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Stabilization via delay feedback for highly nonlinear stochastic time-varying delay systems with Markovian switching and Poisson jump

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Abstract. Little work seems to be known about stabilization results of highly nonlinear stochastic time-varying delay systems (STVDSs) with Markovian switching and Poisson jump. This paper is concerned with the stabilization problem for a class of STVDSs with Markovian switching and Poisson jump. The coefficients of such systems do not satisfy the conventional linear growth conditions, but are subject to high nonlinearity. The aim of this paper is to design a delay feedback controller to make an unstable highly nonlinear STVDSs with Markovian switching and Poisson jump H_∞-stable and asymptotically stable. Besides, an illustrative example is provided to support the theoretical results.

Keywords: stochastic systems, time-varying delay, delay feedback control, Markovian switching, Poisson jump.

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1 Introduction

Many dynamical systems are inevitably influenced by internal and external random disturbance. Such perturbation can drastically alter the deterministic dynamics and even produce new interesting dynamical behavior. Such systems are often described by stochastic differential equations (see monograph [22]) and the stability analysis of stochastic differential equations has received a great deal of attention, see [1, 12, 15, 16, 23, 32, 36, 38] and the references

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therein. In addition, the evolution process of a stochastic system is not only related to the present state, but also to the past states. In this case, stochastic delay systems (SDSs) are introduced, which have been widely applicable to genetic regulatory networks, complex dynamical networks, biological systems, control and so on ([6, 10, 13, 28, 35]). Accordingly, many results on the stability of SDSs have been obtained, see, e.g. [7–9, 14, 29, 31].

It is well known that a Brownian motion is a continuous stochastic process, however, some systems may suffer from the jump type abrupt perturbations and the phenomenon of discontinuous random pulse excitation. In such cases, incorporating jumps into SDSs seems to be necessary, and it is therefore valuable to discuss the SDSs with Poisson jump, see, e.g., [2, 11, 17, 26]. In the case of the SDSs with Poisson jump experiencing abrupt changes in their structure and parameters due to sudden changes of system factors, SDSs with Markovian switching and Poisson jump (SDSwMSPJs) can be applied to model them. This kind of models are more realistic, and the stability research of them has aroused great concern (see, e.g., [19, 21, 34, 37]).

Consider an unstable STVDS with Markovian switching and Poisson jump

$$\begin{aligned} dx(t) &= f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ &\quad + h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned}$$

on $t \geq 0$, where the state $x(t) \in \mathbb{R}^n$, $r(t)$ is a Markov chain, $N(t)$ is a scalar Poisson process, $\delta(\cdot) : \mathbb{R}^+ \to [0, \delta]$ be continuous function with $\delta > 0$. For details, see the system (2.1) below. To make this given unstable system become stable, it is conventional to design a feedback control $u(x(t), r(t), t)$ in the drift term, based on the current state $x(t)$, for the controlled system to become stable. Due to the fact that there exists a time lag $\tau (\tau > 0)$ between the observation of the state is made and the time when the feedback control reaches the system, it is thus more realistic to take into account the control depends on a past state $x(t-\tau)$ (see, e.g. [18, 33]). Therefore, the control should be of the form $u(x(t-\tau), r(t), t)$. In this paper, we assume that $\tau \leq \delta$. Hence, the stabilization problem becomes to design a delay feedback control $u(x(t-\tau), r(t), t)$ for the controlled system

$$\begin{aligned} dx(t) = {} & [f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ & + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ & + h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned}$$

to be stable. In [24], Mao et al. designed a delay feedback controller to stabilize an unstable SDSs with Markovian switching for the first time, where both the drift and diffusion coefficients of the given unstable system meet the linear growth condition. Notice that in many economic and ecological systems, the coefficients of these systems are characterized by nonlinearity, e.g. [4] and [5]. Therefore, the stabilization problems of a class of highly nonlinear stochastic systems or SDSs with Markovian switching via delay feedback control have received considerable research interests.

Recently, Lu et al. [20] used the delay feedback control to make unstable highly nonlinear stochastic systems with Markovian switching asymptotically stable. Later, Li and Mao [18] made a progress and used the delay feedback control to tackle the stabilization problem for a given unstable highly nonlinear SDSs with Markovian switching. Shen et al. [33] explored the stabilization of highly nonlinear neutral SDSs with Markovian switching by delay feedback control. Zhao and Zhu [39] designed a delay feedback control function to study the stability of highly nonlinear switched stochastic systems with time delays. Mei et al. [27] further studied the exponential stabilization problem for a class of highly nonlinear infinite delay stochastic functional differential systems with Markovian switching. It should be noted that, though the coefficients of the given unstable systems in [18,20,27,33] are highly nonlinear, little work has focused on the stabilization problem of SDSs with Markovian switching and Poisson jump simultaneously, not to mention the case where the SDSs under consideration are highly nonlinear and the delay of the SDSs is time-varying. As we know, the increment of Poisson jump has a nonzero mean, which brings significant difficulties for the stabilization of STVDSs with Markovian switching and Poisson jump. Therefore, the motivation of this paper is to overcome the identified difficulties by launching a systematic investigation.

Inspired by the analysis above, this paper investigates the stabilization problems via delay feedback control for a class of highly nonlinear STVDSs with Markovian switching and Poisson jump. Different from the existing literature, a new stabilization problem is studied for a class of highly nonlinear SDSs, where both the Markovian switching and Poisson jump are taken into consideration, which advances the results of the system considered in [24] and covers the results in [18,20,27,33,39]. Moreover, the delay of the SDSs is time-varying, which also covers the results in [18, 20, 27, 33]. The main contributions of this paper are summarized: (1) Very few results seem to be known about the stabilization problem of STVDSs with Markovian switching and Poisson jump simultaneously, not to mention the case where the coefficients of such systems are highly nonlinear. This paper investigates the stabilization of highly nonlinear STVDSs with Markovian switching and Poisson jump; (2) A delay feedback controller is designed to make an unstable highly nonlinear STVDS with Markovian switching and Poisson jump H_∞-stable and asymptotically stable.

In this paper, we first present some notations and preliminaries in Section 2. Then in Section 3, we prove that the controlled highly nonlinear STVDSs with Markovian switching and Poisson jump is H_∞-stable and asymptotically stable, respectively. Finally, an example is provided to illustrate the obtained results in Section 4.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $|x|$ denotes the Euclidean norm of a vector x. $\langle x, y\rangle$ or $x^T y$ represents the inner product of $\forall x, y \in \mathbb{R}^n$. For $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ stand for $\max\{a, b\}$ and $\min\{a, b\}$, respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. its right continuous and $\mathcal{F}_0$ contains all P-null sets). For $\tau > 0$, let $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Let $C^b_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$. For $\forall t \geq 0$ and $\delta > 0$, let $\delta(\cdot) : \mathbb{R}^+ \to [0, \delta]$ be continuous function and $\dot{\delta}(t) = d\delta(t)/dt \leq \bar{\delta} < 1$. In the case when $\delta(t) \equiv$ constant, we assert $\bar{\delta} = 0$. Let $\{r(t)\}_{t\geq 0}$ be a right-continuous Markov chain on the complete probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j for $i \neq j$, and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}^+$ (see [25]).

Consider the following unstable n-dimensional stochastic time-varying delay systems with Markovian switching and Poisson jump

$$\begin{aligned} dx(t) = {} & f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ & + h(x(t), x(t-\delta(t)), r(t), t)dN(t), \qquad t \geq 0 \end{aligned} \tag{2.1}$$

with the initial value

$$x_0 = \varphi = \{x(t) : -\delta \leq t \leq 0\} \in C^b_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2.2}$$

where $f, g, h : \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+ \to \mathbb{R}^n$ are Borel measurable functions, $B(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda > 0$. $\tilde{N}(t) = N(t) - \lambda t$ is a compensated Poisson process satisfying the property of martingale. Moreover, $B(t)$, $N(t)$ and $r(t)$ are assumed to be mutually independent. For the purpose of the stability, we also assume that $f(0,0,i,t) = g(0,0,i,t) = h(0,0,i,t) = 0$ for $\forall (i,t) \in S \times \mathbb{R}^+$. We are required to design a delay feedback $u(x(t-\tau), r(t), t)$ in the drift term so that the controlled system which is described by

$$\begin{aligned} dx(t) = {} & [f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ & + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ & + h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned} \tag{2.3}$$

becomes stable.

For the existence and uniqueness of the global solution, we assume that the local Lipschitz condition and the polynomial growth condition are true. For $\forall x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$ and $(i,t) \in S \times \mathbb{R}^+$, we also impose the following assumptions:

Assumption 1: For any real number $h > 0$, there is a constant $L_h > 0$ such that

$$\begin{aligned} & |f(x,y,i,t) - f(\bar{x},\bar{y},i,t)| \vee |g(x,y,i,t) - g(\bar{x},\bar{y},i,t)| \\ & \qquad \vee |h(x,y,i,t) - h(\bar{x},\bar{y},i,t)| \leq L_h(|x-\bar{x}| + |y-\bar{y}|) \end{aligned} \tag{2.4}$$

with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq h$. Moreover, there exists a positive constant β such that

$$|u(x,i,t) - u(y,i,t)| \leq \beta |x-y|. \tag{2.5}$$

For the stability purpose, we also require that $u(0,i,t) = 0$. Then we can obtain

$$|u(x,i,t)| \leq \beta |x|, \ \forall (x,i,t) \in \mathbb{R}^n \times S \times \mathbb{R}^+. \tag{2.6}$$

Assumption 2: There exist positive constants K and $q_i (i = 1,2,3)$ satisfying

$$\begin{aligned} |f(x,y,i,t)| &\leq K(1 + |x|^{q_1} + |y|^{q_1}), \\ |g(x,y,i,t)| &\leq K(1 + |x|^{q_2} + |y|^{q_2}), \\ |h(x,y,i,t)| &\leq K(1 + |x|^{q_3} + |y|^{q_3}) \end{aligned} \tag{2.7}$$

with $q_1 \geq 1$, $q_2 \geq 1$ and $q_3 \geq 1$.

Remark 2.1. If $q_i = 1 (i = 1, 2, 3)$, the condition (2.7) is the linear growth condition. In this paper, we consider that the coefficients of the stochastic time-varying delay systems (2.1) with Markovian switching and Poisson jump are highly nonlinear, so we refer to the (2.7) as the polynomial growth condition with $\max_{1\le i\le 3}\{q_i\} > 1$.

Let $C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$ be the family of all nonnegative functions $V(x, i, t)$ on $\mathbb{R}^n \times S \times \mathbb{R}^+$, which are continuously twice differentiable in x and once in t. Define an operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned} LV(x, y, i, t) = &V_t(x, i, t) + V_x(x, i, t)f(x, y, i, t) \\ &+ \frac{1}{2}\text{trace}[g^T(x, y, i, t)V_{xx}(x, i, t)g(x, y, i, t)] \\ &+ \lambda[V(x + h(x, y, i, t), i, t) - V(x, i, t)] + \sum_{j=1}^{N} \gamma_{ij} V(x, j, t), \end{aligned}$$

where

$$V_t(x, i, t) = \frac{\partial V(x, i, t)}{\partial t}, \qquad V_x(x, i, t) = \left(\frac{\partial V(x, i, t)}{\partial x_1}, \ldots, \frac{\partial V(x, i, t)}{\partial x_n}\right),$$
$$V_{xx}(x, i, t) = \left(\frac{\partial^2 V(x, i, t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

In the following, we can cite the generalized Itô formula

$$V(x(t), r(t), t) = V(x(0), r(0), 0) + \int_0^t LV(x(s), x(s - \delta(s)), r(s), s) + G_t, \tag{2.8}$$

where

$$\begin{aligned} G_t = &\int_0^t V_x(x(s), r(s), s)g(x(s), x(s - \delta(s)), r(s), s)dB(s) \\ &+ \int_0^t [V_x(x(s) + h(x(s), x(s - \delta(s)), r(s), s), r(s), s) - V_x(x(s), r(s), s)] \times d\tilde{N}(s) \\ &+ \int_0^t \int_{\mathbb{R}} [V_x(x(s), r_0 + k(r(s), u), s) - V_x(x(s), r(s), s)]\mu(ds, du) \end{aligned} \tag{2.9}$$

with $r_0 = r(0)$. The detailed representation of the functions μ and k can be found in [25]. Moreover, $\mu(ds, du)$ is a martingale measure and $\{G_t\}_{t\ge 0}$ is a local martingale.

It is well known that under the Assumption 1 that the (2.3) with the given initial condition (2.2) admits a unique maximal local solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we impose another assumption:

Assumption 3: Let $H(\cdot) \in C(\mathbb{R}^n \times [-\delta, \infty); \mathbb{R}^+)$. There is a function $V \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$, as well as $q \ge 2(q_1 \vee q_2 \vee q_3)$, and positive numbers c_1, c_2, c_3, c_4 such that $c_3 + c_4 < c_2$, $|x|^q \le V(x, i, t) \le H(x, t), \forall (x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}^+$, and $LV(x, y, i, t) + V_x(x, i, t)u(z, i, t) \le c_1 - c_2 H(x, t) + c_3(1 - \bar{\delta})H(y, t - \delta(t)) + c_4 H(z, t - \tau), \forall (x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+, z \in \mathbb{R}^n$.

Theorem 2.2. *Let the Assumptions 1–3 hold. Under the initial value* (2.2)*, the system* (2.3) *admits a unique global solution $x(t)$ on $t \ge -\delta$ and the solution $x(t)$ satisfies*

$$\sup_{-\delta\le t<\infty} \mathbb{E}|x(t)|^q < \infty. \tag{2.10}$$

Proof. (1) Existence and uniqueness. Fix any initial value (2.2). It follows from Assumption 1 that Eq. (2.3) has an unique maximal local solution $x(t)$ on $t \in [-\delta, \sigma_e]$, where σ_e is the explosion time. If we prove that the solution $x(t)$ is global, we only need to show that $\sigma_e = \infty$. Let m_0 be a sufficiently large integer such that $\|x_0\| = \|\varphi\| = \sup_{-\tau \leq s \leq 0} x(s) < m_0$. For each integer $m > m_0$, define the stopping time $\sigma_m = \inf\{t \in [0, \sigma_e) : |x(t)| \geq m\}$. As usual we set $\inf \emptyset = \infty$, here $\emptyset$ is an empty set. Clearly, σ_m's are increasing and $\sigma_\infty = \lim_{m\to\infty} \sigma_m \leq \sigma_e$. By Itô's formula, we can get that for $\forall t > 0$,

$$\begin{aligned}
&\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&\quad = V(x(0), r(0), 0) + \mathbb{E}\int_0^{t\wedge\sigma_m} [LV(x(s), x(s-\delta(s)), r(s), s) \\
&\qquad + V_x(x(s), r(s), s)u(x(s-\tau), r(s), s)]ds. \qquad (2.11)
\end{aligned}$$

Applying Assumption 3, we can obtain that

$$\begin{aligned}
&\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&\quad \leq V(x(0), r(0), 0) + c_1 t - c_2 \int_0^{t\wedge\sigma_m} H(x(s), s)ds \\
&\qquad + c_3(1-\bar{\delta}) \int_0^{t\wedge\sigma_m} H(x(s-\delta(s)), s-\delta(s))ds \\
&\qquad + c_4 \int_0^{t\wedge\sigma_m} H(x(s-\tau), s-\tau)ds. \qquad (2.12)
\end{aligned}$$

Noting that

$$\begin{aligned}
\int_0^{t\wedge\sigma_m} H(x(s-\delta(s)), s-\delta(s))ds &\leq \frac{1}{1-\bar{\delta}} \int_{-\delta(s)}^{t\wedge\sigma_m-\delta(s)} H(x(u), u)du \qquad (2.13) \\
&\leq \frac{1}{1-\bar{\delta}} \int_{-\delta}^{0} H(x(s), s)ds + \frac{1}{1-\bar{\delta}} \int_0^{t\wedge\sigma_m} H(x(s), s)ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{t\wedge\sigma_m} H(x(s-\tau), s-\tau)ds &= \int_{-\tau}^{t\wedge\sigma_m-\tau} H(x(u), u)du \\
&\leq \int_{-\tau}^{0} H(x(s), s)ds + \int_0^{t\wedge\sigma_m} H(x(s), s)ds. \qquad (2.14)
\end{aligned}$$

Substituting (2.13) and (2.14) into (2.12) that we have

$$\begin{aligned}
\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) &\leq V(x(0), r(0), 0) + c_3 \int_{-\delta}^{0} H(x(s), s)ds \\
&\quad - (c_2 - c_3 - c_4) \int_0^{t\wedge\sigma_m} H(x(s), s)ds \\
&\quad + c_4 \int_{-\tau}^{0} H(x(s), s)ds + c_1 t. \qquad (2.15)
\end{aligned}$$

For $c_3 + c_4 < c_2$, we can further get

$$\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \leq M_1 + c_1 t, \qquad (2.16)$$

where $M_1 = V(x(0), r(0), 0) + c_3 \int_{-\delta}^{0} H(x(s), s)ds + c_4 \int_{-\tau}^{0} H(x(s), s)ds$. Therefore,

$$\mathbb{E}[V(x(\sigma_m), r(\sigma_m), \sigma_m) I_{\{\sigma_m \leq t\}}] \leq M_1 + c_1 t.$$

For $|x|^q \le V(x,i,t)$, then we can obtain

$$\mathbb{E}[|x(\sigma_m)|^q I_{\{\sigma_m \le t\}}] \le M_1 + c_1 t,$$

By the definition of σ_m, we have $m^q \mathbb{P}(\sigma_m \le t) \le M_1 + c_1 t$. When $m \to \infty$, we have $\mathbb{P}(\sigma_\infty \le t) \to 0$, that is $\sigma_\infty > t$ a.s. Letting $t \to \infty$, we obtain that $\sigma_\infty = \infty$ a.s.

(2) Prove $\sup_{-\tau \le t \le \infty} \mathbb{E}|x(t)|^q < \infty$. Set $f(u) = c_2 - c_3 e^{u\delta} - c_4 e^{u\tau} - u$ for $\forall u > 0$. Obviously, $f(u)$ is continuous in u. Since $f(0) = c_2 - c_3 - c_4 > 0$, by the local sign preserving property of a continuous function, there is a sufficiently small positive number ε such that $f(\varepsilon) = c_2 - c_3 e^{\varepsilon\delta} - c_4 e^{\varepsilon\tau} - \varepsilon > 0$. For $\forall t > 0$, applying Itô's formula to $e^{\varepsilon t} V(x(t), r(t), t)$, we gain

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)} V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&= V(x(0), r(0), 0) + \mathbb{E}\int_0^{t\wedge\sigma_m} \varepsilon e^{\varepsilon s} V(x(s), r(s), s) ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}[LV(x(s), x(s-\delta(s)), r(s), s) \\
&\quad + V_x(x(s), r(s), s) u(x(s-\tau), r(s), s)] ds. \qquad (2.17)
\end{aligned}$$

Applying Assumption 3, we obtain

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)} V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}[c_1 - c_2 H(x(s), s) + c_3(1-\bar{\delta}) H(x(s-\delta(s)), s-\delta(s)) \\
&\quad + c_4 H(x(s-\tau), s-\tau)] ds \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds + \frac{c_1}{\varepsilon}(e^{\varepsilon t} - 1) \\
&\quad - c_2\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds + c_3(1-\bar{\delta})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon\delta(s)} e^{\varepsilon(s-\delta(s))} \\
&\quad \times H(x(s-\delta(s)), s-\delta(s)) ds \\
&\quad + c_4\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon\tau} e^{\varepsilon(s-\tau)} H(x(s-\tau), s-\tau) ds \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds \\
&\quad + \frac{c_1}{\varepsilon} e^{\varepsilon t} - c_2\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds \\
&\quad + c_3 e^{\varepsilon\delta}(1-\bar{\delta})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\delta(s))} H(x(s-\delta(s)), s-\delta(s)) ds \\
&\quad + c_4 e^{\varepsilon\tau}\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\tau)} H(x(s-\tau), s-\tau) ds. \qquad (2.18)
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\delta(s))} H(x(s-\delta(s)), s-\delta(s)) ds \\
&\le \frac{1}{1-\bar{\delta}} \int_{-\delta(s)}^{t\wedge\sigma_m - \delta(s)} e^{\varepsilon u} H(x(u), u) du \\
&\le \frac{1}{1-\bar{\delta}} \int_{-\delta}^{0} e^{\varepsilon s} H(x(s), s) ds + \frac{1}{1-\bar{\delta}} \int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds \qquad (2.19)
\end{aligned}$$

and

$$\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\tau)}H(x(s-\tau),s-\tau)ds = \int_{-\tau}^{t\wedge\sigma_m-\tau} e^{\varepsilon u}H(x(u),u)du \le \int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds + \int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds. \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18) that we have

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m)\\
&\le V(x(0),r(0),0) + c_3e^{\varepsilon\delta}\int_{-\delta}^{0} e^{\varepsilon s}H(x(s),s)ds\\
&\quad + c_4e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds + \frac{c_1}{\varepsilon}e^{\varepsilon t}\\
&\quad + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}V(x(s),r(s),s)ds\\
&\quad - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds\\
&= M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t} + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}V(x(s),r(s),s)ds\\
&\quad - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds,
\end{aligned}$$

where $M_2 = V(x(0),r(0),0) + c_3e^{\varepsilon\delta}\int_{-\delta}^{0} e^{\varepsilon s}H(x(s),s)ds + c_4e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds$. For $|x|^q \le V(x,i,t) \le H(x,t)$, we further compute

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m)\\
&\le M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t} - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau} - \varepsilon)\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds.
\end{aligned}$$

For $c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau} - \varepsilon > 0$,

$$\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m) \le M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t}.$$

Letting $m \to \infty$. Then

$$\mathbb{E}V(x(t),r(t),t) \le M_2e^{-\varepsilon t} + \frac{c_1}{\varepsilon} < M_2 + \frac{c_1}{\varepsilon}.$$

Applying $|x(t)|^q \le V(x(t),r(t),t)$ again, we can get $\mathbb{E}|x(t)|^q < M_2 + \frac{c_1}{\varepsilon}$, $\forall t > 0$. Together with $t \in [-\delta,0]$, $\sup_{t\in[-\delta,0]}\mathbb{E}|x(t)|^q \le \|\varphi\|^q$, therefore, $\sup_{t\in[-\delta,\infty)}\mathbb{E}|x(t)|^q < \infty$. The proof is complete. □

3 Main results

In this section, we will investigate the H_∞-stabilization and asymptotic stabilization.

To proceed, a Lyapunov functional $\bar{V}(x_t,r_t,t)$ need to be constructed on the segment $x_t := \{x(t+s) : -2\delta \le s \le 0\}$ and $r_t = \{r(t+s) : -2\delta \le s \le 0\}$ for $t \ge 0$. For x_t and r_t to be well defined for $0 \le t < 2\delta$, we set $x(s) = \varphi(-\delta)$ for $s \in [-2\delta,-\delta)$ and $r(s) = r_0$ for $s \in [-2\delta,0)$. Let

$$\bar{V}(x_t,r_t,t) = \bar{U}(x(t),r(t),t) + \theta\int_{-\tau}^{0}\int_{t+s}^{t} Q(v)dvds, \qquad t \ge 0, \tag{3.1}$$

where θ is a positive number to be determined later, $\bar{U} \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$ such that

$$\lim_{|x|\to\infty} \left[\inf_{(i,t)\in S\times\mathbb{R}^+} \bar{U}(x,i,t) \right] = \infty \tag{3.2}$$

and

$$\begin{aligned} Q(t) = \tau &|f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)|^2 \\ &+ |g(x(t), x(t-\delta(t)), r(t), t)|^2 \\ &+ 2\lambda(1+\lambda\tau)|h(x(t), x(t-\delta(t)), r(t), t)|^2. \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned} &f(x,y,i,s) = f(x,y,i,0), \ u(z,i,s) = u(z,i,0), \\ &g(x,y,i,s) = g(x,y,i,0), h(x,y,i,s) = h(x,y,i,0) \end{aligned}$$

for $(x,y,i,s) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times [-2\delta, 0)$. Applying Itô's formula to $\bar{U}(x(t), r(t), t)$, we obtain

$$\begin{aligned} &d\bar{U}(x(t), r(t), t) \\ &\quad = \bar{U}_t(x(t), r(t), t)dt + \bar{U}_x(x(t), r(t), t) \\ &\qquad \times [f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &\qquad + \frac{1}{2}\text{trace}[g^T(x(t), x(t-\delta(t)), r(t), t) \\ &\qquad \times \bar{U}_{xx}(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)]dt \\ &\qquad + \lambda[\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)]dt \\ &\qquad + \sum_{j=1}^{N} \gamma_{r(t)j} \bar{U}(x(t), j, t)dt + \bar{U}_x(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ &\qquad + [\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)]d\tilde{N}(t) \\ &\qquad + \int_{\mathbb{R}} [\bar{U}(x(t), r_0 + h(r(t), u), t) - \bar{U}(x(t), r(t), t)]\mu(dt, du) \\ &\quad = [\bar{U}_x(x(t), r(t), t)(u(x(t-\tau), r(t), t) - u(x(t), r(t), t)) \\ &\qquad + L\bar{U}(x(t), x(t-\delta(t)), r(t), t)]dt + dM(t), \ t \geq 0, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &L\bar{U}(x(t), x(t-\delta(t)), r(t), t) \\ &\quad = \bar{U}_t(x(t), r(t), t) + \bar{U}_x(x(t), r(t), t)[f(x(t), x(t-\delta(t)), r(t), t) \\ &\qquad + u(x(t), r(t), t)] + \frac{1}{2}\text{trace}[g^T(x(t), x(t-\delta(t)), r(t), t) \\ &\qquad \times \bar{U}_{xx}(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)] \\ &\qquad + \lambda[\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)] \\ &\qquad + \sum_{j=1}^{N} \gamma_{r(t)j} \bar{U}(x(t), j, t) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}M(t) = &\int_0^t \bar{U}_x(x(s),r(s),s)g(x(s),x(s-\delta(s)),r(s),s)dB(s)\\ &+\int_0^t\int_{\mathbb{R}}[\bar{U}(x(t),r_0+h(r(t),u),t)-\bar{U}(x(t),r(t),t)]\mu(ds,du)\\ &+\int_0^t[\bar{U}(x(s)+h(x(s),x(s-\delta(s)),r(s),s),r(s),s)-\bar{U}(x(s),r(s),s)]d\tilde{N}(s).\end{aligned}$$

Here $M(t)$ is a local martingale with $M(0)=0$.

To investigate the H_∞-stability and asymptotic stability of system (2.3), the following assumption is also given.

Assumption 4: For all $(x,y,i,t)\in\mathbb{R}^n\times\mathbb{R}^n\times S\times\mathbb{R}^+$, assume that there exist functions $U(x,i,t)\in C^{2,1}(\mathbb{R}^n\times S\times\mathbb{R}^+;\mathbb{R}^+)$, $W(x)\in C(\mathbb{R}^n;\mathbb{R}^+)$ and positive constants α and $\rho_i(i=1,2,\ldots,6)$ such that

$$\begin{aligned}&L\bar{U}(x,y,i,t)+\rho_1|\bar{U}_x(x,i,t)|^2+\rho_2|f(x,y,i,t)|^2\\ &\quad+\rho_3|g(x,y,i,t)|^2+\rho_4|h(x,y,i,t)|^2\\ &\le -\rho_5|x|^2+\rho_6(1-\bar{\delta})|y|^2-W(x)+\alpha(1-\bar{\delta})W(y),\end{aligned}\tag{3.6}$$

where $\alpha<1$ and $\rho_6<\rho_5$.

The following theorem shows that the controlled system (2.3) is stable in the sense of H_∞.

Theorem 3.1. *Suppose that Assumptions 1–2 and Assumption 4 hold, if positive number τ is small enough for*

$$\tau\le\frac{1}{\beta}\sqrt{\frac{2\rho_1\rho_2}{3}}\wedge\frac{4\rho_1\rho_3}{3\beta^2}\wedge\frac{1}{\beta^2}\sqrt{\frac{2\rho_1(\rho_5-\rho_6)}{3}}\tag{3.7}$$

and $\frac{3\lambda(1+\lambda\tau)\tau\beta^2}{2\rho_1}\le\rho_4$. Then for any given initial data (2.2), the solution of the controlled system (2.3) has the property

$$\int_0^\infty\mathbb{E}[|x(t)|^2+W(x(t))]dt<\infty.\tag{3.8}$$

Moreover, there exist positive constants c and $\tilde{p}>2$ such that $c|x|^{\tilde{p}}\le W(x)(\forall(x,t)\in\mathbb{R}^n\times\mathbb{R}^+)$, then the controlled system (2.3) is H_∞-stable, namely

$$\int_0^\infty\mathbb{E}|x(t)|^p dt<\infty,\qquad p\in[2,\tilde{p}]\tag{3.9}$$

for any given initial value (2.2).

Proof. Given any initial value (2.2). Applying Itô's formula to $\bar{V}(x_{t\wedge\sigma_m},r_{t\wedge\sigma_m},t\wedge\sigma_m)$ defined by (3.1) yields

$$\mathbb{E}\bar{V}(x_{t\wedge\sigma_m},r_{t\wedge\sigma_m},t\wedge\sigma_m)=\bar{V}(x_0,r_0,0)+\mathbb{E}\int_0^{t\wedge\sigma_m}L\bar{V}(x_s,r_s,s)ds,\qquad t\ge 0,\tag{3.10}$$

where σ_m is defined as the same as in Theorem 2.2, and

$$\begin{aligned}L\bar{V}(x_t,r_t,t)=&L\bar{U}(x(t),x(t-\delta(t)),r(t),t)\\ &+\bar{U}_x(x(t),r(t),t)[u(x(t-\tau),r(t),t)-u(x(t),r(t),t)]\\ &+\theta\tau Q(t)-\theta\int_{t-\tau}^t Q(r)dr.\end{aligned}\tag{3.11}$$

Using (2.5) that we can gain

$$\begin{aligned}
&\bar{U}_x(x(t),r(t),t)[u(x(t-\tau),r(t),t)-u(x(t),r(t),t)] \\
&\quad\leq \rho_1|U_x(x(t),r(t),t)|^2+\frac{1}{4\rho_1}|u(x(t-\tau),r(t),t)-u(x(t),r(t),t)|^2 \\
&\quad\leq \rho_1|U_x(x(t),r(t),t)|^2+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2.
\end{aligned} \tag{3.12}$$

Then we can obtain

$$\begin{aligned}
L\bar{V}(x_t,r_t,t) \leq\ & L\bar{U}(x(t),x(t-\delta(t)),r(t),t)+\rho_1|\bar{U}_x(x(t),r(t),t)|^2 \\
&+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2+2\theta\tau^2|f(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\theta\tau^2|u(x(t-\tau),r(t),t)|^2+\theta\tau|g(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\lambda(1+\lambda\tau)\theta\tau|h(x(t),x(t-\delta(t)),r(t),t)|^2-\theta\int_{t-\tau}^{t}Q(r)dr.
\end{aligned} \tag{3.13}$$

According to (3.7), we can further gain

$$\begin{aligned}
L\bar{V}(x_t,r_t,t) \leq\ & L\bar{U}(x(t),x(t-\delta(t)),r(t),t)+\rho_1|\bar{U}_x(x(t),r(t),t)|^2 \\
&+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2+\rho_2|f(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+\rho_3|g(x(t),x(t-\delta(t)),r(t),t)|^2+\rho_4|h(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\theta\tau^2\beta^2|x(t-\tau)|^2-\theta\int_{t-\tau}^{t}Q(r)dr \\
\leq\ & -\rho_5|x(t)|^2+\rho_6(1-\bar{\delta})|x(t-\delta(t))|^2-W(x(t)) \\
&+\alpha(1-\bar{\delta})W(x(t-\delta(t)))+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2 \\
&+2\theta\tau^2\beta^2|x(t-\tau)|^2-\theta\int_{t-\tau}^{t}Q(r)dr.
\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.10), we can obtain

$$\mathbb{E}\bar{V}(x_{t\wedge\sigma_m},r_{t\wedge\sigma_m},t\wedge\sigma_m) \leq \bar{V}(x_0,r_0,0)+v_1+v_2+v_3-v_4, \tag{3.15}$$

where

$$\begin{aligned}
v_1 &= \mathbb{E}\int_0^{t\wedge\sigma_m}[-\rho_5|x(r)|^2+\rho_6(1-\bar{\delta})|x(r-\delta(r))|^2+2\theta\tau^2\beta^2|x(r-\tau)|^2]dr, \\
v_2 &= \mathbb{E}\int_0^{t\wedge\sigma_m}[-W(x(r))+\alpha(1-\bar{\delta})W(x(r-\delta(r)))]dr, \\
v_3 &= \frac{\beta^2}{4\rho_1}\mathbb{E}\int_0^{t\wedge\sigma_m}|x(r)-x(r-\tau)|^2dr, \\
v_4 &= \theta\mathbb{E}\int_0^{t\wedge\sigma_m}\int_{r-\tau}^{r}Q(v)dvdr.
\end{aligned}$$

Noting that

$$\begin{aligned}\int_0^{t\wedge\sigma_m}|x(s-\delta(s))|^2ds &\le \frac{1}{1-\bar\delta}\int_{-\delta(0)}^{t\wedge\sigma_m-\delta(t\wedge\sigma_m)}|x(r)|^2dr \le \frac{1}{1-\bar\delta}\int_{-\delta}^{t\wedge\sigma_m}|x(r)|^2dr,\\ \int_0^{t\wedge\sigma_m}|x(s-\tau)|^2ds &\le \int_{-\tau}^{t\wedge\sigma_m-\tau}|x(r)|^2dr \le \int_{-\delta}^{t\wedge\sigma_m}|x(r)|^2dr,\\ \int_0^{t\wedge\sigma_m}W(x(s-\delta(s)))ds &\le \frac{1}{1-\bar\delta}\int_{-\delta(0)}^{t\wedge\sigma_m-\delta(t\wedge\sigma_m)}W(x(r))dr\\ &\le \frac{1}{1-\bar\delta}\int_{-\delta}^{t\wedge\sigma_m}W(x(r))dr,\end{aligned}$$

$$\begin{aligned}v_1 &\le (\rho_6+2\theta\tau^2\beta^2)\int_{-\delta}^0|x(r)|^2dr-(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\sigma_m}|x(r))|^2dr,\\ v_2 &\le \alpha\int_{-\delta}^0 W(x(r))dr-(1-\alpha)\int_0^{t\wedge\sigma_m}W(x(r))dr.\end{aligned}\tag{3.16}$$

Substituting (3.16) into (3.15) that we obtain

$$\begin{aligned}\mathbb{E}\bar V(x_{t\wedge\sigma_m},r_{t\wedge\sigma_m},t\wedge\sigma_m) \le C-(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\sigma_m}|x(r)|^2dr\\ -(1-\alpha)\mathbb{E}\int_0^{t\wedge\sigma_m}W(x(r))dr+v_3-v_4,\end{aligned}\tag{3.17}$$

where $C=\bar V(x_0,r_0,0)+(\rho_6+2\theta\tau^2\beta^2)\int_{-\tau}^0|x(r)|^2dr+\alpha\int_{-\tau}^0 W(x(r))dr$. Letting $m\to\infty$ and applying the classical Fatou lemma, we gain

$$\begin{aligned}\mathbb{E}V(x_t,r_t,t) \le C-(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^t|x(r)|^2dr\\ -(1-\alpha)\mathbb{E}\int_0^t W(x(r))dr+\bar v_3-\bar v_4,\end{aligned}\tag{3.18}$$

where

$$\bar v_3=\frac{\beta^2}{4\rho_1}\mathbb{E}\int_0^t|x(r)-x(r-\tau)|^2dr,\qquad \bar v_4=\theta\mathbb{E}\int_0^t\int_{r-\tau}^r Q(v)dvdr.$$

For $t\in[0,\tau]$, we have

$$\begin{aligned}\bar v_3 &\le \frac{\beta^2}{4\rho_1}\int_0^t\mathbb{E}|x(r)-x(r-\tau)|^2dr\\ &\le \frac{\beta^2}{2\rho_1}\int_0^\tau(\mathbb{E}|x(r)|^2+\mathbb{E}|x(r-\tau)|^2)dr\\ &\le \frac{\beta^2}{\rho_1}\tau\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right).\end{aligned}$$

For $t>\tau$, we gain

$$\bar v_3 \le \frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right)+\frac{\beta^2}{4\rho_1}\mathbb{E}\int_\tau^t|x(r)-x(r-\tau)|^2dr.$$

It follows from (2.3) and Hölder inequality that we can obtain

$$\mathbb{E}\int_{\tau}^{t}|x(r)-x(r-\tau)|^2dr \leq 3\mathbb{E}\int_{\tau}^{t}\int_{r-\tau}^{r}Q(v)dvdr \leq 3\mathbb{E}\int_{0}^{t}\int_{r-\tau}^{r}Q(v)dvdr.$$

Then we can get

$$\bar{v}_3 \leq \left(\frac{\beta^2\tau}{\rho_1}\sup_{-\tau\leq r\leq\tau}\mathbb{E}|x(r)|^2\right)+\frac{3\beta^2}{4\rho_1}\mathbb{E}\int_{0}^{t}\int_{r-\tau}^{r}Q(v)dvdr. \tag{3.19}$$

Substituting (3.19) into (3.18) we obtain

$$\begin{aligned}0 \leq \mathbb{E}\bar{V}(x_t,r_t,t) \leq C &+ \frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\leq r\leq\tau}\mathbb{E}|x(r)|^2\right)\\
&-(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_{0}^{t}|x(r)|^2dr\\
&-(1-\alpha)\mathbb{E}\int_{0}^{t}W(x(r))dr\\
&+\frac{3\beta^2}{4\rho_1}\mathbb{E}\int_{0}^{t}\int_{r-\tau}^{r}Q(v)dvdr-\theta\mathbb{E}\int_{0}^{t}\int_{r-\tau}^{r}Q(v)dvdr. \end{aligned}\tag{3.20}$$

Let $\theta=\frac{3\beta^2}{4\rho_1}$. For $\tau<\frac{1}{\beta^2}\sqrt{\frac{2(\rho_5-\rho_6)\rho_1}{3}}$, then

$$\begin{aligned}&\min\{\rho_5-\rho_6-2\theta\tau^2\beta^2,1-\alpha\}\int_{0}^{t}\mathbb{E}[|x(r)|^2+W(x(r))]dr\\
&\leq(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_{0}^{t}|x(r)|^2dr+(1-\alpha)\mathbb{E}\int_{0}^{t}W(x(r))dr\\
&\leq C+\frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\leq r\leq\tau}\mathbb{E}|x(r)|^2\right)\end{aligned}\tag{3.21}$$

which implies the desired conclusion (3.8). Moreover, for $c|x|^{\bar{p}}\leq W(x)$, applying the inequality $|v|^b\leq|v|^a+|v|^c(\forall 0<a\leq b\leq c)$, we can get for any $p\in[2,\bar{p}]$,

$$\begin{aligned}\min\{1,c\}\int_{0}^{\infty}\mathbb{E}|x(t)|^pdt &\leq \min\{1,c\}\int_{0}^{\infty}\mathbb{E}[|x(t)|^2+|x(t)|^{\bar{p}}]dt\\
&\leq\int_{0}^{\infty}\mathbb{E}[|x(t)|^2+c|x(t)|^{\bar{p}}]dt\\
&\leq\int_{0}^{\infty}\mathbb{E}[|x(t)|^2+W(x(t))]dt<\infty\end{aligned}$$

which implies (3.9) is true. The proof is complete. □

The next theorem illustrates that the controlled system (2.3) is asymptotically stable.

Theorem 3.2. *Let all the conditions of Theorem 3.1 hold. If $p\geq 2$ and $q\geq(p+q_1-1)\vee(p+2q_2-2)\vee pq_3$, then for the any given initial date (2.2), the solution of the system (2.3) satisfies*

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^p=0. \tag{3.22}$$

Namely, the system (2.3) is asymptotically stable.

Proof. Applying Itô's formula to $|x(t)|^p$ that we obtain for any $0 \le t_1 < t_2 < \infty$,

$$\begin{aligned}
&\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \\
&\quad \le \mathbb{E}\int_{t_1}^{t_2} p|x(t)|^{p-2}x^T(t)[f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\
&\qquad + \frac{p(p-1)}{2}\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}|g(x(t), x(t-\delta(t)), r(t), t)|^2 dt \\
&\qquad + \lambda\mathbb{E}\int_{t_1}^{t_2} (|x(t) + h(x(t), x(t-\delta(t)), r(t), t)|^p - |x(t)|^p)\, dt.
\end{aligned} \tag{3.23}$$

It follows from (2.7) that we gain

$$\begin{aligned}
\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \le\ & p\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}[K(1 + |x(t)|^{q_1} + |x(t-\delta(t))|^{q_1}) + \beta|x(t-\tau)|]dt \\
&+ \frac{3p(p-1)K^2}{2}\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}(1 + |x(t)|^{2q_2} + |x(t-\delta(t))|^{2q_2})dt \\
&+ \lambda 6^{p-1}K^p\mathbb{E}\int_{t_1}^{t_2} (1 + |x(t)|^{pq_3} + |x(t-\delta(t))|^{pq_3})dt \\
&+ \lambda(2^{p-1} - 1)\mathbb{E}\int_{t_1}^{t_2} |x(t)|^p dt.
\end{aligned} \tag{3.24}$$

By Young's inequality, we can get

$$\begin{aligned}
&\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}|x(t-\delta(t))|^{q_1} dt \\
&\quad \le \frac{p-1}{p+q_1-1}\int_{t_1}^{t_2} \mathbb{E}|x(t)|^{p+q_1-1}dt + \frac{q_1}{p+q_1-1}\int_{t_1}^{t_2} \mathbb{E}|x(t-\delta(t))|^{p+q_1-1}dt.
\end{aligned} \tag{3.25}$$

Using Theorem 2.2 and $p + q_1 - 1 \le q$ that we get

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}|x(t-\delta(t))|^{q_1} dt \le \sup_{-\delta \le t < \infty} \mathbb{E}|x(t)|^{p+q_1-1}(t_2 - t_1). \tag{3.26}$$

Similarly, according to $p + 2q_2 - 2 \le q$ and $pq_3 \le q$ that we can get

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}|x(t-\delta(t))|^{2q_2} dt \le \sup_{-\delta \le t < \infty} \mathbb{E}|x(t)|^{p+2q_2-2}(t_2 - t_1) \tag{3.27}$$

and

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{pq_3} dt \le \sup_{0 \le t < \infty} \mathbb{E}|x(t)|^{pq_3}(t_2 - t_1). \tag{3.28}$$

It follows from (3.26)–(3.28) that we can obtain

$$\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \le \widetilde{C}(t_2 - t_1) \tag{3.29}$$

where $\widetilde{C}$ is a constant independent t_1, t_2. That is, $\mathbb{E}|x(t)|^p$ is uniformly continuous in t on $\mathbb{R}^+$. Together with (3.9), we can assert $\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0$. The proof is thus complete. □

Remark 3.3. Different from the existing literature, this paper studies the H_∞-stability and asymptotic stability for a class of highly nonlinear SDSs, where both the Markovian switching and Poisson jump are taken into consideration, which advances the results of the system with the coefficients satisfying the linear growth condition in [24] and covers the results in [18,20,27,33,39].

4 An example

In this section, we give an example to illustrate the the obtained results. Let $B(t)$ be a scalar Brown motion and $N(t)$ be a Poisson process with intensity $\lambda = 1$. For the sake of simplicity, here we consider $\delta(t) \equiv \delta$. We thus consider the following scalar system

$$\begin{aligned} dx(t) &= f(x(t), x(t-\delta), r(t), t)dt + g(x(t), x(t-\delta), r(t), t)dB(t) \\ &\quad + h(x(t), x(t-\delta), r(t), t)dN(t), \qquad t \geq 0 \end{aligned} \tag{4.1}$$

with the initial data $x(t) = 3 + 2\cos(t)$, $t \in [-1, 0]$ and $r(0) = 1$, where $f(x, y, 1, t) = x + \frac{1}{2}y^3 - 2x^3 - 2x^7$, $f(x, y, 2, t) = x + y^3 - 2x^3 - x^7$, $g(x, y, 1, t) = g(x, y, 2, t) = \frac{1}{4}y^2$, $h(x, y, 1, t) = h(x, y, 2, t) = \frac{1}{2}x$, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} cc-2 & 2 \\ 1 & -1 \end{bmatrix}.$$

It is easy to see that $q_1 = 7$, $q_2 = 2$ and $q_3 = 1$. The sample paths of the Markov chain and the solution of the system (4.1) are shown in Fig. 4.1. From this figure, we can see that the system (4.1) is unstable. We are in position to design a control function $u : \mathbb{R} \times S \times \mathbb{R}^+ \to \mathbb{R}$ defined by $u(x, 1, t) = -4x, u(x, 2, t) = -5x$ to make the system (4.1) become stable. It is also easy to see that $\beta = 5$. The controlled system of the form

$$\begin{aligned} dx(t) &= [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &\quad + g(x(t), x(t-\delta), r(t), t)dB(t) \\ &\quad + h(x(t), x(t-\delta), r(t), t)dN(t), \qquad t \geq 0. \end{aligned} \tag{4.2}$$

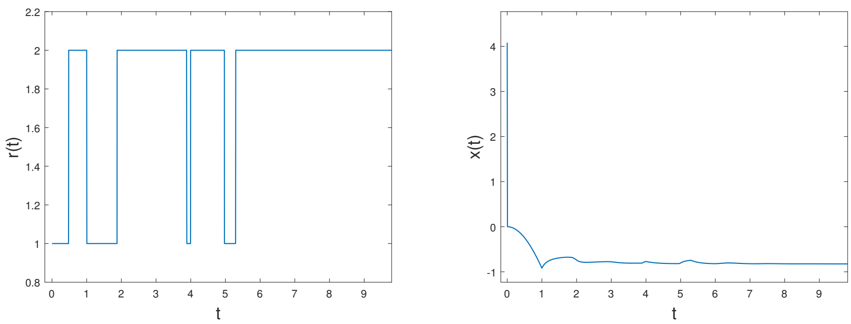

Figure 4.1: The sample paths of the Markov chain (left) and the solution of the system (4.1) (right) with $\delta = 1$.

Let $V(x, i, t) = x^{14} (i = 1, 2)$. By Young inequality, we compute

$$\begin{aligned} &LV(x, y, i, t) + V_x(x, i, t)u(z, i, t) \\ &\quad \leq \begin{cases} -28x^{20} - 18x^{16} + 356.93x^{14} + 2.74y^{16} + 4z^{14}, & i = 1, \\ -14x^{20} - 12x^{16} + 369.93x^{14} + 4.05y^{16} + 5z^{14}, & i = 2, \end{cases} \\ &\quad \leq c_1 - 12(x^{16} + x^{14}) + 4.05(y^{14} + y^{16}) + 5(z^{14} + z^{16}), \end{aligned}$$

where $c_1 = \sup_{x \in R}\{-14x^{20} + 381.93x^{14}\} < \infty$. Therefore, Assumption 3 is fulfilled with $c_2 = 12$, $c_3 = 4.05$, $c_4 = 5$, $H(x, t) = x^{14} + x^{16}$ and $q = 14$.

In the following, we define $\bar{U}(x,1,t) = x^2 + x^4 + x^8$ and $\bar{U}(x,2,t) = \frac{1}{2}(x^2 + x^4 + x^8)$. Then

$$L\bar{U}(x,y,i,t) \leq \begin{cases} -\dfrac{21}{4}x^{12} - \dfrac{195}{16}x^4 - \dfrac{7}{2}x^8 - \dfrac{403}{20}x^{10} - 16x^{14} + \dfrac{13}{16}y^4 + \dfrac{5}{4}y^6 + \dfrac{19}{10}y^{10}, & i = 1, \\ -\dfrac{19}{8}x^2 - \dfrac{215}{8}x^4 - \dfrac{47}{16}x^6 - \dfrac{7}{2}x^8 - \dfrac{267}{40}x^{10} - 4x^{14} + \dfrac{25}{32}y^4 + \dfrac{9}{8}y^6 + \dfrac{31}{20}y^{10}, & i = 2. \end{cases}$$

Moreover,

$$|\bar{U}_x(x,i,t)|^2 \leq \begin{cases} 12x^2 + 48x^6 + 192x^{14}, & i = 1, \\ 3x^2 + 12x^6 + 48x^{14}, & i = 2, \end{cases}$$

and

$$|f(x,y,i,t)|^2 \leq \begin{cases} 4x^2 + y^6 + 16x^6 + 16x^{14}, & i = 1, \\ 4x^2 + 4y^6 + 16x^6 + 4x^{14}, & i = 2, \end{cases}$$

and for $\forall i = 1,2$, $|g(x,y,i,t)|^2 = \frac{1}{16}y^4$, $|h(x,y,i,t)|^2 = \frac{1}{4}x^2$. Let $\rho_1 = \frac{1}{650}$, $\rho_2 = \frac{1}{80}$, $\rho_3 = \frac{1}{5}$ and $\rho_4 = \frac{1}{2}$. Then we can compute

$$\begin{aligned} &L\bar{U}(x,y,i,t) + \rho_1|\bar{U}_x(x,i,t)|^2 + \rho_2|f(x,y,i,t)|^2 + \rho_3|g(x,y,i,t)|^2 + \rho_4|h(x,y,i,t)|^2 \\ &\leq -2.2|x|^2 - 2.7(x^4 + x^6 + x^8 + x^{10} + x^{14}) + 1.9(y^4 + y^6 + y^8 + y^{10} + y^{14}). \end{aligned} \tag{4.3}$$

It follows from (4.3) that we can assert that Assumption 4 is satisfied with $W(x) = 2.7(x^4 + x^6 + x^8 + x^{10} + x^{14})$, $\rho_5 = \frac{11}{5}$, $\rho_6 = 0$ and $\alpha = \frac{19}{27}$. By computing, we can set $\tau = 10^{-5}$ to satisfy (3.7) and $\frac{3\lambda(1+\lambda\tau)\tau\beta^2}{2\rho_1} \leq \rho_4$. Then according to Theorem 3.1, we therefore conclude that the solution of the controlled system (4.2) satisfies the following property

$$\int_0^\infty \mathbb{E}[x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t) + x^{14}(t)]dt < \infty.$$

Thus, we can get

$$\int_0^\infty \mathbb{E}[x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t)]dt < \infty.$$

Moreover, as $|x(t)|^p \leq x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t)$ for any $p \in [2,10]$, we obtain $\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty$.

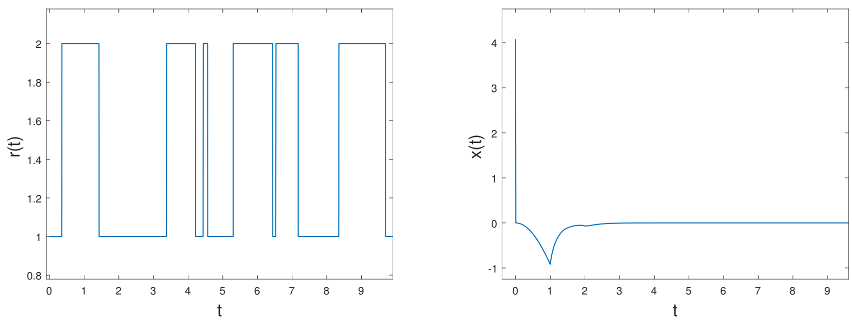

Figure 4.2: The sample paths of the Markov chain (left) and the solution of the system (4.2) (right) with $\delta = 1$ and $\tau = 10^{-5}$.

Let $p = 4$. Recalling $q_1 = 7$, $q_2 = 2$, $q_3 = 1$, then all the conditions of Theorem 3.2 are satisfied, so we can get $\lim_{t\to\infty} \mathbb{E}|x(t)|^4 = 0$.

The sample paths of the Markov chain and the solution of the controlled system (4.2) are shown in Fig. 4.2. The simulation supports the theoretical results.

5 Conclusion

Up to now, very few stabilization results seem to be known about the STVDSs with Markovian switching and Poisson jump, not to mention the case where the coefficients of such systems are highly nonlinear. This paper discussed the stabilization problem of such systems. In this paper, we designed a delay feedback controller to make an unstable highly nonlinear STVDS with Markovian switching and Poisson jump H_∞-stable and asymptotically stable, which enriches the stabilization results on such systems. Moreover, an illustrative example has been presented to verify the effectiveness of the obtained results.

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