensions, or as a 4-dimensional table? The iterative method developed above addresses both issues. Represent $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$, with the coefficients $f_k = \int_D f(y)\varphi_k(y)\, dy$. The solution of (1.1) is

$$u(x) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k} \varphi_k(x) , \quad \text{with } f_k = \int_D f(y)\varphi_k(y)\, dy \, . \tag{4.1}$$

Replacing f_k's in the sum by their expressions as integrals, one can express the solution of (1.1) as

$$u(x) = \int_D G(x, y) f(y)\, dy \, , \tag{4.2}$$

with Green's function

$$G(x, y) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k} \, . \tag{4.3}$$

However, it is easier to use the form (4.1) rather than (4.2) because *Mathematica* cannot handle numerical integration in y variables, when x variables are present (even with the delayed assignment). This is perfectly understandable, because addition of thousands of functions (obtained by interpolation) is an enormous task. One can introduce a mesh, and compute (4.2) in parallel at each point, using as many processors as there are points on the mesh, but this "industrial strength" computational effort is beyond our scope. However, the usefulness of our method probably lies in this direction.

We did try the eigenfunction expansion in both two and three dimensions, using the first 50 eigenfunctions. Conclusion: the method is slow. The method requires either the knowledge or calculation of the eigenvalues and the eigenfuctions of the Laplacian. On a rectangle $R = [0, a] \times [0, b] \times [0, c]$ in three dimensions, the eigenfunctions (vanishing on ∂R) are

$$\varphi = c_0 \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \sin \frac{p\pi}{c} z \, , \qquad c_0 = \sqrt{\frac{8}{abc}} \, ,$$

with $\int_R \varphi^2\, dxdydz = 1$. The corresponding eigenvalues are

$$\lambda = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{c^2} \right).$$

The order and multiplicity of eigenvalues depends on a particular choice of a, b, c. Let us take $a = b = 1$, $c = \sqrt{2}$. Then $\lambda = \pi^2 \left(m^2 + n^2 + \frac{p^2}{2} \right)$. The order of eigenvalues is determined by $(m, n, p) \equiv m^2 + n^2 + \frac{p^2}{2}$.

a. $(1,1,1)$ gives $\lambda_1 = \frac{5}{2}\pi^2$, $\varphi_1 = c_0 \sin \pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$.

b. $(1,1,2)$ gives $\lambda_2 = 4\pi^2$, $\varphi_2 = c_0 \sin \pi x \sin \pi y \sin \frac{2\pi}{\sqrt{2}} z$.

c. $(2,1,1)$ and $(1,2,1)$ give a repeated eigenvalue $\lambda_3 = \lambda_4 = \frac{11}{2}\pi^2$, with the eigenfunctions $\varphi_3 = c_0 \sin 2\pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$ and $\varphi_4 = \sin \pi x \sin 2\pi y \sin \frac{\pi}{\sqrt{2}} z$, and so on.

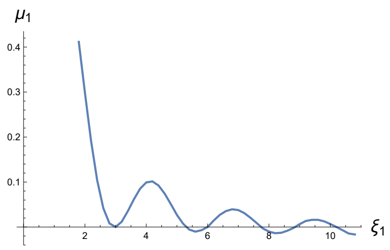

Figure 4.1: The solution curve $\mu_1 = \mu_1(\xi_1)$ of the problem (4.4), oscillating around the ξ_1-axis.

We wrote a code, allowing us to calculate a large number of eigenfunctions automatically. We solved a number of examples for the problem (1.2), obtaining the expected results, but the computations were slow.

Example 4.1. We performed curve-following for the following semilinear problem on a parallelepiped $\Omega = (0,1) \times (0,1) \times (0, \sqrt{2})$ in three dimensions

$$\Delta u + \lambda_1 u + \sin u = \mu_1 \varphi_1(x,y,z) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.4}$$

see Figure 4.1. Here $\lambda_1 = \frac{5}{2}\pi^2$ is the principal eigenvalue of the Laplacian on Ω with zero boundary condition, so that the problem is at resonance. Decompose the solution as $u(x) = \xi_1 \varphi_1(r) + U(x,y,z)$, with $U(x,y,z) \in \varphi_1^{\perp}$ in $L^2(\Omega)$, where $\varphi_1 = \sqrt{\frac{8}{\sqrt{2}}} \sin \pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$. The following facts follow from the results proved in [3] and [4]. The solution set of (4.4) is exhausted by a single continuous curve $(u(x,y,z), \mu_1)(\xi_1)$. Moreover $\mu_1(\xi_1) \to 0$ as $\xi_1 \to \infty$, while $\mu_1(\xi_1)$ changes sign infinitely many times. In particular, the problem (4.4) has infinitely many solutions at $\mu_1 = 0$. Performing the curve following required solving linear problems of the type (1.2) repeatedly. We used eigenfunction expansions, and it took long time to compute the solution curve in Figure 4.1.

Mathematica's NDSolve command can also handle the problem (1.2) in two and three dimensions. It appears that the accuracy is excellent in two dimensions, but not in dimension three.

References

[1] W. Craig, *A course on partial differential equations*, AMS Graduate Studies in Mathematics, Vol. 197, Providence, RI, 2018. https://doi.org/10.1090/gsm/197; MR3839330

[2] P. Korman, *Lectures on differential equations*, AMS/MAA Textbooks, Vol. 54, Providence, RI, 2019. MR3969937

[3] P. Korman, Global solution curves in harmonic parameters, and multiplicity of solutions, *J. Differential Equations* **296**(2021), 186–212. https://doi.org/10.1016/j.jde.2021.05.051; MR4272565

[4] P. Korman, D. S. Schmidt, Infinitely many solutions and asymptotics for resonant oscillatory problems, Special issue in honor of Alan C. Lazer, *Electron. J. Differential Equations*, Special Issue 01(2021), 301–313.

[5] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Institute Lecture Notes, Vol. 6, Courant Institute of Mathematical Sciences, New York University, New York, 1974. MR0488102