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Time dependent evolution inclusions governed by the difference of two subdifferentials

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Abstract

The purpose of this paper is to study evolution inclusions involving time dependent subdifferential operators which are non-monotone. More precisely, we study existence of solutions for the following evolution equation in a real Hilbert space X : $u'(t) + \partial f^t(u(t)) - \partial\varphi^t(u(t)) \ni w(t)$, $u(0) = u_0$, where f^t and φ^t are closed convex proper functions on X.

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1 Introduction

Let $T > 0$ be a real. In this work, we consider a non-convex evolution equation governed by the difference of two subdifferentials. Our interest is the existence of solutions of the following inclusion in a real Hilbert space X:

$$u'(t) + \partial f^t(u(t)) - \partial\varphi^t(u(t)) \ni w(t)\ ,\quad t \in [0, T], \tag{1}$$

$(f^t)_{t\in[0,T]}$ and $(\varphi^t)_{t\in[0,T]}$ being families of lower semi-continuous convex proper functions and w belonging to $L^2(0, T; X)$. For any $t \in [0, T]$, ∂f^t and $\partial\varphi^t$ denote the subdifferential of f^t and those of φ^t in the sense of convex analysis.
The interest of a such problem is explained by the fact that the class of functions which are written as the difference of two convex functions is large. There is a significant literature on the class of functions which are the difference of two continuous convex functions (called DC functions). The difficult is these functions are in general not convex functions.

The existence of solution u to problem (1) was investigated in the case where f^t and φ^t are not dependent on t, in Otani [13] or Koi and Watanabe [12] when X is a real Hilbert space, and in Akagi and Otani [1] when X is a reflexive Banach space. In our case, $\mathrm{dom} f^s \neq \mathrm{dom} f^t$ and $\mathrm{dom}\varphi^{\mathrm{s}} \neq \mathrm{dom}\varphi^{\mathrm{t}}$ as soon as $s \neq t$. The domain of f^t and those of φ^t will depend on t in a suitable way. By using the Moreau Yosida approximation of f^t, and of φ^t, we shall construct some solution of (1).

When X is a real separable Hilbert space, many authors deals with the existence of solutions to more general evolution inclusions:

$$u'(t) + \partial f^t(u(t)) + B(t, u(t)) \ni 0\ , \quad t \in [0, T] \tag{2}$$

$(B(t, .))_{t\in[0,T]}$ being a family of multivalued operators on X, see [10] and references therein. For each t, the operator $B(t, .) : X \rightrightarrows X$ is a multivalued perturbation of ∂f^t, dependent on the time t.
When the perturbation $B(t, .)$ is single valued and monotone, many existence, uniqueness and regularity results have been established, *see* Brezis [5] (if f^t is independent of t), Attouch-Damlamian [4] and Yamada [15]. The study of case $B(t, .)$ nonmonotone and upper-semicontinuous with convex closed values has been developed under some boundedness conditions. For example, Attouch-Damlamian [3] have studied the case f independent of time. Otani [14] has extended this result with more general assumptions (the convex function f^t depends on time t).
This type of inclusion has been studied when the values of $B(t, .)$ are not necessary convex by Cellina and Staicu [7] (if f^t and $B(t, .)$ are independent of t, see also [6] for extended results) and in more general case in [9]. The authors then assume $-B(t, .) \subset \partial g$, $g : X \to \mathbb{R}$ being a convex lsc function.
In this paper we deal with the case where $-B(t, .) = \partial \varphi^t$.

Lastly, the existence result could be applied to some non linear parabolic differential equations in domain with moving boundaries for example.

Definition 1. *A continuous function $u : [0, T] \to X$ is said to be a solution of (1) on $[0, T]$ if the following conditions are satisfied:*

1. *u is an absolutely continuous function on $[0, T]$;*
2. *$u(t) \in \mathrm{dom} f^t \cap \mathrm{dom}\varphi^t$ for all $t \in [0, T]$;*
3. *there exists a section $\alpha(t) \in \partial\varphi^t(u(t))$ satisfying for a.e. $t \in [0, T]$*

$$u'(t) + \partial f^t(u(t)) \ni \alpha(t) + w(t).$$

We shall prove two existence theorems under the following assumptions by using the method of Kenmochi [11] :

$(\mathcal{H}_f)$ For each $r \geqslant 0$, there are absolutely continuous real-valued functions h_r, k_r on $[0, T]$ such that:

(i) $h'_r \in L^2(0, T)$ and $k'_r \in L^1(0, T)$;
(ii) for each $s, t \in [0, T]$ with $s \leqslant t$ and each $x_s \in \mathrm{dom} f^s$ with $\|x_s\| \leqslant r$ there exists $x_t \in \mathrm{dom} f^t$ satisfying

$$\begin{cases} \|x_t - x_s\| \leqslant |h_r(t) - h_r(s)| \left(1 + |f^s(x_s)|^{1/2}\right) \\ f^t(x_t) \leqslant f^s(x_s) + |k_r(t) - k_r(s)| \left(1 + |f^s(x_s)|\right), \end{cases}$$

$(\mathcal{H}_\varphi)$ For each $r \geqslant 0$, there are absolutely continuous real-valued functions a_r, b_r on $[0,T]$ such that:

(i) $a'_r \in L^2(0,T)$ and $b'_r \in L^1(0,T)$;
(ii) for each $s,t \in [0,T]$ and each $x_s \in \mathrm{dom}\varphi^{\mathrm{s}}$ with $\|x_s\| \leqslant r$ there exists $x_t \in \mathrm{dom}\varphi^{\mathrm{t}}$ satisfying

$$\begin{cases} \|x_t - x_s\| \leqslant |a_r(t) - a_r(s)| \left(1 + |\varphi^s(x_s)|^{1/2}\right) \\ \varphi^t(x_t) \leqslant \varphi^s(x_s) + |b_r(t) - b_r(s)| \left(1 + |\varphi^s(x_s)|\right), \end{cases}$$

2 Preliminaries

Let X be a real Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. Let h be a lower semi-continuous convex proper function on X. Set $\mathrm{dom}h = \{x \in X \mid h(x) < \infty\}$ the effective domain of h.

The set $\partial h(x)$, $x \in X$, is the ordinary subdifferential at x of convex analysis, that is

$$\partial h(x) = \{y \in X \mid \forall z \in X \;\; h(z) \geqslant h(x) + \langle y, z - x \rangle\}.$$

For any $x \in X$, $\partial^o h(x)$ stands for its element of minimal norm (that is the minimal section of $\partial h(x)$); if $\partial h(x) = \emptyset$, then $\|\partial^o h(x)\| = \infty$. Set $\mathrm{Dom}\partial h = \{x \in X \mid \partial h(x) \neq \emptyset\}$.

Let $\lambda > 0$. The function h_λ denotes the Moreau-Yosida proximal function of index λ of h which is defined by

$$h_\lambda(x) = \min_{y \in X}\{h(y) + \frac{1}{2\lambda}\|x - y\|^2\}.$$

The operator $J^h_\lambda = (I + \lambda \partial h)^{-1}$ denotes the resolvent of the index λ of ∂h:

$$h_\lambda(x) = h(J^h_\lambda x) + \frac{1}{2\lambda}\|x - J^h_\lambda x\|^2.$$

The function h_λ is convex, Frï¿½chet-differentiable on X and $(h_\lambda)_{\lambda>0}$ converges increasingly to h when λ decreases to 0. The Yosida approximation of index λ of ∂h, is

$$\nabla h_\lambda = \frac{1}{\lambda}(I - J^h_\lambda).$$

It is known that

$$\forall x, y \in X, \; \|J^h_\lambda x - J^h_\lambda y\| \leqslant \|x - y\|$$

and ∇h_λ is a $1/\lambda$-Lipschitz continuous function, *see* Attouch [2] and Brezis [5] for more details. Just recall that we have:

$$\forall x, y \in X \;, \;\; 0 \leqslant h_\lambda(y) - h_\lambda(x) - \langle \nabla h_\lambda(x), y - x \rangle \leqslant \frac{1}{\lambda}\|y - x\|^2. \tag{3}$$

and:

$$\forall x \in X \;, \;\; \nabla h_\lambda(x) \in \partial h(J^h_\lambda x) \quad \text{and} \quad \|\nabla h_\lambda(x)\| \leqslant \|\partial^0 h(x)\|.$$

As usual, the Hilbert space $L^2(0,T;X)$ denotes the space of X-valued measurable functions on $[0,T]$ which are 2^{nd} power integrable, in which $\|.\|_{L^2(0,T;X)}$ and $\langle .,. \rangle_{L^2(0,T;X)}$ are the norm and the scalar product.
The set of the continuous functions from $[0,T]$ to X is denoted by $\mathcal{C}([0,T],X)$. A function is of class $\mathcal{C}^1$ if it is continuously differentiable, it is of class $\mathcal{C}^{1,1}$ if, moreover, its Jacobian is Lipschitz continuous.

3 A global existence theorem

First, by considering the case where $(\varphi^t)_{t\in[0,T]}$ is a family of convex $\mathcal{C}^{1,1}$- functions on X, we give sufficient conditions to ensure existence and uniqueness of global solutions.
In this section, $(f^t)_{t\in[0,T]}$ denotes a family of lower semi-continuous convex proper functions and $(\varphi^t)_{t\in[0,T]}$ denotes a family of convex $\mathcal{C}^{1,1}$-functions on X such that $t\mapsto \nabla\varphi^t(x_0)$ is bounded on $[0,T]$ for some $x_0\in X$ and $(\nabla\varphi^t)_{t\in[0,T]}$ is equi-Lipschitz on X.

3.1 Estimations on the Moreau-Yosida approximate

Let $\lambda>0$. To study the evolution equation (1) we consider the approximate problems:

$$u_\lambda'(t)+\nabla f_\lambda^t(u_\lambda(t))-\nabla\varphi^t(u_\lambda(t))=w(t)\ ,\ \ u_\lambda(0)=u_0.$$

The functions f_λ^t denote, for any $t\in[0,T]$, the Moreau-Yosida proximal function of index λ of f^t, and

$$J_\lambda^t=(I+\lambda\partial f^t)^{-1}\ ,\ \ \nabla f_\lambda^t=\lambda^{-1}(I-J_\lambda^t).$$

Lemma 1. *Under the assumption ($\mathcal{H}_f$), we can find a set $\{z_t : t\in[0,T]\}$ and a real $\rho_0>0$ such that $\|z_t\|\leqslant\rho_0$ and $f^t(z_t)\leqslant\rho_0$ for every $t\in[0,T]$.*

Proof. Let $z_0\in \mathrm{dom} f^0$ and $r>0$ such that $r\geqslant\|z_0\|$. For all $t\in[0,T]$, there exists $z_t\in\mathrm{dom} f^t$ satisfying

$$\begin{cases} \|z_t-z_0\|\leqslant|h_r(t)-h_r(0)|(1+|f^0(z_0)|^{1/2}) \\ f^t(z_t)\leqslant f^0(z_0)+|k_r(t)-k_r(0)|(1+|f^0(z_0)|). \end{cases}$$

The lemma holds with $\rho_0=(\ r+\|h_r'\|_{L^1}(1+|f^0(z_0)|^{1/2})\ )\vee(\ f^0(z_0)+\|k_r'\|_{L^1}(1+|f^0(z_0)|)\ )$.

□

From Kenmochi [11, 26,Chapter 1,Section 1.5, Lemma 1.5.1], there exists some positive number α_f such that for all $t\in[0,T]$ and $x\in X$ we have $f^t(x)\geqslant-\alpha_f(1+\|x\|)$.

Lemma 2. *There exists $M_1>0$ such that*

$$\|J_\lambda^t x\|\leqslant\|x\|+M_1$$

for any $t\in[0,T]$, $x\in X$ and $\lambda\in]0,1]$.
If ($\mathcal{H}_f$) holds, there exists $M_2>0$ such that

$$-M_2(\|x\|+1)\leqslant f^t(J_\lambda^t x)\leqslant f_\lambda^t(x)\leqslant\frac{1}{\lambda}(M_2+\|x\|^2)$$

for any $t\in[0,T]$, $x\in X$ and $\lambda\in]0,1]$.

Proof. Let x_0 be fixed in X. Let $x\in X$. We have for any $t\in[0,T]$ and $\lambda\in]0,1]$

$$\|J_\lambda^t x\|\leqslant\|J_\lambda^t x-J_\lambda^t x_0\|+\|J_\lambda^t x_0\|\leqslant\|x-x_0\|+\|J_\lambda^t x_0\|$$

with

$$\begin{array}{rcl}\|J^t_\lambda x_0\| & \leqslant & \|J^t_\lambda x_0 - J^t_1 x_0\| + \|J^t_1 x_0\| \\ & \leqslant & \|J^t_\lambda x_0 - J^t_\lambda(\lambda x_0 + (1-\lambda)J^t_1 x_0)\| + \|J^t_1 x_0\| \\ & \leqslant & (1-\lambda)\|x_0 - J^t_1 x_0\| + \|J^t_1 x_0\| \\ & \leqslant & 2\|x_0 - J^t_1 x_0\| + \|x_0\|.\end{array}$$

Since $f^t_1(x_0) = f^t(J^t_1 x_0) + \frac{1}{2}\|x_0 - J^t_1 x_0\|^2 \geqslant -\alpha_f(\|J^t_1 x_0\| + 1) + \frac{1}{2}\|x_0 - J^t_1 x_0\|^2$, we obtain

$$f^t_1(x_0) \geqslant \frac{1}{2}(\|x_0 - J^t_1 x_0\| - \alpha_f)^2 - \frac{\alpha_f^2}{2} - \alpha_f(1 + \|x_0\|),$$

which assures

$$\|x_0 - J^t_1 x_0\| \leqslant \alpha_f + \sqrt{2 f^t_1(x_0) + \alpha_f^2 + \alpha_f(1 + \|x_0\|)}.$$

Let us add that $2 f^t_1(x_0) \leqslant 2 f^t(z_t) + \|z_t - x_0\|^2 \leqslant 2\rho_0 + (\rho_0 + \|x_0\|)^2$, and we can conclude

$$\|J^t_\lambda x_0\| \leqslant 2\alpha_f + 2\sqrt{2\rho_0 + (\rho_0 + \|x_0\|)^2 + \alpha_f^2 + \alpha_f(1 + \|x_0\|)} + \|x_0\|.$$

Consequently,

$$\|J^t_\lambda x\| \leqslant \|x\| + M_1$$

where $M_1 = \|x_0\| + 2\alpha_f + 2\sqrt{2\rho_0 + (\rho_0 + \|x_0\|)^2 + \alpha_f^2 + \alpha_f(1 + \|x_0\|)} + \|x_0\|$. Next, $f^t_\lambda(x) \geqslant f^t(J^t_\lambda x) \geqslant -\alpha_f(\|x\| + M_1 + 1)$ and

$$2\lambda f^t_\lambda(x) \leqslant 2\lambda f^t(z_t) + \|z_t - x\|^2 \leqslant 2\lambda\rho_0 + (\rho_0 + \|x\|)^2.$$

□

Let us set now :

$$\forall r > 0\ , \quad \rho = r + M_1.$$

Then, for any $x \in X$ such that $\|x\| \leqslant r$, we have $\|J^t_\lambda(x)\| \vee \|x\| \leqslant \rho$ for any $\lambda > 0$ and $t \in [0, T]$.

Proposition 1. *Assume that ($\mathcal{H}_f$) holds. Let $x \in X$ and $\lambda \in]0, 1]$. Then, $t \longmapsto f^t_\lambda(x)$ is of bounded variation on $[0, T]$ and for any $r \geqslant \|x\|$*

$$\frac{d}{ds} f^s_\lambda(x) \leqslant \|\nabla f^s_\lambda(x)\|\ |h'_\rho(s)|(1 + |f^s(J^s_\lambda x)|^{1/2}) + |k'_\rho(s)|(1 + |f^s(J^s_\lambda x)|). \qquad (4)$$

almost everywhere.

Proof. For all $t \geqslant s$, there exists $w \in \mathrm{dom} f^t$ such that

$$\left\{\begin{array}{l}\|J^s_\lambda x - w\| \leqslant |h_\rho(t) - h_\rho(s)|(1 + |f^s(x_s)|^{1/2}) \\ f^t(w) \leqslant f^s(J^s_\lambda x) + |k_\rho(t) - k_\rho(s)|(1 + |f^s(x_s)|).\end{array}\right.$$

Hence,

$$\begin{array}{rcl}f^t_\lambda(x) - f^s_\lambda(x) & \leqslant & \frac{1}{2\lambda}\|w - x\|^2 + f^t(w) - \frac{1}{2\lambda}\|x - J^s_\lambda x\|^2 - f^s(J^s_\lambda x) \\ & \leqslant & \frac{1}{2\lambda}\left(\|w - x\|^2 - \|x - J^s_\lambda x\|^2\right) + |k_\rho(t) - k_\rho(s)|(1 + |f^s(J^s_\lambda x)|).\end{array}$$

From $\|w-x\|^2 = \|x - J^s_\lambda x\|^2 + \|J^s_\lambda x - w\|^2 + 2\langle x - J^s_\lambda x, J^s_\lambda x - w\rangle$, we deduce

$$\begin{aligned} & f^t_\lambda(x) - f^s_\lambda(x) \\ \leqslant\ & \frac{1}{2\lambda}|h_\rho(t)-h_\rho(s)|^2(1+|f^s(J^s_\lambda x)|^{1/2})^2 + \|\nabla f^s_\lambda(x)\||h_\rho(t)-h_\rho(s)|(1+|f^s(J^s_\lambda x)|^{1/2}) \\ & +|k_\rho(t)-k_\rho(s)|(1+|f^s(J^s_\lambda x)|). \end{aligned}$$

Since $s \mapsto \|\nabla f^s_\lambda(x)\|$ and $s \mapsto |f^s(J^s_\lambda x)|$ are bounded on $[0,T]$, the function $t \longmapsto f^t_\lambda(x)$ is of bounded variation on $[0,T]$ and it is differentiable almost everywhere on $[0,T]$. Its derivative is integrable on $[0,T]$ and satisfies for any $s \leqslant t$ in $[0,T]$

$$f^t_\lambda(x) - f^s_\lambda(x) \leqslant \int_s^t \frac{d}{d\tau} f^\tau_\lambda(x)\, d\tau.$$

Furthermore, we obtain the inequality (4) for a.e. $s \in [0,T]$.

□

Corollary 1. *Assume ($\mathcal{H}_f$) holds. Let $x : [0,T] \to X$ be an absolutely continuous function with $x' \in L^1(0,T;X)$. Set $r \geqslant \|x(t)\|$ for any $t \in [0,T]$. Then, $t \mapsto f^t_\lambda(x(t))$ is of bounded variation and for any $t \geqslant s$ in $[0,T]$*

$$f^t_\lambda(x(t)) - f^s_\lambda(x(s)) \leqslant \int_s^t \frac{d}{d\tau} f^\tau_\lambda(x(\tau))\, d\tau. \tag{5}$$

Its derivative is integrable on $[0,T]$ and satisfies for a.e. $s \in [0,T]$

$$\begin{aligned} & \frac{d}{ds} f^s_\lambda(x(s)) - \langle \nabla f^s_\lambda(x(s)), x'(s)\rangle \\ \leqslant\ & \|\nabla f^s_\lambda(x(s))\|\ |h'_\rho(s)|(1+|f^s(J^s_\lambda x(s))|^{1/2}) + |k'_\rho(s)|(1+|f^s(J^s_\lambda x(s))|). \end{aligned} \tag{6}$$

Proof. Applying the inequality (3) to f^t, we obtain for all $t \geqslant s$ in $[0,T]$

$$\begin{aligned} & f^t_\lambda(x(t)) - f^s_\lambda(x(s)) - \langle \nabla f^s_\lambda(x(s)), x(t)-x(s)\rangle \\ \leqslant\ & f^t_\lambda(x(t)) - f^s_\lambda(x(t)) + f^s_\lambda(x(t)) - f^s_\lambda(x(s)) - \langle \nabla f^s_\lambda(x(s)), x(t)-x(s)\rangle \\ \leqslant\ & \frac{1}{2\lambda}|h_\rho(t)-h_\rho(s)|^2(1+|f^s(J^s_\lambda x(t))|^{1/2})^2 + \|\nabla f^s_\lambda(x(t))\||h_\rho(t)-h_\rho(s)|(1+|f^s(J^s_\lambda x(t))|^{1/2}) \\ & +|k_\rho(t)-k_\rho(s)|(1+|f^s(J^s_\lambda x(t))|) + \frac{1}{\lambda}\|x(t)-x(s)\|^2. \end{aligned}$$

Since $(s,t) \mapsto \|\nabla f^s_\lambda(x(t))\|$ and $(s,t) \mapsto |f^s(J^s_\lambda x(t))|$ are bounded on $[0,T]^2$, the function $t \mapsto f^t_\lambda(x(t))$ is of bounded variation on $[0,T]$ and it is differentiable almost everywhere on $[0,T]$. Its derivative $t \mapsto \frac{d}{dt} f^t_\lambda(x(t))$ is integrable on $[0,T]$ and we get (5). Observe that for a.e. $s \in [0,T]$

$$\lim_{t\to s^+} \frac{1}{t-s}|h_\rho(t)-h_\rho(s)|^2(1+|f^s(J^s_\lambda x(t))|^{1/2})^2 = 0$$

and

$$\lim_{t\to s^+} \frac{1}{t-s}\|x(t)-x(s)\|^2 = 0.$$

Add that the map $x \mapsto f^s(J_\lambda^s x) = f_\lambda^s(x) - \frac{1}{2\lambda}\|x - J_\lambda^s x\|^2$ is continuous on X and

$$\lim_{t\to s^+} \frac{1}{t-s}[\|\nabla f_\lambda^s(x(t))\| |h_\rho(t) - h_\rho(s)|(1 + |f^s(J_\lambda^s x(t))|^{1/2}) + |k_\rho(t) - k_\rho(s)|(1 + |f^s(J_\lambda^s x(t))|)]$$
$$= \|\nabla f_\lambda^s(x(s))\| \, |h'_\rho(s)|(1 + |f^s(J_\lambda^s x(s))|^{1/2}) + |k'_\rho(s)|(1 + |f^s(J_\lambda^s x(s))|).$$

We then obtain (6) almost everywhere.

□

Applying [11, lemma 1.2.2.], under $(\mathcal{H}_f)$, the maps $t \mapsto f_\lambda^t(v(t))$ and $t \mapsto f^t(v(t))$ are measurable on $t \in [0, T]$ for each $\lambda > 0$ and $v \in L^1(0, T; X)$.
Let us consider the function $\tilde{f} : L^2(0, T; X) \to \overline{\mathbb{R}}$ defined by $\tilde{f}(v) = \displaystyle\int_0^T f^t(v(t))\, dt$ and, for each $\lambda > 0$, the function $\psi_\lambda : L^2(0, T; X) \to \overline{\mathbb{R}}$ defined by $\psi_\lambda(v) = \displaystyle\int_0^T f_\lambda^t(v(t))\, dt$. Then, $\tilde{f}$ is proper lower semi-continuous and convex and ψ_λ is finite, continuous and convex on $L^2(0, T; X)$. Furthermore, by [11, Lemma 1.2.3 and Lemma 1.2.4.], for each $\lambda > 0$ and $v \in L^1(0, T; X)$, $t \mapsto \nabla f_\lambda^t(v(t))$ is measurable on $[0, T]$ and $\nabla f_\lambda^t(v(t)) = \nabla[\psi_\lambda(v)](t)$ for a.e. $t \in [0, T]$ since ψ_λ coincides with the Moreau-Yosida approximation of $\tilde{f}$. So we obtain:

Corollary 2. *Assume that $(\mathcal{H}_f)$ holds. For all $v \in L^2(0, T; X)$ and $\lambda > 0$, we have $\tilde{f}_\lambda(v) = \int_0^T f_\lambda^t(v(t))\, dt$. Furthermore, $J_\lambda^{\tilde{f}}(v)(t) = J_\lambda^t v(t)$ and $\nabla \tilde{f}_\lambda(v)(t) = \nabla f_\lambda^t(v(t))$ for a.e. $t \in [0, T]$.*

3.2 Other technical results

Let k denote the uniform Lipschitz constant of $(\nabla\varphi^t)_{t\in[0,T]}$.

Lemma 3. *There exists $M_3 > 0$ such that $\|\nabla\varphi^t(x)\| \leqslant M_3(\|x\| + 1)$ for any $t \in [0, T]$ and $x \in X$.*
If $(\mathcal{H}_\varphi)$ holds, there exists $M_4 > 0$ such that $-M_4(\|x\| + 1) \leqslant \varphi^t(x) \leqslant M_4(\|x\|^2 + 1)$ for any $t \in [0, T]$ and $x \in X$.

Proof. Let $x \in X$. We have

$$\begin{aligned} \|\nabla\varphi^t(x)\| &\leqslant \|\nabla\varphi^t(x) - \nabla\varphi^t(x_0)\| + \|\nabla\varphi^t(x_0)\| \\ &\leqslant k\|x - x_0\| + \|\nabla\varphi^t(x_0)\| \\ &\leqslant k\|x\| + k\|x_0\| + \|\nabla\varphi^t(x_0)\|. \end{aligned}$$

We can conclude thanks to the assumption $t \mapsto \nabla\varphi^t(x_0)$ bounded.
Next, from Kenmochi [11, Lemma 1.5.1], there is a nonnegative constant α_φ such that $\varphi^t(x) \geqslant -\alpha'(\|x\| + 1)$ for all $x \in X$ and $t \in [0, T]$.
In the same way as lemma 1, for any $t \in [0, T]$, there exists $x_t \in X$ satisfying

$$\begin{cases} \|x_t - x_0\| \leqslant |a_r(t) - a_r(0)| \left(1 + |\varphi^0(x_0)|^{1/2}\right) \\ \varphi^t(x_t) \leqslant \varphi^0(x_0) + |b_r(t) - b_r(0)| \left(1 + |\varphi^0(x_0)|\right), \end{cases}$$

where $r \geqslant \|x_0\|$. And, $\varphi^t(x) \leqslant \varphi^t(x_t) - \langle \nabla\varphi^t(x), x_t - x\rangle$ for any $x \in X$.

□

Lemma 4. *Under ($\mathcal{H}_\varphi$), for any $x \in X$, the map $t \longmapsto \nabla\varphi^t(x)$ is continuous on $[0, T]$.*

Proof. Let $x \in X$ and $r \geqslant M_3(\|x\| + 1)$. By assumption ($\mathcal{H}_\varphi$), for each $s, t \in [0, T]$ there exists $x_s \in X$ satisfying

$$\begin{cases} \|\nabla\varphi^t(x) - x_s\| \leqslant |a_r(t) - a_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|^{1/2}) \\ \varphi^s(x_s) \leqslant \varphi^t(\nabla\varphi^t(x)) + |b_r(t) - b_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|). \end{cases}$$

Since φ^s is convex, we have

$$\varphi^s(\nabla\varphi^s(x)) + \langle\nabla\varphi^s(x), x_s - \nabla\varphi^s(x)\rangle \leqslant \varphi^s(x_s) \leqslant \varphi^t(\nabla\varphi^t(x)) + |b_r(t) - b_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|).$$

Hence, for any s, t, we have

$$\begin{aligned} & \langle\nabla\varphi^s(x), \nabla\varphi^t(x) - \nabla\varphi^s(x)\rangle \\ \leqslant\ & \langle\nabla\varphi^s(x), \nabla\varphi^t(x) - x_s\rangle + \varphi^t(\nabla\varphi^t(x)) - \varphi^s(\nabla\varphi^s(x)) + |b_r(t) - b_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|) \\ \leqslant\ & \|\nabla\varphi^s(x)\||a_r(t) - a_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|^{1/2}) + \varphi^t(\nabla\varphi^t(x)) - \varphi^s(\nabla\varphi^s(x)) \\ & + |b_r(t) - b_r(s)|(1 + |\varphi^t(\nabla\varphi^t(x))|). \end{aligned}$$

By symmetry, we have for any t, s in $[0, T]$

$$\begin{aligned} & \langle\nabla\varphi^t(x), \nabla\varphi^s(x) - \nabla\varphi^t(x)\rangle \\ \leqslant\ & \|\nabla\varphi^t(x)\||a_r(t) - a_r(s)|(1 + |\varphi^s(\nabla\varphi^s(x))|^{1/2}) + \varphi^s(\nabla\varphi^s(x)) - \varphi^t(\nabla\varphi^t(x)) \\ & + |b_r(t) - b_r(s)|(1 + |\varphi^s(\nabla\varphi^s(x))|). \end{aligned}$$

Adding these two inequalities we obtain

$$\begin{aligned} & \|\nabla\varphi^t(x) - \nabla\varphi^s(x)\|^2 \\ \leqslant\ & [\ \|\nabla\varphi^t(x)\| + \|\nabla\varphi^s(x)\|\ ]\ |a_r(t) - a_r(s)|(1 + |\varphi^s(\nabla\varphi^s(x))|^{1/2} \vee |\varphi^t(\nabla\varphi^t(x))|^{1/2}) \\ & + 2|b_r(t) - b_r(s)|(1 + |\varphi^s(\nabla\varphi^s(x))| \vee |\varphi^t(\nabla\varphi^t(x))|). \end{aligned}$$

Since both $t \mapsto \|\nabla\varphi^t(x)\|$ and $t \mapsto |\varphi^t(\nabla\varphi^s(x))|$ are bounded on $[0, T]$, $t \mapsto \nabla\varphi^t(x)$ is continuous on $[0, T]$.

□

Proposition 2. *Under ($\mathcal{H}_\varphi$), for any $x \in X$, the map $t \longmapsto \varphi^t(x)$ is absolutely continuous on $[0, T]$ and*

$$\begin{aligned} |\varphi^t(x) - \varphi^s(x)| \leqslant (\|\nabla\varphi^t(x)\| \vee \|\nabla\varphi^s(x)\|)\ |a_r(t) - a_r(s)|(1 + |\varphi^t(x)|^{1/2} \vee |\varphi^s(x)|^{1/2}) \\ + |b_r(t) - b_r(s)|(1 + |\varphi^t(x)| \vee |\varphi^s(x)|) \end{aligned}$$

for any $r \geqslant \|x\|$ and $0 \leqslant s, t \leqslant T$

Proof. Let $s, t \in [0, T]$ and $x \in X$ with $\|x\| \leqslant r$. There exists $x_t \in X$ satisfying

$$\begin{cases} \|x_t - x\| \leqslant |a_r(t) - a_r(s)| \left(1 + |\varphi^s(x)|^{1/2}\right) \\ \varphi^t(x_t) \leqslant \varphi^s(x) + |b_r(t) - b_r(s)| \left(1 + |\varphi^s(x)|\right), \end{cases}$$

We have

$$\begin{aligned}\varphi^t(x)-\varphi^s(x) &= \varphi^t(x)-\varphi^t(x_t)+\varphi^t(x_t)-\varphi^s(x)\\ &\leqslant \langle\nabla\varphi^t(x),x-x_t\rangle+|b_r(t)-b_r(s)|(1+|\varphi^s(x)|)\\ &\leqslant \|\nabla\varphi^t(x)\|\;|a_r(t)-a_r(s)|(1+|\varphi^t(x)|^{1/2})+|b_r(t)-b_r(s)|(1+|\varphi^t(x)|).\end{aligned}$$

If s and t are exchanged the above inequality still holds.

□

Corollary 3. *Assume $(\mathcal{H}_\varphi)$ holds. Let $x:[0,T]\to X$ be an absolutely continuous function and $r\geqslant\|x(t)\|$ for any $t\in[0,T]$. Then, $t\mapsto\varphi^t(x(t))$ is absolutely continuous and we have for a.e. $t\in[0,T]$*

$$\left|\langle\nabla\varphi^t(x(t)),x'(t)\rangle-\frac{d}{dt}\varphi^t(x(t))\right|\leqslant\|\nabla\varphi^t(x(t))\|\;|a_r'(t)|(1+|\varphi^t(x(t))|^{1/2})+|b_r'(t)|(1+|\varphi^t(x(t))|).$$

Proof. Since $\nabla\varphi^t:X\to X$ is k-Lipschitz continuous, it's easy to see that

$$\forall x,y\in X,\;0\leqslant\varphi^t(y)-\varphi^t(x)-\langle\nabla\varphi^t(x),y-x\rangle\leqslant k\|y-x\|^2.$$

For all t,s in $[0,T]$

$$\begin{aligned}&|\langle\nabla\varphi^s(x(s)),x(t)-x(s)\rangle-\varphi^t(x(t))+\varphi^s(x(s))|\\ \leqslant\;&|\langle\nabla\varphi^s(x(s)),x(t)-x(s)\rangle+\varphi^s(x(s))-\varphi^s(x(t))|+|\varphi^s(x(t))-\varphi^t(x(t))|\\ \leqslant\;&k\|x(s)-x(t)\|^2++|b_r(t)-b_r(s)|(1+|\varphi^s(x(t))|\vee|\varphi^t(x(t))|)\\ &(\|\nabla\varphi^s(x(t))\|\vee\|\nabla\varphi^t(x(t))\|)\;|a_r(t)-a_r(s)|(1+|\varphi^s(x(t))|^{1/2}\vee|\varphi^t(x(t))|^{1/2}).\end{aligned}$$

Dividing by $t-s>0$ and letting $t\to s^+$, we obtain

$$\begin{aligned}&|\langle\nabla\varphi^s(x(s)),x'(s)\rangle-\frac{d}{ds}\varphi^s(x(s))|\\ \leqslant\;&\|\nabla\varphi^s(x(s))\|\;|a_r'(s)|(1+|\varphi^s(x(s))|^{1/2})+|b_r'(s)|(1+|\varphi^s(x(s))|).\end{aligned}$$

□

Let us consider the function $\tilde\varphi:L^2(0,T;X)\to\mathbb{R}$ defined by $\tilde\varphi(v)=\int_0^T\varphi^t(v(t))\,dt$. The function $\tilde\varphi$ are proper convex of class $\mathcal{C}^{1,1}$ with $\nabla\tilde\varphi(v)(t)=\nabla\varphi^t(v(t))$ for all $v\in L^2(0,T;X)$ and for a.e. $t\in[0,T]$.

3.3 Existence results

Let $\lambda>0$ be fixed. By applying [5, Theorem 1.4] we obtain the existence of $u_\lambda:[0,T]\to X$, the unique solution to the problem

$$u_\lambda'(t)+\nabla(f_\lambda^t-\varphi^t)(u_\lambda(t))=w(t)\text{ a.e. }t\in[0,T]\;\;,\;\;u_\lambda(0)=u_0.$$

The curve u_λ is absolutely continuous on $[0,T]$ and $u_\lambda'\in L^2(0,T;X)$. We now study the approximate solution u_λ.

Lemma 5. *Under ($\mathcal{H}_f$) the following estimates hold*

$$\begin{array}{l} r := \sup\{\|u_\lambda(t)\| \mid t\in[0,T], \lambda\in]0,1]\} < \infty, \\ \sup\{\|J^t_\lambda u_\lambda(s)\| \mid s,t\in[0,T], \lambda\in]0,1]\} < \infty, \\ \sup\{\lambda|f^t_\lambda(u_\lambda(s))| \mid t,s\in[0,T], \lambda\in]0,1]\} < \infty, \\ \sup\{\int_0^T |f^s_\lambda(u_\lambda(s))|\,ds \mid \lambda\in]0,1]\} < \infty. \end{array}$$

Proof. According to the results in [11, Chap. 1], for each $\lambda > 0$, there exists a unique absolutely continuous function $v_\lambda : [0,T] \to X$ such that $v_\lambda(0) = u_0$ and $v'_\lambda(t)+\nabla f^t_\lambda(v_\lambda(t)) \ni w(t)$ for a.e. $t\in[0,T]$. Furthermore $(v_\lambda)_\lambda$ uniformly converges to v on $[0,T]$, v being the unique solution of $v'(t) + \partial f^t(v(t)) \ni w(t)$ a.e. $t \in [0,T]$, $v(0) = u_0$.
We have for a.e. $t\in[0,T]$:

$$\begin{array}{rl} & \frac{d}{dt}\frac{1}{2}\|u_\lambda(t)-v_\lambda(t)\|^2 \\ = & \langle u'_\lambda(t)-v'_\lambda(t), u_\lambda(t)-v_\lambda(t)\rangle \\ = & -\langle \nabla f^t_\lambda(u_\lambda(t)) - \nabla f^t_\lambda(v_\lambda(t)), u_\lambda(t)-v_\lambda(t)\rangle + \langle \nabla\varphi^t(u_\lambda(t)), u_\lambda(t)-v_\lambda(t)\rangle \\ \leqslant & \langle \nabla\varphi^t(u_\lambda(t)), u_\lambda(t)-v_\lambda(t)\rangle \\ \leqslant & k\|u_\lambda(t)-v_\lambda(t)\|^2 + \langle \nabla\varphi^t(v_\lambda(t)), u_\lambda(t)-v_\lambda(t)\rangle \\ \leqslant & k\|u_\lambda(t)-v_\lambda(t)\|^2 + +M_3(\|v_\lambda(t)\|+1)\,\|u_\lambda(t)-v_\lambda(t)\|. \end{array}$$

We thus obtain for a.e. $t\in[0,T]$

$$\frac{d}{dt}\|u_\lambda(t)-v_\lambda(t)\|^2 \leqslant 2(k+1)\|u_\lambda(t)-v_\lambda(t)\|^2 + \frac{M_3^2}{2}(\|v_\lambda(t)\|+1)^2.$$

Gronwall's lemma yields for any $t\in[0,T]$

$$\|u_\lambda(t)-v_\lambda(t)\|^2 \leqslant \frac{M_3^2}{2}\int_0^t (\|v_\lambda(s)\|+1)^2 \exp[2(t-s)(k+1)]\,ds.$$

$(v_\lambda)_{\lambda\in]0,1]}$ being uniformly bounded on $[0,T]$, $(u_\lambda)_{\lambda\in]0,1]}$ is uniformly bounded on $[0,T]$. Thanks to lemma 2, we have $\|J^t_\lambda u_\lambda(s)\| \leqslant \|u_\lambda(s)\| + M_1$ and

$$-M_2(\|u_\lambda(s)\|+1) \leqslant f^t(J^t_\lambda u_\lambda(s)) \leqslant f^t_\lambda(u_\lambda(s)) \leqslant \frac{1}{\lambda}(M_2 + \|u_\lambda(s)\|^2)$$

for any $t\in[0,T]$, $x\in X$ and $\lambda\in]0,1]$. To conclude,

$$\begin{array}{rcl} f^t_\lambda(u_\lambda(t)) & \leqslant & f^t_\lambda(v(t)) + \langle \nabla f^t_\lambda(u_\lambda(t)), u_\lambda(t)-v(t)\rangle \\ & \leqslant & f^t(v(t)) + k\|u_\lambda(t)-v(t)\|^2 + \langle \nabla f^t_\lambda(v(t)), u_\lambda(t)-v(t)\rangle \\ & \leqslant & f^t(v(t)) + k\|u_\lambda(t)-v(t)\|^2 + \|v'(t)-w(t)\|\|u_\lambda(t)-v(t)\|. \end{array}$$

Notice that $t\mapsto f^t(v(t))$ is bounded on $[0,T]$ and $v'\in L^2(0,T;X)$. Furthermore, $f^t_\lambda(u_\lambda(t)) \geqslant -M_2(\|u_\lambda(t)\|+1)$ for any $t\in[0,T]$. Thus,

$$\sup\{\int_0^T |f^s_\lambda(u_\lambda(s))|\,ds \mid \lambda\in]0,1]\} < \infty.$$

□

For $r > 0$ chosen as in Lemma 5, we always set $\rho = r + M_1$. Then,

$$\rho \geqslant \|u_\lambda(t)\| \vee \|J_\lambda^t u_\lambda(s)\|$$

for any $t, s \in [0, T]$ and $\lambda \in]0, 1]$.

Proposition 3. *Under ($\mathcal{H}_f$) and ($\mathcal{H}_\varphi$), $t \mapsto f_\lambda^t(u_\lambda(t))$ is of bounded variation on $[0, T]$, $t \mapsto \varphi^t(u_\lambda(t))$ is absolutely continuous on $[0, T]$ and the following inequality holds for a.e. $t \in [0, T]$*

$$\begin{aligned} & \frac{d}{dt} f_\lambda^t(u_\lambda(t)) + \frac{1}{2}\|u_\lambda'(t)\|^2 + \frac{1}{2}\|u_\lambda'(t) - w(t)\|^2 \\ \leqslant\ & \frac{d}{dt}\varphi^t(u_\lambda(t)) + \frac{1}{2}\|w(t)\|^2 + \|\nabla f_\lambda^t(u_\lambda(t))\| |h_\rho'(t)| (1 + |f^t(J_\lambda^t u_\lambda(t))|^{1/2}) \\ & + |k_\rho'(t)| \left(1 + |f^t(J_\lambda^t u_\lambda(t))|\right) \\ & + \|\nabla \varphi^t(u_\lambda(t))\| |a_\rho'(t)| (1 + |\varphi^t(u_\lambda(t))|^{1/2}) + |b_\rho'(t)| \left(1 + |\varphi^t(u_\lambda(t))|\right). \end{aligned}$$

Proof. Let $\lambda > 0$. From Corollary 1, $t \mapsto f_\lambda^t(u_\lambda(t))$ is of bounded variation on $[0, T]$. Its derivative is integrable on $[0, T]$ and satisfies for a.e. $t \in [0, T]$

$$\begin{aligned} & \frac{d}{ds} f_\lambda^s(u_\lambda(s)) - \langle \nabla f_\lambda^s(u_\lambda(s)), u_\lambda'(s)\rangle \\ \leqslant\ & \|\nabla f_\lambda^s(u_\lambda(s))\| |h_\rho'(s)| (1 + |f^s(J_\lambda^s u_\lambda(s))|^{1/2}) + |k_\rho'(s)| (1 + |f^s(J_\lambda^s u_\lambda(s))|). \end{aligned}$$

But $u_\lambda'(s) + \nabla f_\lambda^s(u_\lambda(s)) - \nabla\varphi^s(u_\lambda(s)) = w(s)$, so

$$\begin{aligned} & \frac{d}{ds} f_\lambda^s(u_\lambda(s)) + \langle u_\lambda'(s) - w(s), u_\lambda'(s)\rangle \\ \leqslant\ & \langle \nabla\varphi^s(u_\lambda(s)), u_\lambda'(s)\rangle + \|\nabla f_\lambda^s(u_\lambda(s))\| |h_\rho'(s)| (1 + |f^s(J_\lambda^s u_\lambda(s))|^{1/2}) \\ & + |k_\rho'(s)| (1 + |f^s(J_\lambda^s u_\lambda(s))|). \end{aligned}$$

with

$$\langle u_\lambda'(s) - w(s), u_\lambda'(s)\rangle = \frac{1}{2}\|u_\lambda'(s)\|^2 + \frac{1}{2}\|u_\lambda'(s) - w(s)\|^2 - \frac{1}{2}\|w(s)\|^2.$$

We conclude thanks to Corollary 3.

□

Corollary 4. *Under ($\mathcal{H}_f$) and ($\mathcal{H}_\varphi$),*

$$\sup_{0<\lambda\leqslant 1, t\in[0,T]} |f_\lambda^t(u_\lambda(t))| < \infty,$$

$$M_5 := \sup_{0<\lambda\leqslant 1} \int_0^T \|u_\lambda'(t)\|^2\, dt < \infty,$$

and

$$\sup_{\lambda\in[0,1[} \int_0^T \|\nabla f_\lambda^t(u_\lambda(t))\|^2\, dt < \infty\,, \qquad \sup_{\lambda\in[0,1[, t\in[0,T]} \sqrt{\lambda}\|\nabla f_\lambda^t(u_\lambda(t))\| < \infty. \tag{7}$$

Hence,

$$\|u_\lambda(t) - u_\lambda(s)\| \leqslant \sqrt{M_5(t-s)} \tag{8}$$

for any $\lambda \in]0, 1]$ and $0 \leqslant s \leqslant t \leqslant T$.

Proof. According to Proposition 3, we have for a.e. $t \in [0,T]$

$$\begin{aligned}
& \frac{d}{dt} f_\lambda^t(u_\lambda(t)) + \frac{1}{2}\|u_\lambda'(t)\|^2 + \frac{1}{2}\|u_\lambda'(t) - w(t)\|^2 \\
\leqslant\ & \frac{d}{dt}\varphi^t(u_\lambda(t)) + \frac{1}{2}\|w(t)\|^2 + \|u_\lambda'(t) - \nabla\varphi^t(u_\lambda(t)) - w(t)\||h_\rho'(t)|(1 + |f^t(J_\lambda^t u_\lambda(t))|^{1/2}) \\
& + |k_\rho'(t)| \left(1 + |f^t(J_\lambda^t u_\lambda(t))|\right) + \|\nabla\varphi^t(u_\lambda(t))\||a_\rho'(t)|(1 + |\varphi^t(u_\lambda(t))|^{1/2}) \\
& + |b_\rho'(t)| \left(1 + |\varphi^t(u_\lambda(t))|\right) \\
\leqslant\ & \frac{d}{dt}\varphi^t(u_\lambda(t)) + \frac{1}{2}\|w(t)\|^2 + [\|u_\lambda'(t) - w(t)\| + C]\,|h_\rho'(t)|(1 + |f^t(J_\lambda^t u_\lambda(t))|^{1/2}) \\
& + |k_\rho'(t)| \left(1 + |f^t(J_\lambda^t u_\lambda(t))|\right) + C\left[|a_\rho'(t)| + |b_\rho'(t)|\right].
\end{aligned}$$

where C is a suitable nonnegative constant. Thus,

$$\begin{aligned}
& \frac{d}{dt} f_\lambda^t(u_\lambda(t)) + \frac{1}{2}\|u_\lambda'(t)\|^2 \\
\leqslant\ & \frac{d}{dt}\varphi^t(u_\lambda(t)) + \frac{1}{2}\|w(t)\|^2 + \frac{1}{2}|h_\rho'(t)|^2(1 + |f^t(J_\lambda^t u_\lambda(t))|^{1/2})^2 \\
& + C|h_\rho'(t)|(1 + |f^t(J_\lambda^t u_\lambda(t))|^{1/2}) + |k_\rho'(t)| \left(1 + |f^t(J_\lambda^t u_\lambda(t))|\right) + C\left[|a_\rho'(t)| + |b_\rho'(t)|\right].
\end{aligned}$$

By integrating we obtain for all $t \in [0,T]$

$$\begin{aligned}
& f_\lambda^t(u_\lambda(t)) - f^0(u_0) + \frac{1}{2}\int_0^t \|u_\lambda'(s)\|^2\,ds \\
\leqslant\ & \varphi^t(u_\lambda(t)) - \varphi^0(u_0) + \frac{1}{2}\|w\|_{L^2}^2 + (1 + CT^{1/2})\|h_\rho'\|_{L^2} + \frac{1}{2}C^2T + \|k_\rho'\|_{L^1} + C\|a_\rho'\|_{L^1} \\
& + C\|b_\rho'\|_{L^1} + \int_0^t \left(\frac{3}{2}|h_\rho'(s)|^2 + |k_\rho'(s)|\right)|f^s(J_\lambda^s u_\lambda(s))|\,ds.
\end{aligned}$$

By Lemma 2, since $f_\lambda^t(u_\lambda(t)) \geqslant f^t(J_\lambda^t u_\lambda(t)) \geqslant -M_2(\|u_\lambda(t)\| + 1)$, there exists a suitable nonnegative constant K which satisfies :

$$\begin{aligned}
& |f^t(J_\lambda^t u_\lambda(t))| + \frac{1}{2}\int_0^t \|u_\lambda'(s)\|^2\,ds \\
\leqslant\ & K + \int_0^t \left(\frac{3}{2}|h_\rho'(s)|^2 + |k_\rho'(s)|\right)|f^s(J_\lambda^s u_\lambda(s))|\,ds.
\end{aligned}$$

By Gronwall's lemma it follows:

$$|f^t(J_\lambda^t u_\lambda(t))| \leqslant K\exp\int_0^t \left(\frac{3}{2}|h_\rho'(\tau)|^2 + |k_\rho'(\tau)|\right)d\tau \leqslant K\exp\left(\frac{3}{2}\|h_\rho'\|_{L^2}^2 + \|k_\rho'\|_{L^1}\right),$$

which implies

$$C_1 := \sup_{0<\lambda\leqslant 1, t\in[0,T]} |f^t(J_\lambda^t u_\lambda(t))| < \infty.$$

It ensures that

$$f_\lambda^t(u_\lambda(t)) + \frac{1}{2}\int_0^t \|u_\lambda'(s)\|^2\,ds \leqslant K + C_1\left(\frac{3}{2}\|h_\rho'\|_{L^2}^2 + \|k_\rho'\|_{L^1}\right)$$

Consequently,

$$\sup_{0<\lambda\leqslant 1, t\in[0,T]} |f_\lambda^t(u_\lambda(t))| < \infty,$$

$$\sup_{0<\lambda\leqslant 1} \int_0^T \|u_\lambda'(t)\|^2\, dt < \infty.$$

Since $u_\lambda'(t) + \nabla f_\lambda^t(u_\lambda(t)) - \nabla\varphi^t(u_\lambda(t)) = w(t)$, we obtain

$$\sup_{0<\lambda\leqslant 1} \int_0^T \|\nabla f_\lambda^t(u_\lambda(t))\|^2\, dt < \infty.$$

□

We then obtain the convergence of approximate problems :

Proposition 4. *Under ($\mathcal{H}_f$) and ($\mathcal{H}_\varphi$), there exists an absolutely continuous curve $u : [0,T] \to X$ such that $(u_\lambda)_\lambda$ converges uniformly to u on $[0,T]$ in X and $(u_\lambda')_\lambda$ converges strongly to u' in $L^2(0,T;X)$ as λ goes to 0_+. Furthermore, $u(0) = u_0$ and $u(t) \in \operatorname{dom} f^t$ for any $t \in [0,T]$.*

Proof. Let $\lambda, \mu > 0$ and $t \in [0,T]$, then

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|^2 = \langle u_\lambda'(t) - u_\mu'(t), u_\lambda(t) - u_\mu(t)\rangle \\ =\ & -\langle \nabla f_\lambda^t(u_\lambda(t)) - \nabla f_\mu^t(u_\mu(t)), u_\lambda(t) - u_\mu(t)\rangle + \langle \nabla\varphi^t(u_\lambda(t)) - \nabla\varphi^t(u_\mu(t)), u_\lambda(t) - u_\mu(t)\rangle. \end{aligned}$$

Since $u_\lambda(t) = J_\lambda^t u_\lambda(t) + \lambda\nabla f_\lambda^t(u_\lambda(t))$ and also for μ, by monotonicity of ∂f^t, it follows

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|^2 \\ \leqslant\ & -\left\langle \nabla f_\lambda^t(u_\lambda(t)) - \nabla f_\mu^t(u_\mu(t)), \lambda\nabla f_\lambda^t(u_\lambda(t)) - \mu\nabla f_\mu^t(u_\mu(t))\right\rangle + k\|u_\lambda(t) - u_\mu(t)\|^2 \end{aligned}$$

k being the ratio of the Lipschitz continuous function $\nabla\varphi^t$. Let us set

$$\theta_{\lambda,\mu}(t) = -(\lambda+\mu)\|\nabla f_\lambda^t(u_\lambda(t)) - \nabla f_\mu^t(u_\mu(t))\|^2 - (\lambda-\mu)\left[\|\nabla f_\lambda^t(u_\lambda(t))\|^2 - \|\nabla f_\mu^t(u_\mu(t))\|^2\right].$$

Notice $\theta_{\lambda,\mu} \in L^1(0,T;\mathbb{R})$ and:

$$-2\left\langle \nabla f_\lambda^t(u_\lambda(t)) - \nabla f_\mu^t(u_\mu(t)), \lambda\nabla f_\lambda^t(u_\lambda(t)) - \mu\nabla f_\mu^t(u_\mu(t))\right\rangle = \theta_{\lambda,\mu}(t).$$

By Gronwall's lemma, we obtain

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leqslant \int_0^t \theta_{\lambda,\mu}(s)\exp(2k(t-s))\, ds,$$

which ensures

$$\begin{aligned} 0 &\leqslant \|u_\lambda(t) - u_\mu(t)\|^2 + (\lambda+\mu)\textstyle\int_0^t \|\nabla f_\lambda^s(u_\lambda(s)) - \nabla f_\mu^s(u_\mu(s))\|^2 \exp(2k(t-s))\, ds \quad (9) \\ &\leqslant -(\lambda-\mu)\textstyle\int_0^t \left[\|\nabla f_\lambda^s(u_\lambda(s))\|^2 - \|\nabla f_\mu^s(u_\mu(s))\|^2\right]\exp(2k(t-s))\, ds. \quad (10) \end{aligned}$$

Therefore, the sequence $(\int_0^t \|\nabla f_\lambda^s(u_\lambda(s))\|^2 \exp(2k(t-s))\,ds)_\lambda$ is nondecreasing as $\lambda \downarrow 0$ and bounded from above, thus converges. Furthermore,

$$\begin{aligned}
& \int_0^t \|\nabla f_\lambda^s(u_\lambda(s)) - \nabla f_\mu^s(u_\mu(s))\|^2\,ds \\
\leqslant\ & \int_0^t \|\nabla f_\lambda^s(u_\lambda(s)) - \nabla f_\mu^s(u_\mu(s))\|^2\, \mathrm{e}^{2k(t-s)}\,ds \\
\leqslant\ & -\frac{\lambda-\mu}{\lambda+\mu}\int_0^t [\, \|\nabla f_\lambda^s(u_\lambda(s))\|^2 - \|\nabla f_\mu^s(u_\mu(s))\|^2\,]\, \mathrm{e}^{2k(t-s)}\,ds \\
\leqslant\ & \left|\int_0^t [\, \|\nabla f_\lambda^s(u_\lambda(s))\|^2 - \|\nabla f_\mu^s(u_\mu(s))\|^2\,]\, \mathrm{e}^{2k(t-s)}\,ds\right|.
\end{aligned}$$

Consequently, $(s \mapsto \nabla f_\lambda^s(u_\lambda(s)))_{\lambda>0}$ is a Cauchy sequence with respect to the norm $\|.\|_{L^2}$ and it converges to some ξ in $L^2(0,T;X)$ when λ goes to 0_+.

We also deduce that $(u_\lambda)_\lambda$ converges uniformly to some continuous curve u on $[0,T]$. It is clear that $u(0) = u_0$. By using Corollary 4, $\|u(t) - u(s)\| \leqslant M_5\sqrt{t-s}$ for any $0 \leqslant s \leqslant t \leqslant T$ and $\sup_{t\in[0,T]} f^t(u(t)) < +\infty$. So, u is an absolutely continuous function from $[0,T]$ to X and $u(t) \in \mathrm{domf^t}$.

The sequence $(u'_\lambda)_\lambda$ converges weakly in L^2 to u' as $\lambda \downarrow 0$. We have in $L^2(0,T;X)$

$$u'_\lambda = -\nabla\tilde{f}_\lambda(u_\lambda) + \nabla\tilde{\varphi}(u_\lambda) + w,$$

which converges in L^2. Thus, $(u'_\lambda)_\lambda$ converges to u' in $L^2(0,T;X)$ as λ goes to 0_+.

□

We now prove the uniqueness and existence theorem:

Theorem 1. *Let $(f^t)_{t\in[0,T]}$ be a family of lower semi-continuous convex proper functions and $(\varphi^t)_{t\in[0,T]}$ be a family of convex $\mathcal{C}^{1,1}$-functions on X. Assume that $t \mapsto \nabla\varphi^t(x_0)$ is bounded on $[0,T]$ for some $x_0 \in X$ and $(\nabla\varphi^t)_{t\in[0,T]}$ is equi-Lipschitz on X.*

If $(\mathcal{H}_f)$ and $(\mathcal{H}_\varphi)$ hold, for any $u_0 \in \mathrm{dom} f^0$, the problem (1) has a unique solution $u : [0,T] \to X$ such that $u(0) = u_0$.

Proof. For uniqueness result, we consider $u_1, u_2 : [0,T] \to X$ two solutions of (1). By monotonicity of ∂f^t, we obtain for a.e. $s \in [0,T]$:

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2}\|u_1(s) - u_2(s)\|^2 &= \langle u_1'(t) - u_2'(t), u_1(t) - u_2(t)\rangle \\
&\leqslant \langle \nabla\varphi^t(u_1(t)) - \nabla\varphi^t(u_2(t)), u_1(t) - u_2(t)\rangle \\
&\leqslant k\|u_1(s) - u_2(s)\|^2,
\end{aligned}$$

k being the Lipschitz constant of $\nabla\varphi^t$. By Gronwall's lemma, it follows for any $t \in [0,T]$

$$\|u_1(t) - u_2(t)\|^2 \leqslant \|u_1(0) - u_2(0)\|^2 \exp(2kt).$$

So, $u_1(0) = u_2(0) = u_0$ yields $u_1(t) = u_2(t)$ for every $t \in [0,T]$.

For existence result, we use above propositions. Since $\nabla\tilde{f}_\lambda(u_\lambda) \in \partial\tilde{f}(J_\lambda^{\tilde{f}}(u_\lambda))$ and $\partial\tilde{f}$ is a maximal monotone set-valued map, by letting λ to 0_+, ξ belongs to $\partial\tilde{f}(u)$. Lastly, the equality $u' + \xi - \nabla\tilde{\varphi}(u) = w$ holds in $L^2(0,T;X)$. We deduce $u'(t) + \xi(t) - \nabla\varphi^t(u(t)) = w(t)$ with $\xi(t) \in \partial f^t(u(t)$ for a.e. $t \in [0,T]$.

□

Examples: Let us consider

$$\varphi^t(x) = \frac{\varepsilon(t)}{2}\|x\|^2.$$

Assume that $\varepsilon : [0,T] \to \mathbb{R}$ is an absolutely continuous function such that $\varepsilon' \in L^1(0,T)$. According to Theorem 1, for any $u_0 \in \mathrm{dom} f^0$, the problem

$$u'(t) + \partial f^t(u(t)) \ni \varepsilon(t)u(t) + w(t)$$

has a unique solution $u : [0,T] \to X$ such that $u(0) = u_0$.
More generally we can consider

$$\varphi^t(x) = \frac{1}{2}\langle A^t x, x\rangle$$

where $A^t : X \to X$ is a linear positive symmetric continuous operator. Assume that there exists $b : [0,T] \to \mathbb{R}$ an absolutely continuous function such that $b' \in L^1(0,T)$ and $\|A^t - A^s\| \leqslant |b(t) - b(s)|$ for any $t, s \in [0,T]$. According to Theorem 1, for any $u_0 \in \mathrm{dom} f^0$, the problem

$$u'(t) + \partial f^t(u(t)) \ni A(t)u(t) + w(t)$$

has a unique solution $u : [0,T] \to X$ such that $u(0) = u_0$.

3.4 Properties of evolution curve

Lemma 6. *Under the assumptions of Theorem 1, we have for any $t \in [0,T]$:*

$$\lim_{\lambda \to 0_+} \frac{1}{\lambda}\|u_\lambda(t) - u(t)\|^2 = 0 \quad \textit{and} \quad f^t(u(t)) = \lim_{\lambda \to 0_+} f^t_\lambda(u_\lambda(t)).$$

Proof. Let $t \in [0,T]$ and $\lambda \in]0,1]$. By the proof of Proposition 4 and letting μ to 0_+ in (9) we obtain

$$\begin{aligned} &\|u_\lambda(t) - u(t)\|^2 + \lambda \textstyle\int_0^t \|\nabla f^s_\lambda(u_\lambda(s)) - \xi(s)\|^2 \exp(2k(t-s))\, ds \\ &\qquad \leqslant -\lambda \textstyle\int_0^t [\|\nabla f^s_\lambda(u_\lambda(s))\|^2 - \|\xi(s)\|^2] \exp(2k(t-s))\, ds. \end{aligned}$$

So,

$$\frac{1}{\lambda}\|u_\lambda(t) - u(t)\|^2 \leqslant -\int_0^t \left[\|\nabla f^s_\lambda(u_\lambda(s))\|^2 - \|\xi(s)\|^2\right] \exp(2k(t-s))\, ds,$$

which converges to 0 as λ goes to 0_+. Furthermore,

$$|\langle \nabla f^t_\lambda(u_\lambda(t)), u(t) - u_\lambda(t)\rangle| \leqslant \sqrt{\lambda}\|\nabla f^t_\lambda(u_\lambda(t))\| \times \frac{1}{\sqrt{\lambda}}\|u(t) - u_\lambda(t)\|,$$

which ensures that

$$\lim_{\lambda \to 0_+} \langle \nabla f^t_\lambda(u_\lambda(t)), u(t) - u_\lambda(t)\rangle = 0.$$

We also can establish following inequalities by convexity of f^t_λ :

$$f^t_\lambda(u_\lambda(t)) + \langle \nabla f^t_\lambda(u_\lambda(t)), u(t) - u_\lambda(t)\rangle \leqslant f^t_\lambda(u(t)) \leqslant f^t(u(t)).$$

We know that $\lim_{\lambda\to 0_+} f_\lambda^t(u(t)) = f^t(u(t))$. By epiconvergency of $(f_\lambda^t)_{\lambda\to 0_+}$ to f^t, we have $f^t(u(t)) \leqslant \liminf_{\lambda\to 0_+} f_\lambda^t(u_\lambda(t))$, hence

$$f^t(u(t)) = \lim_{\lambda\to 0_+} f_\lambda^t(u_\lambda(t))$$

for any $t \in [0,T]$.

□

Theorem 2. *Under the assumptions of Theorem 1, the function $t \mapsto f^t(u(t))$ is of bounded variation on $[0,T]$ and we have the inequality for $t, s \in [0,T]$, $t \geqslant s$:*

$$\begin{aligned} & f^t(u(t)) - f^s(u(s)) + \frac{1}{2}\int_s^t \|u'(\tau)\|^2\,d\tau + \frac{1}{2}\int_s^t \|u'(\tau) - w(\tau)\|^2\,d\tau \\ \leqslant\; & \varphi^t(u(t)) - \varphi^s(u(s)) + \frac{1}{2}\int_s^t \|w(\tau)\|^2\,d\tau + \int_s^t c_\rho(\tau)\,d\tau. \end{aligned}$$

where we set

$$\begin{aligned} c_\rho(t) \;=\; & \|\xi(t)\||h'_\rho(t)|(1+|f^t(u(t))|^{1/2}) + |k'_\rho(t)|\left(1+|f^t(u(t))|\right) + \\ & \|\nabla\varphi^t(u(t))\||a'_\rho(t)|(1+|\varphi^t(u(t))|^{1/2}) + |b'_\rho(t)|\left(1+|\varphi^t(u(t))|\right). \end{aligned}$$

Proof. Following Proposition 3, we have for a.e. $s \in [0,T]$

$$\begin{aligned} & \frac{d}{dt} f_\lambda^t(u_\lambda(t)) + \frac{1}{2}\|u'_\lambda(t)\|^2 + \frac{1}{2}\|u'_\lambda(t) - w(t)\|^2 \\ \leqslant\; & \frac{d}{dt}\varphi^t(u_\lambda(t)) + \frac{1}{2}\|w(t)\|^2 + \|\nabla f_\lambda^t(u_\lambda(t))\||h'_\rho(t)|(1+|f^t(J_\lambda^t u_\lambda(t))|^{1/2}) \\ & + |k'_\rho(t)|\left(1+|f^t(J_\lambda^t u_\lambda(t))|\right) \\ & + \|\nabla\varphi^t(u_\lambda(t))\||a'_\rho(t)|(1+|\varphi^t(u_\lambda(t))|^{1/2}) + |b'_\rho(t)|\left(1+|\varphi^t(u_\lambda(t))|\right). \end{aligned}$$

By integrating, we obtain for any $t \leqslant s$

$$\begin{aligned} & f_\lambda^t(u_\lambda(t)) - f_\lambda^s(u_\lambda(s)) + \frac{1}{2}\int_s^t \|u'_\lambda(\tau)\|^2\,d\tau + \frac{1}{2}\int_s^t \|u'_\lambda(\tau) - w(\tau)\|^2\,d\tau \\ \leqslant\; & \varphi^t(u_\lambda(t)) - \varphi^s(u_\lambda(s)) + \frac{1}{2}\int_s^t \|w(\tau)\|^2\,d\tau \\ & + \int_s^t \|\nabla f_\lambda^\tau(u_\lambda(\tau))\|\;|h'_\rho(\tau)|(1+|f^\tau(J_\lambda^\tau u_\lambda(\tau))|^{1/2}) + |k'_\rho(\tau)|(1+|f^\tau(J_\lambda^\tau u_\lambda(\tau))|)\,d\tau \\ & + \int_s^t \|\nabla\varphi^\tau(u_\lambda(\tau))\|\;\left[|a'_\rho(\tau)|(1+|\varphi^\tau(u_\lambda(\tau))|^{1/2}) + |b'_\rho(\tau)|(1+|\varphi^\tau(u_\lambda(\tau))|)\right]\,d\tau. \end{aligned}$$

Letting λ to 0_+ it follows

$$\begin{aligned} & f^t(u(t)) - f^s(u(s)) + \frac{1}{2}\int_s^t \|u'(\tau)\|^2\,d\tau + \frac{1}{2}\int_s^t \|u'(\tau) - w(\tau)\|^2\,d\tau \\ \leqslant\; & \varphi^t(u(t)) - \varphi^s(u(s)) + \frac{1}{2}\int_s^t \|w(\tau)\|^2\,d\tau + \int_s^t c_\rho(\tau)\,d\tau. \end{aligned}$$

□

4 A local existence theorem

We now consider the general case. In this section, $(f^t)_{t\in[0,T]}$ and $(\varphi^t)_{t\in[0,T]}$ denote families of lower semi-continuous convex proper functions on X. By adding a compactness assumption, we shall obtain a local existence result.

4.1 Approximate problems

To solve the evolution equation (1), we regularize the convex function φ^t by considering the $\mathcal{C}^{1,1}$-function φ^t_λ which denotes the Moreau-Yosida approximate of index $\lambda > 0$ of φ^t. We then obtain the nonconvex evolution equation

$$u'_\lambda(t) + \partial f^t(u_\lambda(t)) - \nabla\varphi^t_\lambda(u_\lambda(t)) \ni w(t)\ , \quad t \in [0, T] \tag{11}$$

We use the notations:

$$J^t_\lambda = (I + \lambda\partial\varphi^t)^{-1}\ , \quad \nabla\varphi^t_\lambda = \lambda^{-1}(I - J^t_\lambda).$$

According to Lemma 2, under the assumption $(\mathcal{H}_\varphi)$, there exist non negative constants N_1 and N_2 such that

$$\|J^t_\lambda x\| \leqslant \|x\| + N_1$$

$$-N_2(\|x\| + 1) \leqslant \varphi^t(J^t_\lambda x) \leqslant \varphi^t_\lambda(x) \leqslant \frac{1}{\lambda}(N_2 + \|x\|^2)$$

for any $t \in [0, T]$, $x \in X$ and $\lambda \in]0, 1]$. Let us set now :

$$\forall r > 0\ , \quad \rho = r + \max(M_1, N_1).$$

Then, for any $x \in X$ such that $\|x\| \leqslant r$, we have $\|J^t_\lambda x\| \vee \|x\| \leqslant \rho$ for any $\lambda > 0$ and $t \in [0, T]$.

Lemma 7. *[15, Proposition 3.1]. Assume $(\mathcal{H}_\varphi)$ holds. Let $x \in X$ and $\lambda > 0$. The maps $t \longmapsto J^t_\lambda x$ and $t \longmapsto \nabla\varphi^t_\lambda(x)$ are continuous on $[0, T]$.*

Proof. Let $x \in X$ with $\|x\| \leqslant r$ and $\lambda \in]0, 1]$. By assumption $(\mathcal{H}_\varphi)$, for each $s, t \in [0, T]$ there exists $x_s \in \mathrm{dom}\varphi^s$ satisfying

$$\begin{cases} \|J^t_\lambda x - x_s\| \leqslant |a_\rho(t) - a_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|^{1/2}) \\ \varphi^s(x_s) \leqslant \varphi^t(J^t_\lambda x) + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|). \end{cases}$$

Since $\lambda^{-1}(x - J^s_\lambda x) \in \partial\varphi^s(J^s_\lambda x)$, we have

$$\varphi^s(J^s_\lambda x) + \frac{1}{\lambda}\langle x - J^s_\lambda x, x_s - J^s_\lambda x\rangle \leqslant \varphi^s(x_s) \leqslant \varphi^t(J^t_\lambda x) + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|).$$

Hence, for any s, t, we have

$$\begin{aligned} & \frac{1}{\lambda}\langle x - J^s_\lambda x, J^t_\lambda x - J^s_\lambda x\rangle \\ \leqslant\ & \frac{1}{\lambda}\langle x - J^s_\lambda x, J^t_\lambda x - x_s\rangle + \varphi^t(J^t_\lambda x) - \varphi^s(J^s_\lambda x) + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|) \\ \leqslant\ & \|\nabla\varphi^s_\lambda(x)\||a_\rho(t) - a_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|^{1/2}) + \varphi^t(J^t_\lambda x) - \varphi^s(J^s_\lambda x) \\ & + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^t(J^t_\lambda x)|). \end{aligned}$$

By symmetry, we have for any t, s in $[0, T]$

$$\frac{1}{\lambda}\langle x - J^t_\lambda x, J^s_\lambda x - J^t_\lambda x\rangle \leqslant \|\nabla\varphi^t_\lambda(x)\||a_\rho(t) - a_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|^{1/2}) + \varphi^s(J^s_\lambda x) - \varphi^t(J^t_\lambda x) \\ +|b_\rho(t) - b_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|).$$

Adding these two inequalities we obtain

$$\begin{aligned} & \frac{1}{\lambda}\|J^s_\lambda x - J^t_\lambda x\|^2 \\ \leqslant\ & [\ \|\nabla\varphi^t_\lambda(x)\| + \|\nabla\varphi^s_\lambda(x)\|\ ]\ |a_\rho(t) - a_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|^{1/2} \vee |\varphi^t(J^t_\lambda x)|^{1/2}) \\ & +2|b_\rho(t) - b_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)| \vee |\varphi^t(J^t_\lambda x)|). \end{aligned}$$

Since both $t \mapsto \|\nabla\varphi^t_\lambda(x)\|$ and $t \mapsto |\varphi^t(J^t_\lambda x)|$ are bounded on $[0, T]$, $t \mapsto J^t_\lambda x$ is continuous on $[0, T]$.

□

In the same way of Proposition 1, we obtain:

Lemma 8. *Assume ($\mathcal{H}_\varphi$) holds. Let $x \in X$ and $\lambda > 0$. The function $t \longmapsto \varphi^t_\lambda(x)$ is absolutely continuous on $[0, T]$ and for any $r \geqslant \|x\|$*

$$\frac{d}{ds}\varphi^s_\lambda(x) \leqslant \|\nabla\varphi^s_\lambda(x)\|\ |a'_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|^{1/2}) + |b'_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|). \tag{12}$$

almost everywhere.
If $x : [0, T] \to X$ is an absolutely continuous function and $r \geqslant \|x(t)\|$ for any $t \in [0, T]$, then $t \mapsto \varphi^t_\lambda(x(t))$ is absolutely continuous and we have for a.e. $t \in [0, T]$

$$\begin{aligned} & \left|\langle\nabla\varphi^t_\lambda(x(t)), x'(t)\rangle - \frac{d}{dt}\varphi^t_\lambda(x(t))\right| \\ \leqslant\ & \|\nabla\varphi^t_\lambda(x(t))\|\ |a'_\rho(t)|(1 + |\varphi^t(J^t_\lambda x(t))|^{1/2}) + |b'_\rho(t)|(1 + |\varphi^t(J^t_\lambda x(t))|). \end{aligned}$$

Proof. Let $x \in X$ and $\lambda > 0$. For all t, s, there exists $w \in \mathrm{dom}\varphi^t$ such that

$$\begin{cases} \|J^s_\lambda x - w\| \leqslant |a_\rho(t) - a_\rho(s)|(1 + |\varphi^s(x_s)|^{1/2}) \\ \varphi^t(w) \leqslant \varphi^s(J^s_\lambda x) + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^s(x_s)|). \end{cases}$$

Hence,

$$\begin{aligned} \varphi^t_\lambda(x) - \varphi^s_\lambda(x) &\leqslant \frac{1}{2\lambda}\|w - x\|^2 + \varphi^t(w) - \frac{1}{2\lambda}\|x - J^s_\lambda x\|^2 - \varphi^s(J^s_\lambda x) \\ &\leqslant \frac{1}{2\lambda}\left(\|w - x\|^2 - \|x - J^s_\lambda x\|^2\right) + |b_\rho(t) - b_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|). \end{aligned}$$

From $\|w - x\|^2 = \|x - J^s_\lambda x\|^2 + \|J^s_\lambda x - w\|^2 + 2\langle x - J^s_\lambda x, J^s_\lambda x - w\rangle$, we deduce

$$\begin{aligned} & \varphi^t_\lambda(x) - \varphi^s_\lambda(x) \\ \leqslant\ & \frac{1}{2\lambda}|a_\rho(t) - a_\rho(s)|^2(1 + |\varphi^s(J^s_\lambda x)|^{1/2})^2 + \|\nabla\varphi^s_\lambda(x)\||a_\rho(t) - a_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|^{1/2}) \\ & +|b_\rho(t) - b_\rho(s)|(1 + |\varphi^s(J^s_\lambda x)|). \end{aligned}$$

If s and t are exchanged the above inequality still holds. Since $s \mapsto \|\nabla\varphi^s_\lambda(x)\|$ is bounded on $[0,T]$ according to Lemma 7, the function $t \longmapsto \varphi^t_\lambda(x)$ is absolutely continuous on $[0,T]$.
By the inequality (3) applied to φ^t we have for all t, s in $[0,T]$

$$\begin{aligned}
& |\langle \nabla\varphi^s_\lambda(x(s)), x(t)-x(s)\rangle - \varphi^t_\lambda(x(t)) + \varphi^s_\lambda(x(s))| \\
\leqslant\ & |\langle \nabla\varphi^s_\lambda(x(s)), x(t)-x(s)\rangle + \varphi^s_\lambda(x(s)) - \varphi^s_\lambda(x(t))| + |\varphi^s_\lambda(x(t)) - \varphi^t_\lambda(x(t))| \\
\leqslant\ & \frac{1}{\lambda}\|x(s)-x(t)\|^2 + \frac{1}{2\lambda}|a_\rho(t)-a_\rho(s)|^2(1+|\varphi^s(J^s_\lambda x)|^{1/2})^2 \\
& + \max[\|\nabla\varphi^t_\lambda(x(t))\|, \|\nabla\varphi^s_\lambda(x(s))\|]|a_\rho(t)-a_\rho(s)|(1+\max[|\varphi^t(J^t_\lambda x(t))|, |\varphi^s(J^s_\lambda x(s))|]^{1/2}) \\
& + |b_\rho(t)-b_\rho(s)|(1+\max[|\varphi^t(J^t_\lambda x(t))|, |\varphi^s(J^s_\lambda x(s))|]).
\end{aligned}$$

Dividing by $t-s>0$ and letting $t \to s^+$, we obtain

$$\begin{aligned}
& |\langle \nabla\varphi^s_\lambda(x(s)), x'(s)\rangle - \frac{d}{ds}\varphi^s_\lambda(x(s))| \\
\leqslant\ & \|\nabla\varphi^s_\lambda(x(s))\|\ |a'_r(s)|(1+|\varphi^s(J^s_\lambda x(s))|^{1/2}) + |b'_r(s)|(1+|\varphi^s(J^s_\lambda x(s))|).
\end{aligned}$$

□

As in the above section we can prove that equation (11) has a unique solution $u_\lambda : [0,T] \to X$ satisfying $u_\lambda(0) = u_0$. Yet, there exists $\xi_\lambda \in L^2(0,T;X)$ such that for a.e. $t \in [0,T]$

$$\xi_\lambda(t) \in \partial f^t(u_\lambda(t)) \quad , \quad u'_\lambda(t) + \xi_\lambda(t) - \nabla\varphi^t_\lambda(u_\lambda(t)) = w(t).$$

The curve u_λ is absolutely continuous on $[0,T]$ and $u'_\lambda \in L^2(0,T;X)$.

4.2 Convergence of approximate problems

We now study the approximate solution $u_\lambda : [0,T] \to X$. This curve is absolutely continuous on $[0,T]$ and $u'_\lambda \in L^2(0,T;X)$. Let $r>0$. Let us set for any $\lambda \in]0,1]$

$$T_\lambda = \sup\{t \in [0,T] \mid \forall s \in [0,t],\ \|u_\lambda(s)-u_0\| \leqslant r\}.$$

Let $\lambda \in]0,1]$. According to Theorem 2, the function $t \mapsto f^t \circ u_\lambda(t)$ is of bounded variation on $[0,T]$ and the following inequality holds for any $t \in [0,T_\lambda]$:

$$\begin{aligned}
f^t(u_\lambda(t)) - f^0(u_0) + \frac{1}{2}\int_0^t \|u'_\lambda(\tau)\|^2\,d\tau + \frac{1}{2}\int_0^t \|u'_\lambda(\tau)-w(\tau)\|^2\,d\tau \\
\leqslant \varphi^t_\lambda(u_\lambda(t)) - \varphi^0_\lambda(u_0) + \frac{1}{2}\int_0^t \|w(\tau)\|^2\,d\tau + \int_0^t c_{\lambda,\rho}(\tau)\,d\tau
\end{aligned}$$

where $\rho \geqslant r + \|u_0\| + M_1 \vee N_1$ and

$$\begin{aligned}
c_{\lambda,\rho}(t) \ =\ & \|\xi_\lambda(t)\||h'_\rho(t)|(1+|f^t(u_\lambda(t))|^{1/2}) + |k'_\rho(t)|\left(1+|f^t(u_\lambda(t))|\right) + \\
& \|\nabla\varphi^t_\lambda(u_\lambda(t))\||a'_\rho(t)|(1+|\varphi^t_\lambda(u_\lambda(t))|^{1/2}) + |b'_\rho(t)|\left(1+|\varphi^t_\lambda(u_\lambda(t))|\right).
\end{aligned}$$

Let us prove that $\inf_{\lambda\in]0,1]} T_\lambda > 0$.

Lemma 9. *Assume there exist* $c_2 \in [0,1[$, $\eta \in \mathbb{R}_+$ *and* $\sigma \in L^2(0,T;\mathbb{R}_+)$ *such that*

$$\|\partial^o \varphi^t(x)\| \leqslant c_2 \|\partial^o f^t(x)\| + \eta |f^t(x)|^{1/2} + \sigma(t)$$

for all $x \in \mathrm{Dom}\partial f^t$, $\|x - u_0\| \leqslant r$ *and* $t \in [0,T]$.
Let $\lambda \in]0,1]$. *For a.e.* $t \in [0,T_\lambda]$, *we have* $c_{\lambda,\rho}(t) \leqslant \tilde{c}_{\lambda,\rho}(t)$ *where we set*

$$\begin{aligned}
\tilde{c}_{\lambda,\rho}(t) \;=\; & \frac{1}{1-c_2}\|u'_\lambda(t) - w(t)\| \left[|h'_\rho(t)|(1 + |f^t(u_\lambda(t))|^{1/2}) + c_2|a'_\rho(t)|(1 + |\varphi^t_\lambda(u_\lambda(t))|^{1/2})\right] + \\
& \frac{1}{1-c_2}(\eta|f^t(u_\lambda(t))|^{1/2} + \sigma(t)) \left[|h'_\rho(t)|(1 + |f^t(u_\lambda(t))|^{1/2}) + |a'_\rho(t)|(1 + |\varphi^t_\lambda(u_\lambda(t))|^{1/2})\right] \\
& + |k'_\rho(t)| \left(1 + |f^t(u_\lambda(t))|\right) + |b'_\rho(t)| \left(1 + |\varphi^t_\lambda(u_\lambda(t))|\right).
\end{aligned}$$

Proof. Since $u'_\lambda(t) + \xi_\lambda(t) - \nabla\varphi^t_\lambda(u_\lambda(t)) = w(t)$, it follows :

$$\|\xi_\lambda(t)\| \leqslant \|\nabla\varphi^t_\lambda(u_\lambda(t))\| + \|u'_\lambda(t) - w(t)\|.$$

From this

$$\begin{aligned}
c_{\lambda,\rho}(t) \;\leqslant\; & \|u'_\lambda(t) - w(t)\| |h'_\rho(t)|(1 + |f^t(u_\lambda(t))|^{1/2}) + |k'_\rho(t)| \left(1 + |f^t(u_\lambda(t))|\right) + \\
& \|\nabla\varphi^t_\lambda(u_\lambda(t))\| \left[|h'_\rho(t)|(1 + |f^t(u_\lambda(t))|^{1/2}) + |a'_\rho(t)|(1 + |\varphi^t_\lambda(u_\lambda(t))|^{1/2})\right] \\
& + |b'_\rho(t)| \left(1 + |\varphi^t_\lambda(u_\lambda(t))|\right).
\end{aligned}$$

On the other hand, we have for $t \in [0,T_\lambda]$

$$\begin{aligned}
\|\nabla\varphi^t_\lambda(u_\lambda(t))\| \;&\leqslant\; \|\partial^o\varphi^t(u_\lambda(t))\| \leqslant c_2\|\xi_\lambda(t)\| + \eta|f^t(u_\lambda(t))|^{1/2} + \sigma(t) \\
&\leqslant\; c_2\|\nabla\varphi^t_\lambda(u_\lambda(t))\| + c_2\|u'_\lambda(t) - w(t)\| + \eta|f^t(u_\lambda(t))|^{1/2} + \sigma(t).
\end{aligned}$$

Thus,

$$\|\nabla\varphi^t_\lambda(u_\lambda(t))\| \leqslant \frac{c_2}{1-c_2}\|u'_\lambda(t) - w(t)\| + \frac{1}{1-c_2}(\eta|f^t(u_\lambda(t))|^{1/2} + \sigma(t)). \tag{13}$$

Consequently, $c_{\lambda,\rho}(t) \leqslant \tilde{c}_{\lambda,\rho}(t)$.

□

Lemma 10. *Assume that:*

1. *there exist* $c_1 \in [0,1[$ *and* $c_0 \in \mathbb{R}$ *such that* $\varphi^t(x) \leqslant c_1|f^t(x)| + c_0$ *for all* $x \in \mathrm{dom} f^t$, $\|x - u_0\| \leqslant r$ *and* $t \in [0,T]$;

2. *there exist* $c_2 \in [0,1[$, $\eta \in \mathbb{R}_+$ *and* $\sigma \in L^2(0,T;\mathbb{R}_+)$ *such that*

$$\|\partial^o \varphi^t(x)\| \leqslant c_2 \|\partial^o f^t(x)\| + \eta |f^t(x)|^{1/2} + \sigma(t)$$

   for all $x \in \mathrm{Dom}\partial f^t$, $\|x - u_0\| \leqslant r$ *and* $t \in [0,T]$.

Under ($\mathcal{H}_f$) *and* ($\mathcal{H}_\varphi$), *we have:*

$$\sup_{0<\lambda\leqslant 1, t\in[0,T_\lambda]} |f^t(u_\lambda(t))| < \infty,$$

$$M_6 := \sup_{0<\lambda\leqslant 1} \int_0^{T_\lambda} \|u_\lambda'(t)\|^2\, dt < \infty,$$

and

$$\sup_{0<\lambda\leqslant 1} \int_0^{T_\lambda} \|\xi_\lambda(t)\|^2\, dt < \infty. \tag{14}$$

Hence,

$$\|u_\lambda(t) - u_\lambda(s)\| \leqslant \sqrt{M_6(t-s)}$$

for any $\lambda \in]0,1]$ *and* $0 \leqslant s \leqslant t \leqslant T_\lambda$.

Proof. Let $\lambda \in]0,1]$. We have for all $t \in [0, T_\lambda]$

$$\begin{aligned} & f^t(u_\lambda(t)) + \frac{1}{2}\int_0^t \|u_\lambda'(\tau)\|^2\, d\tau + \frac{1}{2}\int_0^t \|u_\lambda'(\tau) - w(\tau)\|^2\, d\tau \\ \leqslant\ & f^0(u_0) + \varphi_\lambda^t(u_\lambda(t)) - \varphi_1^0(u_0) + \frac{1}{2}\int_0^T \|w(\tau)\|^2\, d\tau + \int_0^t \tilde{c}_{\lambda,\rho}(\tau)\, d\tau. \end{aligned}$$

So, there exists a suitable real constant K which satisfies :

$$f^t(u_\lambda(t)) + \frac{1}{2}\int_0^t \|u_\lambda'(\tau)\|^2\, d\tau \leqslant K + \varphi_\lambda^t(u_\lambda(t)) + \int_0^t d_{\lambda,\rho}(\tau)\, d\tau$$

where

$$\begin{aligned} & d_{\lambda,\rho}(t) \\ = \ & \frac{1}{2(1-c_2)^2}\left[|h_\rho'(t)|(1+|f^t(u_\lambda(t))|^{1/2}) + c_2|a_\rho'(t)|(1+|\varphi_\lambda^t(u_\lambda(t))|^{1/2})\right]^2 \\ & + \frac{1}{1-c_2}\left(\eta|f^t(u_\lambda(t)\right)|^{1/2} + \sigma(t))\left[|h_\rho'(t)|(1+|f^t(u_\lambda(t))|^{1/2}) + |a_\rho'(t)|(1+|\varphi_\lambda^t(u_\lambda(t))|^{1/2})\right] \\ & + |k_\rho'(t)|\left(1+|f^t(u_\lambda(t))|\right) + |b_\rho'(t)|\left(1+|\varphi_\lambda^t(u_\lambda(t))|\right). \end{aligned}$$

By assumption, $\varphi_\lambda^t(u_\lambda(t)) \leqslant c_1|f^t(u_\lambda(t))| + c_0$ for any $t \in [0, T_\lambda]$, which ensures

$$(1-c_1)f^t(u_\lambda(t)) + \frac{1}{2}\int_0^t \|u_\lambda'(\tau)\|^2\, d\tau \leqslant K + c_0 + \int_0^t d_{\lambda,\rho}(\tau)\, d\tau.$$

Since $\varphi_\lambda^t(u_\lambda(t)) \geqslant -N_2(r + \|u_0\| + 1)$ for any $t \in [0, T_\lambda]$, we can find $\tilde{c_0} > 0$ such that $|\varphi_\lambda^t(u_\lambda(t))| \leqslant c_1|f^t(u_\lambda(t))| + \tilde{c_0}$ for any $t \in [0, T_\lambda]$. It follows:

$$\begin{aligned} d_{\lambda,\rho}(t) \ \leqslant\ & \frac{2}{(1-c_2)^2}\left[|h_\rho'(t)|^2 + c_2^2|a_\rho'(t)|^2(1+\tilde{c_0}) + (|h_\rho'(t)|^2 + c_1c_2^2|a_\rho'(t)|^2)|f_\lambda^t(u_\lambda(t))|\right] \\ & + \frac{1}{1-c_2}\sigma(t)\left[|h_\rho'(t)| + |a_\rho'(t)|(1+\tilde{c_0}^{1/2}) + |f^t(u_\lambda(t))|\left[|h_\rho'(t)| + |a_\rho'(t)|c_1^{1/2}\right]\right] \\ & + \frac{1}{1-c_2}|f^t(u_\lambda(t))|^{1/2}\left[\sigma(t)(|h_\rho'(t)| + |a_\rho'(t)|c_1^{1/2}) + \eta|h_\rho'(t)| + \eta|a_\rho'(t)|(1+\tilde{c_0}^{1/2})\right] \\ & + |k_\rho'(t)| + |b_\rho'(t)|(1+\tilde{c_0}) + |f^t(u_\lambda(t))|\left(|k_\rho'(t)| + |b_\rho'(t)|c_1\right) \end{aligned}$$

Setting

$$\psi_1(t) = \frac{2}{(1-c_2)^2}(|h'_\rho(t)|^2 + c_1 c_2^2 |a'_\rho(t)|^2) + \frac{1}{1-c_2}(|h'_\rho(t)| + |a'_\rho(t)| c_1^{1/2}) + |k'_\rho(t)| + |b'_\rho(t)| c_1$$

we obtain

$$\begin{aligned} d_{\lambda,\rho}(t) \leqslant\ & \frac{2}{(1-c_2)^2}\left[|h'_\rho(t)|^2 + c_2^2|a'_\rho(t)|^2(1+\tilde{c_0})\right] + \frac{1}{1-c_2}\sigma(t)\left[|h'_\rho(t)| + |a'_\rho(t)|(1+\tilde{c_0}^{1/2})\right] + \\ & \frac{1}{1-c_2}\left[\frac{1}{2}|f^t(u_\lambda(t))|(\eta+\sigma(t)^2) + |h'_\rho(t)|^2 + |a'_\rho(t)|^2 c_1 + \eta|h'_\rho(t)|^2 + \eta|a'_\rho(t)|^2(1+\tilde{c_0}^{1/2})^2\right] \\ & + |k'_\rho(t)| + |b'_\rho(t)|(1+\tilde{c_0}) + |f^t_\lambda(u_\lambda(t))| \times \psi_1(t) \end{aligned}$$

which ensures:

$$\begin{aligned} d_{\lambda,\rho}(t) \leqslant\ & \frac{2}{(1-c_2)^2}\left[|h'_\rho(t)|^2 + c_2^2|a'_\rho(t)|^2(1+\tilde{c_0})\right] + \frac{1}{1-c_2}\sigma(t)\left[|h'_\rho(t)| + |a'_\rho(t)|(1+\tilde{c_0}^{1/2})\right] \\ & + \frac{1}{1-c_2}\left[(1+\eta)|h'_\rho(t)|^2 + |a'_\rho(t)|^2(c_1+\eta(1+\tilde{c_0}^{1/2})^2)\right] + |k'_\rho(t)| \\ & + |b'_\rho(t)|(1+\tilde{c_0}) + |f^t_\lambda(u_\lambda(t))| \times \psi_2(t) \end{aligned}$$

where $\psi_2(t) = \psi_1(t) + \dfrac{\eta+\sigma(t)^2}{2(1-c_2)}$. Integrating on $[0,t]$, we obtain

$$\begin{aligned} & \int_0^t d_{\lambda,\rho}(\tau)\,d\tau \\ \leqslant\ & \frac{2}{(1-c_2)^2}\left[\|h'_\rho\|^2_{L^2} + c_2^2\|a'_\rho\|_{L^2}(1+\tilde{c_0})\right] + \frac{1}{1-c_2}\|\sigma\|_{L^2}\left[\|h'_\rho\|_{L^2} + \|a'_\rho\|_{L^2}(1+\tilde{c_0}^{1/2})\right] \\ & + \frac{1}{1-c_2}\left[(1+\eta)\|h'_\rho\|^2_{L^2} + \|a'_\rho\|^2_{L^2}(c_1+\eta(1+\tilde{c_0}^{1/2})^2)\right] + \|k'_\rho\|_{L^1} + \|b'_\rho\|_{L^1}(1+\tilde{c_0}) \\ & + \int_0^t |f^s_\lambda(u_\lambda(s))|\psi_2(s)\,ds. \end{aligned}$$

We notice that there exists a positive constant K_1 such that for any $\lambda \in]0,1]$

$$\forall t \in [0,T_\lambda]\ ,\ \ (1-c_1)f^t(u_\lambda(t)) + \frac{1}{2}\int_0^t \|u'_\lambda(\tau)\|^2\,d\tau \leqslant K_1 + \int_0^t |f^s_\lambda(u_\lambda(s))|\psi_2(s)\,ds.$$

Since $f^t(u_\lambda(t)) \geqslant -\alpha_f(r+\|u_0\|+1)$, we obtain for some positive constant K_2 and some function $\Psi \in L^1(0,T)$

$$\forall t \in [0,T_\lambda]\ ,\ \ |f^t(u_\lambda(t))| \leqslant K_2\left(1 + \int_0^t |f^s_\lambda(u_\lambda(s))|\Psi(s)\,ds\right).$$

Gronwall's lemma assures

$$\forall t \in [0,T_\lambda]\ ,\ \ |f^t(u_\lambda(t))| \leqslant K_2 \exp\left(\int_0^t \Psi(s)\,ds\right) \leqslant K_2 \exp\|\Psi\|_{L^1}.$$

Thus:

$$\sup_{0<\lambda\leqslant 1, t\in[0,T_\lambda]} |f^t(u_\lambda(t))| < \infty, \qquad \sup_{0<\lambda\leqslant 1, t\in[0,T_\lambda]} |\varphi^t_\lambda(u_\lambda(t))| < \infty,$$

which implies

$$\sup_{0<\lambda\leqslant 1}\int_0^{T_\lambda}\|u_\lambda'(t)\|^2\,dt<\infty,\qquad \sup_{0<\lambda\leqslant 1}\int_0^{T_\lambda}\|\nabla\varphi_\lambda^t(u_\lambda(t))\|^2\,dt<\infty.$$

Since $u_\lambda'(t)+\xi_\lambda(t)-\nabla\varphi_\lambda^t(u_\lambda(t))=w(t)$, we obtain

$$\sup_{\lambda\in[0,1[}\int_0^{T_\lambda}\|\xi_\lambda(t)\|^2\,dt<\infty.$$

□

Proposition 5. *Under the assumptions of the above lemma, there exists some $T'>0$ such that $T'\leqslant T_\lambda$ for any $\lambda\in]0,1]$.*

Proof. Let T' belong to $[0,\inf_{\lambda\in]0,1]}T_\lambda]$. Assume that $T'=0$. There exists $\Lambda\in]0,1]$ such that $0<T_\lambda\leqslant\dfrac{r^2}{4M_6}$ for any $\lambda\in]0,\Lambda[$. It follows for such λ:

$$\forall t,s\in[0,T_\lambda]\;,\;\;\|u_\lambda(t)-u_\lambda(s)\|\leqslant\sqrt{M_6(t-s)}\leqslant\sqrt{M_6T_\lambda}\leqslant\frac{r}{2}.$$

Particularly, for $s=0$ and $t=T_\lambda$, we obtain $\|u_\lambda(T_\lambda)-u_0\|\leqslant\frac{r}{2}$ which contradicts the definition of T_λ since we could find some $T_\lambda'>T_\lambda$ such that $\|u_\lambda(t)-u_0\|\leqslant r$, $t\in[0,T_\lambda']$, by continuity of u_λ. Consequently, $T'>0$ and $T'\leqslant T_\lambda$ for any $\lambda\in]0,1]$.

□

Proposition 6. *Assume the assumptions of the above lemma holds and add the compactness asumption:*

> *for each $t\in[0,T]$, the function f^t is of compact type around u_0, that is the set*
>
> $$\{x\in X\mid\|x-u_0\|\leqslant r,\;|f^t(x)|\leqslant c\}$$
>
> *is compact at each level c.*

If $(\mathcal{H}_f)$ and $(\mathcal{H}_\varphi)$ hold, there exists an absolutely continuous curve $u:[0,T']\to X$ such that a subsequence $(u_{\lambda_n})_n$ converges uniformly to u on $[0,T']$ and $(u_{\lambda_n}')_n$ converges weakly to u' in $L^2(0,T';X)$ as λ goes to 0_+. Furthermore, $u(0)=u_0$ and $u(t)\in\mathrm{dom}f^t$ for any $t\in[0,T']$.

Proof. Under the compactness assumption on each f^t, $\{u_\lambda(t)\mid\lambda\in]0,1]\}$ is relatively compact in X for each $t\in[0,T']$. Furthermore, $\|u_\lambda(t)-u_\lambda(s)\|\leqslant\sqrt{M_6|t-s|}$ for any $\lambda\in]0,1]$ and $0\leqslant s,t<T'$. By Ascoli's theorem $\{u_\lambda\mid\lambda\in]0,1]\}$ is relatively compact in $\mathcal{C}([0,T'],X)$ and $(u_\lambda)_\lambda$ admits a subsequence which converges uniformly to some $u\in\mathcal{C}([0,T'],X)$ on $[0,T']$.
From Lemma 10 there exist subsequences $(u_{\lambda_n}')_n$, $(\xi_{\lambda_n})_n$ and $(\nabla\tilde{\varphi_{\lambda_n}}(u_{\lambda_n}))_n$ which weakly converge in $L^2(0,T',X)$ respectively to u', ξ and β.

□

We are now ready to prove the following local existence result:

Theorem 3. *Let $(f^t)_{t\in[0,T]}$ and $(\varphi^t)_{t\in[0,T]}$ be families of lower semi-continuous convex proper functions. Let $u_0 \in \mathrm{dom} f^0 \cap \mathrm{dom}\varphi^0$. Assume the existence of $r > 0$ satisfying:*

1. *there exist $c_1 \in [0, 1[$ and $c_0 \in \mathbb{R}$ such that*

$$\varphi^t(x) \leqslant c_1|f^t(x)| + c_0$$

for all $x \in \mathrm{dom} f^t$, $\|x - u_0\| \leqslant r$ and $t \in [0, T]$;

2. *there exist $c_2 \in [0, 1[$, $\eta \in \mathbb{R}_+$ and $\sigma \in L^2(0, T; \mathbb{R}_+)$ such that*

$$\|\partial^o\varphi^t(x)\| \leqslant c_2\|\partial^o f^t(x)\| + \eta|f^t(x)|^{1/2} + \sigma(t)$$

for all $x \in \mathrm{Dom}\partial f^t$, $\|x - u_0\| \leqslant r$ and $t \in [0, T]$;

3. *each $t \in [0, T]$, the function f^t is of compact type around u_0, that is the set*

$$\{x \in X \mid \|x - u_0\| \leqslant r,\ |f^t(x)| \leqslant c\}$$

is compact at each level c.

If $(\mathcal{H}_f)$ and $(\mathcal{H}_\varphi)$ hold, there exists some $T' \in]0, T]$ such that the problem (1) has a solution $u : [0, T'] \to X$ satisfying $u(0) = u_0$.

Proof. We have $u' + \xi_\lambda = \nabla\tilde{\varphi}_\lambda(u_\lambda) + w$ in $L^2(0, T'; X)$. By weak convergency, we have $\xi \in \partial\tilde{f}(u)$, that is $\xi(t) \in \partial f^t(u(t))$ for a.e. $t \in [0, T']$. Yet, $\beta \in \partial\tilde{\varphi}(u)$, that is $\beta(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0, T']$. So, $u'(t) + \partial f^t(u(t)) - \partial\varphi^t(u(t)) \ni w(t)$ for a.e. $t \in [0, T']$.

□

Notice under assumption 1., if $x \in \mathrm{dom} f^t$ and $\|x - u_0\| \leqslant r$, then $x \in \mathrm{dom}\varphi^t$. And, under assumption 2., if $x \in \mathrm{Dom}\partial f^t$ and $\|x - u_0\| \leqslant r$, then $x \in \mathrm{Dom}\partial\varphi^t$.
Furthermore by Lemma 3, the assumptions 1 and 2 of Theorem 3 are trivially satisfied when $(\varphi^t)_{t\in[0,T]}$ is a family of convex $\mathcal{C}^{1,1}$- functions on X.

5 Example

Let Ω be a regular ($\mathcal{C}^1$) bounded open subset of $\mathbb{R}^n$ and $w \in L^2(0, T \times \Omega)$. We denote by Γ the smooth boundary of Ω.
For each $t \in [0, T]$, let $\beta^t : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $\beta^t(x, .)$ is (continuous) increasing function from $\mathbb{R}$ to $\mathbb{R}$.
Let $(\gamma^t)_{t\in(0,T]}$ be a family of maximal monotone operators from $\mathbb{R}$ into $\mathbb{R}$ such that $\gamma^t(0) \ni 0$.
We apply the results of the above section to a problem of parabolic type :

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + \beta^t(x, u(x,t)) + w(x,t)\ , & (x,t) \in \Omega\times]0,T[\\ -\frac{\partial u}{\partial \eta}(x,t) \in \gamma^t(u(x,t))\ , & (x,t) \in \Gamma\times]0,T[\\ u(x,0) = u_0(x)\ , & x \in \Omega \end{cases}$$

where $\frac{\partial u}{\partial \eta}$ denotes the outward normal derivative.

We define proper lower semi-continuous convex functions $\ell^t(x,.) : \mathbb{R} \to \mathbb{R}$ and $j^t : \mathbb{R} \to \mathbb{R}_+$ by $\ell^t(x,s) = \displaystyle\int_0^s \beta^t(x,\tau)\,d\tau$ and $\partial j^t = \gamma^t$. Then, $\ell^t(x,.)$ is a continuously differentiable convex function, and $j^t(s) \geqslant 0$ for any $s \in \mathbb{R}$. We then define $f^t, \varphi^t : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$f^t(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + \int_\Gamma j^t(u(x))\,d\Gamma & \text{if } u \in H^1(\Omega) \text{ and } j^t \circ u \in L^1(\Gamma) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\varphi^t(u) = \begin{cases} \int_\Omega \ell^t(x,u(x))\,dx & \text{if } \ell^t \circ u \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

The function φ^t is proper lower semi-continuous and convex on $L^2(\Omega)$. The function f^t is proper and convex on $L^2(\Omega)$. Introducing a slight modification, the functional f^t becomes lower semi-continuous on some $\{u \in L^2(\Omega) \mid \|u_0 - u\|_{L^2} \leqslant R\}$, see [8]. Furthermore, for a given function $\alpha \in L^2(\Omega)$,

$$\alpha \in \partial\varphi^t(u) \iff \alpha(x) = \beta^t(x,u(x))\,, \quad \text{a.e. } x \in \Omega$$

and $\partial f^t(u) = -\Delta u$ with $\mathrm{Dom}\partial f^t = \{u \in H^1(\Omega) \mid -\frac{\partial u}{\partial \eta} \in \gamma^t(u) \text{ a.e. on } \Gamma\}$. So, the above problem can be written in the form of (1).
We shall consider the following assumptions :

$(\mathcal{H}_j)$ There is an absolutely continuous real-valued function k on $[0,T]$ such that $k' \in L^1(0,T)$ and for each $s,t \in [0,T]$ with $s \leqslant t$ we have

$$j^t \leqslant j^s + |k(t) - k(s)|\,(1 + |j^s|)\,,$$

$(\mathcal{H}_\ell)$ There is an absolutely continuous real-valued function b on $[0,T]$ such that $b' \in L^1(0,T)$ and for each $s,t \in [0,T]$ and a.e. $x \in \Omega$ we have

$$\ell^t(x,.) \leqslant \ell^s(x,.) + |b(t) - b(s)|\,(1 + |\ell^s(x,.)|)\,,$$

We obtain the local existence result:

Theorem 4. *Let $u_0 \in H^1(\Omega)$ with $j^0 \circ u_0 \in L^1(\Gamma)$ and $\ell^0 \circ u_0 \in L^1(\Omega)$. Assume the existence of $K_2 \geqslant 0$ and $K_3 \in L^2(\Omega, \mathbb{R}_+)$ such that for almost every $x \in \Omega$ and any $t \in [0,T]$, $s \in \mathbb{R}$:*

$$|\beta^t(x,s)| \leqslant K_2|s| + K_3(x).$$

If $(\mathcal{H}_j)$ and $(\mathcal{H}_\ell)$ hold, there exists $T' \in]0,T]$ such that the above problem of parabolic form has a solution $u : [0,T'] \to X$ satisfying $u(0) = u_0$.

Proof. We want to apply Theorem 3 with X the Hilbert space $L^2(\Omega)$ endowed with the usual scalar product. The families $(f^t)_{t\in[0,T]}$ and $(\varphi^t)_{t\in[0,T]}$ are above defined and satisfy $(\mathcal{H}_f)$ and $(\mathcal{H}_\varphi)$ respectively.
Let $t \in [0,T]$. By assumption on β^t, we have for any $s \in [0,T]$ and almost every $x \in \Omega$

$$|\ell^t(x,s)| \leqslant \frac{1}{2}K_2 s^2 + K_3|s|.$$

If $u \in L^2(\Omega)$, Cauchy-Schwarz inequality ensures $\ell^t \circ u \in L^1(\Omega)$ and

$$|\varphi^t(u)| \leqslant \frac{1}{2} K_2 \|u\|_{L^2}^2 + \|K_3\|_{L^2} \|u\|_{L^2}.$$

The condition *1.* of Theorem 3 is satisfied with $c_1 = 0$.
In the same way, the minimal section of $\partial\varphi^t(u)$ at u being $\beta^t(., u(.))$ almost everywhere on Ω we have

$$\|\partial^0 \varphi^t(u)\|_{L^2} \leqslant K_2 \|u\|_{L^2}^2 + \|K_3\|_{L^2}.$$

The condition *2.* of Theorem 3 is satisfied with $c_2 = 0$.
The condition *3.* is satisfied thanks to Rellich theorem. Theorem 3 ensures existence of solutions.

□

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