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Existence of positive ground state solutions of critical nonlinear Klein–Gordon–Maxwell systems

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Abstract. In this paper we study the following nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega+\varphi)^2]u = f(u) & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega+\varphi)u & \text{in } \mathbb{R}^3, \end{cases}$$

where $0 < \omega < m_0$. Based on an abstract critical point theorem established by Jeanjean, the existence of positive ground state solutions is proved, when the nonlinear term $f(u)$ exhibits linear near zero and a general critical growth near infinity. Compared with other recent literature, some different arguments have been introduced and some results are extended.

Keywords: Klein–Gordon–Maxwell system, general critical growth, positive ground state solutions, variational methods.

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1 Introduction

This article is concerned with the following Klein–Gordon–Maxwell equations

$$\begin{cases} -\Delta u + [m_0^2 - (\omega+\varphi)^2]u = f(u) & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega+\varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{KGME}$$

where $0 < \omega < m_0$. We assume that the followings hold for f:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd;

(f_2) $\lim_{s\to 0} \frac{f(s)}{s} = -m < 0$;

(f_3) $\lim_{|s|\to+\infty} \frac{f(s)}{s^5} = K > 0$;

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(f_4) there exist $D > 0$ and $q \in (2, 6)$ such that $f(s) + ms \geq Ks^5 + Ds^{q-1}$ for all $s > 0$;

(f_5) there exists constant $\gamma > 2$ such that $f(s)s - \gamma F(s) \geq 0$ for all $s \in \mathbb{R}$, where $F(s) = \int_0^s f(t)dt$.

This system is well known as a model describing the interaction between the nonlinear Klein–Gordon field and the electrostatic field. The presence of nonlinear term $f(u)$ simulates the interaction between many particles or external nonlinear perturbations.

In recent years, there is large quality works devoted to the system (KGME), and we would like to recall some of them. In a remarkable work, V. Benci and D. Fortunato [4] are the first to study the following system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \varphi)^2]u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

using the variational method, the authors proved the existence of infinitely many radially symmetric solutions when $m_0 > \omega > 0$ and $4 < q < 6$. In [15,16], D'Aprile and Mugnai considered the case $2 < p \leq 4$ and established some non-existence results for $p > 6$. Afterwards, there are also more literatures focusing on the existence and multiplicity of solutions for the problem (KGME). See [12,13,19] and the references therein.

There are some results related the critical case. In [11], Cassani considered the following system with the critical term:

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \varphi)^2]u = \mu|u|^{p-2}u + |u|^{2^*-2}u & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$. He showed that system (1.2) possesses a radially symmetric solution under one of the following conditions:

(i) $4 < p < 6$ and $|m_0| > |\omega|$;

(ii) $p = 4$, $|m_0| > |\omega|$ and μ large enough.

Soon afterwards, the authors of [9] studied the following critical Klein–Gordon–Maxwell system with external potential:

$$\begin{cases} -\Delta u + \mu V(x)u - (2\omega + \varphi)\varphi^2]u = \lambda f(u) + |u|^5 & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Provided $f(u)$ satisfying assumptions:

(f_2') $\lim_{s\to 0} \frac{f(s)}{s} = 0$;

(f_3') $\lim_{s\to\infty} \frac{f(s)}{s^5} = 0$;

(f_4') $\frac{1}{4}f(u)u - F(u) \geq 0$.

They obtained a nontrivial solution for (1.3). For more related results, we refer the readers to [3,24].

The existence of ground state solutions, that is, couples (u, φ) which solve (KGME) and minimize the action functional associated to (KGME) among all possible nontrivial solutions, has been investigated by many authors. Inspired by the approach of Benci and Fortunato, Azzollini and Pomponio [10] proved that (1.1) admits a ground state solution provided one of the following assumptions:

(i) $3 \le p < 5$ and $m_0 > \omega$;

(ii) $1 < p < 3$, $m_0\sqrt{p-1} > \omega\sqrt{6-p}$.

Soon afterwards, Carrião *et al.* [22] dealt with the critical Klein–Gordon–Maxwell system (1.2) with potentials. Combining the minimization of the corresponding Euler–Lagrange functional on the Nehari manifold, they proved the existence of positive ground state solutions for system (1.2). Very recently, Moura, Miyagaki *et al.* [14] considered quasicritical Klein–Gordon–Maxwell systems with potential, and obtained positive ground state solutions. For other related results about Klein–Gordon–Maxwell systems the authors maybe see [7,17,25].

Here we also mention that the papers [2,6], Berestycki and Lions studied the following elliptic equation

$$-\Delta u = f(u), \qquad u \in H^1(\mathbb{R}^N). \tag{1.4}$$

Under the following conditions on $f(u)$:

(A_1) $f(u) \in C(\mathbb{R}, \mathbb{R})$ is odd;

(A_2) $-\infty < \liminf_{u\to 0^+} \frac{f(u)}{u} \le \limsup_{u\to 0^+} \frac{f(u)}{u} = -m < 0$ for $N \ge 3$ and $\lim_{u\to 0} \frac{f(u)}{u} = -m < 0$ for $N = 2$;

(A_3) when $N \ge 3$, $-\infty < \limsup_{u\to\infty} \frac{f(u)}{u^{\frac{N+2}{N-2}}} \le 0$; when $N = 2$ for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $f(u) \le C_\alpha \exp(\alpha u^2)$ for $u > 0$;

(A_4) there exists $\varsigma > 0$ such that$F(\varsigma) = \int_0^\varsigma f(s)ds > 0$,

Berestycki and Lions [6] proved the existence of a positive least energy solution when $N \ge 3$ and Berestycki *et al.* [2] investigated the existence of infinitely many bound state solutions when $N = 2$. Under the above assumptions, Azzollini, d'Avenia and Pomponio [1] obtained the existence of at least a radial positive solution to a class of Schrödinger–Poisson problems, and Azzollini [28] proved the existence of ground state solutions for Kirchhoff-type problems, and soon after Zhang and Zou [27] investigated the existence of ground state solutions of the problem (1.4) with the critical growth assumption on $f(u)$.

Under the assumptions (f_1)–(f_5), Zhang [21] studied a class of Schrödinger–Poisson problems and established the existence of ground state solutions for $q \in (2, 4]$ with D large enough, or $q \in (4, 6)$, where $m = 0$; Liu [20] considered a Kirchhoff-type problem and obtained the existence of ground state solutions without (f_5).

Motivated by the above mentioned works, in particular by [9,20,21,27], the main purpose of this paper is to consider the existence of positive least energy solutions of (KGME) with a general nonlinearity in the critical growth. To our best knowledge, under the assumptions (f_1)–(f_5), there is no work on the the existence of positive ground state solutions for problem (KGME). Precisely, we have the following results.

Theorem 1.1. *If (f_1)–(f_5) hold. Assume that either $q \in (2, 4]$ with D sufficiently large, or $q \in (4, 6)$, then the problem (KGME) possesses a positive radial solution if one of the following conditions is satisfied:*

(i) $2 < \gamma < 3$ *and* $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$;

(ii) $3 \le \gamma < \infty$ *and* $0 < \omega < m_0$.

Theorem 1.2. *If (f_1)–(f_5) hold. Assume that either $q \in (2,4]$ with D sufficiently large, or $q \in (4,6)$, then the problem (KGME) possesses a positive ground state solution provided one of the following conditions holds:*

(i) $2 < \gamma < 3$ *and* $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$*;*

(ii) $3 \leq \gamma < \infty$ *and* $0 < \omega < m_0$.

Theorem 1.3. *If we replace the condition (f_5) by the following condition:*

(f_6) *there exists $\gamma > 2$ such that $t :\to \frac{f(x,t)}{t^{\gamma-1}}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$.*

Then the conclusions of Theorems 1.1 and 1.2 remain true.

Remark 1.4. Assumptions (f_1)–(f_4) were used by [20,21]. Since the problem in [20] is different from ours, the methods used in [20] do not work here. The similar hypotheses on $f(u)$ as above (f_1)–(f_5) are introduced in [21], where the authors used a cut-off functional to obtain bounded (PS) sequences. However, our device is different from the main arguments of [21]. Moreover, the results of [21] hold under $\gamma > 3$, and in our case, $\gamma > 2$.

Remark 1.5. The condition (f_4) plays a crucial role to ensure the existence of ground state solution to the problem (KGME). And the condition (f_5) is a technical condition to overcome the difficulty caused by the critical exponential growth case.

In our paper, due to the presence of a nonlocal term φ and the effect of the nonlinearity in the critical growth, there exist several difficulties to solve. In the first place, the lack of the following Ambrosetti–Rabinowitz growth hypothesis on f :

$$\exists\, \mu > 4 \text{ s. t. } 0 < \mu F(s) \leq s f(s), \quad \forall t \in \mathbb{R}$$

brings a obstacle in proving the boundedness of (PS) sequence. To overcome this difficulty, we will use approaches developed by Jeanjean [23] to obtain the boundedness. In the next place, since we deal with the critical case, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is not compact, and the functional I does not satisfy $(PS)_c$ condition at every energy level c. To avoid the difficulty, we try to pull the energy level down below some critical level c_1^* (Section 3). In the end, we apply the Strauss' compactness result [5] to obtain the convergence of $(PS)_c$ sequence.

An outline of the paper is as follows. In Section 2, we give some preliminary lemmas. Section 3 is devoted to the existence of the mountain pass solution and positive ground state solution. Throughout the paper we denote by C the various positive constants. Let $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ be the Sobolev space equipped with the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. $S = \inf\limits_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}}$ denotes the best Sobolev constant.

2 Preliminaries

In this section we give notations and prove some preliminary lemmas. Let us define an equivalent norm on $H^1(\mathbb{R}^3)$, that is

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + mu^2) dx \quad \text{for fixed } m > 0.$$

For any $1 \leq s < \infty$, we denote that $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm $\|u\|_{L^s}^s = \int_{\mathbb{R}^3} |u|^s dx$. Then we have that, for $2 \leq s \leq 2^*$, $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ continuously. Let $H := H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radial functions}\}$. Then $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $2 < s < 2^*$.

According to the variational nature of (KGME), we define its the energy functional as follows:

$$\Phi(u,\varphi) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 - |\nabla \varphi|^2 + [m_0^2 - (\omega+\varphi)^2]u^2)dx - \int_{\mathbb{R}^3} F(u)dx. \tag{2.1}$$

Under the assumptions (f_1)–(f_2), by standard arguments, we can prove that $\Phi(u,\varphi)$ is a well defined C^1 function on $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and that the weak solutions of (KGME) is critical points of the functional Φ. Obviously, the functional Φ is the strongly indefiniteness, which means that it is unbounded both from below and from above on infinite-dimensional subspaces. In order to avoid this indefiniteness, we apply the reduction method developed by Benci and Fortunato [8]. For deducing our results, we introduce the following results whose idea of proof comes from [15,16].

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there is a unique $\varphi = \varphi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies the following equation*

$$-\Delta\varphi + \varphi u^2 = -\omega u^2. \tag{2.2}$$

Furthermore the map $\Psi : u \in H^1(\mathbb{R}^3) \to \varphi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable and

(i) in the set $\{x : u(x) \neq 0\}$, for $\omega > 0$,

$$-\omega \leq \varphi \leq 0;$$

(ii) $\|\varphi_u\|_{D^{1,2}} \leq C\|u\|^2$ and $\int_{\mathbb{R}^3} |\varphi_u| u^2 dx \leq C\|u\|_{\frac{12}{5}}^4$.

Multiplying (2.2) by φ_u and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\varphi_u|^2 dx = -\int_{\mathbb{R}^3} u^2\varphi_u^2 dx - \int_{\mathbb{R}^3} \omega u^2 \varphi_u dx. \tag{2.3}$$

Lemma 2.2. *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to subsequence, $\varphi_{u_n} \rightharpoonup \varphi_u$ in $D^{1,2}(\mathbb{R}^3)$. As a consequence, $\Psi'(u_n) \to \Psi'(u)$ in the sense of distributions.*

By the definition of Φ and (2.3) the functional $I(u) = \Phi(u,\varphi)$ may be rewritten as the following form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (m_0^2 - \omega^2)u^2 - \omega\varphi_u u^2)dx - \int_{\mathbb{R}^3} F(u)dx. \tag{2.4}$$

In view of Lemmas 2.1 and 2.2, the conditions (f_1)–(f_3) imply $I(u) \in C^1$ and its Gateaux derivative is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3}[\nabla u \nabla v + (m_0^2 - \omega^2)uv - (2\omega + \varphi_u)\varphi_u uv]dx - \int_{\mathbb{R}^3} f(u)v dx \tag{2.5}$$

for all $u, v \in H$. Then (u,φ) is a weak solution of (KGME) if and only if $\varphi = \varphi_u$ and u is a critical point of I on H.

For simplicity, in this paper we may assume that $K = 1$. Set $g(t) = f(t) + mt$, so the functional I is reduced as

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx - \int_{\mathbb{R}^3}G(u)dx, \tag{2.6}$$

where $G(s) = \int_0^s g(t)dt$. In the following we give the abstract result established by Jeanjean [23].

Lemma 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathrm{h} \subset R_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in \mathrm{h}$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $J_\lambda(0) = 0$. For any $\lambda \in \mathrm{h}$, we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) < 0\}.$$

If for every $\lambda \in \mathrm{h}$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0.$$

Then for every almost $\lambda \in \mathrm{h}$ there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*

(ii) *$J_\lambda(u_n) \to c_\lambda$;*

(iii) *$J'_\lambda(u_n) \to 0$ in the dual X^{-1} of X.*

In our case, $X = H$, $h = [\frac{1}{2}, 1]$,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx, \qquad B(u) = \int_{\mathbb{R}^3}G(u)dx,$$

and so the family of functionals we study is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx - \lambda\int_{\mathbb{R}^3}G(u)dx. \tag{2.7}$$

and for every $u, v \in H$,

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = &\int_{\mathbb{R}^3}(\nabla u\nabla v + muv)dx + \int_{\mathbb{R}^3}(m_0^2 - \omega^2)uvdx \\ &- \int_{\mathbb{R}^3}(2\omega + \varphi_u)\varphi_u uvdx - \lambda\int_{\mathbb{R}^3}g(u)vdx. \end{aligned}\tag{2.8}$$

We shall use the following Pohožaev type identity. Its proof can be done as in [16].

Lemma 2.4. *For $\lambda \in [\frac{1}{2}, 1]$, let $u \in H$ be a critical point of I_λ, then*

$$\begin{aligned}P_\lambda(u) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}mu^2dx + \frac{3}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(5\omega + 2\varphi_u)\varphi_u u^2dx - 3\lambda\int_{\mathbb{R}^3}G(u)dx = 0.\end{aligned}$$

If $\lambda = 1$, the above Pohožaev equality turns to be the following

$$\begin{aligned}P(u) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}mu^2dx + \frac{3}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(5\omega + 2\varphi_u)\varphi_u u^2dx - 3\int_{\mathbb{R}^3}G(u)dx = 0.\end{aligned}$$

Next we shall cite a variant of the Strauss compactness result [5], which plays a fundamental tool in our arguments:

Lemma 2.5. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s\to\infty} \frac{P(s)}{Q(s)} = 0.$$

Let $\{v_n\}_n$, v and ψ be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$, with z bounded, such that

$$\sup_n \int_{\mathbb{R}^N} |Q(v_n(x))|\psi dx < \infty,$$

$$P(v_n(x)) \to v(x) \quad \text{a.e. in } \mathbb{R}^N, \text{ as } n \to +\infty.$$

Then for any bounded Borel set B one has $\|(P(v_n) - v)\psi\|_{L^1(B)} \to 0$. Moreover, if

$$\lim_{s\to 0} \frac{P(s)}{Q(s)} = 0,$$

$$\lim_{|x|\to+\infty} \sup_n |v_n(x)| = 0,$$

then

$$\|(P(v_n) - v)\psi\|_{L^1(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

3 Proof of main results

In this section we will look for a positive ground state solutions of (KGME). First, we will prove the existence of a mountain pass solution. Now, we give several lemmas which imply that I_λ satisfies the conditions of Lemma 2.3.

Lemma 3.1. *Assume that (f_1)–(f_4) hold. Then*

(i) $\Gamma_\lambda \neq \emptyset$ for every $\lambda \in h$;

(ii) there exists a constant $\tilde{c}$ such that $c_\lambda \geq \tilde{c} > 0$.

Proof. (i) For any $\lambda \in h$, it follows from Lemma 2.1, (2.7) and (f_4) that

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} (m_0^2 - \omega^2)u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} \omega\varphi_u u^2 dx - \frac{1}{12}\int_{\mathbb{R}^3} |u|^6 dx - \frac{D}{2q}\int_{\mathbb{R}^3} |u|^q dx.$$

Then

$$I_\lambda(tu) \leq \frac{1}{2}t^2\|u\|^2 + \frac{1}{2}t^2\int_{\mathbb{R}^3} m_0^2 u^2 dx - \frac{1}{12}t^6\int_{\mathbb{R}^3} |u|^6 dx - \frac{D}{2q}t^q\int_{\mathbb{R}^3} |u|^q dx.$$

Then we can choose $t_0 > 0$ large and $u \in H\backslash\{0\}$ such that $I_\lambda(t_0 u) < 0$ for every $\lambda \in h$. Define $\gamma_1 : [0, 1] \to H$ in the following way

$$\gamma_1(t) = tt_0 u, \qquad 0 \leq t \leq 1.$$

It is easy to see γ_1 a continuous path from $t_0 u$. Moreover, for every $\lambda \in h$, $I_\lambda(\gamma_1(1)) < 0$ and $I_\lambda(\gamma_1(0)) = 0$. The proof is completed.

(ii) Using (f_1)–(f_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|g(u)| \leq \varepsilon|u| + C_\varepsilon|u|^5$. Then by Sobolev's embedding theorem, one has

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2dx - \frac{C_\varepsilon}{6}\int_{\mathbb{R}^3}|u|^6dx \geq \frac{m-\varepsilon}{2m}\|u\|^2 - \frac{CC_\varepsilon}{6}\|u\|^6.$$

For fixed $\varepsilon \in (0, m)$, there exists $\tilde{c} > 0$ such that $I_\lambda(u) \geq \tilde{c} > 0$ for any $\lambda \in h$ and $u \in H$ with $\|u\| = \rho$ small enough. Now fix $\lambda \in h$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0$ and $I_\lambda(\gamma(1)) < 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore for every $\lambda \in h$

$$c_\lambda \geq \inf_{\gamma\in\Gamma_\lambda} I_\lambda(\gamma(t_\gamma)) \geq \tilde{c} > 0,$$

which implies (ii) of Lemma 3.1. □

It follows from Lemma 3.1 that the conclusions of Lemma 2.3 hold.

Lemma 3.2. *Assume that (f_1)–(f_4) hold. If $q \in (4, 6)$ or $q \in (2, 4]$ with D is large enough, then $c_\lambda < c^*_\lambda := \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}$.*

Proof. For $\varepsilon, r > 0$, define $u_\varepsilon(x) = \frac{\phi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x|^2)^{\frac{1}{2}}}$, where $\phi \in C_0^\infty(B_{2r}(0)), 0 \leq \phi \leq 1$ and $\phi|_{B_r(0)} \equiv 1$. And it is well known that the best Sobolev constant S is attained by the functions $\frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x|^2)^{\frac{1}{2}}}$. Direct calculation yields that

$$\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3}|u_\varepsilon|^6dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{3.1}$$

and

$$\int_{\mathbb{R}^3}|u_\varepsilon|^tdx = \begin{cases} K\varepsilon^{\frac{t}{4}}, & t \in [2,3), \\ K\varepsilon^{\frac{3}{4}}|In\varepsilon|, & t = 3, \\ K\varepsilon^{\frac{6-t}{4}}, & t \in (3,6), \end{cases} \tag{3.2}$$

where K_1, K_2, K are positive constants. Moreover, $S = K_1K_2^{-\frac{1}{3}}$. Using (3.1) and (3.2), we have

$$\frac{\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx}{(\int_{\mathbb{R}^3}u_\varepsilon^6dx)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}).$$

Set

$$g(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2(m_0^2 - \omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - \frac{\lambda}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6dx.$$

It is easy to see that $g(t)$ attains its maximum at $t_0 = [\frac{\|u_\varepsilon\|^2+(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{\frac{\lambda}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}]^{\frac{1}{4}}$ and then

$$\begin{aligned}\max_{t\geq 0} g(t) = &\frac{1}{2}\lambda^{-\frac{1}{2}}\sqrt{[\frac{\|u_\varepsilon\|^2)}{(\int_{\mathbb{R}^3}|u_\varepsilon|^6dx)^{\frac{1}{3}}}]^3 + \frac{(m_0^2-\omega^2)\|u_\varepsilon\|^4\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}} \\ &+ \frac{1}{2}(m_0^2-\omega^2)\lambda^{-\frac{1}{2}}\sqrt{\frac{\|u_\varepsilon\|^2(\int_{\mathbb{R}^3}|u_\varepsilon|^2dx)^4 + (m_0^2-\omega^2)(\int_{\mathbb{R}^3}|u_\varepsilon|^2dx)^6}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}} \\ &- \frac{\lambda^{-\frac{1}{2}}}{6}\sqrt{[\frac{\|u_\varepsilon\|^2 + (m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{(\int_{\mathbb{R}^3}|u_\varepsilon|^6dx)^{\frac{1}{3}}}]^3} = \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}\end{aligned}$$

for $\varepsilon > 0$ small enough. Obviously, there exists $0 < t' < 1$ such that, for $\varepsilon < 1$, one has

$$\max_{t'\geq t\geq 0} I_\lambda(tu_\varepsilon(x)) \leq \max_{t'\geq t\geq 0}\left(\frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2 - w^2)t^2\int_{\mathbb{R}^3}|u_\varepsilon|^2dx + Ct^2\int_{\mathbb{R}^3}|u_\varepsilon|^2dx\right)$$
$$\leq Ct^2\|u_\varepsilon\|^2 < \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}. \tag{3.3}$$

Using (f_4), (2.7) and Lemma 2.1, one has

$$\begin{aligned} I_\lambda(tu_\varepsilon(x)) &= \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_\varepsilon^2dx - \frac{1}{2}t^2\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx - \lambda\int_{\mathbb{R}^3}G(tu_\varepsilon)dx \\ &\leq \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_\varepsilon^2dx - \frac{1}{2}t^2\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx - \frac{\lambda}{6}t^2\int_{\mathbb{R}^3}|u_\varepsilon|^6dx \\ &\quad - \frac{\lambda D}{q}\int_{\mathbb{R}^3}|u_\varepsilon|^qdx \\ &= g(t) - \frac{1}{2}t^2\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx - \frac{\lambda D}{q}\int_{\mathbb{R}^3}|u_\varepsilon|^qdx \\ &\leq g(t) + Ct^2\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - CDt^q\int_{\mathbb{R}^3}|u_\varepsilon|^qdx. \end{aligned} \tag{3.4}$$

It follows from (3.4) and Lemma 3.1 that

$$\lim_{t\to\infty} I_\lambda(tu_\varepsilon(x)) = -\infty \tag{3.5}$$

and

$$I_\lambda(tu_\varepsilon(x)) > 0 \tag{3.6}$$

as t is close to 0. Now we prove that there exists $0 < \varepsilon_0 < 1$ such that $\lim_{t\to\infty} I_\lambda(tu_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. Set

$$\eta(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - \frac{1}{2}t^2\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx$$
$$- \frac{\lambda}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6dx - \frac{\lambda D}{q}t^q\int_{\mathbb{R}^3}|u_\varepsilon|^qdx. \tag{3.7}$$

Following from (3.5) and (3.6), (3.7) means that there exists $t_\varepsilon > 0$ such that $\eta(t_\varepsilon) = 0$ and for $t > t_\varepsilon,\ \eta(t) < 0$. Then we get

$$0 = \eta(t_\varepsilon) = t_\varepsilon^2\bigg(\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx$$
$$- \frac{\lambda}{6}t_\varepsilon^4\int_{\mathbb{R}^3}|u_\varepsilon|^6dx - \frac{\lambda D}{q}t_\varepsilon^{q-2}\int_{\mathbb{R}^3}|u_\varepsilon|^qdx\bigg), \tag{3.8}$$

thus (3.1) and (3.2) mean that for $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} \frac{\lambda}{6}t_\varepsilon^4 &\leq \frac{1}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}\left[\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_{tu_\varepsilon}u_\varepsilon^2dx\right] \\ &\leq \frac{1}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}\left[\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx + C\int_{\mathbb{R}^3}|u_\varepsilon|^2dx\right] \\ &\leq \frac{1}{2}\frac{K_1 + O(\varepsilon^{\frac{1}{2}})}{K_2 + O(\varepsilon^{\frac{3}{2}})} + O(\varepsilon^{\frac{1}{2}}) \leq \frac{1}{2}\frac{K_1 + O(\varepsilon_0^{\frac{1}{2}})}{K_2} + O(\varepsilon_0^{\frac{1}{2}}), \end{aligned} \tag{3.9}$$

where ε_0 is small enough. (3.9) implies that for some $t^* > 0$, t_ε is bounded from above uniformly for $\varepsilon \in (0, \varepsilon_0)$, where t^* is independent of ε. Using (3.5) and (3.9) we easily get that there exists $0 < \varepsilon_0 < 1$ such that $\lim_{t\to\infty} I_\lambda(tu_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. Thus there exists $t'' > t^*$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\max_{t\geq t''} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.10}$$

It follows from (3.1), (3.2) and (3.4) that

$$\begin{aligned}\max_{t''\geq t\geq t'} I_\lambda(tu_\varepsilon) &\leq g(t_0) + C\int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - CD\int_{\mathbb{R}^3} |u_\varepsilon|^q dx \\ &= \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) - CD\int_{\mathbb{R}^3} |u_\varepsilon|^q dx.\end{aligned} \tag{3.11}$$

For $q \in (2, 4]$ and D sufficiently large, $\varepsilon \in (0, \varepsilon_0)$ fixed, we derive from (3.11) that

$$\max_{t''\geq t\geq t'} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.12}$$

For $q \in (4, 6)$, observe that $\frac{6-q}{4} < \frac{1}{2}$, then it follows from (3.2) and (3.11) that, there exists $0 < \varepsilon_1 < \varepsilon_0$ small enough such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\max_{t''\geq t\geq t'} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.13}$$

It follows from (3.3), (3.10), (3.12) and (3.13) that $c_\lambda < c_\lambda^* := \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$. □

Lemma 3.3. *Assume that (f_1)–(f_3) and (f_5) hold. Let $\{u_\lambda\}$ be a critical point for $I_\lambda(u_\lambda)$ at level c_λ. Then $I_\lambda(u_\lambda) \geq 0$.*

Proof. If $\gamma \geq 4$ in (f_5), then it follows from (f_5), (2.7) and (2.8) that

$$\begin{aligned}I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{1}{\gamma}\langle I'_\lambda(u_\lambda), u_\lambda\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3} \left[|\nabla u_\lambda|^2 + (m_0^2 - \omega^2)u_\lambda^2\right] dx \\ &\quad + \int_{\mathbb{R}^3} \left[\left(\frac{2}{\gamma} - \frac{1}{2}\right)\omega\varphi_{u_\lambda} u_\lambda^2 + \frac{1}{\gamma}\varphi_{u_\lambda}^2 u_\lambda^2\right] dx + \int_{\mathbb{R}^3}\left[\frac{1}{\gamma} f(u_\lambda)u_\lambda - F(u_\lambda)\right] dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx \geq 0.\end{aligned} \tag{3.14}$$

Now, we consider $2 < \gamma < 4$ in (f_5). By (f_5), (2.7) and (2.8) and Lemma 2.4, we obtain that

$$\begin{aligned} I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{2}{6-\gamma}\langle I'_\lambda(u_\lambda), u_\lambda\rangle - \frac{2-\gamma}{2(6-\gamma)}P_\lambda(u_\lambda) \\ &= I_\lambda(u_\lambda) - \Big\{ \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u_\lambda|^2 dx + \frac{10-3\gamma}{2(6-\gamma)}\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_\lambda^2 dx \\ &\qquad - \int_{\mathbb{R}^3}\left[\frac{18-5\gamma}{2(6-\gamma)}\omega\varphi_{u_\lambda} + \frac{8-2\gamma}{2(6-\gamma)}\varphi_{u_\lambda}^2 u_\lambda^2\right]dx \\ &\qquad - \lambda\int_{\mathbb{R}^3}\left[\frac{2}{6-\gamma}f(u_\lambda)u_\lambda + \frac{6(2-\gamma)}{2(6-\gamma)}F(u_\lambda)\right]dx\Big\} \\ &= \frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_\lambda^2 dx + \int_{\mathbb{R}^3}\left[\frac{2(3-\gamma)}{6-\gamma}\omega\varphi_{u_\lambda} + \frac{4-\gamma}{6-\gamma}\varphi_{u_\lambda}^2\right]u_\lambda^2 dx \\ &\quad + \frac{2}{6-\gamma}\lambda\int_{\mathbb{R}^3}[f(u_\lambda)u_\lambda - \gamma F(u_\lambda)]dx. \end{aligned} \tag{3.15}$$

Set $h(t) = (4-\gamma)t^2 + 2(3-\gamma)\omega t$. We distinguish two cases:

Case 1. $3 \le \gamma < 4$ and $0 < \omega < m_0$. In this case, one has

$$h(t) \ge 0, \qquad \forall -\omega \le t \le 0. \tag{3.16}$$

Note that $-\omega \le \varphi_{u_\lambda} \le 0$. From (f_5), (3.15), (3.16), we have

$$I_\lambda(u_\lambda) \ge \frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_\lambda^2 dx \ge 0. \tag{3.17}$$

Case 2. $2 < \gamma < 3$ and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$. For $\forall -\omega \le t \le 0$, an elementary computation means that

$$\begin{aligned} (\gamma-2)(m_0-\omega^2) + h(t) &= (\gamma-2)(m_0-\omega^2) + (4-\gamma)(t^2 + \frac{2(3-\gamma)}{4-\gamma}\omega t) \\ &\ge (\gamma-2)(m_0-\omega^2) - \frac{(\gamma-3)^2}{4-\gamma}\omega^2 \\ &= \frac{(\gamma-2)(4-\gamma)m_0^2-\omega^2}{4-\gamma} > 0. \end{aligned} \tag{3.18}$$

Then from (f_5), (3.15) and (3.18), we get

$$I_\lambda(u_\lambda) \ge \frac{1}{(6-\gamma)(4-\gamma)}[(\gamma-2)(4-\gamma)m_0^2-\omega^2]\int_{\mathbb{R}^3}u_\lambda^2 dx \ge 0. \tag{3.19}$$

It follows from (3.14), (3.17) and (3.19) that $I_\lambda(u_\lambda) \ge 0$. □

Lemma 3.4. *Assume that (f_1)–(f_5). For almost every $\lambda \in [\frac{1}{2}, 1]$, there is $u_\lambda \in H \setminus \{0\}$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.*

Proof. By Lemma 2.3 and Lemma 3.1, for almost every $\lambda \in [\frac{1}{2}, 1]$, there exists a bounded (PS) sequence $\{u_n\} \subset H$ such that

$$I_\lambda(u_n) \to c_\lambda, \qquad I'_\lambda(u_n) = 0 \quad \text{in } H', \tag{3.20}$$

where H' is the dual space of H. Using Lemma 2.2, up to a subsequence, we can suppose that there exists $u \in H$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda && \text{weakly in } H, \\ u_n &\to u_\lambda && \text{in } L^p(\mathbb{R}^3),\ 2 < p < 6, \\ u_n &\to u_\lambda && \text{a.e. in } \mathbb{R}^3, \\ \varphi_{u_n} &\rightharpoonup \varphi_{u_\lambda} && \text{weakly in } D^{1,2}(\mathbb{R}^3). \end{aligned} \tag{3.21}$$

If we apply Lemma 2.5 for $P(t) = g(t) - t^5$, $Q(t) = t^5$,$\{v_n\}_n = \{u_n\}_n$, $v = g(u_\lambda) - u_\lambda^5$ and $\psi \in C_0^\infty(\mathbb{R}^3)$. By (f_2)–(f_4) and (3.21), we have

$$\int_{\mathbb{R}^3} (g(u_n) - u_n^5)\psi dx \to \int_{\mathbb{R}^3} (g(u_\lambda) - u_\lambda^5)\psi dx. \tag{3.22}$$

If we apply Lemma 2.5 for $P(t) = F(t) + \frac{1}{2}mt^2 - \frac{1}{6}t^6 = G(t) - \frac{1}{6}t^6$, $Q(t) = t^2 + t^6$, $\{v_n\}_n = \{u_n\}_n$, $v = F(u_\lambda) + \frac{1}{2}mu_\lambda^2 - \frac{1}{6}u_\lambda^6 = G(u_\lambda) - \frac{1}{6}u_\lambda^6$, and $\psi = 1$. By (f_2)–(f_4) and (3.21), we have

$$\int_{\mathbb{R}^3} \left(G(u_n) - \frac{1}{6}u_n^6\right) dx \to \int_{\mathbb{R}^3} \left(G(u_\lambda) - \frac{1}{6}u_\lambda^6\right) dx. \tag{3.23}$$

Similarly, we also have

$$\int_{\mathbb{R}^3} (g(u_n)u_n - u_n^6) dx \to \int_{\mathbb{R}^3} (g(u_\lambda)u_\lambda - u_\lambda^6) dx. \tag{3.24}$$

Introduce the notation $\mathrm{Y} = \mathrm{Supp}(\psi)$. Using (3.21) and the Sobolev inequality, one has

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n} u_n^2 \psi - \varphi_{u_\lambda} u_\lambda^2 \psi) dx \right| &\le \int_{\mathbb{R}^3} |\varphi_{u_n}||u_n^2 - u_\lambda^2||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n} - \varphi_{u_\lambda}||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx\right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n^2 - u_\lambda^2|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n} - \varphi_{u_\lambda}|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda^2|^6 dx\right)^{\frac{1}{6}} \sup|\psi| = o(1), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n} u_n \psi - \varphi_{u_\lambda} u_\lambda \psi) dx \right| &\le \int_{\mathbb{R}^3} |\varphi_{u_n}||u_n - u_\lambda||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n} - \varphi_{u_\lambda}||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx\right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n - u_\lambda|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n} - \varphi_{u_\lambda}|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda|^6 dx\right)^{\frac{1}{6}} \sup|\psi| = o(1), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n}^2 u_n \psi - \varphi_{u_\lambda}^2 u_\lambda \psi) dx \right| &\le \int_{\mathbb{R}^3} \varphi_{u_n}^2 |u_n - u_\lambda||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n}^2 - \varphi_{u_\lambda}^2||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx\right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n - u_\lambda|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n}^2 - \varphi_{u_\lambda}^2|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda|^6 dx\right)^{\frac{1}{6}} \sup|\psi| = o(1). \end{aligned} \tag{3.27}$$

It follows from $\langle I'_\lambda(u_n), \psi\rangle = 0$, (3.21), (3.22), (3.26) and (3.27) that

$$\int_{\mathbb{R}^3} (\nabla u_\lambda \nabla \psi + m u_\lambda \psi + (m_0^2 - w^2) u_\lambda \psi) dx - \int_{\mathbb{R}^3} (2w + \varphi_{u_\lambda}) \varphi_{u_\lambda} u_\lambda \psi dx \\ - \lambda \int_{\mathbb{R}^3} (g(u_\lambda)\psi - u_\lambda^5 \psi) dx - \lambda \int_{\mathbb{R}^3} u_\lambda^5 \psi dx = 0,$$

i.e. $J'_\lambda(u_\lambda) = 0$, where

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} (m_0^2 - \omega^2) u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} \omega \varphi_u u^2 dx \\ - \lambda \int_{\mathbb{R}^3} (G(u) - \frac{1}{6} u^6) dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} u^6 dx.$$

Set $v_n = u_n - u_\lambda$. Then $v_n \rightharpoonup 0$ in H. Following from the well-known Brezis–Lieb lemma [18], we get

$$\begin{aligned} \|v_n\|_2^2 &= \|u_n\|_2^2 - \|u_\lambda\|_2^2 + o(1), \\ \|\nabla v_n\|_2^2 &= \|\nabla u_n\|_2^2 - \|\nabla u_\lambda\|_2^2 + o(1), \\ \|v_n\|_6^6 &= \|u_n\|_6^6 - \|u_\lambda\|_6^6 + o(1). \end{aligned} \tag{3.28}$$

Then, by Lemma 3.3, (3.24), (3.26), (3.27), (3.28), $J'_\lambda(u_n) = 0$ and $J'_\lambda(u_\lambda) = 0$, we have

$$\begin{aligned} o(1) &= \langle J'_\lambda(u_n), u_n\rangle - \langle J'_\lambda(u_\lambda), u_\lambda\rangle \\ &= \|v_n\|^2 + (m_0^2 - \omega^2)\|v_n\|_2^2 - \lambda \int_{\mathbb{R}^3} |v_n|^6 dx. \end{aligned} \tag{3.29}$$

Up to a subsequence, we may assume that $\|v_n\|^2 + (m_0^2 - \omega^2)\|v_n\|_2^2 \to l \geq 0$. By (3.29), $\lambda \int_{\mathbb{R}^3} |v_n|^6 dx \to l$. If $l > 0$, then the Sobolev embedding theorem means that $S \leq \frac{\int_{\mathbb{R}^3} |\nabla v_n|^2 dx}{(\int_{\mathbb{R}^3} |v_n|^6 dx)^{\frac{1}{3}}} \leq \frac{\|v_n\|^2 + (m_0^2 - \omega^2)\|v_n\|_2^2}{(\int_{\mathbb{R}^3} |v_n|^6 dx)^{\frac{1}{3}}}$, which implies that

$$l \geq \lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.30}$$

By (3.20), (3.21), (3.23), (3.25) and (3.28), we get

$$\begin{aligned} c_\lambda - I_\lambda(u_\lambda) &= I_\lambda(u_n) - I_\lambda(u_\lambda) + o(1) \\ &= \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} (m_0^2 - \omega^2)|v_n|^2 dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} |v_n|^6 dx + o(1). \end{aligned} \tag{3.31}$$

Then, by (3.30)–(3.31), we have $c_\lambda - I_\lambda(u_\lambda) = \frac{1}{3} l > \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$, by Lemma 3.3, which contradicts with $c_\lambda - I_\lambda(u_\lambda) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$ since that $I_\lambda(u_\lambda) \geq 0$. Therefore, $l = 0$, i.e. $\|v_n\|^2 = o(1)$, hence $u_n \to u_\lambda$ in H. This completes Lemma 3.4. □

Proof of Theorem 1.1. According to Lemma 3.4, there exists sequences $\{\lambda_n\} \subset h$ with $\lambda_n \to 1$, $c_{\lambda_n} \in (0, \frac{1}{3}\lambda_n^{-\frac{1}{2}} S^{\frac{3}{2}})$ and a sequence of $\{u_{\lambda_n}\}$, denoted by $\{u_n\}$ such that $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$. Next we show $\{u_n\}$ is bounded. The proof will be developed in several steps: Indeed, by Lemma 2.4, $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$, we have

$$\begin{cases} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_n^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} (5\omega + 2\varphi_{u_n})\varphi_{u_n} u_n^2 dx - 3\lambda_n \int_{\mathbb{R}^3} F(u_n) dx = 0, \\ \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla u_n|^2 + (m_0^2 - \omega^2) u_n^2 - \omega \varphi_{u_n} u^2] dx - \lambda_n \int_{\mathbb{R}^3} F(u_n) dx = c_{\lambda_n} \leq c_{\frac{1}{2}}, \\ \int_{\mathbb{R}^3} [|\nabla u_n|^2 + (m_0^2 - \omega^2) u_n^2 - (2\omega + \varphi_{u_n})\varphi_{u_n} u_n^2] dx - \lambda_n \int_{\mathbb{R}^3} f(u_n) u_n dx = 0. \end{cases}$$

Step 1. If $\gamma \geq 4$ in (f_5), then it follows from (3.14) that

$$\begin{aligned} c_{\frac{1}{2}} \geq c_{\lambda_n} &= I_{\lambda_n}(u_n) = I_{\lambda_n}(u_n) - \frac{1}{\gamma}\langle I'_{\lambda_n}(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla u_\lambda|^2 + (m_0^2 - \omega^2)u_\lambda^2]dx \geq C\|u_n\|^2. \end{aligned} \tag{3.32}$$

Thus, we deduce from (3.32) that $\{u_n\}$ is bounded in H if $\gamma \geq 4$.

Step 2. If $2 < \gamma < 4$ in (f_5), we distinguish two cases:

Case 1. $3 \leq \gamma < 4$ and $0 < \omega < m_0$. Following from (3.17), we have

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_n) \geq \frac{\gamma - 2}{6 - \gamma}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u_n^2 dx. \tag{3.33}$$

Case 2. $2 < \gamma < 3$ and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$. From (3.19), we have

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_\lambda) \geq \frac{1}{(6-\gamma)(4-\gamma)}[(\gamma-2)(4-\gamma)m_0^2 - \omega^2]\int_{\mathbb{R}^3} u_n^2 dx. \tag{3.34}$$

Deriving from (3.33) and (3.34), we get the boundedness of $\|u_n\|_2$. Then by Lemma 2.1, one has

$$0 \leq \int_{\mathbb{R}^3} -\omega\varphi_{u_n}u_n^2 dx \leq \int_{\mathbb{R}^3}\omega^2 u_n^2 dx \leq C. \tag{3.35}$$

Thus from Lemma 2.4 and (3.35) we deduce that

$$\begin{aligned} c_{\frac{1}{2}} \geq c_{\lambda_n} &= I_{\lambda_n}(u_n) - \frac{1}{6}P_{\lambda_n}(u_n) = \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}(\omega + \varphi_{u_n})\varphi_{u_n}u_n^2 dx \\ &\geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}\omega\varphi_{u_n}u_n^2 dx \geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx - C, \end{aligned}$$

which means the boundedness of $\{\|\nabla u_n|_2\}$. This completes the proof. □

Note that $I(u_n) = I_{\lambda_n}(u_n) - (\lambda_n - 1)\int_{\mathbb{R}^3} F(u_n)dx$ and $I'(u_n) = I'_{\lambda_n}(u_n) - (\lambda_n - 1)\int_{\mathbb{R}^3} f(u_n)u_n dx$. By using the fact that the map $\lambda \to c_\lambda$ is left-continuous (see [23]), $\lambda_n \to 1$, the boundedness of $\{u_n\}$, we can show that

$$\lim_{n\to\infty} I(u_n) = c_1, \qquad \lim_{n\to\infty} I'(u_n) = 0.$$

Lemma 3.4 yields that there exists $u_0 \in H \setminus \{0\}$ being a critical point of I and $I(u_0) = c_1$. Set

$$\begin{aligned} I^+(u) = &\frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2 dx \\ &- \left(\int_{\mathbb{R}^3} G(u)dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6 dx\right) - \frac{1}{6}\int_{\mathbb{R}^3}|u^+|^6 dx, \end{aligned}$$

where $u^+ = \max\{u, 0\}$. Repeating all the calculations above word by word, there is nonzero function u_0 solving the equation

$$-\Delta u + (m_0^2 - \omega^2)u + mu - \omega\varphi_u u = (g(u) - u^5) + (u^+)^5. \tag{3.36}$$

Using $u^- = \max\{-u_0, 0\}$ as a text function and integrating (3.36) by parts, we obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} (|\nabla u_0^-|^2 + m|u_0^-|^2 + (m_0^2 - \omega^2)|u_0^-|^2)dx \\ &- \int_{\mathbb{R}^3} \omega\varphi_{u_0}|u_0^-|^2)dx - \int_{\mathbb{R}^3} (g(u_0) - u_0^5)u_0^- dx. \end{aligned} \tag{3.37}$$

We deduce from (f_1) and (f_4) that $g(t) - t^5$ is an odd function and $g(t) - t^5 > 0$ for $t > 0$. So from (3.37) one has

$$0 = \int_{\mathbb{R}^3} (|\nabla u_0^-|^2 + m|u_0^-|^2 + (m_0^2 - \omega^2)|u_0^-|^2)dx - \int_{\mathbb{R}^3} \omega\varphi_{u_0}|u_0^-|^2)dx.$$

From Lemma 2.1, we obtain that $u_0^- = 0$ and $u_0 \geq 0$. Then u_0 is a nonnegative solution of the problem (KGME). Deducing from Harnack's inequality (see [26]), we can obtain that $u_0 > 0$ for all $x \in \mathbb{R}^3$, and u_0 is a positive critical point of the functional $I(u)$. Then by Lemma 2.1, we have $\varphi = \varphi_{u_0}$. From (2.1), (2.3) and (2.4) that (u_0, φ_{u_0}) is a positive solution of (KGME). The proof is complete. In what follows, we prove the existence of a positive ground state solution for the problem (KGME).

Proof of Theorem 1.2. Set $\tilde{m} := \inf\{I(u) : u \in H \setminus \{0\}, I'(u) = 0\}$. According to the arguments as above, we know that $0 < \tilde{m} \leq c < c_1^* := \frac{1}{3}S^{\frac{3}{2}}$. By the definition of $\tilde{m}$, there exists a sequence $\{u_n\} \subset H$ such that $u_n \neq 0$, $I(u_n) \to \tilde{m}$ and $I'(u_n) = 0$. Similar to the arguments as **Step 1** and **Step 2** in Theorem 1.1, we obtain the boundedness of $\{u_n\}$ in H. Since $I'(u_n) = 0$, we deduce from (2.3), (2.5), (f_1)–(f_3) and the Sobolev embedding inequality that

$$\begin{aligned} \|u_n\|^2 &= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + m|u_n|^2 + (m_0^2 - \omega^2)|u_n|^2)dx + \int_{\mathbb{R}^3} [2|\nabla\varphi_{u_n}|^2 + \varphi_{u_n}^2 u_n^2]dx \\ &= \int_{\mathbb{R}^3} g(u_n)u_n dx \leq \varepsilon \int_{\mathbb{R}^3} (|u_n|^2 + |u_n|^6)dx, \end{aligned}$$

and so, there exists $C > 0$ such that $\|u_n\| \geq C$. Then we can claim that there exists $\sigma > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx \geq \sigma > 0. \tag{3.38}$$

Otherwise, $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx = 0$. Using Lemma I.1 of [26], it follows that, for $2 < s < 6$, $\int_{\mathbb{R}^3} |u_n|^s dx \to 0$ in $L^s(\mathbb{R}^3)$. Using the same arguments as Lemma 3.4, we can obtain $\tilde{m} \geq c_1^* := \frac{1}{3}S^{\frac{3}{2}}$, which contradicts $\tilde{m} < c_1^* := \frac{1}{3}S^{\frac{3}{2}}$. Then (3.38) holds. Going if necessary to a subsequence, by (3.37), we may assume the existence of $y_n \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 dx \geq \frac{\sigma}{2} > 0.$$

Set $v_n(x) = u_n(x + y_n)$. Then

$$\|v_n\| = \|u_n\|, \qquad \int_{B_1(0)} |v_n|^2 dx \geq \frac{\sigma}{2} > 0.$$

Since $\varphi_{u_n}(x + y_n) = \varphi_{v_n}(x)$, by (2.4), (2.5), we have

$$I(v_n) \to \tilde{m}, \qquad I'(v_n) = 0.$$

From the boundedness of $\{u_n\}$ in H, $\{v_n\}$ is also bounded. Then there exits $v_0 \neq 0$ such that $v_n \rightharpoonup v_0$ weakly in H. In view of Lemma 3.4, one can conclude that

$$\langle I'(v_0), v_0\rangle = 0, \qquad I(v_0) \geq \tilde{m}. \tag{3.39}$$

On the other hand, we will prove that $\tilde{m} \geq I(v_0)$.

In fact, if $\gamma \geq 4$ in (f_5), by (2.3), (2.5), (3.39) and Fatou's lemma, one has

$$\begin{aligned}
\tilde{m} &= \lim_{n\to\infty}\left\{I(v_n) - \frac{1}{\gamma}\langle I'(v_n), v_n\rangle\right\} \\
&= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla v_n|^2 + (m_0^2-\omega^2)v_n^2]dx\right. \\
&\qquad \left. + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\gamma}-\frac{1}{2}\right)\omega\varphi_{v_n}v_n^2 + \frac{1}{\gamma}\varphi_{v_n}^2 v_n^2\right]dx + \int_{\mathbb{R}^3}\left[\frac{1}{\gamma}f(v_n)v_n - F(v_n)\right]dx\right\} \\
&\geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla v_0|^2 + (m_0^2-\omega^2)v_0^2]dx \\
&\quad + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\gamma}-\frac{1}{2}\right)\omega\varphi_{v_0}v_0^2 + \frac{1}{\gamma}\varphi_{v_0}^2 v_0^2\right]dx + \int_{\mathbb{R}^3}\left[\frac{1}{\gamma}f(v_0)v_0 - F(v_0)\right]dx \\
&= I(v_0) - \frac{1}{\gamma}\langle I'(v_0), v_0\rangle = I(v_0).
\end{aligned} \tag{3.40}$$

If $2 < \gamma < 3$ in (f_5) and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$ or $3 \leq \gamma < 4$ in (f_5) and $0 < \omega < m_0$, by (3.15) and Fatou's lemma, one also has

$$\begin{aligned}
\tilde{m} &= \lim_{n\to\infty}\left\{I(v_n) - \frac{2}{6-\gamma}\langle I'(v_n), v_n\rangle - \frac{2-\gamma}{2(6-\gamma)}P(v_n)\right\} \\
&= \lim_{n\to\infty}\left\{\frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)v_n^2 dx + \int_{\mathbb{R}^3}\left[\frac{2(3-\gamma)}{6-\gamma}\omega\varphi_{v_n} + \frac{4-\gamma}{6-\gamma}\varphi_{v_n}^2\right]v_n^2 dx\right. \\
&\qquad \left. + \frac{2}{6-\gamma}\lambda\int_{\mathbb{R}^3}[f(v_n)v_n - \gamma F(v_n)]dx\right\} \\
&\geq \frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)v_0^2 dx + \int_{\mathbb{R}^3}\left[\frac{2(3-\gamma)}{6-\gamma}\omega\varphi_{v_0} + \frac{4-\gamma}{6-\gamma}\varphi_{v_0}^2\right]v_0^2 dx \\
&\quad + \frac{2}{6-\gamma}\lambda\int_{\mathbb{R}^3}[f(v_0)v_0 - \gamma F(v_0)]dx \\
&= I(v_0) - \frac{2}{6-\gamma}\langle I'(v_0), v_0\rangle - \frac{2-\gamma}{2(6-\gamma)}P(v_0) = I(v_0).
\end{aligned} \tag{3.41}$$

Combining (3.39), (3.40) with (3.41), we derive that $I(v_0) = \tilde{m} = \inf\{I(u) : u \in H \setminus \{0\}\} > 0$. Arguments as Theorem 1.1, we get $v_0 > 0$. Thus, by Lemma 2.1, (2.1), (2.3) and (2.4), $(v_0, \varphi_{v_0}) \in H \times D^{1,2}(\mathbb{R}^3)$ is a positive ground state solution of problem (KGME). The proof is complete. □

Proof of Theorem 1.3. It is sufficient to prove (f_5). Indeed, by (f_6), whenever $u > 0$,

$$F(x,u) = \int_0^1 f(x,ut)u\,dt = \int_0^1 \frac{f(x,ut)}{(ut)^{\gamma-1}}u^\gamma t^{\gamma-1}dt \leq \int_0^1 \frac{f(x,u)}{u^{\gamma-1}}u^\gamma t^{\gamma-1}dt = \frac{1}{\gamma}uf(u),$$

and whenever $u < 0$,

$$F(x,u) = \int_0^1 f(x,ut)udt = -\int_0^1 \frac{f(x,ut)}{(-ut)^{\gamma-1}}(-u)^{\gamma}t^{\gamma-1}dt$$
$$= -\int_0^1 \frac{f(x,ut)}{|ut|^{\gamma-1}}|u|^{\gamma}t^{\gamma-1}dt \leq -\int_0^1 \frac{f(x,u)}{|u|^{\gamma-1}}|u|^{\gamma}t^{\gamma-1}dt = \frac{1}{\gamma}uf(u).$$

The above results mean (f_5) holds. The proof is complete. □

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References

[1] A. Azzollini, The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, *Differential Integral Equations* **25**(2012), No. 5–6, 543–554. https://doi.org/10.48550/arXiv.1001.0269; MR2951740; Zbl 1265.35069

[2] A. Azzollini, P. d'Avenia, A. Pomponio, On the Schrödinger–Maxwell equations under the effect of a general nonlinear term, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**(2010), No. 2, 779–791. https://doi.org/10.1016/J.ANIHPC.2009.11.012; MR2595202; Zbl 1187.35231

[3] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Klein–Gordon–Maxwell equations, *Topol. Methods Nonlinear Anal.* **35**(2010), No. 1, 33–42. https://doi.org/10.1007/s11721-009-0037-5; MR2677428; Zbl 1203.35274

[4] V. Benci, D. Fortunato, The nonlinear Klein–Gordon equation coupled with the Maxwell equations, in: *Proceedings of the Third World Congress of Nonlinear Analysts Part 9 (Catania, 2000), Nonlinear Anal.* **47**(2001), No. 9, 6065–6072. https://doi.org/10.1016/S0362-546X(01)00688-5; MR1970778; Zbl 1042.78500

[5] V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations, *Rev. Math. Phys.* **14**(2002), No. 4, 409–420. https://doi.org/10.1142/S0129055X02001168; MR1901222; Zbl 1037.35075

[6] H. Berestycki, T. Gallouët, O. Kavian, Équations de champs scalaires euclidiens non linéaire dans le plan (in French) [Nonlinear Euclidean scalar field equations in the plane], *C. R. Acad. Sci. Paris Paris Sér. I Math.* **297**(1983), No. 5, 307–310. MR0734575

[7] H. Berestycki, P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* **82**(1983), No. 4, 313–346. https://doi.org/10.1007/BF00250555; MR0695535; Zbl 0533.35029

[8] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**(1993), No. 3, 486–490. https://doi.org/10.2307/2044999; MR0699419; Zbl 0526.46037

[9] P. C. Carrião, P. L. Cunha, O. H. Miyagaki, Existence results for the Klein–Gordon–Maxwell equations in higher dimensions with critical exponents, *Commun Pure Appl. Anal.* **10**(2011), No. 2, 709–718. https://doi.org/cpaa.2011.10.709; MR2754298; Zbl 1231.35247

[10] P. C. Carrião, P. L. Cunha, O. H. Miyagaki, Positive ground state solutions for the critical Klein–Gordon–Maxwell system with potentials, *Nonlinear. Anal.* **75**(2012), No. 10, 4068–4078. https://doi.org/10.1016/j.na.2012.02.023; MR2914593; Zbl 1238.35109

[11] D. Cassani, Existence and non-existence of solitary waves for the critical Klein–Gordon equation coupled with Maxwell's equations, *Nonlinear Anal.* **58**(2004), No. 7-8, 733–747. https://doi.org/10.1016/j.na.2003.05.001; MR2085333; Zbl 1057.35041

[12] S. J. Chen, S. Z. Song, Multiple solutions for nonhomogeneous Klein–Gordon–Maxwell equations on $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.* **22**(2015), 259–271. https://doi.org/10.1016/j.nonrwa.2014.09.006; Zbl 1306.35024

[13] S. T. Chen, X. H. Tang, Improved results for Klein–Gordon–Maxwell syetem with general nonlinearity, *Discrete Contin. Dyn. Syst.* **38**(2018), No. 5, 2333–2348. https://doi.org/10.3934/dcds.2018096; MR3809038; Zbl 1398.35026

[14] P. L. Cunha, Subcritical and supercritical Klein–Gordon–Maxwell equations without Ambrosetti–Rabinowitz condition, *Differential Integral Equations* **27**(2014), No. 3–4, 387–399. https://doi.org/10.1007/s10589-013-9591-2; MR3161609; Zbl 1324.35021

[15] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A* **134**(2004), No. 5, 893–906.https://doi.org//10.1017/S030821050000353X; MR2099569; Zbl 1064.35182

[16] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein–Gordon–Maxwell equations, *Adv. Nonlinear Stud.* **4**(2004), No. 3, 307–322. https://doi.org/10.1515/ans-2004-0305; MR2079817; Zbl 1142.35406

[17] P. d'Avenia, L. Pisani, G. Siciliano, Dirichlet and Neumann problems for Klein–Gordon–Maxwell systems, *Nonlinear Anal.* **71**(2009), No. 12, 1985–1995. https://doi.org/10.1016/j.na.2009.02.111; MR2671970; Zbl 1238.35115

[18] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin, 1983. https://doi.org/10.1007/978-3-642-61798-0; MRMR737190; Zbl 0562.35001

[19] X. M. He, Multiplicity of solutions for a nonlinear Klein–Gordon–Maxwell system, *Acta Appl. Math.* **130**(2014), 237–250. https://doi.org/10.1007/s10440-013-9845-0; MR3180946; Zbl 1301.35175

[20] L. Jeanjean, On the existence of bounded Palais–Smale sequence and application to a Landesman–Lazer type problem set on $\mathbb{R}^3$, *Proc. Roy. Soc. Edinburgh Sect. A* **129**(1999), No. 4, 789–804. https://doi.org/10.1017/S0308210500013147; MR1718530; Zbl 0935.35044

[21] Z. S. Liu, S. J. Shang, On ground states for the Kirchhoff-type problem with a general critical nonlinearity, *J. Math. Anal. Appl.* **426**(2015), No. 1, 267–287. https://doi.org/10.1016/j.jmaa.2015.01.044; MR3306373; Zbl 1312.35109

[22] E. L. Moura, O. H. Miyagaki, R. Ruviaro, Positive ground state solution for quasicritical Klein–Gordon–Maxwell type systems with potential vanishing at infinity, *Electron. J. Differential Equations* **2017**, No. 154, 1–11. MR3690181; Zbl 1370.35012

[23] W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55**(1977), No. 2, 149–162. https://doi.org/10.1007/BF01626517; MR0454365; Zbl 0356.35028

[24] F. Z. Wang, Ground state solutions for the electrostatic nonlinear Klein–Gordon–Maxwell system, *Nonlinear Anal.* **74**(2011), No. 14, 4796–4803. https://doi.org/10.1016/j.na.2011.04.050; MR2810718; Zbl 1225.35232

[25] L. X. Wang, X. M. Wang, L. L. Zhang, Ground state solutions for the critical Klein–Gordon–Maxwell system, *Acta Math. Sci. Ser. B (Engl. Ed.)* **39**(2019), No. 5, 1451–1460. https://doi.org/10.1007/s10473-019-0521-y; MR4068831; Zbl 07550993

[26] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[27] J. Zhang, On the Schrödinger–Poisson equations with a general nonlinearity in the critical growth, *Nonlinear Anal.* **75**(2012), No. 18, 6391–6401. https://doi.org/10.1016/j.na.2012.07.008;MR2965225; Zbl 1254.35065

[28] J. J. Zhang, W. M. Zou, A Berestycki–Lions theorem revisted, *Commun. Contemp. Math.* **14**(2012), No. 5, 1–14. https://doi.org/10.1142/S0219199712500332; MR2972523