Exact solution of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic model

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Abstract. Exact solution of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic model is established, and various properties of solution are derived directly from the exact solution. The exact solution of an initial value problem for SIRD epidemic model is represented in an explicit form, and it is shown that the parametric form of the exact solution is a solution of some linear differential system.

Keywords: exact solution, SIRD epidemic model, initial value problem, linear differential system.

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1   Introduction

Recently there is an increasing requirement for mathematical approach to epidemic models. It goes without saying that a vast literature and research papers, dealing with epidemic models has been published so far (see, e.g., [2–4,7]). It seems that little is known about exact solutions of epidemic models. Exact solutions of the Susceptible-Infectious-Recovered (SIR) epidemic model were studied by Bohner, Streipert and Torres [1], Harko, Lobo and Mak [5], Shabbir, Khan and Sadiq [9] and Yoshida [11]. However there appears to be no known results about exact solutions of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic models. The objective of this paper is to obtain an exact solution of SIRD differential system, and to derive various properties of the exact solution. Furthermore we show that the parametric form of the exact solution satisfies some linear differential system.

The differential system called Susceptible–Infectious–Recovered–Deceased (SIRD) epidem-

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ic model is the following:

$$\frac{dS(t)}{dt} = -\beta S(t)I(t) + \nu R(t),$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t),$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \nu R(t),$$

$$\frac{dD(t)}{dt} = \mu I(t)$$

(see, e.g., [8]). If $\nu = 0$, we obtain the simplified SIRD differential system

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1.1}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t), \tag{1.2}$$

$$\frac{dR(t)}{dt} = \gamma I(t), \tag{1.3}$$

$$\frac{dD(t)}{dt} = \mu I(t) \tag{1.4}$$

for $t > 0$, where β, γ and μ are positive constants. We consider the following initial condition:

$$S(0) = \tilde{S}, \qquad I(0) = \tilde{I}, \qquad R(0) = \tilde{R}, \qquad D(0) = \tilde{D}, \tag{1.5}$$

where $\tilde{S} + \tilde{I} + \tilde{R} + \tilde{D} = N$ (positive constant). Since

$$\frac{d}{dt}\left(S(t) + I(t) + R(t) + D(t)\right) = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} + \frac{dD(t)}{dt} = 0$$

by (1.1)–(1.4), it follows that

$$S(t) + I(t) + R(t) + D(t) = k \qquad (t \geq 0)$$

for some constant k. In view of the fact that

$$k = S(0) + I(0) + R(0) + D(0) = \tilde{S} + \tilde{I} + \tilde{R} + \tilde{D} = N,$$

we conclude that

$$S(t) + I(t) + R(t) + D(t) = N \qquad (t \geq 0).$$

It is assumed throughout this paper that:

(A$_1$) $\tilde{S} > \dfrac{\gamma + \mu}{\beta}$;

(A$_2$) $\tilde{I} > 0$;

(A$_3$) $\tilde{R} \geq 0$ satisfies

$$N - \tilde{D} > \tilde{S}e^{(\beta/\gamma)\tilde{R}} + \tilde{R}; \tag{1.6}$$

(A$_4$) $\tilde{D} \geq 0$.

In Section 2 we show that a positive solution of the SIRD differential system can be represented in a parametric form, and we derive an exact solution of the SIRD differential system (1.1)–(1.4). Section 3 is devoted to the investigation of various properties of the exact solution.

2 Exact solution of SIRD differential system

First we need the following important lemma.

Lemma 2.1. *If $S(t) > 0$ for $t > 0$, then the following holds:*

$$R'(t) = \gamma \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(t)} - \left(1 + \frac{\mu}{\gamma} \right) R(t) \right) \quad (2.1)$$

for $t > 0$.

Proof. From (1.1) and (1.3) we see that

$$R'(t) = \gamma I(t) = \gamma \left(\frac{S'(t)}{-\beta S(t)} \right) = -\frac{\gamma}{\beta} (\log S(t))',$$

and integrating the above on $[0, t]$ yields

$$R(t) - \tilde{R} = -\frac{\gamma}{\beta} (\log S(t) - \log \tilde{S}).$$

Therefore we obtain

$$\log S(t) = -\frac{\beta}{\gamma} (R(t) - \tilde{R}) + \log \tilde{S}$$

and hence

$$S(t) = \exp \left(\log \tilde{S} - \frac{\beta}{\gamma} R(t) + \frac{\beta}{\gamma} \tilde{R} \right) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(t)}. \quad (2.2)$$

It follows from (1.3) and (1.4) that

$$D'(t) = \mu I(t) = \frac{\mu}{\gamma} (\gamma I(t)) = \frac{\mu}{\gamma} R'(t)$$

and hence we get

$$D(t) = \frac{\mu}{\gamma} R(t) + C$$

for some constant C. The initial condition (1.5) implies

$$C = \tilde{D} - \frac{\mu}{\gamma} \tilde{R}.$$

Consequently we obtain

$$D(t) = \frac{\mu}{\gamma} R(t) + \tilde{D} - \frac{\mu}{\gamma} \tilde{R}. \quad (2.3)$$

Taking account of (2.2), (2.3) and $I(t) = N - S(t) - R(t) - D(t)$, we observe that

$$\begin{aligned} R'(t) &= \gamma I(t) \\ &= \gamma (N - S(t) - R(t) - D(t)) \\ &= \gamma \left(N - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(t)} - R(t) - \frac{\mu}{\gamma} R(t) - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} \right) \\ &= \gamma \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(t)} - \left(1 + \frac{\mu}{\gamma} \right) R(t) \right), \end{aligned}$$

which is the desired identity (2.1). □

By a *solution* of the SIRD differential system (1.1)–(1.4) we mean a vector-valued function $(S(t), I(t), R(t), D(t))$ of class $C^1(0, \infty) \cap C[0, \infty)$ which satisfies (1.1)–(1.4). Associated with every continuous function $f(t)$ on $[0, \infty)$, we define

$$f(\infty) := \lim_{t \to \infty} f(t).$$

Lemma 2.2. *Let $(S(t), I(t), R(t), D(t))$ be a solution of the SIRD differential system (1.1)–(1.4) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then there exist the limits $S(\infty)$, $I(\infty)$, $R(\infty)$ and $D(\infty)$.*

Proof. Since $S(t) > 0$ and $I(t) > 0$, we see that $S'(t) < 0$, and therefore $S(t)$ is decreasing on $[0, \infty)$. It is trivial that $S(t)$ is bounded from below because $S(t) > 0$. Hence, there exists the limit $S(\infty)$. We observe that $R(t)$ is increasing on $[0, \infty)$ and bounded from above in view of the fact that $R'(t) = \gamma I(t) > 0$ and $R(t) < N$. Therefore there exists $R(\infty)$. Similarly there exists $D(\infty)$. Since $I(t) = N - S(t) - R(t) - D(t)$ and there exist $S(\infty)$, $R(\infty)$ and $D(\infty)$, it follows that there exists $I(\infty)$.   □

Theorem 2.3. *Let $(S(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.5) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then $(S(t), I(t), R(t), D(t))$ can be represented in the following parametric form*:

$$S(t) = S(\varphi(u)) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} u, \tag{2.4}$$

$$I(t) = I(\varphi(u)) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} u + \frac{\gamma + \mu}{\beta} \log u, \tag{2.5}$$

$$R(t) = R(\varphi(u)) = -\frac{\gamma}{\beta} \log u, \tag{2.6}$$

$$D(t) = D(\varphi(u)) = -\frac{\mu}{\beta} \log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \tag{2.7}$$

for $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$, where $t = \varphi(u)$ is given in the proof.

Proof. Define the function $u(t)$ by

$$u(t) := e^{-(\beta/\gamma)R(t)}.$$

We note that there exists the limit $R(\infty)$ by Lemma 2.2. Then $u = u(t)$ is decreasing on $[0, \infty)$, $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$ and $\lim_{t \to \infty} u(t) = e^{-(\beta/\gamma)R(\infty)}$ since $R(t)$ is increasing on $[0, \infty)$ and $\tilde{R} \le R(t) < R(\infty)$. It is clear that $u(t)$ is of class $C^1(0, \infty)$ in view of $e^{-(\beta/\gamma)R(t)} \in C^1(0, \infty)$. Therefore, there exists the inverse function $\varphi(u)$ of $u = u(t)$ such that

$$t = \varphi(u) \quad \left(e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}} \right),$$

$$\varphi(u) \in C^1(e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}),$$

$$\varphi(u) \text{ is decreasing in } (e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}],$$

$$\varphi\big(e^{-(\beta/\gamma)\tilde{R}}\big) = 0,$$

$$\lim_{u \to e^{-(\beta/\gamma)R(\infty)}+0} \varphi(u) = \infty.$$

Substituting $t = \varphi(u)$ into (2.1) in Lemma 2.1 yields

$$R'(\varphi(u)) = \gamma \left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\varphi(u))} - \left(1 + \frac{\mu}{\gamma}\right) R(\varphi(u)) \right) \tag{2.8}$$

for $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$. Differentiating both sides of $u = e^{-(\beta/\gamma)R(\varphi(u))}$ with respect to u, we get

$$
\begin{aligned}
1 &= -\frac{\beta}{\gamma} R'(\varphi(u))\varphi'(u)e^{-(\beta/\gamma)R(\varphi(u))} \\
&= -\frac{\beta}{\gamma} R'(\varphi(u))\varphi'(u)u,
\end{aligned}
$$

and therefore

$$
R'(\varphi(u)) = -\frac{\gamma}{\beta}\frac{1}{\varphi'(u)u}. \tag{2.9}
$$

It is obvious that

$$
R(\varphi(u)) = -\frac{\gamma}{\beta}\log u \tag{2.10}
$$

in view of $u = e^{-(\beta/\gamma)R(\varphi(u))}$. Combining (2.8)–(2.10), we have

$$
-\frac{\gamma}{\beta}\frac{1}{\varphi'(u)u} = \gamma N - \gamma\tilde{D} + \mu\tilde{R} - \gamma\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma}{\beta}(\gamma+\mu)\log u
$$

and therefore

$$
\begin{aligned}
-\varphi'(u) &= \frac{\gamma}{\beta}\frac{1}{u\left(\gamma N - \gamma\tilde{D} + \mu\tilde{R} - \gamma\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + (\gamma/\beta)(\gamma+\mu)\log u\right)} \\
&= \frac{1}{u\left(\beta N - \beta\tilde{D} + (\beta\mu/\gamma)\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + (\gamma+\mu)\log u\right)}. \tag{2.11}
\end{aligned}
$$

Integrating (2.11) over $[u, e^{-(\beta/\gamma)\tilde{R}}]$ and taking account of $\varphi(e^{-(\beta/\gamma)\tilde{R}}) = 0$, we get

$$
\varphi(u) = \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi(\xi)},
$$

where

$$
\psi(\xi) = \beta N - \beta\tilde{D} + \frac{\beta\mu}{\gamma}\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi. \tag{2.12}
$$

It follows from (2.2), (2.3) and (2.10) that

$$
\begin{aligned}
S(t) &= S(\varphi(u)) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\varphi(u))} = \tilde{S}e^{(\beta/\gamma)\tilde{R}}u, \\
R(t) &= R(\varphi(u)) = -\frac{\gamma}{\beta}\log u, \\
D(t) &= D(\varphi(u)) = \frac{\mu}{\gamma}R(\varphi(u)) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = -\frac{\mu}{\beta}\log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}, \\
I(t) &= I(\varphi(u)) = N - S(\varphi(u)) - R(\varphi(u)) - D(\varphi(u)) \\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u,
\end{aligned}
$$

which is the desired solution (2.4)–(2.7). Since $\lim_{u \to e^{-(\beta/\gamma)R(\infty)}+0}\varphi(u) = \infty$, it is necessary that

$$
\begin{aligned}
\lim_{\xi \to e^{-(\beta/\gamma)R(\infty)}+0}\psi(\xi) &= \lim_{\xi \to e^{-(\beta/\gamma)R(\infty)}+0}\left(\beta N - \beta\tilde{D} + \frac{\beta\mu}{\gamma}\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi\right) \\
&= \lim_{x \to R(\infty)-0}\beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x} - \frac{\gamma+\mu}{\gamma}x\right) \\
&= \beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)} - \frac{\gamma+\mu}{\gamma}R(\infty)\right) \\
&= 0,
\end{aligned}
$$

which implies

$$R(\infty) = \frac{\gamma}{\gamma + \mu} N - \frac{\gamma}{\gamma + \mu} \tilde{D} + \frac{\mu}{\gamma + \mu} \tilde{R} - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\infty)}. \tag{2.13}$$

We find that $\psi'(\xi) = 0$ for $\xi = \bar{\xi} = \big((\gamma + \mu)/(\beta\tilde{S})\big) e^{-(\beta/\gamma)\tilde{R}}$, and that $e^{-(\beta/\gamma)R(\infty)} < \bar{\xi} < e^{-(\beta/\gamma)\tilde{R}}$ if $(\gamma + \mu)/\beta < \tilde{S} < ((\gamma + \mu)/\beta) e^{(\beta/\gamma)(R(\infty) - \tilde{R})}$. Since $\psi'(\xi) > 0$ for $e^{-(\beta/\gamma)R(\infty)} < \xi < \bar{\xi}$ and $\psi'(\xi) < 0$ for $\bar{\xi} < \xi < e^{-(\beta/\gamma)\tilde{R}}$, we observe that $\psi(\xi)$ is increasing in $(e^{-(\beta/\gamma)R(\infty)}, \bar{\xi})$ and is decreasing in $(\bar{\xi}, e^{-(\beta/\gamma)\tilde{R}})$. In view of the fact that $\psi\big(e^{-(\beta/\gamma)\tilde{R}}\big) = \beta\left(N - \tilde{S} - \tilde{R} - \tilde{D}\right) = \beta\tilde{I} > 0$ and $\lim_{\xi \to e^{-(\beta/\gamma)R(\infty)}+0} \psi(\xi) = 0$, we see that

$$\psi(\xi) > 0 \quad \text{in } (e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}]$$

under the condition $(\gamma + \mu)/\beta < \tilde{S} < ((\gamma + \mu)/\beta) e^{(\beta/\gamma)(R(\infty) - \tilde{R})}$. Moreover, we get

$$\lim_{t \to \infty} \frac{\beta}{\gamma} R'(t) = \lim_{t \to \infty} \beta \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(t)} - \left(1 + \frac{\mu}{\gamma}\right) R(t) \right)$$

$$= \beta \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\infty)} - \left(1 + \frac{\mu}{\gamma}\right) R(\infty) \right)$$

$$= 0,$$

which implies $I(\infty) = 0$ in light of (1.3). □

Lemma 2.4. *Under the hypothesis* (A$_3$), *the transcendental equation*

$$x = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)x}$$

has a unique solution $x = \alpha$ *such that*

$$\tilde{R} < \alpha < N,$$

where

$$F(N, \tilde{D}, \tilde{R}, \gamma, \mu) := \frac{\gamma}{\gamma + \mu} N - \frac{\gamma}{\gamma + \mu} \tilde{D} + \frac{\mu}{\gamma + \mu} \tilde{R}$$

(*cf.* Figure 2.1).

Proof. First we note that

$$F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} = \frac{\gamma}{\gamma + \mu} \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} \right) > 0 \tag{2.14}$$

in view of (1.6). We define the sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 = \tilde{a} \left(0 < \tilde{a} \leq F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \right),$$

$$a_{n+1} = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)a_n} \quad (n = 1, 2, \dots). \tag{2.15}$$

It is easily seen that

$$a_1 = \tilde{a} \leq F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}}$$

$$\leq F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)a_1}$$
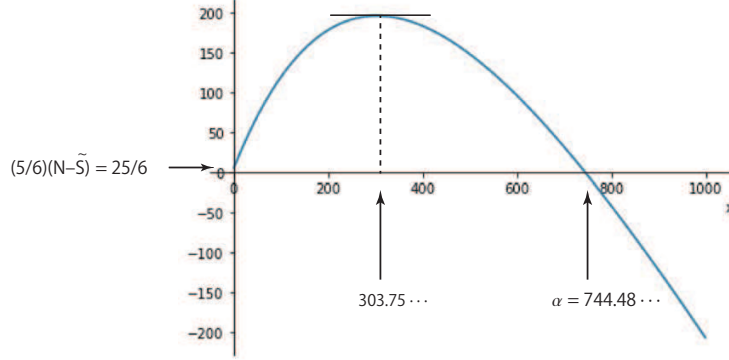
$$= a_2.$$

Figure 2.1: Variation of $F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - (\gamma/(\gamma + \mu))\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x} - x$ for $N = 1000, \tilde{S} = 995, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.15/1000, \gamma = 0.05$ and $\mu = 0.01$. In this case we see that $F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - (\gamma/(\gamma + \mu))\tilde{S} = (5/6)(N - \tilde{S}) = 25/6$ and $0 < \alpha = 744.48 \cdots < 1000$.

If $a_{n+1} \geq a_n$, then

$$a_{n+2} - a_{n+1} = \frac{\gamma}{\gamma + \mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\left(e^{-(\beta/\gamma)a_n} - e^{-(\beta/\gamma)a_{n+1}}\right)$$

$$\geq 0.$$

Therefore we find that $a_{n+2} \geq a_{n+1}$, and hence the sequence $\{a_n\}$ is nondecreasing by the mathematical induction. We observe that the sequence $\{a_n\}$ is bounded because

$$|a_{n+1}| \leq F(N, \tilde{D}, \tilde{R}, \gamma, \mu) + \frac{\gamma}{\gamma + \mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}.$$

Since $\{a_n\}$ is nondecreasing and bounded, there exists $\lim_{n \to \infty} a_n = \alpha$. Taking the limit as $n \to \infty$ in (2.15), we have

$$\alpha = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha}. \tag{2.16}$$

The straight line $y = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - x$ and the exponential curve $y = (\gamma/(\gamma+\mu))\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x}$ has only one intersecting point in $0 < x < N$ by virtue of (2.14), and so the uniqueness of α follows. We claim that $\tilde{R} < \alpha < N$. Since

$$F(N, \tilde{D}, \tilde{R}, \gamma, \mu) = \frac{\gamma}{\gamma + \mu}N - \frac{\gamma}{\gamma + \mu}\tilde{D} + \frac{\mu}{\gamma + \mu}\tilde{R}$$

$$= N - \frac{\mu}{\gamma + \mu}(N - \tilde{R}) - \frac{\gamma}{\gamma + \mu}\tilde{D}$$

$$\leq N,$$

we obtain

$$\alpha = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha}$$

$$< F(N, \tilde{D}, \tilde{R}, \gamma, \mu) \leq N.$$

The inequality $\alpha > \tilde{R}$ follows from the following inequality

$$
\begin{aligned}
\alpha &\geq F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \\
&= \frac{\gamma}{\gamma + \mu} \left(N - \tilde{D} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} \right) + \frac{\mu}{\gamma + \mu} \tilde{R} \\
&> \frac{\gamma}{\gamma + \mu} \tilde{R} + \frac{\mu}{\gamma + \mu} \tilde{R} = \tilde{R}
\end{aligned}
$$

in view of (1.6).   □

We assume that the following hypothesis

(A$_5$) $\tilde{S} < \dfrac{\gamma + \mu}{\beta} e^{(\beta/\gamma)(\alpha - \tilde{R})}$

holds in the rest of this paper. We note that (A$_5$) is equivalent to the following

(A$_5'$) $\dfrac{\gamma + \mu}{\beta} > N - \tilde{D} + \dfrac{\mu}{\gamma} \tilde{R} - \dfrac{\gamma + \mu}{\gamma} \alpha$

in view of $\tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)\alpha} = N - \tilde{D} + (\mu/\gamma)\tilde{R} - ((\gamma + \mu)/\gamma)\alpha$.

Theorem 2.5. *The initial value problem* (1.1)–(1.5) *has the solution*

$$
S(t) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi^{-1}(t), \tag{2.17}
$$

$$
I(t) = N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi^{-1}(t) + \frac{\gamma + \mu}{\beta} \log \varphi^{-1}(t), \tag{2.18}
$$

$$
R(t) = -\frac{\gamma}{\beta} \log \varphi^{-1}(t), \tag{2.19}
$$

$$
D(t) = -\frac{\mu}{\beta} \log \varphi^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma} \tilde{R}, \tag{2.20}
$$

where $\varphi^{-1}(t)$ *denotes the inverse function of* $\varphi : (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}] \to [0, \infty)$ *such that*

$$
t = \varphi(u) := \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi \psi(\xi)},
$$

where $\psi(\xi)$ *is given by* (2.12).

Proof. We note that $\varphi(u) \in C^1(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}})$, $\varphi(u)$ is decreasing in $(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$, $\varphi(e^{-(\beta/\gamma)\tilde{R}}) = 0$. We claim that $\lim_{u \to e^{-(\beta/\gamma)\alpha}+0} \varphi(u) = \infty$. A little calculation yields

$$
\begin{aligned}
\lim_{\xi \to e^{-(\beta/\gamma)\alpha}+0} \psi(\xi) &= \lim_{\xi \to e^{-(\beta/\gamma)\alpha}+0} \left(\beta N - \beta \tilde{D} + \frac{\beta \mu}{\gamma} \tilde{R} - \beta \tilde{S} e^{(\beta/\gamma)\tilde{R}} \xi + (\gamma + \mu) \log \xi \right) \\
&= \lim_{x \to \alpha - 0} \beta \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)x} - \frac{\gamma + \mu}{\gamma} x \right) \\
&= \beta \left(N - \tilde{D} + \frac{\mu}{\gamma} \tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)\alpha} - \frac{\gamma + \mu}{\gamma} \alpha \right) \\
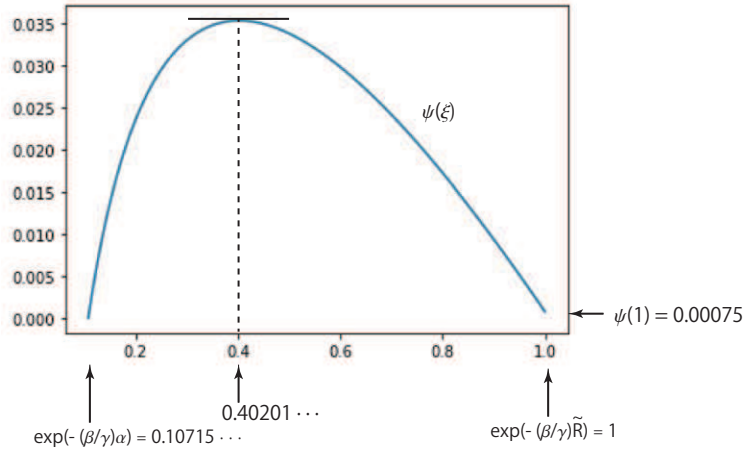&= 0 \tag{2.21}
\end{aligned}
$$

Figure 2.2: Variation of $\psi(\xi) = \beta N - \beta \tilde{D} + (\beta \mu / \gamma) \tilde{R} - \beta \tilde{S} e^{(\beta/\gamma)\tilde{R}} \xi + (\gamma + \mu) \log \xi$ for $N = 1000, \tilde{S} = 995, \tilde{I} = 5, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.15/1000, \gamma = 0.05$ and $\mu = 0.01$. In this case we observe that $e^{-(\beta/\gamma)\alpha} = 0.10715\ldots, e^{-(\beta/\gamma)\tilde{R}} = 1,$ $\psi(1) = \beta N - \beta \tilde{S} = \beta \tilde{I} = 0.00075, \psi'(\xi) = 0$ for $\xi = 0.40201\ldots, (\gamma + \mu)/\beta = 40,$ $((\gamma + \mu)/\beta)e^{(\beta/\gamma)\alpha} > 2000,$ and $(\gamma + \mu)/\beta < \tilde{S} < ((\gamma + \mu)/\beta)e^{(\beta/\gamma)\alpha}.$

in view of (2.16). Taking account of the hypotheses (A$_1$) and (A$_5$), we find that $\psi(\xi) > 0$ in $(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$ by the same arguments as in the proof of Theorem 2.3. Since

$$\frac{1}{\xi \psi(\xi)} = \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \frac{1}{\psi(\xi)} + \frac{1}{\gamma + \mu} \frac{-\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}} + (\gamma + \mu)/\xi}{\psi(\xi)},$$

we get

$$\varphi(u) = \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi \psi(\xi)}$$

$$= \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma + \mu} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{\psi'(\xi)}{\psi(\xi)} d\xi$$

$$= \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma + \mu} \left(\log \psi(e^{-(\beta/\gamma)\tilde{R}}) - \log \psi(u) \right)$$

$$= \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma + \mu} \left(\log (\beta \tilde{I}) - \log \psi(u) \right). \tag{2.22}$$

Therefore, we see from (2.21) and (2.22) that

$$\lim_{u \to e^{-(\beta/\gamma)\alpha} + 0} \varphi(u) = \lim_{u \to e^{-(\beta/\gamma)\alpha} + 0} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi \psi(\xi)} = \infty.$$

Then we conclude that $\varphi^{-1}(t) \in C^1(0, \infty)$, $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$, and that

$$\varphi^{-1}(0) = e^{-(\beta/\gamma)\tilde{R}},$$

$$\lim_{t \to \infty} \varphi^{-1}(t) = e^{-(\beta/\gamma)\alpha}.$$

It is easily verified that

$$\left(\varphi^{-1}(t) \right)' = \frac{1}{\varphi'(u)} \Big|_{u = \varphi^{-1}(t)} = -u \psi(u) \Big|_{u = \varphi^{-1}(t)}$$

$$= -\varphi^{-1}(t) \psi(\varphi^{-1}(t)) = -\beta I(t) \varphi^{-1}(t), \tag{2.23}$$

in light of

$$\psi(\varphi^{-1}(t)) = \beta I(t). \tag{2.24}$$

We observe, using (2.17)–(2.20) and (2.23), that

$$S'(t) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\left(\varphi^{-1}(t)\right)' = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\left(-\beta I(t)\varphi^{-1}(t)\right)$$

$$= -\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)I(t) = -\beta S(t)I(t),$$

$$I'(t) = -\tilde{S}e^{(\beta/\gamma)\tilde{R}}\left(\varphi^{-1}(t)\right)' + \frac{\gamma+\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)}$$

$$= -(-\beta S(t)I(t)) + \frac{\gamma+\mu}{\beta}(-\beta I(t)) = \beta S(t)I(t) - \gamma I(t) - \mu I(t),$$

$$R'(t) = -\frac{\gamma}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}(-\beta I(t)) = \gamma I(t),$$

$$D'(t) = -\frac{\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}(-\beta I(t)) = \mu I(t),$$

and consequently (2.17)–(2.20) satisfy (1.1)–(1.4), respectively. It is easy to check that

$$S(0) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(0) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\tilde{R}} = \tilde{S},$$

$$R(0) = -\frac{\gamma}{\beta}\log\varphi^{-1}(0) = -\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} = \tilde{R},$$

$$D(0) = -\frac{\mu}{\beta}\log\varphi^{-1}(0) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}$$

$$= -\frac{\mu}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}$$

$$= -\frac{\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R}\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = \tilde{D},$$

$$I(0) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(0) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(0)$$

$$= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} + \frac{\gamma+\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R}\right)$$

$$= N - \tilde{D} - \tilde{S} - \tilde{R} = \tilde{I}. \qquad \square$$

3 Various properties of the exact solution

We can derive various properties of solutions of SIRD epidemic model via the differential system qualitatively, however we obtain more detailed properties directly from the exact solution of the SIRD differential system.

Theorem 3.1. *We observe that $I(\infty) = 0$ and $I(t) > 0$ on $[0,\infty)$, and that $I(t)$ has the maximum*

$$\max_{t\geq 0} I(t) = N - \tilde{D} - \tilde{R} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right)$$

at

$$t = T := \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Furthermore, $I(t)$ is increasing in $[0,T)$ and is decreasing in (T,∞).

Proof. Taking account of (2.16), we easily check that

$$I(\infty) = \lim_{t\to\infty} I(t)$$

$$= \lim_{u\to e^{-(\beta/\gamma)\alpha}+0} \left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u \right)$$

$$= \frac{\gamma+\mu}{\gamma} \left(F(N,\tilde{D},\tilde{R},\gamma,\mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} - \alpha \right)$$

$$= 0.$$

Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$ for $t \ge 0$ and $\psi(\xi) > 0$ for $e^{-(\beta/\gamma)\alpha} < \xi \le e^{-(\beta/\gamma)\tilde{R}}$, we find that $I(t) = (1/\beta)\psi(\varphi^{-1}(t)) > 0$ on $[0,\infty)$. Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$ for $t > 0$ and $\psi(\xi) > 0$ for $e^{-(\beta/\gamma)\alpha} < \xi \le e^{-(\beta/\gamma)\tilde{R}}$, we see that $I(t) = (1/\beta)\psi(\varphi^{-1}(t))$ given by (2.24) is positive for $t > 0$. Differentiating both sides of (2.24), we arrive at

$$I'(t) = \frac{1}{\beta}\psi'(\varphi^{-1}(t))(\varphi^{-1}(t))'$$

$$= \frac{1}{\beta}\left(-\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma+\mu}{\varphi^{-1}(t)} \right)(\varphi^{-1}(t))'$$

$$= \frac{1}{\beta}(-\beta S(t) + \gamma + \mu)\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)}$$

$$= \frac{1}{\beta}(-\beta S(t) + \gamma + \mu)(-\beta I(t))$$

$$= (\beta S(t) - (\gamma+\mu))I(t) \tag{3.1}$$

in view of (2.23). It is obvious that $I'(t) = 0$ holds if and only if

$$\varphi^{-1}(t) = \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}$$

or

$$S(t) = \frac{\gamma+\mu}{\beta},$$

which yield

$$t = T = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} \right) = S^{-1}\left(\frac{\gamma+\mu}{\beta} \right).$$

We note that

$$e^{-(\beta/\gamma)\alpha} < \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} = \frac{\gamma+\mu}{\beta\tilde{S}}e^{-(\beta/\gamma)\tilde{R}} < e^{-(\beta/\gamma)\tilde{R}}$$

in light of the hypotheses (A$_1$) and (A$_5$). Since $(\varphi^{-1}(t))' < 0$ and $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we observe that $I'(t) > 0$ [resp. < 0] if and only if $t < T$ [resp. $> T$]. Hence, $I(t)$ is increasing in $[0,T)$ and is decreasing in (T,∞). We find that the maximum of $I(t)$ on $[0,\infty)$ is given by

$$\frac{1}{\beta}\psi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} \right) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} + \frac{\gamma+\mu}{\beta}\log\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} \right)$$

$$= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \frac{\gamma+\mu}{\beta} + \frac{\gamma+\mu}{\beta}\left(\log\frac{\gamma+\mu}{\beta} - \log\tilde{S} - \frac{\beta}{\gamma}\tilde{R} \right)$$

$$= N - \tilde{D} - \tilde{R} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta} \right). \qquad \square$$

Corollary 3.2. *The function $I(t)$ has the maximum at*

$$T = \varphi\left(\frac{\gamma + \mu}{\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}}}\right)$$

$$= \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_{(\gamma+\mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)}$$

$$+ \frac{1}{\gamma + \mu} \left(\log\left(\beta\tilde{I}\right) - \log\left(\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)\right)\right),$$

and T satisfies the following inequality

$$\tau_1 \leq T \leq \tau_2,$$

where

$$\tau_1 = \frac{(\beta/(\gamma+\mu))\tilde{S} - 1}{\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma)} + \frac{1}{\gamma + \mu}\left(\log\left(\beta\tilde{I}\right) - \log\left(\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)\right)\right)$$

and

$$\tau_2 = \frac{(\beta/(\gamma+\mu))\tilde{S} - 1}{\beta\tilde{I}} + \frac{1}{\gamma + \mu}\left(\log\left(\beta\tilde{I}\right) - \log\left(\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)\right)\right)$$

with $H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)$ being

$$H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu) := N - \tilde{D} - \tilde{R} - \frac{\gamma + \mu}{\beta}\left(1 + \log \tilde{S} - \log \frac{\gamma + \mu}{\beta}\right).$$

Proof. It follows from (2.22) that

$$T = \varphi\left(\frac{\gamma + \mu}{\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}}}\right)$$

$$= \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_{(\gamma+\mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)}$$

$$+ \frac{1}{\gamma + \mu}\left(\log\left(\beta\tilde{I}\right) - \log\left(\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)\right)\right)$$

because of

$$\psi\left(\frac{\gamma + \mu}{\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}}}\right) = \beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu).$$

From (2.12) we see that

$$\psi'(\xi) = -\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma + \mu}{\xi},$$

and that $\psi'\left((\gamma + \mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}})\right) = 0$, $\psi(e^{-(\beta/\gamma)\tilde{R}}) = \beta\tilde{I}$, and $\psi(\xi)$ is decreasing on $[(\gamma + \mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}}), e^{-(\beta/\gamma)\tilde{R}}]$. Then we get

$$\beta\tilde{I} \leq \psi(\xi) \leq \beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu),$$

and hence

$$\frac{1}{\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)} \leq \frac{1}{\psi(\xi)} \leq \frac{1}{\beta\tilde{I}}.$$

Integrating the above inequality over $[(\gamma + \mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}}), e^{-(\beta/\gamma)\tilde{R}}]$, and then multiplying by $(\beta/(\gamma + \mu))\tilde{S} e^{(\beta/\gamma)\tilde{R}}$, we are led to

$$\frac{(\beta/(\gamma + \mu))\tilde{S} - 1}{\beta H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)} \leq \frac{\beta}{\gamma + \mu} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \int_{(\gamma+\mu)/(\beta \tilde{S} e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)}$$

$$\leq \frac{(\beta/(\gamma + \mu))\tilde{S} - 1}{\beta \tilde{I}},$$

which yields the desired inequality. □

Theorem 3.3. *We find that $R(\infty) = \alpha$,*

$$R(\infty) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\infty)} - \frac{\mu}{\gamma}R(\infty), \tag{3.2}$$

and that $R(t)$ is an increasing function on $[0, \infty)$ such that

$$\tilde{R} \leq R(t) < \alpha = R(\infty).$$

Proof. It follows from (2.19) that

$$R(\infty) = \lim_{t \to \infty} R(t) = \lim_{t \to \infty} -\frac{\gamma}{\beta} \log \varphi^{-1}(t)$$

$$= \lim_{u \to e^{-(\beta/\gamma)\alpha} + 0} -\frac{\gamma}{\beta} \log u$$

$$= \alpha.$$

Since $\alpha = R(\infty)$, the identity (3.2) follows from (2.16). Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \leq e^{-(\beta/\gamma)\tilde{R}}$, we obtain

$$-\frac{\gamma}{\beta} \log e^{-(\beta/\gamma)\tilde{R}} \leq R(t) < -\frac{\gamma}{\beta} \log e^{-(\beta/\gamma)\alpha},$$

or

$$\tilde{R} \leq R(t) < \alpha = R(\infty).$$

It is obvious that $R(t)$ is increasing on $[0, \infty)$ in view of the fact that $\varphi^{-1}(t)$ is decreasing $[0, \infty)$. □

Theorem 3.4. *We see that*

$$S(\infty) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\infty)}, \tag{3.3}$$

and that $S(t)$ is a decreasing function on $[0, \infty)$ such that

$$\tilde{S} \geq S(t) > \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)\alpha} = S(\infty).$$

Proof. The identity (3.3) follows from

$$S(\infty) = \lim_{t \to \infty} S(t) = \lim_{t \to \infty} \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi^{-1}(t)$$

$$= \lim_{u \to e^{-(\beta/\gamma)\alpha} + 0} \tilde{S} e^{(\beta/\gamma)\tilde{R}} u$$

$$= \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)\alpha}$$

$$= \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\gamma)R(\infty)}.$$

Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$, we have

$$\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} < \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) \le \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\tilde{R}}.$$

Hence we obtain

$$\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} < S(t) \le \tilde{S}.$$

Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we deduce that $S(t)$ is also decreasing on $[0,\infty)$. $\square$

Theorem 3.5. *The following holds:*

$$D(\infty) = \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}, \tag{3.4}$$

and $D(t)$ is an increasing function on $[0,\infty)$ such that

$$\tilde{D} \le D(t) < D(\infty).$$

Proof. Taking account of (2.20), we obtain

$$
\begin{aligned}
\lim_{t\to\infty} D(t) &= \lim_{t\to\infty}\left(-\frac{\mu}{\beta}\log\varphi^{-1}(t)\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\
&= \lim_{u\to e^{-(\beta/\gamma)\alpha}+0}\left(-\frac{\mu}{\beta}\log u\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\
&= \frac{\mu}{\gamma}\alpha + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\
&= \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R},
\end{aligned}
$$

which is the desired identity (3.4). Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$, we get

$$-\frac{\beta}{\gamma}\alpha < \log\varphi^{-1}(t) \le -\frac{\beta}{\gamma}\tilde{R},$$

and hence

$$\frac{\mu}{\gamma}\tilde{R} \le -\frac{\mu}{\beta}\log\varphi^{-1}(t) < \frac{\mu}{\gamma}\alpha,$$

which implies

$$\tilde{D} \le D(t) < \frac{\mu}{\gamma}\alpha + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.$$

We conclude that $D(t)$ is increasing on $[0,\infty)$ since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$. $\square$

Theorem 3.6. *If*

$$\tilde{S} \le \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right), \tag{3.5}$$

then there exists a number T_1 ($T < T_1$) such that $I(t)$ is concave in $(0, T_1)$, and is convex in (T_1, ∞). If

$$\tilde{S} > \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right), \tag{3.6}$$

then there exist two numbers T_2 and T_3 ($0 < T_2 < T < T_3$) such that $I(t)$ is convex in $(0, T_1) \cup (T_3, \infty)$, and is concave in (T_2, T_3) (cf. Figures 3.1, 3.2).
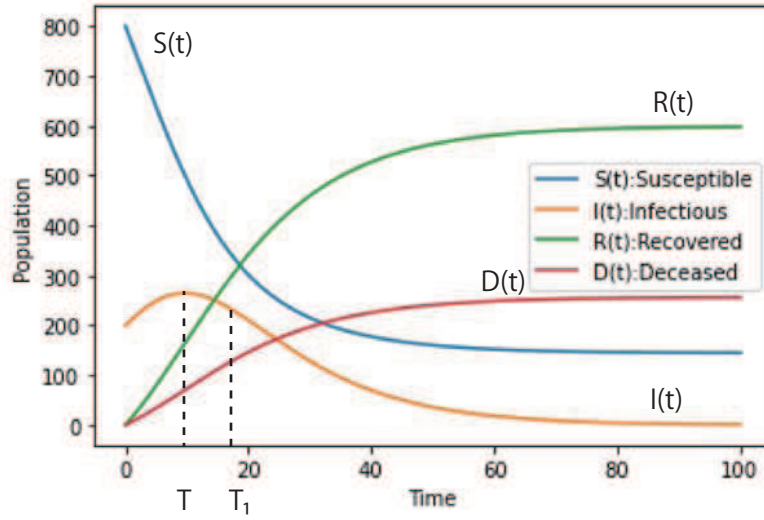
Figure 3.1: Variation of $S(t)$, $I(t)$, $R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 800, \tilde{I} = 200, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.07$ and $\mu = 0.03$. In this case $\tilde{S}(= 800) > (\gamma + \mu)/\beta(= 500) > S(\infty)(= 144.57\ldots)$, and the condition (3.5) is satisfied because $(\gamma + \mu)/\beta = 500$ and $\tilde{S}(= 800) < (\gamma + \mu)/\beta + (1/2)\left(\tilde{I} + \sqrt{(4(\gamma + \mu)/\beta)\tilde{I} + \tilde{I}^2}\right)(= 931.66\ldots)$.

Proof. First we note that the hypotheses (A$_1$) and (A$_5'$) imply that

$$\tilde{S} > \frac{\gamma + \mu}{\beta} > N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \frac{\gamma + \mu}{\gamma}\alpha$$

$$= N - R(\infty) - \left(\frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}\right)$$

$$= N - R(\infty) - D(\infty) = S(\infty)$$

in light of $\alpha = R(\infty)$ and $I(\infty) = 0$. Differentiating (3.1) with respect to t and taking (1.1), (1.2) into account, we obtain

$$I''(t) = (\beta S'(t))I(t) + (\beta S(t) - (\gamma + \mu))I'(t)$$
$$= \beta(-\beta S(t)I(t))I(t) + (\beta S(t) - (\gamma + \mu))(\beta S(t)I(t) - \gamma I(t) - \mu I(t))$$
$$= \left(-\beta^2 S(t)I(t) + \beta^2 S(t)^2 - 2\beta(\gamma + \mu)S(t) + (\gamma + \mu)^2\right)I(t)$$
$$= \beta^2\left(S(t)^2 - S(t)I(t) - \frac{2(\gamma + \mu)}{\beta}S(t) + \frac{(\gamma + \mu)^2}{\beta^2}\right)I(t)$$
$$= \beta^2\left(S(t) - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2}\frac{1}{S(t)} - I(t)\right)S(t)I(t).$$

Now we investigate the sign of $I''(t)$. We define

$$G(t) := S(t) - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2}\frac{1}{S(t)} - I(t)$$
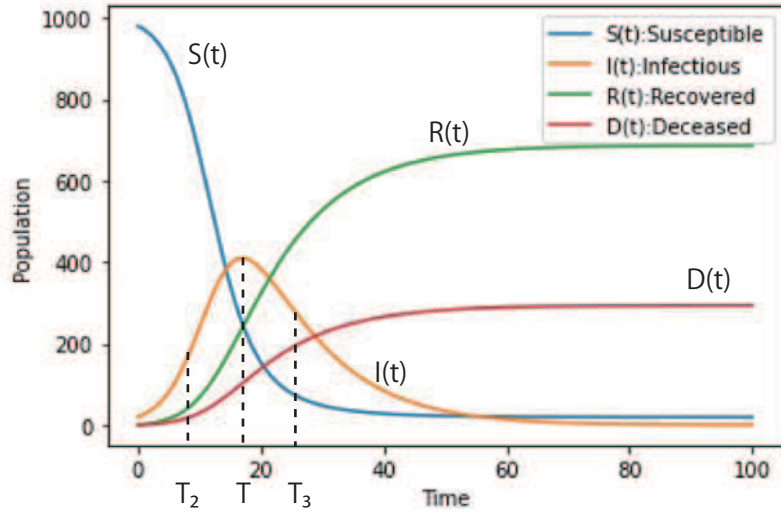
Figure 3.2: Variation of $S(t)$, $I(t)$, $R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 980, \tilde{I} = 20, \tilde{R} = \tilde{D} = 0, \beta = 0.4/1000, \gamma = 0.07$ and $\mu = 0.03$. In this case $\tilde{S}(= 980) > (\gamma + \mu)/\beta (= 250) > S(\infty)(= 19.39\dots)$, and the condition (3.6) is satisfied because $\tilde{S}(= 980) > (\gamma + \mu)/\beta + (1/2)(\tilde{I} + \sqrt{(4(\gamma + \mu)/\beta)\tilde{I} + \tilde{I}^2})(= 331.41\dots)$.

and differentiate both sides of the above with respect to t to obtain

$$
\begin{aligned}
G'(t) &= S'(t) - \frac{(\gamma + \mu)^2}{\beta^2} \frac{S'(t)}{S(t)^2} - I'(t) \\
&= S'(t) - \frac{(\gamma + \mu)^2}{\beta^2} \frac{-\beta S(t) I(t)}{S(t)^2} - I'(t) \\
&= -\beta S(t) I(t) + \frac{(\gamma + \mu)^2}{\beta} \frac{I(t)}{S(t)} - \left(\beta S(t) I(t) - \gamma I(t) - \mu I(t)\right) \\
&= -2\beta S(t) I(t) + \frac{(\gamma + \mu)^2}{\beta} \frac{I(t)}{S(t)} + (\gamma + \mu) I(t) \\
&= -2\beta \left(S(t)^2 - \frac{(\gamma + \mu)}{2\beta} S(t) - \frac{(\gamma + \mu)^2}{2\beta^2} \right) \frac{I(t)}{S(t)} \\
&= -2\beta \left(S(t) - \frac{\gamma + \mu}{\beta} \right) \left(S(t) + \frac{\gamma + \mu}{2\beta} \right) \frac{I(t)}{S(t)}.
\end{aligned}
$$

Since $S(t) + (\gamma + \mu)/(2\beta) > 0$, it follows that $G'(t) = 0$ for $t = T = S^{-1}((\gamma + \mu)/\beta)$, and that $G'(t) < 0$ [resp. > 0] if $t < T$ [resp. $> T$]. Therefore, $G(t)$ is decreasing in $[0, T)$ and increasing in (T, ∞). It is readily seen that

$$
\begin{aligned}
G(0) &= \tilde{S} - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2} \frac{1}{\tilde{S}} - \tilde{I} \\
&= \frac{1}{\tilde{S}} \left(\tilde{S}^2 - \left(\frac{2(\gamma + \mu)}{\beta} + \tilde{I} \right) \tilde{S} + \frac{(\gamma + \mu)^2}{\beta^2} \right) \\
&= \frac{1}{\tilde{S}} (\tilde{S} - s_1)(\tilde{S} - s_2),
\end{aligned}
$$

where

$$s_1 = \frac{\gamma + \mu}{\beta} + \frac{1}{2}\left(\tilde{I} - \sqrt{\frac{4(\gamma + \mu)}{\beta}\tilde{I} + \tilde{I}^2}\right) \quad \left(< \frac{\gamma + \mu}{\beta}\right),$$

$$s_2 = \frac{\gamma + \mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma + \mu)}{\beta}\tilde{I} + \tilde{I}^2}\right) \quad \left(> \frac{\gamma + \mu}{\beta}\right).$$

Moreover we observe that

$$G(T) = S(T) - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2}\frac{1}{S(T)} - I(T)$$

$$= \frac{\gamma + \mu}{\beta} - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2}\frac{\beta}{\gamma + \mu} - \max_{t \geq 0} I(t)$$

$$= -\max_{t \geq 0} I(t) < 0,$$

and that

$$\lim_{t \to \infty} G(t) = S(\infty) - \frac{2(\gamma + \mu)}{\beta} + \frac{(\gamma + \mu)^2}{\beta^2}\frac{1}{S(\infty)} - I(\infty)$$

$$= \frac{1}{S(\infty)}\left(\frac{\gamma + \mu}{\beta} - S(\infty)\right)^2 > 0.$$

If (3.5) is satisfied, then $G(0) \leq 0$, and therefore there exists a number $T_1 > T$ such that $G(T_1) = 0$, $G(t)$ is negative in $(0, T_1)$, and $G(t)$ is positive in (T_1, ∞). Since $I''(t) = \beta^2 G(t)S(t)I(t)$, we deduce that $I(t)$ is concave in $(0, T_1)$, and is convex in (T_1, ∞). If (3.6) is satisfied, then $G(0) > 0$, and hence there exist two numbers T_2 and T_3 $(0 < T_2 < T < T_3)$ such that $G(T_2) = G(T_3) = 0$, $G(t)$ is positive in $(0, T_2) \cup (T_3, \infty)$, and $G(t)$ is negative in (T_2, T_3). Consequently we conclude that $I(t)$ is convex in $(0, T_2) \cup (T_3, \infty)$, and is concave in (T_2, T_3). $\qquad\square$

Theorem 3.7. *The following identity holds:*

$$S(\infty) = \tilde{S} + \tilde{I} + \frac{\gamma + \mu}{\beta}\log\frac{S(\infty)}{\tilde{S}}.$$

Proof. Since $I(\infty) = 0$, we observe, using (3.4), that

$$S(\infty) = N - R(\infty) - D(\infty)$$

$$= N - R(\infty) - \frac{\mu}{\gamma}R(\infty) - \tilde{D} + \frac{\mu}{\gamma}\tilde{R}$$

$$= N - \tilde{D} - \tilde{R} + \left(1 + \frac{\mu}{\gamma}\right)\tilde{R} - \left(1 + \frac{\mu}{\gamma}\right)R(\infty)$$

$$= \tilde{S} + \tilde{I} + \left(1 + \frac{\mu}{\gamma}\right)\frac{\gamma}{\beta}\left(\frac{\beta}{\gamma}\tilde{R} - \frac{\beta}{\gamma}R(\infty)\right)$$

$$= \tilde{S} + \tilde{I} + \frac{\gamma + \mu}{\beta}\log\left(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}\right)$$

$$= \tilde{S} + \tilde{I} + \frac{\gamma + \mu}{\beta}\log\frac{S(\infty)}{\tilde{S}}$$

by virtue of (3.3). $\qquad\square$

Theorem 3.8. *It follows that*

$$S'(\infty) = I'(\infty) = R'(\infty) = D'(\infty) = 0.$$

Proof. Since $I(\infty) = 0$, we conclude that

$$\begin{aligned} S'(\infty) &= -\beta S(\infty)I(\infty) = 0, \\ I'(\infty) &= \beta S(\infty)I(\infty) - \gamma I(\infty) - \mu I(\infty) = 0, \\ R'(\infty) &= \gamma I(\infty) = 0, \\ D'(\infty) &= \mu I(\infty) = 0, \end{aligned}$$

by taking account of (1.1)–(1.4).   □

Theorem 3.9. *Let $(S(t), I(t), R(t), D(t))$ be the exact solution (2.17)–(2.20) of the initial value problem (1.1)–(1.5), and let*

$$\left(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right) := \left(S(\varphi(u)), I(\varphi(u)), R(\varphi(u)), D(\varphi(u))\right).$$

Then $\left(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right)$ is a solution of the initial value problem for the linear differential system

$$\frac{d\hat{S}(u)}{du} = \frac{\hat{S}(u)}{u}, \tag{3.7}$$

$$\frac{d\hat{I}(u)}{du} = -\frac{\hat{S}(u)}{u} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}, \tag{3.8}$$

$$\frac{d\hat{R}(u)}{du} = -\frac{\gamma}{\beta}\frac{1}{u}, \tag{3.9}$$

$$\frac{d\hat{D}(u)}{du} = -\frac{\mu}{\beta}\frac{1}{u} \tag{3.10}$$

for $u \in (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}})$, with the initial condition

$$\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{S}, \tag{3.11}$$

$$\hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{I}, \tag{3.12}$$

$$\hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{R}, \tag{3.13}$$

$$\hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{D}. \tag{3.14}$$

Proof. First we remark that

$$\hat{I}(u) = I(\varphi(u)) = \frac{1}{\beta}\psi(u) \tag{3.15}$$

in light of (2.24). Noting

$$S'(\varphi(u)) = -\beta S(\varphi(u))I(\varphi(u)) = -\beta\hat{S}(u)\hat{I}(u),$$

we are led to

$$\begin{aligned} \frac{d\hat{S}(u)}{du} &= \frac{dS(t)}{dt}\bigg|_{t=\varphi(u)}\varphi'(u) = S'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\ &= \left(-\beta\hat{S}(u)\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right) = \beta\hat{S}(u)\hat{I}(u)\left(\frac{1}{u\psi(u)}\right) \\ &= \frac{\hat{S}(u)}{u} \end{aligned}$$

in view of (3.15). Similarly we obtain

$$
\begin{aligned}
\frac{d\hat{R}(u)}{du} &= \left.\frac{dR(t)}{dt}\right|_{t=\varphi(u)} \varphi'(u) = R'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= (\gamma\hat{I}(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\gamma}{\beta}\frac{1}{u},
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\hat{D}(u)}{du} &= \left.\frac{dD(t)}{dt}\right|_{t=\varphi(u)} \varphi'(u) = D'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= (\mu\hat{I}(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\mu}{\beta}\frac{1}{u},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\hat{I}(u)}{du} &= \left.\frac{dI(t)}{dt}\right|_{t=\varphi(u)} \varphi'(u) = I'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= (\beta\hat{S}(u)\hat{I}(u) - \gamma\hat{I}(u) - \mu\hat{I}(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\hat{S}(u)}{u} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}.
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= S\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = S(0) = \tilde{S}, \\
\hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= I\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = I(0) = \tilde{I}, \\
\hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= R\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = R(0) = \tilde{R}, \\
\hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= D\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = D(0) = \tilde{D}.
\end{aligned}
$$

Hence, $(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u))$ is a solution of the initial value problem (3.7)–(3.14). □

Theorem 3.10. *Solving the initial value problem (3.7)–(3.14), we obtain the solution (2.4)–(2.7) for* $u \in (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$.

Proof. Since (3.7) is equivalent to

$$
\frac{d}{du}\left(\frac{1}{u}\hat{S}(u)\right) = 0,
$$

we derive

$$
\hat{S}(u) = ku
$$

for some constant k. It follows from (3.11) that

$$
\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = ke^{-(\beta/\gamma)\tilde{R}} = \tilde{S}
$$

and hence

$$k = \tilde{S} e^{(\beta/\gamma)\tilde{R}},$$

which yields

$$\hat{S}(u) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} u.$$

Solving (3.9) yields

$$\hat{R}(u) = -\frac{\gamma}{\beta} \log u + k$$

for some constant k. The initial condition (3.13) implies

$$\hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\frac{\gamma}{\beta} \log e^{-(\beta/\gamma)\tilde{R}} + k = \tilde{R} + k = \tilde{R}$$

and hence $k = 0$. Consequently we have

$$\hat{R}(u) = -\frac{\gamma}{\beta} \log u.$$

We solve (3.10) to obtain

$$\hat{D}(u) = -\frac{\mu}{\beta} \log u + k$$

for some constant k. The initial condition (3.14) implies

$$\hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\frac{\mu}{\beta} \log e^{-(\beta/\gamma)\tilde{R}} + k = \frac{\mu}{\gamma}\tilde{R} + k = \tilde{D}$$

and hence $k = \tilde{D} - (\mu/\gamma)\tilde{R}$. Consequently we have

$$\hat{D}(u) = -\frac{\mu}{\beta} \log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.$$

Since

$$\frac{\hat{S}(u)}{u} = \tilde{S} e^{(\beta/\gamma)\tilde{R}},$$

we obtain

$$\frac{d\hat{I}(u)}{du} = -\tilde{S} e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}.$$

Hence we get

$$\hat{I}(u) = -\tilde{S} e^{(\beta/\gamma)\tilde{R}} u + \frac{\gamma}{\beta} \log u + \frac{\mu}{\beta} \log u + k$$

for some constant k. From the initial condition (3.12) it follows that

$$\hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\tilde{S} + \frac{\gamma}{\beta} \log e^{-(\beta/\gamma)\tilde{R}} + \frac{\mu}{\beta} \log e^{-(\beta/\gamma)\tilde{R}} + k = -\tilde{S} - \tilde{R} - \frac{\mu}{\gamma}\tilde{R} + k = \tilde{I},$$

which implies

$$k = \tilde{S} + \tilde{I} + \tilde{R} + \frac{\mu}{\gamma}\tilde{R} = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R}.$$

Therefore we deduce that

$$\hat{I}(u) = -\tilde{S} e^{(\beta/\gamma)\tilde{R}} u + \frac{\gamma + \mu}{\beta} \log u + N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R}. \qquad \square$$
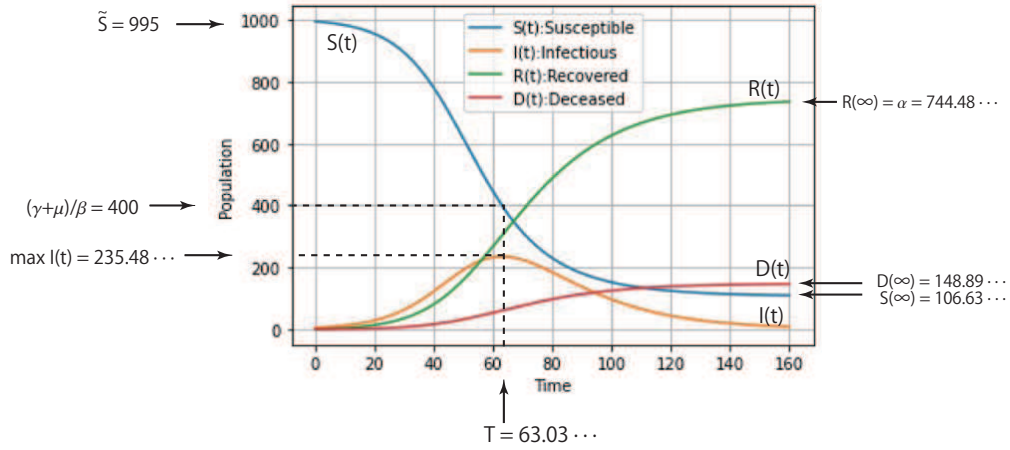
Figure 3.3: Variation of $S(t), I(t), R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 995, \tilde{I} = 5, \tilde{R} = \tilde{D} = 0, \beta = 0.15/1000, \gamma = 0.05$ and $\mu = 0.01$. In this case we obtain $R(\infty) = \alpha = 744.48\ldots$, $I(\infty) = 0$, $D(\infty) = 148.89\ldots$, $S(\infty) = N - R(\infty) - D(\infty) = 106.63\ldots$, $(\gamma + \mu)/\beta = 400$, $\tilde{S}(= 995) > (\gamma + \mu)/\beta(= 400) > S(\infty)(= 106.63\ldots)$, $\max_{t \geq 0} I(t) = 235.48\ldots$ and $T = 63.03\ldots$, where T is calculated by

$$T = \varphi\big((\gamma + \mu)/(\beta\tilde{S})\big) = \int_{400/995}^{1} \frac{d\xi}{\xi\psi(\xi)}$$
$$= \int_{400/995}^{1} \frac{d\xi}{\xi(0.15 - (0.15/1000) \times 995\,\xi + 0.06\log\xi)} = 63.03\ldots$$

Remark 3.11. The hypothesis (A$_3$) is satisfied if $\tilde{R} = 0$, since $N > \tilde{S} + \tilde{D}$.

Remark 3.12. The right differential coefficient $I'_{+}(0)$ is positive because

$$I'_{+}(0) = \lim_{t \to +0} I'(t) = \lim_{t \to +0} \big(\beta S(t)I(t) - \gamma I(t) - \mu I(t)\big)$$
$$= \beta\tilde{S}\tilde{I} - \gamma\tilde{I} - \mu\tilde{I} = \big(\beta\tilde{S} - \gamma - \mu\big)\tilde{I} > 0$$

in view of the hypotheses (A$_1$) and (A$_2$).

Remark 3.13. In the case where $(S(\infty) <)\,\tilde{S} \leq (\gamma + \mu)/\beta$ (i.e., $I'_{+}(0) \leq 0$) we deduce that $\psi(\xi)$ is increasing in $\big(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}\big]$, $\lim_{\xi \to e^{-(\beta/\gamma)\alpha}+0} \psi(\xi) = 0$ and $\psi\big(e^{-(\beta/\gamma)\tilde{R}}\big) = \beta\tilde{I}$. Since $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$, $\varphi^{-1}(0) = e^{-(\beta/\gamma)\tilde{R}}$ and $\lim_{t \to \infty} \varphi^{-1}(t) = e^{-(\beta/\gamma)\alpha}$, it follows that $I(t) = (1/\beta)\psi\big(\varphi^{-1}(t)\big)$ is decreasing on $[0, \infty)$, and that $I(0) = (1/\beta)\psi\big(\varphi^{-1}(0)\big) = \tilde{I}$ and $I(\infty) = \lim_{t \to \infty} I(t) = \lim_{\xi \to e^{-(\beta/\gamma)\alpha}+0}(1/\beta)\psi(\xi) = 0$ (*cf.* Figure 3.4).

Remark 3.14. The constant $H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)$ defined in Corollary 3.2 is equal to $\max_{t \geq 0} I(t)$ given in Theorem 3.1.

Remark 3.15. It follows from Theorems 3.1, 3.3–3.5 that $S(t) > 0, I(t) > 0$ for $t \geq 0$ and $R(t) > 0, D(t) > 0$ for $t > 0$.
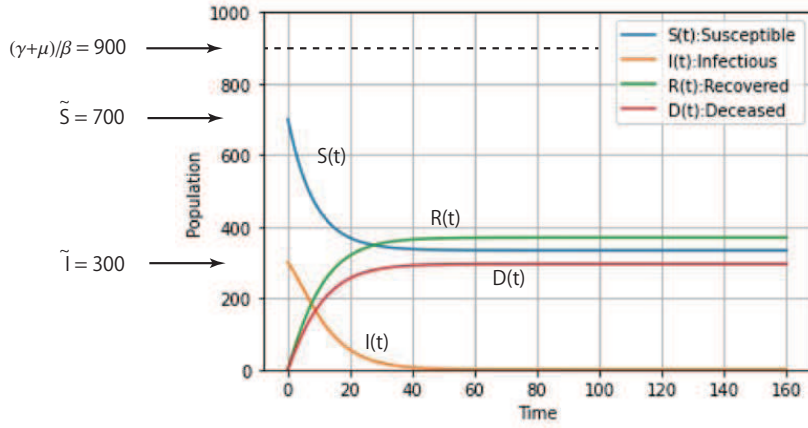
Figure 3.4: Variation of $S(t), I(t), R(t), D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 700, \tilde{I} = 300, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.1$ and $\mu = 0.08$. In this case we see that $(\gamma + \mu)/\beta (= 900) > \tilde{S}(= 700)$, $I(\infty) = 0$ and $I(t)$ is decreasing on $[0, \infty)$.

Remark 3.16. Under the hypothesis

(A$_3'$) $\tilde{D} \geq 0$ satisfies

$$N - \tilde{R} > \tilde{S} e^{(\beta/\mu)\tilde{D}} + \tilde{D},$$

the transcendental equation

$$y = \frac{\mu}{\mu + \gamma} N - \frac{\mu}{\mu + \gamma} \tilde{R} + \frac{\gamma}{\mu + \gamma} \tilde{D} - \frac{\mu}{\mu + \gamma} \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)y} \tag{3.16}$$

has a unique solution $y = \alpha_*$ such that

$$\tilde{D} < \alpha_* < N$$

by the same arguments as in Lemma 2.4. Since the equation (3.16) reduces to the transcendental equation in Lemma 2.4 by the transformation $y = \tilde{D} - (\mu/\gamma)(\tilde{R} - x)$, we see that $\alpha_* = \tilde{D} - (\mu/\gamma)(\tilde{R} - \alpha)$. We define

$$\varphi_*(w) := \int_w^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi \psi_*(\xi)}$$

for $e^{-(\beta/\mu)\alpha_*} < w \leq e^{-(\beta/\mu)\tilde{D}}$, where

$$\psi_*(\xi) = \beta N - \beta \tilde{R} + \frac{\beta \gamma}{\mu} \tilde{D} - \beta \tilde{S} e^{(\beta/\mu)\tilde{D}} \xi + (\mu + \gamma) \log \xi.$$

It follows from the transformation

$$\xi = e^{-(\beta/\mu)\tilde{D}} e^{(\beta/\gamma)\tilde{R}} \eta$$

that

$$\varphi_*(w) = \int_{e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\gamma)\tilde{R}} w}^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\eta}{\eta \psi(\eta)},$$

where $e^{-(\beta/\gamma)\alpha} < e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w \le e^{-(\beta/\gamma)\tilde{R}}$. Then there exist the inverse functions $\varphi_*^{-1}(t)$ and $\varphi^{-1}(t)$ of the functions

$$t = \varphi_*(w), \quad t = \varphi\big(e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w\big),$$

respectively, and the following hold:

$$w = \varphi_*^{-1}(t), \quad e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w = \varphi^{-1}(t) \quad (0 \le t < \infty).$$

Hence we obtain

$$\varphi_*^{-1}(t) = e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) \quad (0 \le t < \infty).$$

Let $\big(S_*(t), I_*(t), R_*(t), D_*(t)\big)$ be the exact solution of the initial value problem (1.1)–(1.5) by starting our arguments utilizing (1.4) instead of (1.3). Then we observe that

$$
\begin{aligned}
S_*(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi_*^{-1}(t)\\
&= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\\
&= \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) = S(t),\\
I_*(t) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)\\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) = I(t),\\
R_*(t) &= -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\\
&= -\frac{\gamma}{\beta}\log\big(e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\big) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\\
&= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) = R(t),\\
D_*(t) &= -\frac{\mu}{\beta}\log\varphi_*^{-1}(t)\\
&= -\frac{\mu}{\beta}\log\big(e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\big)\\
&= \tilde{D} - \frac{\mu}{\gamma}\tilde{R} - \frac{\mu}{\beta}\log\varphi^{-1}(t) = D(t)
\end{aligned}
$$

for $0 \le t < \infty$. Consequently we conclude that

$$\big(S_*(t), I_*(t), R_*(t), D_*(t)\big) \equiv \big(S(t), I(t), R(t), D(t)\big) \quad \text{on } [0, \infty).$$

Remark 3.17. In this paper we derived the explicit formula for the exact solution of the SIRD epidemic model, and obtained various properties of the exact solution including the maximum of $I(t)$, the concavity and convexity of $I(t)$, time T which attains $\max_{t\ge 0} I(t)$ and the linear differential system which is satisfied by the parametric form of the exact solution. If $\mu = 0$ and $D(t) \equiv 0$, then the SIRD epidemic model reduces to the SIR epidemic model. We note that our results can be applied to the SIR epidemic model if we set $\mu = 0$ and $D(t) \equiv 0$.

Acknowledgements

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