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Sobolev inequality with non-uniformly degenerating gradient

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Abstract. In this paper we prove the following weighted Sobolev inequality in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, of a homogeneous space $(\mathbb{R}^n, \rho, w dx)$, under suitable compatibility conditions on the positive weight functions $(v, w, \omega_1, \omega_2, \ldots, \omega_n)$ and on the quasi-metric ρ,

$$\Big(\int_\Omega |f|^q v\, w dz\Big)^{\frac{1}{q}} \leq C \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i M_S w\, dz\Big)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}),$$

where $q \geq p > 1$ and M_S denotes the strong maximal operator. Some corollaries on non-uniformly degenerating gradient inequalities are derived.

Keywords: Sobolev's inequality, homogeneous space, non-uniformly degenerating gradient.

2020 Mathematics Subject Classification: 26D10, 35B45, 42B25.

1 Introduction

In this paper we aim to prove the following weighted Sobolev type inequality in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, of a homogeneous space $(\mathbb{R}^n, \rho, w dx)$

$$\Big(\int_\Omega |f|^q v\, w dz\Big)^{\frac{1}{q}} \leq C \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i M_S w\, dz\Big)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}), \tag{1.1}$$

where $q \geq p > 1$ and M_S denotes the strong maximal operator. This can be done under suitable compatibility conditions on the positive weight functions $(v, w, \omega_1, \omega_2, \ldots, \omega_n)$ and on the quasi-metric ρ.

We say that (1.1) is a non-uniform weighted Sobolev inequality since the functions $\omega_i \omega_j^{-1}$, $i, j = 1, \ldots, n$, are not assumed to be neither bounded nor bounded away from zero in any compact subset of Ω.

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Poincaré–Sobolev type inequalities are essential in many contexts of the theory of elliptic and parabolic partial differential equations such as the Harnack's inequality, the regularity of solutions, the continuation of differential inequalities, the absence of positive eigenvalues, the estimation of negative eigenvalues, the spectrum discreetness of Schrödinger operator etc. (see, e.g., [1,3,5,11,12,23–27,30,33,36,38,40]).

The study of the above mentioned qualitative properties of second order elliptic equations in absence of uniform ellipticity condition and in lack of uniform degeneration relies on Poincaré–Sobolev type weighted inequalities having non-uniformly degenerating gradient. Meanwhile, the theory has also been extended to more general contexts, such as that of Carnot–Carathéodory metrics associated with families of vector fields (see, e.g., [8–10,13]).

To deduce the inequality (1.1) one could first derive a suitable representation formula in terms of integral operators of potential type, and then use some continuity results for these operators in proper metric spaces, endowed with doubling measures (see, for example, [18]). In this paper, we show a new different approach to obtain the inequality (1.1). The arguments of our proofs are inspired by those of [31] (see, also the recent papers [29,32]), where the Euclidean metric was considered. However, the ideas of [31,32] cannot be simply adapted to homogeneous spaces and to non-uniformly degenerating gradients, since not all the homogeneous spaces posses the Besicovitch covering property (see, e.g., [35]). To overcome this difficulty, we use the "$5B$" covering lemma that holds in any homogeneous space, see e.g., [7,39]. We refer to [37], and to the references therein where the Euclidean metric and equal weights ω_i, , $i = 1, \ldots, n$, are considered.

In general, when dealing with multi-weighted Sobolev inequalities the task is to find sufficient (and hopefully necessary) conditions on the measures $\omega_i(x)dx$, $i = 1, \ldots, n$, and $v(x)dx$ which give

$$\Big(\int_\Omega |f|^q v\,dz\Big)^{\frac{1}{q}} \leq C\sum_{i=1}^{N}\Big(\int_\Omega |f_{z_i}|^p \omega_i\,dz\Big)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}), \tag{1.2}$$

where $1 \leq p \leq q < \infty$ and the constant C does not depend on f and Ω. For equal weights ω_i, $i = 1, \ldots, n$, sharp sufficient conditions can be found in [4,15] and in the papers [20,32,34]. Though this subject has been extensively studied in the last years it is still far from its full characterization (see, [6, 14–22]). Some progresses in deriving sufficient conditions for the Sobolev–Poincaré type inequalities with Grushin type weights were made in the works [15,29]. In this article we give sufficient conditions for the inequality (1.2) to hold and we show some generalizations.

2 Notation and main results

We say that $(\mathbb{R}^n, \rho)$ is a quasi-metric space if the function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ satisfies the following properties:

1) $\rho(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$; $\rho(x, y) = 0$ if and only if $x = y$;

2) $\rho(x, y) \leq K_0\Big(\rho(x, z) + \rho(y, z)\Big)$ for all $x, y, z \in \mathbb{R}^n$, with K_0 positive constant;

3) $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathbb{R}^n$.

A useful result by Macías and Segovia (see [28]) asserts that, every quasi-metric space is metrizable, i.e. there exist a distance d and a positive number $\alpha > 0$ such that ρ^α is equivalent to d.

Now, let us denote by $B(x,r) = \{y \in \mathbb{R}^n : \rho(y,x) < r\}$ the ρ-metric ball with center in $x \in \mathbb{R}^n$ and radius $r > 0$, and let μ be a nonnegative Borel measure on $\mathbb{R}^n$ satisfying the doubling condition. We say the measure μ is a doubling measure if there exists C_1 such that

$$\mu\left(B(x,2r)\right) \leq C_1 \mu\left(B(x,r)\right) \quad \text{for all } x \in \mathbb{R}^n,\ r > 0.$$

The quasi-metric space $(\mathbb{R}^n, \rho)$ equipped with a doubling measure μ is called a homogeneous space and it is denoted by $(\mathbb{R}^n, \rho, d\mu)$ (see [7]). In Section 3 we will give an example of homogeneous space.

In sequel, the notation $Q_n(x,r)$ (or simply $Q(x,r)$) denotes the n-dimensional Euclidean ball $Q(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ centered in x and of radius r. For $i = 1, \ldots, n$, we denote by $\ell_i(B(x,r)) = \sup\left\{|z_i - y_i| : z = (z_1, \ldots, z_n), y = (y_1, \ldots, y_n) \in B(x,r)\right\}$ and by $d(\Omega) = \sup\{\rho(x,y) : x, y \in \Omega\}$ the ρ-diameter of the domain Ω. We also let Σ be the collection of ρ-metric balls with center in Ω and radius less then $d(\Omega)$.

Given an integrable function f and a measurable set $E \subset \mathbb{R}^n$ we denote by $f(E) = \int_E f(x)dx$ the weighted measure of E, while $|E|$ denotes the Lebesgue measure of E. Denote by p' the conjugate number of $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$

A measurable function taking a.e. finite positive values is called a weight. A weight function $f : \mathbb{R}^n \to (0, \infty)$ belongs to the A_p-Muckenhoupt weight class, $1 < p < \infty$, with respect to the quasi-metric ρ, if for any ρ-metric ball $B = B(x,r) \subset \mathbb{R}^n$, one has

$$\left(\frac{1}{|B|}\int_B f(z)dz\right)\left(\frac{1}{|B|}\int_B f^{-\frac{1}{p-1}}(z)dz\right)^{p-1} \leq C, \tag{2.1}$$

while it belongs to the A_1-class if

$$\frac{1}{|B|}\int_B f(z)dz \leq C \inf_B f(z),$$

where the constants $C > 0$ do not depend on $x \in \mathbb{R}^n$ and $r > 0$.

A weight function $f : \mathbb{R}^n \to (0, \infty)$ belongs to the A_∞-Muckenhoupt weight class $A_\infty(dx)$ if there exist two constants $C, \delta > 0$ such that for any ρ-metric ball $B = B(x,r)$ and any measurable subset $E \subset B$ it holds that

$$\frac{f(E)}{f(B)} \leq C\left(\frac{|E|}{|B|}\right)^{\delta}. \tag{2.2}$$

Let $g : \mathbb{R}^n \to (0, \infty)$ be a weight function and μ be a doubling measure. We say g belongs to the $A_\infty(\mu)$ weight class if there exist two constants $C, \delta > 0$ such that for any ρ quasi-metric ball $B = B(x,r)$ and any measurable subset $E \subset B$ one has

$$\frac{\int_E g d\mu}{\int_B g d\mu} \leq C\left(\frac{\mu(E)}{\mu(B)}\right)^{\delta}, \tag{2.3}$$

For the main properties of the A_p-Muckenhoupt's weight classes, we refer the reader, for instance, to [7]. It is well-known that $A_p \subset A_\infty$ for any fixed $1 \leq p \leq \infty$ and moreover $A_\infty = \cup_{1 \leq p < \infty} A_p$. Furthermore, $A_p \subset A_{p-\varepsilon}$, for some $\varepsilon > 0$ depending on the constant C in the A_p class definition.

In the statement and proof of Theorem 2.5 below, we use the strong maximal function $M_S w$. For sake of completeness, let us recall its definition. Let $\mathcal{R}$ denote the collection of rectangles R in $\mathbb{R}^n$ with sides parallel to the coordinate axes, we define the strong maximal function M_S as

$$M_S f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad f \in L^{1,loc}$$

In Theorem 2.6, we make use of the classical fractional maximal operator $\mathcal{M}_\varepsilon$ defined as

$$\mathcal{M}_\varepsilon f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{(n-\varepsilon)/n}} \int_Q |f(y)| dy,$$

where the supremum is taken all over the Euclidean balls $\{Q\}$ containing the point x.

In the proofs of our main results we avail ourselves of the so called "$5B$" covering lemma below. This lemma, unlike Besicovich covering property, is valid in any homogeneous space.

Lemma 2.1 ([1, Covering Lemma, p. 270]). *Let (X, ρ, μ) be a homogeneous space. Let $B = \{B_\alpha = B(x_\alpha, r_\alpha) : \alpha \in \Gamma\}$ be a family of balls in X such that $\cup_{\alpha \in \Gamma} B_\alpha$ is bounded. Then there exists a sequence of disjoint balls $\{B(x_i, r_i)\}_{i \in \mathbb{N}} \subset B$ such that for every $\alpha \in \Gamma$ there exists i satisfying $r_\alpha \le 2r_i$ and $B_\alpha \subset B(x_i, 5K_0^2 r_i)$.*

Definition 2.2. Throughout this paper, we consider the quasi-metrics ρ satisfying the following "S"-condition

$$|B^\#| \le C|B|, \tag{2.4}$$

for all the ρ quasi-metric balls B (see next section for their definition). Here $B^\#$ is the smallest parallelepiped with edges parallel to coordinate axes containing the ρ quasi-metric ball B.

Definition 2.3. Moreover, we assume that there exists a constant $c > 0$ such that for every B and every $x, y \in B$, $t \in (0,1)$ one has

$$x + t(y - x) \in cB. \tag{2.5}$$

Now, we are ready to state our main results.

Theorem 2.4. *Let $q \ge p \ge 1$, $(\mathbb{R}^n, \rho, dx)$ be a homogeneous space and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). Let $v \in A_\infty(dx)$ and $\omega_i^{1-p'}$, $i = 1, 2, \ldots, n$, be doubling functions on Σ. If*

$$(\ell_i(B)/|B|) \Big(\int_{B \cap \Omega} v\, dx \Big)^{\frac{1}{q}} \Big(\int_{B \cap \Omega} \omega_i^{1-p'}\, dx \Big)^{\frac{1}{p'}} \le \tilde{A}, \tag{2.6}$$

$i = 1, 2, \ldots, n$, on any $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, dx \Big)^{\frac{1}{q}} \le C_0 \tilde{A} \sum_{i=1}^n \Big(\int_\Omega |f_{z_i}|^p \omega_i\, dx \Big)^{\frac{1}{p}}, \tag{2.7}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending only p, q, n and on C, δ in (2.2).

Theorem 2.4 is an easy consequence of the next assertion.

Theorem 2.5. *Let $q \ge p \ge 1$, $(\mathbb{R}^n, \rho, dx)$ be a homogeneous space and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). Let*

$v : \mathbb{R}^N \to (0,\infty)$ be an $A_\infty(wdx)$ function and $\omega_i^{1-p'} M_S w$, $i = 1,2,\ldots,n$, be doubling functions on Σ. If

$$\ell_i(B)\left(\int_B vw\,dy\right)^{1/q}\left(\int_B \omega_i^{1-p'} M_S w(y)\,dy\right)^{1/p'} \leq A\int_B w(y)\,dy, \tag{2.8}$$

$i = 1,2,\ldots,n$, on any $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, w(z)\,dz\Big)^{\frac{1}{q}} \leq C_0 A \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i\, M_S w(z)\,dz\Big)^{\frac{1}{p}}, \tag{2.9}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending only on p, q, n and on C, δ in (2.3).

We remark that, in Theorem 2.5, the Sobolev type weight inequality (2.9) is proven with different weights for the partial derivatives. This is due to the fact that the weights and the metric must be in a balance with the geometry of the quasi-metric balls. Taking $v \equiv \omega_i \equiv 1$ in (2.9) we get the measure $w(x)dx$ to be a doubling function on Σ, hence we obtain the inequality

$$\Big(\int_\Omega |f|^q w(x)\,dx\Big)^{\frac{1}{q}} \leq C_0 \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p\, M_S w(x)dx\Big)^{\frac{1}{p}}.$$

Moreover, let us mention that the doubling condition on the weights $\omega_i^{1-p'} M_S w$ in Theorem 2.5 is motivated by the use Lemma 4 of [39, Chapter 8].

In the next Theorem 2.6 we give a better estimate. In order to do that, the sufficiency condition (2.8) needs to be suitably strengthened (see (2.11)). Theorem 2.6 below gives, locally, a finer inequality since

$$d(\Omega)^\varepsilon \sup_{B\in\Sigma, B\ni x} w(B)/|B| \geq \sup_{B\in\Sigma, B\ni x} w(B)/|B|^{1-\varepsilon/n}.$$

Theorem 2.6. *Let $q \geq p \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $(\mathbb{R}^N, \rho, wdx)$ be a homogeneous space and assume that there exists a positive constant C_1 such that*

$$C_1|x-y| \leq \rho(x,y) \tag{2.10}$$

for all $x, y \in \Omega$. Let $v : \mathbb{R}^N \to (0,\infty)$ be an $A_\infty(wdx)$ weight function and $\omega_i^{1-p'}\mathcal{M}_\varepsilon w$, $i = 1,2,\ldots,n$, be doubling functions on Σ. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). If

$$\ell_i(B)\left(\frac{r(B)^{n-\varepsilon}}{|B|}\right)\left(\int_B vw\,dy\right)^{1/q}\left(\int_B \omega_i^{1-p'}\mathcal{M}_\varepsilon w\,dy\right)^{1/p'} \leq \bar{A}\int_B w(y)\,dy \tag{2.11}$$

$i = 1,2,\ldots,n$, with $\varepsilon \in [0,1)$ uniformly with respect to $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, w(z)\,dz\Big)^{\frac{1}{q}} \leq C_0 \bar{A} \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i\, \mathcal{M}_\varepsilon w(z)\,dz\Big)^{\frac{1}{p}}, \tag{2.12}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending on p, q, n and on C, δ in (2.3).

3 An example of homogeneous space

Let $\omega : \mathbb{R}^n \to (0,\infty)$ be a positive measurable function, such that $\sigma(x) = \frac{1}{\omega(x)}$ is in the Muckenhoupt A_2-weight class all over the n-dimensional Euclidean balls. This condition used in proofs of the corollaries below. Observe that this gives that also ω is in the Muckenhoupt's A_2-class all over the n-dimensional Euclidean balls.

For $x \in \mathbb{R}^n$, define a function $h_x : t \in [0,\infty) \to h_x(t) \in [0,\infty)$ as

$$h_x(t) = t\left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)} \sigma(s)\,ds\right)^{\frac{1}{2}}, \quad t > 0$$

and assume that $h_x(0) = 0$, $\lim_{t\to+\infty} h_x(t) = +\infty$ for a fixed $x \in \mathbb{R}^n$. Then we may consider an inverse function $h_x^{-1} : s \in [0,\infty) \to h_x^{-1}(s) \in [0,\infty)$ defined as

$$h_x^{-1}(s) = \inf\Big\{t > 0 : \; h_x(t) \geq s\Big\}, \quad s > 0.$$

and $h_x^{-1}(0) = 0$. We can define a quasi-metric ρ on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m = \{z = (x,y)\,|x \in \mathbb{R}^n\,, y \in \mathbb{R}^m\}$ as follows: for any $z_1 = (x_1,y_1), z_2 = (x_2,y_2) \in \mathbb{R}^N$ we put

$$\rho(z_1,z_2) = \max\Big\{|x_1 - x_2|,\; h_{x_1}^{-1}(|y_2 - y_1|),\; h_{x_2}^{-1}(|y_2 - y_1|)\Big\}. \tag{3.1}$$

The function $\rho : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is a quasi-metric satisfying the triangle inequality

$$\rho(z_1,z_2) \leq K_0\Big(\rho(z_1,z_3) + \rho(z_2,z_3)\Big) \tag{3.2}$$

with a constant $K_0 \geq 1$ independent of $z_1, z_2, z_3 \in \mathbb{R}^N$, (see, e.g., [1,15]). Therefore, the above defined quasi-metric space $(\mathbb{R}^N, \rho)$ endowed with the Lebesgue measure is a homogeneous space.

In general, the balls of a homogeneous space are not convex, therefore the conditions (2.4), (2.5) may be failed. The condition (2.4) means that the Lebesgue measure of a metric ball comparable with Lebesgue measure of its circumscribed parallelepiped. Also as we have noted the balls of a metric space are not convex the line segment connecting any two points of a ball may get out of that ball. The meaning of condition (2.5) is that, although the points on a line segment get out of the ball its points are contained on the comparable ball. It easily seen that the balls of metric (3.1) are convex and conditions (2.4), (2.5) are satisfied for that.

4 Applications

In this section, we give two examples of applications of Theorem 2.4. To this aim, let ρ be the quasi-metric defined in (3.1). It is not difficult to see that the ball $B(z_0,R)$ with center in $z_0 = (x_0,y_0) \in \mathbb{R}^N$ and radius $R > 0$ of this quasi-metric is given by

$$\begin{aligned} B(z_0,R) = \Bigg\{ & z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \; |x - x_0| < R, \\ & |y - y_0| < R\left(\frac{1}{|Q(x_0,R)|}\int_{Q(x_0,R)} \sigma(t)dt\right)^{\frac{1}{2}} \Bigg\} \end{aligned} \tag{4.1}$$

Let ω as in the beginning of Section 3 and $f : (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \to f(x,y) \in \mathbb{R}$ be a Lipschitz continuous function. The degenerated gradient of f is given by

$$|\nabla_\omega f|^2 = \omega(x)|\nabla_x f|^2 + |\nabla_y f|^2.$$

For $m \geq 2$ we can prove the following result:

Corollary 4.1. *Let $n+m \geq 3$, $q = \frac{2(n+m)}{n+m-2}$, $t = \frac{n}{n+m-2}$ and let $\omega \in A_2$-Muckenhoupt class function on $\mathbb{R}^n$. Then,*

$$\Big(\int_{B(z_0,R)} \omega^t \, |f|^q \, dz \Big)^{\frac{1}{q}} \leq C_0 \Big(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \Big)^{\frac{1}{2}} \tag{4.2}$$

for any function f, Lipschitz continuous in the ball $B(z_0,R) \subset \mathbb{R}^N$, vanishing on $\partial B(z_0,R)$. The positive constant C_0 in (4.2) depends on n,m and on the constants in the A_2-condition from (2.1).

For $m = 1$, we have:

Corollary 4.2. *Let $n > 1$, $q = \frac{2(n+1)}{n-1}$, and let ω^{-1} be a classical $A_{1+\frac{1}{n'}}$-Muckenhoupt class function on $\mathbb{R}^n$. Then,*

$$\Big(\int_{B(z_0,R)} \omega^{\frac{n}{n-1}} \, |f|^q dz \Big)^{\frac{1}{q}} \leq C_0 \Big(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \Big)^{\frac{1}{2}} \tag{4.3}$$

for any function f, Lipschitz continuous in the ball $B(z_0,R) \subset \mathbb{R}^N$ and vanishing on $\partial B(z_0,R)$. The positive constant C_0 in (4.2) depends on n and on the constants in the $A_{1+\frac{1}{n'}}$-condition from (2.1).

Corollary 4.3. *Let $q \in [2, 2N/(N-2)]$ and let v, $\omega : \mathbb{R}^n \to (0,\infty)$ be functions of the variable x only of classes A_∞ and A_2, respectively. Let*

$$\left(\frac{r}{R}\right)^{1-\frac{(N-n)(n+2)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{v\left(Q_r^x\right)}{v\left(Q_R^x\right)}\right)^{\frac{1}{q}} \leq C \left(\frac{\omega\left(Q_r^x\right)}{\omega\left(Q_R^x\right)}\right)^{\frac{1}{2}-\frac{N-n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \tag{4.4}$$

for any $x \in \mathbb{R}^n$ and $r > 0$. Then for all $f \in Lip_0\left(B_R^{z_0}\right)$

$$\Big(\int_{B(z_0,R)} v \, |f|^q \, dz \Big)^{1/q} \leq C_0 A(x_0,R) \, R \left(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \right)^{1/2} \tag{4.5}$$

holds with

$$A(x_0,R) = R^{-\frac{(N-n)(n+2)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} v\left(Q_R^{x_0}\right)^{\frac{1}{q}} \Big/ \omega\left(Q_R^{x_0}\right)^{\frac{1}{2}-\frac{N-n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)},$$

C_0 depends on the A_∞, A_2 conditions for v, ω and n, q.

The given above corollaries generalize the two-weight Sobolev inequalities to the case of non-uniformly degenerate gradient $\nabla_\omega f$. Therefore, those inequalities are of the well-known inequalities type by Chanillo–Wheeden, Fabes–Kenig–Serapioni with $\omega \equiv 1$. Such inequalities may be applied to the study of equations with Grushin type operator $\partial_{x_i}\big(\omega(x)\partial_{x_i}\big) + \partial^2_{y_j}$ or its generalizations $\partial_{x_i}\big(\omega(x)w\partial_{x_i}\big) + \partial_{y_j}\big(w\partial_{y_j}\big)$ when $w(x,y)$ is a function of two variables x,y obligated to satisfy some conditions.

Note that, the condition (4.4) is a balance condition of Chanillo–Wheeden type [4] for the case of non-uniformly degenerate gradient inequality of the Sobolev type. Note again, the function v depends only the variable x while the function f is dependent of two variables $z = (x,y)$.

5 Proofs of the main results

Let us start proving Theorem 2.5.

5.1 Proof of Theorem 2.5

Assume that f is not equal to zero almost everywhere in Ω, otherwise the result of Theorem 2.5 is trivial. For $\alpha > 0$ set $\Omega_\alpha = \{x \in \Omega : |f(x)| > \alpha\}$. Since f is continuous the set Ω_α is open. Let a fixed α be such that the set $\Omega_{3\alpha}$ is nonempty. Choose a countable covering of Ω_α made up of connected components $\Omega_{\alpha,j} \subset \Omega_\alpha$, $j \in \mathbb{N}$. Denote the parts of $\Omega_{3\alpha}$ and $\Omega_{2\alpha}$ contained in $\Omega_{\alpha,j}$ by $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$, respectively (note that the sets $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$ need not to be connected).

For the reader's convenience, let us recall that the weight function w of the homogeneous space $(\mathbb{R}^n, \rho, w dx)$ satisfies the doubling condition on the ρ quasi-metric balls. Let $b \in \Omega_{3\alpha,j}$ be a fixed point. Let us show that there exists a ρ-quasi metric ball $B = B(b, r(b))$ such that

$$w\left(B \setminus \Omega_{\alpha,j}\right) = \gamma\, w(B), \tag{5.1}$$

where γ is a small positive number that will be chosen later on. To this aim, let $\gamma > 0$ and define the function

$$F(t) = \frac{1}{\gamma} w\left(B(b,t) \setminus \Omega_{\alpha,j}\right) - w\left(B(b,t)\right),$$

which is continuous and negative for sufficiently small $t > 0$ since b is an interior point of $\Omega_{3\alpha,j}$.

From the doubling property of w on the ρ-quasimetric balls it follows that there exists a positive real number τ such that

$$w\left(B\left(b, d(\Omega)\right) \setminus \Omega\right) \geq \tau w\left(B\left(b, d(\Omega)\right)\right).$$

Let us choose the constant $\gamma > 0$ so that the function $F(t)$ is positive for $t = d(\Omega)$. Observe, that is always possible since

$$\begin{aligned} F(d(\Omega)) &= \frac{1}{\gamma} w\left(B\left(b, d(\Omega)\right) \setminus \Omega_{\alpha,j}\right) - w\left(B(b, d(\Omega))\right) \\ &\geq \frac{1}{\gamma} w\left(B\left(b, d(\Omega)\right) \setminus \Omega\right) - w\left(B(b, d(\Omega))\right) \\ &\geq \left(\frac{\tau}{\gamma} - 1\right) w\left(B\left(b, d(\Omega)\right)\right), \end{aligned}$$

thus it suffices to choose γ such that $\frac{\tau}{\gamma} - 1 > 0$ in order to get $F(d(\Omega)) \geq 0$. Hence, by the Bolzano–Cauchy theorem for continuous functions we get that there exists a $t^* \in (0, d(\Omega))$ such that $F(t^*) = 0$. Therefore, if we take $r(b) = t^*$ we achieve equality (5.1).

Now, there are two possibilities:

Case 1)

$$w\left(B^* \cap \Omega_{3\alpha,j}\right) \leq \gamma w(B^*), \tag{5.2}$$

Case 2)

$$w\left(B^* \cap \Omega_{3\alpha,j}\right) > \gamma w(B^*), \tag{5.3}$$

where $B^* = B(b, 5K_0^2 r(b))$.

In case 1), denoted by $\lambda = vw$, using the doubling property of the function $v \in A_\infty(wdx)$, it follows

$$\lambda\left(B^* \cap \Omega_{3\alpha,j}\right) \le C\gamma^\delta \lambda(B^*) \le CC_1\gamma^\delta\lambda(B). \tag{5.4}$$

By (5.1) and since $v \in A_\infty(wdx)$ we have again

$$\lambda(B) = \lambda\left(B \cap \Omega_{\alpha,j}\right) + \lambda\left(B \setminus \Omega_{\alpha,j}\right) \le \lambda\left(B \cap \Omega_{\alpha,j}\right) + C\gamma^\delta\lambda(B),$$

therefore, eventually reducing γ

$$\lambda(B) \le \frac{1}{1 - C\gamma^\delta}\lambda\left(B \cap \Omega_{\alpha,j}\right).$$

Thus, by (5.4) we get

$$\lambda\left(B^* \cap \Omega_{3\alpha,j}\right) \le \frac{CC_1\gamma^\delta}{1 - C\gamma^\delta}\lambda\left(B \cap \Omega_{\alpha,j}\right) \tag{5.5}$$

In case 2), we have two possibilities:

2a)

$$\left|B^* \setminus \Omega_{2\alpha,j}\right| \ge \frac{1}{2}|B^*| \tag{5.6}$$

and

2b)

$$\left|B^* \cap \Omega_{2\alpha,j}\right| > \frac{1}{2}|B^*|. \tag{5.7}$$

If 2a) takes place, let us show that

$$1 \le \frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)|(B^*)^\#|}{|B^*|w(B^*)}\int_{B^{**}\cap\left(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j}\right)}|f_{x_i}(z)|M_S w(z)\,dz, \tag{5.8}$$

where $M_S w$ denotes the strong maximal function of w, B^{**} is the ρ-metric ball $B^{**} = cB^*$ and $(B^*)^\#$ denotes the smallest rectangular with edges parallel to coordinate axes containing B^*.

To prove inequality (5.8), we follow an idea of [31], formula (3.7). Denote $\hat{A} = B^* \setminus \Omega_{2\alpha,j}$ and $Z = B^* \cap \Omega_{3\alpha,j}$. Let the points $x \in \hat{A}$ and $y \in Z$ be arbitrary fixed. Since the quasi-metric balls are not assumed to be convex, the line segment $\overline{xy} = \{x + t(y - x) : t \in [0,1]\}$ connecting x, y may get out of the ball B^* as t varies in $(0, 1)$. But, due to hypothesis (2.5) it will stay in the congruent ball $B^{**} = cB^*$.

Also, the line segment $\overline{xy}$ intersects the surfaces $\{z' \in \Omega_{\alpha,j} : |f(z')| = \alpha\}$ and $\{z'' \in \Omega^j_\alpha : |f(z'')| = 2\alpha\}$ in some points $z' = x + t_1(y - x)$ and $z'' = x + t_2(y - x)$ where $t_1, t_2 \in [0, 1]$, with $t_2 > t_1$ depend on x, y. Here, t_2 corresponds to the value of t for which $\overline{xy}$ meets for the first time the surface $\partial\Omega_{2\alpha,j}$ after leaving $\partial\Omega_{\alpha,j}$ while t_1 corresponds to the value of t when $\overline{xy}$ intersects the surface $\partial\Omega_{\alpha,j}$.

Having this in mind and using (5.1), (5.6) it follows that

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha}\int_{\hat{A}}\Big(\int_Z |f(z'') - f(z')|dy\Big)w(x)dx. \tag{5.9}$$

Whence,

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha}\int_{\hat{A}}\Big(\int_Z\Big(\int_{t_1(z,y)}^{t_2(z,y)}\Big|\frac{\partial f}{\partial t}(x + t(y - x))\Big|dt\Big)dy\Big)w(x)dx.$$

By Fubini's theorem,

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{n} \frac{\ell_i(B^*)}{\alpha} \int_{\hat{A}} \Big(\int_0^1 \Big(\int_{\{y\in B^*:\, x+t(y-x)\in G\}} \Big|\frac{\partial f}{\partial z_i}\big(x+t(y-x)\big)\Big| dy\Big) dt\Big) w(x)dx,$$

where $G = B^{**} \cap (\Omega_{2\alpha,j} \setminus \Omega_{3\alpha,j})$.

Let us now make the change of variable $z = x + t(y-x)$ in the interior integral to pass from y to z. Since $dy = t^{-n}dz$, one has

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{n} \frac{\ell_i(B^*)}{\alpha} \int_{\hat{A}} \Big(\int_0^1 \Big(\int_{\{z\in G:\, \frac{z-x}{t}+x\in Z\}} \Big(\Big|\frac{\partial f}{\partial z_i}(z)\Big| dz\Big) \frac{dt}{t^n}\Big) w(x)dx. \tag{5.10}$$

For $t \in (0,1)$ and $z \in G$ it follows $|x_s - z_s| < tl_s(B^*)$, $s = 1,2,\ldots,n$, therefore applying Fubini's formula again, we get

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{N} \frac{\ell_i(B^*)}{\alpha} \int_0^1 \Big(\int_G \Big|\frac{\partial f}{\partial z_i}(z)\Big| \Big(\int_{\{z:\, |z_s-x_s|<tl_s(B^*),\, s=1,2,\ldots,N\}} w(x)dx\Big) dz\Big) \frac{dt}{t^n}, \tag{5.11}$$

where $G = B^{**} \cap \left(\Omega_{2\alpha,j} \setminus \Omega_{3\alpha,j}\right)$.

Then

$$1 \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*) |\,(B^*)^{\#}\,|}{|B^*| w(B^*)} \int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz,$$

where M_S is the strong maximal operator. Therefore,

$$\lambda\left(\Omega_{3\alpha,j} \cap B^*\right) \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*)\lambda(B^*) |\,(B^*)^{\#}\,|}{|B^*| w(B^*)} \int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz. \tag{5.12}$$

In the case 2b) we can argue as in case 2a) by putting $\hat{A} = B^* \setminus \Omega_{\alpha,j}$ and $Z = \Omega_{2\alpha,j} \cap B^*$. Thus, we have

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha} \int_{B^*\setminus\Omega_{\alpha,j}} \Big(\int_{\Omega_{2\alpha,j}\cap B^*} |f(z'') - f(z')| dy\Big) w(x)dx.$$

In this case the line segment $\overline{xy}$ intersects the surfaces $\{z' \in \Omega_{\alpha,j} : |f(z')| = \alpha\}$ and $\{z'' \in \Omega_\alpha^j : |f(z'')| = 2\alpha\}$ in points that can be expressed as $z' = x + t_1(y-x)$ and $z'' = x + t_2(y-x)$ where $t_1, t_2 \in [0,1]$, with $t_2 > t_1$ depend on x, y. Here, t_2 corresponds to the value of t for which $\overline{xy}$ meets for the first time the surface $\partial\Omega_{2\alpha,j}$ after leaving $\partial\Omega_{\alpha,j}$ while t_1 corresponds to the value of t when $\overline{xy}$ intersects the surface $\partial\Omega_{\alpha,j}$.

In this case, in place of (5.11), we get the following inequality

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{N} \frac{\ell_i(B^*)}{\alpha} \int_0^1 \Big(\int_G \Big|\frac{\partial f}{\partial z_i}(z)\Big| \Big(\int_{\left\{z:\, |z_s-x_s|<tl_s(B),\, s=1,2,\ldots,N\right\}} w(x)dx\Big) dz\Big) \frac{dt}{t^n},$$

where $G = B^{**} \cap \left(\Omega_{\alpha,j} \setminus \Omega_{2\alpha,j}\right)$.

Therefore,

$$1 \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*) |\,(B^*)^{\#}\,|}{|B^*| w(B^*)} \int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz$$

and then

$$\lambda\left(\Omega_{2\alpha,j}\cap B^*\right)\le\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})}|f_{z_i}(z)|M_Sw(z)\,dz. \tag{5.13}$$

Now, since $\Omega_{3\alpha,j}\subset\Omega_{2\alpha,j}$, combining (5.5), (5.12), and (5.13) we have

$$\begin{aligned}\lambda\left(\Omega_{3\alpha,j}\cap B^*\right)\le{}&\frac{CC_1\gamma^{\delta}}{1-C\gamma^{\delta}}\lambda\left(B\cap\Omega_{\alpha}^{j}\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})}|f_{z_i}(z)|M_Sw(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})}|f_{z_i}(z)|M_Sw(z)\,dz.\end{aligned} \tag{5.14}$$

Summing up over $j=1,2,\ldots,$ we obtain

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\cap B^*\right)\le{}&\frac{CC_1\gamma^{\delta}}{1-C\gamma^{\delta}}\lambda\left(B^*\cap\Omega_{\alpha}\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha}\setminus\Omega_{3\alpha})}|f_{z_i}(z)|M_Sw(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha}\setminus\Omega_{2\alpha})}|f_{z_i}(z)|M_Sw(z)\,dz.\end{aligned} \tag{5.15}$$

Recall that the balls system $\left\{B^*=B(b,5K_0^2r(b))\right\}_{b\in\Omega_{3\alpha}}$ covers $\Omega_{3\alpha}$. Using Lemma 2.1, from those balls one can choose a countable subcover $\left\{B^*_m=B(x_m,5K_0^2r(x_m))\right\}_{m\in\mathbb{N}}$ such that

$$\Omega_{3\alpha}\subset\bigcup_m B^*_m. \tag{5.16}$$

Moreover, the balls $\left\{B_m=B(x_m,r(x_m))\right\}_{m\in\mathbb{N}}$ are disjoint, i.e.

$$\bigcap_m B_m=\emptyset. \tag{5.17}$$

Writing (5.15) for the system of balls B^*_m, we get

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\cap B^*_m\right)\le{}&\frac{CC_1\gamma^{\delta}}{1-C\gamma^{\delta}}\lambda\left(B_m\cap\Omega_{\alpha}\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*_m)\lambda(B^*_m)|\left(B^*_m\right)^{\#}|}{|B^*_m|w(B^*_m)}\int_{B^{**}_m\cap(\Omega_{2\alpha}\setminus\Omega_{3\alpha})}|f_{z_i}(z)|M_Sw(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*_m)\lambda(B^*_m)|\left(B^*_m\right)^{\#}|}{|B^*_m|w(B^*_m)}\int_{B^{**}_m\cap(\Omega_{\alpha}\setminus\Omega_{2\alpha})}|f_{z_i}(z)|M_Sw(z)\,dz.\end{aligned} \tag{5.18}$$

Summing up over $m=1,2,\ldots,$ we get

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\right)\le{}&\frac{CC_1\gamma^{\delta}}{1-C\gamma^{\delta}}\lambda\left(\Omega_{\alpha}\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\sum_{m}\frac{\ell_i(B^*_m)\lambda(B^*_m)|\left(B^*_m\right)^{\#}|}{|B^*_m|w(B^*_m)}\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}}\chi_{B^{**}_m}(z)\,|f_{z_i}(z)|M_Sw(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\sum_{m}\frac{\ell_i(B^*_m)\lambda(B^*_m)|\left(B^*_m\right)^{\#}|}{|B^*_m|w(B^*_m)}\int_{\Omega_{\alpha}\setminus\Omega_{2\alpha}}|\chi_{B^{**}_m}(z)\,f_{z_i}(z)|M_Sw(z)\,dz.\end{aligned} \tag{5.19}$$

Denote

$$c_m = \frac{\ell_i(B_m^*)\lambda(B_m^*)|\,(B_m^*)^{\#}\,|}{|B_m^*|w(B_m^*)},$$

then

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\right) \le \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m^{**}}(z)\right) |f_{z_i}(z)| M_S w(z)\, dz \\ &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m \chi_{B_m^{**}}(z)\right) |f_{z_i}(z)| M_S w(z)\, dz.\end{aligned} \tag{5.20}$$

Using Hölder's inequality, this implies

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\right) \le \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\ &\times \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m^{**}}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'} \\ &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\ &\times \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m \chi_{B_m^{**}}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'},\end{aligned}$$

where $\sigma_i = \omega_i^{1-p'}$. Now, using Lemma of 4 in [39, Chapter 8], we have

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\right) \le \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2C_2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{x_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\ &\times \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'} \\ &+ \frac{2C_2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\ &\times \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m \chi_{B_m}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'}.\end{aligned} \tag{5.21}$$

By the property (5.17) of the covering $\{B_m\}$ and by the doubling assumption on $\sigma_i M_S w$ on the ρ quasi-metric balls, we get

$$\begin{aligned}\left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'} &= \left(\sum_m c_m^{p'} \kappa_i(B_m)\right)^{1/p'} \\ &\le CA\left(\sum_m \lambda(B_m)^{p'/q'}\right)^{1/p'},\end{aligned} \tag{5.22}$$

where $\kappa_i = \sigma_i M_S w$. Note that in (5.22) we have used that the condition (2.4) and (2.8) and the doubling assumption on the measures yield

$$c_m^{p'} \kappa_i(B_m) \leq C_3 A^{p'} \left(\lambda(B_m)\right)^{p'/q'}.$$

Now, by (5.1) and since $p'/q' \geq 1$,

$$\begin{aligned}
&\left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'} \\
&\quad \leq C/(1-\gamma)^{1/q'} A \left(\sum_m \lambda\left(B_m \cap \Omega_\alpha\right)^{p'/q'}\right)^{1/p'} \\
&\quad \leq CA/(1-\gamma)^{1/q'} \lambda(\Omega_\alpha)^{1/q'}.
\end{aligned} \tag{5.23}$$

Observe that the same inequality can be obtained also for integrals over the sets $\Omega_\alpha \setminus \Omega_{2\alpha}$. Thus, by (5.21), we get

$$\begin{aligned}
\lambda\left(\Omega_{3\alpha}\right) &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \lambda\left(\Omega_\alpha\right) \\
&+ \frac{2C_3 A}{(1-\gamma)^{1/q'}\gamma\alpha} \lambda(\Omega_\alpha)^{1/q'} \sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\
&+ \frac{2C_3 A}{(1-\gamma)^{1/q'}\gamma\alpha} \lambda(\Omega_\alpha)^{1/q'} \sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p}, \quad \alpha > 0
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \lambda(\Omega_{3\alpha}) d\alpha^q &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \int_0^\infty \lambda(\Omega_\alpha) d\alpha^q \\
&+ \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{j=1}^N A \int_0^\infty \lambda(\Omega_\alpha)^{1/q'} \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \frac{\alpha^{q-1} d\alpha}{\alpha} \\
&+ \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{j=1}^N A \int_0^\infty \lambda(\Omega_\alpha)^{1/q'} \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \frac{\alpha^{q-1} d\alpha}{\alpha}.
\end{aligned} \tag{5.24}$$

Notice that

$$\int_0^\infty \lambda(\Omega_{3\alpha}) d\alpha^q = \frac{1}{3^q} \int_\Omega |f|^q\, v\, w dx \quad \text{and} \quad \int_0^\infty \lambda(\Omega_\alpha) d\alpha^q = \int_\Omega |f|^q v\, w dx. \tag{5.25}$$

Therefore, from (5.25) and Hölder's inequality, we get

$$\begin{aligned}
\frac{1}{3^q} \int_\Omega |f|^q\, v\, w dx &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \int_\Omega |f|^q v\, w dx \\
&+ \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{i=1}^n A \left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right) \frac{d\alpha}{\alpha}\right)^{1/p} \\
&\times \left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'} \alpha^{(q-1)p'-1} d\alpha\right)^{1/p'} \\
&+ \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{i=1}^n A \left(\int_0^\infty \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right) \frac{d\alpha}{\alpha}\right)^{1/p} \\
&\times \left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'} \alpha^{(q-1)p'-1} d\alpha\right)^{1/p'}.
\end{aligned} \tag{5.26}$$

Now, by Fubini's theorem,

$$\left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\,dz\right)\frac{d\alpha}{\alpha}\right)^{1/p} = \left(\ln\frac{3}{2}\right)^{1/p} \|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega},$$

$$\left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p \omega_i M_S w(z)\,dz\right)\frac{d\alpha}{\alpha}\right)^{1/p} = (\ln 2)^{1/p}\,\|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega}.$$

On the other hand, Minkowski's inequality gives

$$\begin{aligned}\left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'}\alpha^{(q-1)p'-1}d\alpha\right)^{1/p'} &\le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \left\|\int_{\Omega_{(\cdot)}} v\,w dx\right\|^{1/q'}_{p'/q',d\alpha^{(q-1)p'}}\\ &\le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \|f\|^{1/q'}_{q,vw,\Omega}.\end{aligned}$$

Using the last inequalities and choosing

$$\frac{1}{3^q} - \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} > 0, \tag{5.27}$$

from (5.26) we get

$$\|f\|_{q,vw,\Omega} \le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \frac{2C_3 q 2^{1/p'}(\ln 3)^{1/p}}{(1-\gamma)^{1/q'}\gamma} A \sum_{i=1}^{n} \|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega}. \tag{5.28}$$

This completes the proof of Theorem 2.5

5.2 Proof of Theorem 2.6

To prove Theorem 2.6 we may argue following along the lines the proof of Theorem 2.5 until formula (5.10). Then, from hypothesis (2.10), for $t \in (0,1)$ and $z = x + t(y-x) \in G$, using the condition (2.10) it follows

$$|x-z| < t|x-y| \le \rho(x,y) \le 2K_0 r(B^*)\,t,$$

therefore applying Fubini's formula again,

$$\begin{aligned}&\frac{1}{2}\gamma w(B^*)|B^*|\\ &\quad\le \sum_{i=1}^{N}\frac{\ell_i(B^*)}{\alpha}\int_G \left|\frac{\partial f}{\partial x_i}(z)\right| \left(\int_0^1 \left(\frac{1}{t^{n-\varepsilon}}\int_{\{x\in B:\,|z-x|<2K_0 r(B^*)\,t\}} w(x)dx\right)\frac{dt}{t^\varepsilon}\right)dz.\end{aligned} \tag{5.29}$$

Now, by the definition of the fractional order Hardy–Littlewood maximal operator over Euclidean balls and since $B(x, 2K_0 r(B^*)t) \ni z$ it follows

$$\int_{\{x\in B:\,|z-x|<2K_0 r(B^*)\,t\}} w(x)dx \le M_\varepsilon w(z)\,(2K_0 r(B^*)\,t)^{n-\varepsilon}.$$

By (5.29), one has

$$1 \le \frac{2^{n+1-\varepsilon}K_0^{n-\varepsilon}}{(1-\varepsilon)\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)r(B^*)^{n-\varepsilon}}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)| M_\varepsilon w(z)\,dz.$$

Arguing further as in Theorem 2.5 we obtain estimate (5.28) with $\bar{A}$ in place of A. The proof of Theorem 2.6 is then complete. □

5.3 Proof of Theorem 2.4

Theorem 2.4 is a corollary of Theorem 2.5 for $w \equiv 1$.

5.4 Proof of Corollary 4.1

The result follows from Theorem 2.4. It is enough to choose $(x,y) \in \mathbb{R}^N$, with $N = n+m$, $v(x,y) = \omega(x)^t$, $t = \frac{n}{n+m-2}$, and $\omega_1 = \cdots = \omega_n = \omega(x)$, $\omega_i \equiv 1$, $i = n+1, n+2, \ldots, n+m$ in the statement of Theorem 2.4. Observe that the A_∞-condition on the ρ-quasimetric balls on $\omega(x)^{\frac{n}{n+m-2}}$ as well as the A_2-condition on the ρ-quasimetric balls for ω are satisfied, in view of (3.1) and (4.1). Indeed, it is well-known that the A_p condition for some $p \geq 1$ implies the A_∞ condition. Therefore, in order to show that ω^t belongs to A_∞ let us show that it belongs to A_p, for some $p \geq 1$. To this aim, observe that, by our assumptions, $\sigma \in A_2$ hence

$$\sigma(Q) \int_Q \omega \, dx \leq C|Q|^2. \tag{5.30}$$

Using the Hölder inequality with powers $\frac{n+m-2}{n}$ and $\frac{n+m-2}{m-2}$,

$$\int_Q \omega^{\frac{n}{n+m-2}} \, dx \leq \left(\int_Q \omega \, dx \right)^{\frac{n}{n+m-2}} |Q|^{\frac{m-2}{n+m-2}},$$

thus, by (5.30), we get

$$\sigma(Q) \left(\int_Q \omega^{\frac{n}{n+m-2}} \, dx \right)^{\frac{n+m-2}{n}} \leq C|Q|^{2+\frac{m-2}{n}}.$$

The last inequality implies $\omega^t \in A_p$ with $p = 1 + \frac{n}{n+m-2}$.

For what concerns hypothesis (2.6), by the definition of the quasimetric ρ given in Section 3, in this case it can be derived by the following inequality

$$Cr|B(z,r)|^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega^t \, ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}} \leq A, \tag{5.31}$$

where $B(z,r)$ is a ρ-quasimetric ball of center z and radius r, $0 < r < R$, while $Q(x,r)$ is the projection of $B(z,r)$ on $\mathbb{R}^n$. Thus, in order to satisfy condition (2.6) we need to estimate the left hand side of (5.31) from above. To this aim, observe that

$$\begin{aligned}
& r|B(z,r)|^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega^t \, ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}} \\
& \leq \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)^t ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} r^{1-(n+m)\left(\frac{1}{2}-\frac{1}{q}\right)} \\
& = C\left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)^t ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \\
& \leq C_1\left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s) ds \right)^{\frac{t}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)},
\end{aligned}$$

where we used the fact that $q = \frac{2(n+m)}{n+m-2}$ gives $1 - (n+m)\left(\frac{1}{2} - \frac{1}{q}\right) = 0$ and Hölder's inequality.

Now, since $\frac{t}{q} = \frac{1}{2} - \frac{m}{2}\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{n}{2(n+m)}$, taking into account assumption $\omega \in A_2$ we get

$$C_1 \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)ds \right)^{\frac{t}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s)ds \right)^{\frac{1}{2} - \frac{m}{2}\left(\frac{1}{2} - \frac{1}{q}\right)}$$
$$= C_1 \left[\left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)ds \right) \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s)ds \right) \right]^{\frac{n}{2(n+m)}} \leq C_2$$

Hence condition (2.6) of Theorem 2.4 satisfied. This completes the proof of Corollary 4.1. □

5.5 Proof of Corollary 4.2

To prove this result, one can follow along the lines the proof of Corollary 4.1, for $t = \frac{n}{n-1}$, with suitable modifications.

5.6 Proof of Corollary 4.3

The proof of Corollary 4.3 is obtained from Theorem 2.4 similarly to that of Corollary 4.1, so we leave the proof to the Reader.

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