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Strong solutions for singular Dirichlet elliptic problems

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Abstract. We prove an existence result for strong solutions $u \in W^{2,q}(\Omega)$ of singular semilinear elliptic problems of the form $-\Delta u = g(\cdot, u)$ in Ω, $u = \tau$ on $\partial\Omega$, where $1 < q < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 \leq \tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, and with $g : \Omega \times (0, \infty) \to [0, \infty)$ belonging to a class of nonnegative Carathéodory functions, which may be singular at $s = 0$ and also at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$. In addition, we give results concerning the uniqueness and regularity of the solutions. A related problem on punctured domains is also considered.

Keywords: singular elliptic problems, strong solutions, Schauder's fixed point theorem, approximation method.

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1 Introduction and statement of the main results

Our aim in this paper is to state existence and uniqueness results for strong solutions $u \in W^{2,q}(\Omega)$ of singular elliptic problems of the form

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $1 < q < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 \leq \tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, with the boundary condition understood in the sense of the trace, and where $g : \Omega \times (0, \infty) \to [0, \infty)$ is a suitable nonnegative Carathéodory function which may be singular at $s = 0$ and at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$.

Singular elliptic problems appear in the study of nonlinear phenomena such as non-Newtonian fluids, the temperature of some electrical conductors, thin films, micro electromechanicals devices, and chemical catalysts process, (see e.g., [6, 15, 19, 20, 28] and the references therein).

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Existence of classical solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (1.1) were obtained, in the pioneering works [11,41] (in both cases for a general second order linear operator instead of the Laplacian, but in [11] with homogeneous boundary condition), and in [9,15,20]. Cases where g has the form $g(x,s) = a(x)s^{-\alpha}$, $\alpha \in (0,\infty)$, and $\tau = 0$ were studied in [26] and [14], and more recently, in [16,40], and [29]. Let us mention also that in [15], problem (1.1) was studied when $\tau = 0$ and $g(x,s) = -\frac{1}{s^\gamma} + f(x)$ for some $\gamma > 0$ and $f \in L^1(\Omega)$.

Existence results for classical solutions of Lane–Emden–Fowler equations with convection and singular potential were obtained in [17], and related problems were studied in [8] and [22]. Problem (1.1) was studied, again in a classical sense, in [1,27,31,34,35,42], and [43], in some cases where $g = g(x,s)$ is singular at $s = 0$, and with some kind of singularity at $x \in \partial\Omega$. Related problems can be found also in [37], [38], and [39].

In [30] it was studied the existence, uniqueness, and regularity properties of the weak solutions of problems of the form $-\operatorname{div}(A(x)\nabla u) = \frac{f(x)}{u^\gamma} + \mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when $A(x)$ is a uniformly elliptic and bounded matrix, $\gamma > 0$, $0 \leq f \in L^1(\Omega)$ in Ω, and μ is a nonnegative bounded Radon measure.

Existence and nonexistence of solutions of problems of the form $-\operatorname{div}(A(x)\nabla u) = fu^{-\gamma}$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, was studied in [4], in the case where A is a bounded elliptic matrix and f is, either a nonnegative function in a suitable $L^p(\Omega)$ or a nonnegative and bounded Radon measure. The existence and uniqueness of solutions of problem of the form $-\operatorname{div}(A(x)\nabla u) = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, was studied in [31] in the case when μ a bounded Radon measure, $A(x)$ is a uniformly elliptic and bounded matrix with Lipschitz continuous coefficients, and $H : (0,\infty) \to (0,\infty)$ satisfies some suitable conditions which allow that $\lim_{s\to 0^+} H(s) = \infty$.

Problems of the form $-\Delta u = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, with $H : (0,\infty) \to (0,\infty)$ allowed to be singular at the origin, in the sense that $\lim_{s\to 0^+} H(s) = \infty$, and where μ is a bounded Radon measure were studied, under different assumptions, in [13] and [32], and the analogous problem $-\Delta_p u = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ (where Δ_p is the usual p-Laplacian operator $\Delta_p(u) := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$), was studied in [12].

In [18] it was proved, via a comparison principle, the uniqueness of the weak solutions of problems of the form $-\Delta_p u = F(\cdot,u)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when F is a nonnegative Carathéodory function on $\Omega \times (0,\infty)$ such that $s \to s^{1-p}F(x,s)$ is decreasing on $(0,\infty)$ for *a.e.* $x \in \Omega$. In addition, again in [18], it was proved the existence of weak solutions of problems of the form $-\Delta_p u = fu^{-\gamma} + gu^q$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when $\gamma \geq 0$, $0 \leq q \leq p-1$; f and g are nonnegative functions belonging to suitable Lebesgue spaces.

The existence of weak solutions in $W_0^{1,q}(\Omega)$ of problem (1.1) was studied in [7] in some cases where $\tau = 0$, $g(x,s) = a(x)s^{-\alpha(x)}$. In [24] it was studied the existence of weak solutions, in $H_0^1(\Omega)$, for problems of the form $-\Delta u = g(\cdot,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, including some cases where $g(x,s)$ is singular at $s = 0$, and also at $x \in \partial\Omega$.

Singular problems on punctured domains were studied in [3]. There it was proved that, if $x_0 \in \Omega$ and if $a : \Omega \to \mathbb{R}$ satisfies certain condition related to the Karamata class, then the problem $-\Delta u = au^{-\alpha}$ in $\Omega \setminus \{x_0\}$, $u > 0$ in $\Omega \setminus \{x_0\}$, $u = 0$ on $\partial\Omega$ has at least one solution such that , $\lim_{x\to x_0} |x - x_0|^{n-2} u(x) = 0$.

The interested reader will find an updated account, concerning the topic of singular elliptic problems, as well as additional references, in the research books [36], and [21].

We assume, from now on, that $n \geq 2$ and that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary. Let $q \in (1,\infty)$, which we fix from now on. We recall that (see, e.g., [25, Theorem 2.4.2.5]), for $f \in L^q(\Omega)$ and $\tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, there exists a unique strong solution $u \in W^{2,q}(\Omega)$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with the boundary condition understood in the sense of the trace, and that u satisfies $\|u\|_{W^{2,q}(\Omega)} \leq c\left(\|f\|_{L^q(\Omega)} + \|\tau\|_{W^{2-\frac{1}{q},q}(\partial\Omega)}\right)$, where c is a positive constant independent of u. We will write $(-\Delta)^{-1}$ for the solution operator $(-\Delta)^{-1} : L^q(\Omega) \to W^{2,q}(\Omega)$ of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

i.e., for the operator defined by $(-\Delta)^{-1} f := u$, where $u \in W^{2,q}(\Omega)$ is the unique strong solution u of problem (1.3).

We will write d_Ω for the function $d_\Omega : \Omega \to \mathbb{R}$, defined by $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. With these notations, our first result reads as follows:

Theorem 1.1. *Let $n \geq 2$, let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and let τ be a nonnegative function in $W^{2-\frac{1}{q},q}(\partial\Omega) \cap C(\partial\Omega)$. Let $g : \Omega \times (0,\infty) \to \mathbb{R}$ satisfying the following three conditions H1)–H3):*

- *H1) g is a Carathéodory function (that is $g(\cdot, s)$ is measurable for any $s > 0$ and $g(x, \cdot)$ is continuous on $(0,\infty)$ for any $x \in \Omega$) and such that, for any $x \in \Omega$, $g(x, \cdot)$ is nonnegative and nonincreasing on $(0,\infty)$.*
- *H2) There exists $A \subset \Omega$ such that $|A| > 0$ and $g(x,s) > 0$ for all $(x,s) \in A \times (0,\infty)$.*
- *H3) $g(\cdot, cd_\Omega) \in L^q(\Omega)$ for all $c \in (0,\infty)$.*

Then problem (1.1) has a strong solution $u \in W^{2,q}(\Omega)$ which satisfies $\tau^* + cd_\Omega \leq u \leq \tau^* + (-\Delta)^{-1}(g(\cdot, cd_\Omega))$ a.e. in Ω, where c is a positive constant and $\tau^* \in W^{2,q}(\Omega)$ is the (unique) strong solution of the problem

$$\begin{cases} -\Delta z = 0 & \text{in } \Omega, \\ z = \tau & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Remark 1.2. For τ as in the statement of Theorem 1.1, since $\tau \in C(\partial\Omega)$, problem (1.4) has a classical solution $\zeta \in C^2(\Omega) \cap C(\overline{\Omega})$ (see e.g., [23, Theorem 2.14]) which, by the classical maximum principle (as stated e.g., in [23, Theorem 3.1]), satisfies $\zeta \geq 0$ in Ω. On the other hand, since $\tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, ([2, Theorem 15.2]) gives that $\zeta \in W^{2,q}(\Omega)$ and that ζ is the strong solution of (1.4). Then $\tau^* \geq 0$ in Ω and $\tau^* \in C(\overline{\Omega})$. Moreover, τ^* is harmonic in Ω, then $\tau^* \in C^\infty(\Omega)$, and so $\tau^* \in W^{2,p}_{loc}(\Omega)$ for any $p \in [1,\infty)$.

The next result states that, if H1)–H3) hold, and if some additional assumptions on g are fulfilled, then the solution u of problem (1.1) is unique and has additional regularity properties:

Theorem 1.3. *Assume the hypothesis of Theorem 1.1 and that, in addition, the following conditions H4)–H5) hold:*

- *H4) g is continuous on $\Omega \times (0,\infty)$,* *H5)* $(-\Delta)^{-1}(g(\cdot, cd_\Omega)) \in C(\overline{\Omega})$ *for any* $c > 0$.

Then problem (1.1) *has a unique strong solution* $u \in W^{2,q}(\Omega)$, *and it belongs to* $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. *In particular,* $u \in C^1(\Omega)$.

Our third result refers to the punctured domain $U := \Omega \setminus \{x_0\}$, where $x_0 \in \Omega$, and reads as follows:

Theorem 1.4. *Let* $x_0 \in \Omega$, $U := \Omega \setminus \{x_0\}$ *and, for* $\delta > 0$, *let*

$$A_\delta := \left\{ x \in \Omega : \frac{\delta}{2} \leq |x - x_0| \leq \delta \right\}. \tag{1.5}$$

Let $h : \Omega \times (0, \infty) \to \mathbb{R}$ *and let* $w \in W^{2,q}(U)$. *Assume that* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = h(\cdot, w) & \text{in } U, \\ w = \tau & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

(with the boundary condition understood in the sense of the trace). If either $w \in C(\Omega)$ *or* $\limsup_{\delta \to 0^+} \frac{1}{\delta^2} \int_{A_\delta} |w| = 0$, *then* $w \in W^{2,q}(\Omega)$ *and* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = h(\cdot, w) & \text{in } \Omega, \\ w = \tau & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We have also the following:

Theorem 1.5. *Assume the hypothesis of Theorem* 1.3. *Let* $x_0 \in \Omega$, $U := \Omega \setminus \{x_0\}$, *and let* $w \in W^{2,q}(U)$. *If* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = g(\cdot, w) & \text{in } U, \\ w = \tau & \text{on } \partial\Omega, \\ w > 0 & \text{in } U. \end{cases}$$

Then:

i) If $\limsup_{x \to x_0} |x - x_0|^{n-2} w(x) = 0$ *then, after redefining* w *in a set with zero measure, it hold that* $w \in W^{2,q}(\Omega) \cap C(\overline{\Omega}) \cap C^1(\Omega)$ *and* w *is the unique solution of problem* (1.1)

ii) If $\|w\|_{L^\infty(U)} = \infty$, *then* $\limsup_{x \to x_0} |x - x_0|^{n-2} w(x) > 0$.

The paper is organized as follows: in Section 2 we study, for $M \geq 1$ and $\varepsilon \in (0, 1]$, the approximated problems

$$\begin{cases} -\Delta u = g_M(\cdot, \varepsilon + u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases}$$

where $g_M(x, s) := \min\{M, g(x, s)\}$. By using Schauder's fixed point theorem, we show that this problem has a unique solution $u_{M,\varepsilon} \in \cap_{1<p<\infty} W^{2,p}(\Omega)$ (see Lemmas 2.2 and 2.4). Lemma 2.6 states that $\varepsilon \to u_{M,\varepsilon}$ is nonincreasing, $M \to u_{M,\varepsilon}$ is nondecreasing, and that $\tau^* + c_0 d_\Omega \leq u_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(\cdot, c_0 d_\Omega)$ in Ω, with c_0 a positive constant independent of M

and ε, and where τ^* is the strong solution of (1.4). Lemma 2.7 shows that if $u_M := \lim_{\varepsilon \to 0^+} u_{M,\varepsilon}$, then $u_M \in W^{2,q}(\Omega)$ and u_M is a strong solution of the problem

$$\begin{cases} -\Delta u_M = g_M(\cdot, u_M) & \text{in } \Omega, \\ u_M = \tau & \text{on } \partial\Omega. \end{cases}$$

The main results are proved in Section 3. To prove Theorem 1.1 we define $\boldsymbol{u} := \lim_{M\to\infty} u_M$ and we show that $\boldsymbol{u}$ is a strong solution of problem (1.1) with the desired properties. This is achieved from thanks to Lemma 2.7 by showing that $g(\cdot, \boldsymbol{u}) := \lim_{M\to\infty} g_M(\cdot, u_M)$ with convergence in $L^q(\Omega)$. To prove Theorem 1.3 we show that, for any strong solution u of problem (1.1), there exists a positive constant c such that $\tau^* + cd_\Omega \leq u \leq \tau^* + (-\Delta)^{-1}(\cdot, cd_\Omega)$ in Ω, which will give the continuity of u at $\partial\Omega$, next we show, by a suitable bootstrap argument, that $u \in W^{2,n}_{loc}(\Omega)$, which gives that $u \in C^1(\Omega)$. Proved that $\boldsymbol{u} \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$, the uniqueness assertion of Theorem 1.3 will follow from the fact that $s \to g(x,s)$ is nonincreasing, combined with the application of an appropriate maximum principle. Finally, Theorem 1.4 is proved by showing that, if $w \in W^{2,q}(\Omega \setminus \{x_0\})$ satisfies the conditions of Theorem 1.4, then w, viewed as a distribution on Ω, belongs to $W^{2,q}(\Omega)$.

2 Preliminaries

Let $g : \Omega \times (0,\infty) \to \mathbb{R}$ be a function satisfying the conditions H1)–H3) of Theorem 1.1 and, for $M \in [1,\infty)$, $\varepsilon \in (0,1]$, let $g_M : \Omega \times (0,\infty) \to \mathbb{R}$ be defined by

$$g_M(x,s) := \min\{M, g(x,s)\}.$$

let $K_M := \|\tau^*\|_{L^q(\Omega)} + M\|(-\Delta)^{-1}(\mathbf{1})\|_{L^q(\Omega)}$, where τ^* is the strong solution of problem (1.4), and let

$$C_M := \{v \in L^q(\Omega) : 0 \leq v \leq K_M\}.$$

For $v \in C_M$, since g is a Carathéodory function, $g_M(\cdot, \varepsilon + v)$ is a measurable function. Let η be a positive and small enough number such that $\eta d_\Omega \leq \varepsilon$ in Ω. Then, since g is nonincreasing in the second variable and $v \geq 0$ in Ω, we have $0 \leq g(\cdot, \varepsilon + v) \leq g(\cdot, \varepsilon) \leq g(\cdot, \eta d_\Omega)$ in Ω. By H3), $g(\cdot, \eta d_\Omega) \in L^q(\Omega)$, then $0 \leq g_M(\cdot, \varepsilon + v) \leq g_M(\cdot, \eta d_\Omega) \in L^q(\Omega)$ and thus $g_M(\cdot, \varepsilon + v) \in L^q(\Omega)$. Then $(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v))$ is a well defined element in $W^{2,q}(\Omega)$. Let $T_{M,\varepsilon} : C_M \to W^{2,q}(\Omega)$ be the operator defined by

$$T_{M,\varepsilon}(v) := \tau^* + (-\Delta)^{-1}(g_M(\cdot, \varepsilon + v)).$$

Remark 2.1.

i) Let us recall the following form of the Aleksandrov maximum principle (which is a particular case of [23], Theorem 9.1): If U is a bounded domain in $\mathbb{R}^n$ and if $u \in W^{2,n}_{loc}(U) \cap C(\overline{U})$ satisfies $-\Delta u \geq 0$ in U (respectively $-\Delta u \leq 0$ in U) and $u \geq 0$ on ∂U (resp. $u \leq 0$ on ∂U), then $u \geq 0$ in U (resp. $u \leq 0$ in U).

ii) If $0 \leq f \in L^q(\Omega)$ then $(-\Delta)^{-1} f \geq 0$ in Ω (note that we do not assume $q \geq n$). Indeed, let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension by zero of f. Then $0 \leq \widetilde{f} \in L^q(\mathbb{R}^n)$ and so $\widetilde{f}$ can be approximated, in the $L^q(\mathbb{R}^n)$ norm, by a sequence $\{\widetilde{f}_j\}_{j\in N} \subset C^\infty(\mathbb{R}^n)$ obtained by convolving $\widetilde{f}$ with suitable mollifiers (see [33, Proposition 1.1.3]). Thus, for each j, $0 \leq \widetilde{f}_{j|\Omega} \in L^{\infty}(\Omega)$, and so the solution u_j of the problem

$$\begin{cases} -\Delta u_j = \widetilde{f}_{j|\Omega} & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $W^{2,p}(\Omega)$ for any $p \in [1,\infty)$ and, since $\{\widetilde{f}_{j|\Omega}\}_{j\in N}$ converges to f in $L^q(\mathbb{R}^n)$, it follows that $\{u_j\}_{j\in N}$ converges to u in $W^{2,q}(\Omega)$. Now, by i), $u_j \geq 0$ in Ω, and then $u \geq 0$ in Ω.

iii) From ii), it follows immediately that if f and h belong to $L^q(\Omega)$ and $f \leq h$ in Ω, then $(-\Delta)^{-1} f \leq (-\Delta)^{-1} h$ in Ω.

Lemma 2.2. *Assume the conditions H1)–H3) of Theorem 1.1, let τ be a nonnegative function in $W^{2-\frac{1}{q},q}(\partial\Omega)$, and let $\tau^* \in W^{2,q}(\Omega)$ be the strong solution of problem (1.4). Then, for $M \in [1,\infty)$ and $\varepsilon \in (0,1]$,*

i) C_M is a closed and convex subset of $L^q(\Omega)$.

ii) $T_{M,\varepsilon}(C_M) \subset C_M$.

iii) $T_{M,\varepsilon} : C_M \to C_M$ is continuous.

iv) $T_{M,\varepsilon} : C_M \to C_M$ is a compact operator.

Proof. i) is immediate. To prove ii), observe that, for $v \in C_M$, since $g(\cdot,\varepsilon+v)$ is nonnegative, Remark 2.1 iii) gives that $(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v)) \geq 0$ and so, since $\tau^* \geq 0$ in Ω, we have $T_{M,\varepsilon}(v) \geq 0$ in Ω. Also,

$$\begin{aligned} \|T_{M,\varepsilon}(v)\|_q &\leq \|\tau^*\|_q + \left\|(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v))\right\|_q \\ &\leq \|\tau^*\|_q + M\left\|(-\Delta)^{-1}(\mathbf{1})\right\|_q = K_M. \end{aligned}$$

Then $T_{M,\varepsilon}(v) \in C_M$.

To show iii), it is enough to see that if $v \in C_M$ and if $\{v_j\}_{j\in\mathbb{N}}$ is a sequence in C_M such that $\{v_j\}_{j\in\mathbb{N}}$ converges to v in $L^q(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{T_{M,\varepsilon}(v_{j_k})\}_{k\in\mathbb{N}}$ converges to $T_{M,\varepsilon}(v)$ in $L^q(\Omega)$.

Let $v \in C_M$ and let $\{v_j\}_{j\in\mathbb{N}} \subset C_M$ be such that $\{v_j\}_{j\in\mathbb{N}}$ converges to v in $L^q(\Omega)$. Then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{v_{j_k}\}_{k\in\mathbb{N}}$ converges to v *a.e.* in Ω. Then, since g_M is a Carathéodory function, $\{g_M(\cdot,\varepsilon+v_{j_k})\}_{k\in\mathbb{N}}$ converges to $g_M(\cdot,\varepsilon+v)$ *a.e.* in Ω. Thus $\lim_{k\to\infty}\left|g_M(\cdot,\varepsilon+v_{j_k}) - g_M(\cdot,\varepsilon+v)\right|^q = 0$ *a.e.* in Ω. Also, $\left|g_M(\cdot,\varepsilon+v_{j_k}) - g_M(\cdot,\varepsilon+v)\right|^q \leq (2M)^q$ and then, by Lebesgue's dominated convergence theorem, $\{g_M(\cdot,\varepsilon+v_{j_k})\}_{k\in\mathbb{N}}$ converges to $g_M(\cdot,\varepsilon+v)$ in $L^q(\Omega)$. Thus $\{(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v_{j_k}))\}_{k\in\mathbb{N}}$ converges to $(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v))$ in $W^{2,q}(\Omega)$. Then iii) holds.

To prove iv), consider a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_M$. Then $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $L^q(\Omega)$, and thus $\{(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v_{j_k}))\}_{k\in\mathbb{N}}$ is bounded in $W^{2,q}(\Omega)$. Then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{(-\Delta)^{-1}(g_M(\cdot,\varepsilon+v_{j_k}))\}_{k\in\mathbb{N}}$ converges in $L^q(\Omega)$, and so iv) holds. □

Lemma 2.3. *Let $h : \Omega \times (0, \infty) \to \mathbb{R}$ be a function such that $h(x, \cdot)$ is nonincreasing on $(0, \infty)$ for any $x \in \Omega$, and let u, v be two functions in $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. If u, v satisfy $-\Delta u = h(\cdot, u)$ in Ω, $-\Delta v = h(\cdot, v)$ in Ω, and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Proof. Let $U := \{x \in \Omega : u(x) > v(x)\}$ and let $V := \{x \in \Omega : u(x) < v(x)\}$. Then U and V are open subsets of Ω. Suppose that $U \neq \varnothing$, Then

$$-\Delta(u - v) = h(\cdot, u) - h(\cdot, v) \leq 0 \quad \text{in } U. \tag{2.1}$$

Also,

$$u - v = 0 \quad \text{on } \partial U. \tag{2.2}$$

Indeed, if $x \in \partial U \cap \partial\Omega$ then $u(x) - v(x) = 0$, and if $x \in \partial U \cap \Omega$ then $u(x) - v(x) \geq 0$ (because $u - v > 0$ in U and $u - v$ is continuous in $\overline{\Omega}$), but if $u(x) - v(x) > 0$ we would have $u - v > 0$ in a neighborhood of x, in contradiction with the fact that $x \in U$. Then $u(x) - v(x) = 0$ also in the case when $x \in \partial U \cap \Omega$. Thus (2.2) holds. Now, from (2.1), (2.2) and Remark 2.1, we obtain $u - v \leq 0$ in U. which is impossible. Thus $U = \varnothing$. Similarly, $V = \varnothing$, and so $u = v$ in Ω. □

Lemma 2.4. *Assume the hypothesis of Theorem 1.1. Then, for $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$*

i) The problem

$$\begin{cases} -\Delta u = g_M(\cdot, \varepsilon + u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique strong solution $u_{M,\varepsilon} \in W^{2,q}(\Omega) \cap C_M$.

ii) The problem

$$\begin{cases} -\Delta v = g_M(\cdot, \varepsilon + \tau^* + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

has a unique strong solution $v_{M,\varepsilon} \in \cap_{1<p<\infty} W^{2,p}(\Omega)$ and $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$.

Proof. Taking into account Lemma 2.2 and Schauder's fixed point theorem (as stated, e.g., in [23, Corollary 11.2]), $T_{M,\varepsilon}$ has a fixed point $u_{M,\varepsilon} \in C_M$, which, by the definition of $T_{M,\varepsilon}$, belongs also to $W^{2,q}(\Omega)$ and that is a strong solution of problem (2.3). Clearly a function $u \in W^{2,q}(\Omega)$ is solution of (2.3) if and only if $v := u - \tau^*$ is a solution of (2.4), and so (2.4) has, at least, a solution $v_{M,\varepsilon} \in W^{2,q}(\Omega)$. Moreover, if v is a solution of (2.4), since $g_M(\cdot, \varepsilon + \tau^* + v) \in L^\infty(\Omega)$ and $v = 0$ on $\partial\Omega$, it follows that $v \in \cap_{1 \leq p < \infty} W^{2,p}(\Omega)$. In particular $v \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Suppose now that v and w are two solutions of (2.4). Then v and w belong to $C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ and $v = w = 0$ on $\partial\Omega$. Since $s \to g(x, \varepsilon + \tau^*(x) + s)$ is nonincreasing for any $x \in \Omega$, the function $h(x, s) := g_M(x, \varepsilon + \tau^*(x) + s)$ is also nonincreasing for any $x \in \Omega$. Then, by Lemma 2.3, $v = w$ in Ω and so the solution of (2.4) is unique. Now, from the equivalence of problems (2.3) and (2.4), the solution of (2.3) is also unique. □

For $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$ we will denote by $u_{M,\varepsilon}$ and $v_{M,\varepsilon}$ the solutions of problems (2.3) and (2.4) given by Lemma 2.4.

Remark 2.5.

i) Let us recall the following form of the Hopf maximum principle (see [5], Lemma 3.2): Suppose that $\rho \geq 0$ belongs to $L^{\infty}(\Omega)$. Let v be the solution of $-\Delta v = \rho$ in Ω, $v = 0$ on $\partial\Omega$. Then
$$v(x) \geq c d_{\Omega}(x) \int_{\Omega} \rho d_{\Omega} \quad a.e. \text{in } \Omega, \tag{2.5}$$
where c is a positive constant depending only on Ω.

ii) Suppose that $\rho \geq 0$ belongs to $L^{\infty}(\Omega)$. If $h \in L^{q}(\Omega)$ and $h \geq \rho$ in Ω, then, from Remark 2.1 iii) and (2.5) it follows immediately that $(-\Delta)^{-1} h \geq c d_{\Omega}(x) \int_{\Omega} \rho d_{\Omega}$ *a.e.* in Ω, where c is the constant given in (2.5).

iii) We recall also Hardy's inequality (see e.g., [33], Theorem 1.10.15): There exists a positive constant c such that $\left\| \frac{\varphi}{d_{\Omega}} \right\|_2 \leq c \left\| \nabla\varphi \right\|_2$ for any $\varphi \in H_0^1(\Omega)$.

Lemma 2.6. *Assume the hypothesis of Theorem 1.1. Then*

i) For each $M \in [1, \infty)$ the map $\varepsilon \to u_{M,\varepsilon}$ is nonincreasing on $(0,1]$.

ii) For each $\varepsilon \in (0,1]$ the map $M \to u_{M,\varepsilon}$ is nondecreasing on $[1, \infty)$.

iii) There exists a positive constant c_0 such that, for any $\varepsilon \in (0,1]$ and $M \in [1,\infty)$, $\tau^* + c_0 d_{\Omega} \leq u_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(\cdot, c_0 d_{\Omega})$ in Ω.

Proof. To see i), suppose that $0 < \varepsilon \leq \eta \leq 1$. Let $U := \left\{x \in \Omega : v_{M,\varepsilon}(x) < v_{M,\eta}(x)\right\}$ and suppose that $U \neq \varnothing$. Since g is nonincreasing in the second variable, the same is true for g_M and so,

$$\begin{aligned} -\Delta(v_{M,\varepsilon}) &= g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}) \geq g_M(\cdot, \eta + \tau^* + v_{M,\varepsilon}) \\ &\geq g_M(\cdot, \eta + \tau^* + v_{M,\eta}) = -\Delta(v_{M,\eta}) \text{ in } U. \end{aligned}$$

Also, as in the proof of Lemma 2.3, we have $v_{M,\varepsilon} = v_{M,\eta}$ on ∂U. Then, by Remark 2.1 iii), $v_{M,\varepsilon} \geq v_{M,\eta}$ in U, which is impossible. Then $U = \varnothing$ and so $v_{M,\varepsilon} \geq v_{M,\eta}$ in Ω, which implies $u_{M,\varepsilon} \geq u_{M,\eta}$ in Ω. Thus i) holds.

To see ii), suppose $1 \leq M_1 \leq M_2$ and $\varepsilon \in (0,1]$. Let $U := \{v_{M_1,\varepsilon} > v_{M_2,\varepsilon}\}$. If $U \neq \varnothing$, then

$$\begin{aligned} -\Delta(v_{M_2,\varepsilon}) &= g_{M_2}(\cdot, \varepsilon + \tau^* + v_{M_2,\varepsilon}) \geq g_{M_1}(\cdot, \varepsilon + \tau^* + v_{M_2,\varepsilon}) \\ &\geq g_{M_1}(\cdot, \varepsilon + \tau^* + v_{M_1,\varepsilon}) = -\Delta(v_{M_1,\varepsilon}) \quad \text{in } U. \end{aligned}$$

Also, $v_{M_1,\varepsilon} = v_{M_2,\varepsilon}$ on ∂U. Then, by Remark 2.1 iii), $v_{M_1,\varepsilon} \leq v_{M_2,\varepsilon}$ in U, which is impossible. Therefore $U = \varnothing$ and so $v_{M_1,\varepsilon} \leq v_{M_2,\varepsilon}$ in Ω, which implies $u_{M_1,\varepsilon} \leq u_{M_2,\varepsilon}$ in Ω. Thus ii) holds.

To prove iii), observe that by i) and ii) we have, for $M \in [1,\infty)$ and $\varepsilon \in (0,1]$,

$$v_{M,\varepsilon} \geq v_{M,1} \geq v_{1,1} \quad \text{in } \Omega. \tag{2.6}$$

Now,

$$\begin{cases} -\Delta v_{1,1} = g_1(\cdot, 1 + \tau^* + v_{1,1}) & \text{in } \Omega, \\ v_{1,1} = 0 & \text{on } \partial\Omega \end{cases}$$

and $0 \leq g_1(\cdot, 1+\tau^* + v_{1,1}) \in L^\infty(\Omega)$. Note that $g_1(\cdot, 1+\tau^* + v_{1,1}) \not\equiv 0$ in Ω (that is: $|\{x \in \Omega : g_1(x, 1+\tau^*(x) + v_{1,1}(x)) > 0\}| > 0$) because if $g_1(\cdot, 1+\tau^* + v_{1,1}) \equiv 0$ in Ω then $g(\cdot, 1+\tau^* + v_{1,1}) \equiv 0$ in Ω, which contradicts H2). Then

$$\int_\Omega d_\Omega g_1(\cdot, 1+\tau^* + v_{1,1}) > 0,$$

and so, taking into account Remark 2.5, there exists a positive constant c', depending only on Ω, such that

$$v_{1,1} \geq c' d_\Omega \int_\Omega g_1(\cdot, 1+\tau^* + v_{1,1})\, d_\Omega \quad a.e. \text{ in } \Omega.$$

Then, from (2.6), $v_{M,\varepsilon} \geq c_0 d_\Omega$ with

$$c_0 := c' \int_\Omega g_1(\cdot, 1+\tau^* + v_{1,1})\, d_\Omega > 0. \tag{2.7}$$

and so, since $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$, we get that $u_{M,\varepsilon} \geq \tau^* + c_0 d_\Omega$ in Ω.

On the other hand, $v_{M,\varepsilon} = (-\Delta)^{-1}(g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}))$. Now, $v_{M,\varepsilon} \geq v_{1,\varepsilon} \geq v_{1,1}$ in Ω, and so $g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}) \leq g_M(\cdot, v_{1,1}) \leq g_M(\cdot, c_0 d_\Omega) \leq g(\cdot, c_0 d_\Omega)$, with c_0 given by (2.7). Then, by Remark 2.1 iii), $(-\Delta)^{-1} g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}) \leq (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω, that is, $v_{M,\varepsilon} \leq (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω. Thus $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω, which completes the proof of the lemma. □

For $M \in [1, \infty)$, let u_M and v_M be the functions, defined on Ω by

$$u_M(x) := \lim_{\varepsilon \to 0^+} u_{M,\varepsilon}(x), \qquad v_M(x) := \lim_{\varepsilon \to 0^+} v_{M,\varepsilon}(x). \tag{2.8}$$

Note that, by Lemma 2.6, $u_M(x)$ is well defined and finite for *a.e.* $x \in \Omega$ and so, since $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$, the same assertion holds also for v_M.

Lemma 2.7. *Assume the hypothesis of Theorem 1.1 and let c_0 be the constant given by Lemma 2.6 iii). Then:*

i) The map $M \to u_M$ is nondecreasing on $[1, \infty)$.

ii) $\tau^* + c_0 d_\Omega \leq u_M \leq \tau^* + (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω,for any $M \geq 1$ (in particular $u_M > 0$ in Ω)

iii) For each $M > 0$, $u_M \in W^{2,q}(\Omega)$ and u_M is a strong solution of the problem

$$\begin{cases} -\Delta u_M = g_M(\cdot, u_M) & \text{in } \Omega, \\ u_M = \tau & \text{on } \partial\Omega. \end{cases}$$

Proof. If $1 \leq M_1 \leq M_2$ and $\varepsilon \in (0, 1]$ then, by Lemma 2.6, $u_{M_1,\varepsilon} \leq u_{M_2,\varepsilon}$, and so, by taking $\lim_{\varepsilon \to 0^+}$, we get $u_{M_1} \leq u_{M_2}$. Thus i) holds. Also, taking $\lim_{\varepsilon \to 0^+}$ in the inequalities of Lemma 2.6 iii) we get ii).

To prove iii) note that, by Lemma 2.4 ii), we have, for $\varepsilon \in (0, 1]$ and $M \in [1, \infty)$,

$$u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}, \tag{2.9}$$

where $v_{M,\varepsilon} = (-\Delta)^{-1}(g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}))$. From (2.9),

$$\lim_{\varepsilon \to 0^+} (\tau^* + v_{M,\varepsilon}) = u_M \quad a.e. \text{ in } \Omega,$$

and so, since g_M is a Carathéodory function,

$$\lim_{\varepsilon\to 0^+} g_M\left(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}\right)=g_M\left(\cdot,u_M\right)\quad a.e.\text{ in }\Omega.$$

Then $\lim_{\varepsilon\to 0^+}\left|g_M\left(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}\right)-g_M\left(\cdot,u_M\right)\right|^q=0$ *a.e.* in Ω. Also,

$$\left|g_M\left(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}\right)-g_M\left(\cdot,u_M\right)\right|^q\le(2M)^q$$

for any $\varepsilon\in(0,1]$. Then, by the Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+} g_M\left(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}\right)=g_M\left(\cdot,u_M\right)$$

with convergence in $L^q(\Omega)$. Then

$$\lim_{\varepsilon\to 0^+}(-\Delta)^{-1}\left(g_M\left(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}\right)\right)=(-\Delta)^{-1}\left(g_M\left(\cdot,u_M\right)\right)$$

with convergence in $W^{2,q}(\Omega)$, and so, in particular, $(-\Delta)^{-1}\left(g_M\left(\cdot,u_M\right)\right)\in W^{2,q}(\Omega)$. Therefore $\lim_{\varepsilon\to 0^+} v_{M,\varepsilon}=(-\Delta)^{-1}\left(g_M\left(\cdot,u_M\right)\right)$ with convergence in $W^{2,q}(\Omega)$, and thus $u_M=\lim_{\varepsilon\to 0^+} u_{M,\varepsilon}=\lim_{\varepsilon\to 0^+}\left(\tau^*+v_{M,\varepsilon}\right)=\tau^*+(-\Delta)^{-1}\left(g_M\left(\cdot,u_M\right)\right)$, with convergence in $W^{2,q}(\Omega)$. Then $-\Delta u_M=g_M\left(\cdot,u_M\right)$ in Ω and $u_M=\tau$ on $\partial\Omega$. □

3 Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Define $\boldsymbol{u}:\Omega\to\mathbb{R}$ by

$$\boldsymbol{u}:=\lim_{M\to\infty} u_M. \tag{3.1}$$

Note that, by Lemma 2.7 i), the map $M\to u_M$ is nondecreasing on $(0,\infty)$, and so $\boldsymbol{u}$ is well defined. By Lemma 2.7 we have, for any $M\ge 1$,

$$\tau^*+c_0 d_\Omega\le u_M\le\tau^*+(-\Delta)^{-1}\left(g\left(\cdot,c_0 d_\Omega\right)\right)\quad\text{in }\Omega, \tag{3.2}$$

with τ^* given by by (1.4). Then

$$\tau^*+c_0 d_\Omega\le\boldsymbol{u}\le\tau^*+(-\Delta)^{-1}\left(g\left(\cdot,c_0 d_\Omega\right)\right)\quad\text{in }\Omega \tag{3.3}$$

Also, by Lemma 2.7,

$$\begin{cases}-\Delta u_M=g_M\left(\cdot,u_M\right) & \text{in }\Omega,\\ u_M=\tau & \text{on }\partial\Omega.\end{cases} \tag{3.4}$$

Note that

$$\lim_{M\to\infty} g_M\left(\cdot,u_M\right)=g\left(\cdot,\boldsymbol{u}\right)\quad a.e.\text{ in }\Omega. \tag{3.5}$$

Indeed, for $k\in\mathbb{N}$ let

$$\Omega_k:=\left\{x\in\Omega:\frac{1}{k}d_\Omega(x)<\boldsymbol{u}(x)<k\right\},$$

and let $E:=\Omega\setminus\cup_{k\in\mathbb{N}}\Omega_k$. Thus $E=\{x\in\Omega:\boldsymbol{u}(x)=0\}\cup\{x\in\Omega:\boldsymbol{u}(x)=\infty\}$ and so, from (3.3) and taking into account that $\tau^*\ge 0$ in Ω and that $(-\Delta)^{-1}\left(g\left(\cdot,c_0 d_\Omega\right)\right)<\infty$ *a.e.* in Ω, we get that $|E|=0$. Then

$$\Omega=\cup_{k\in\mathbb{N}}\Omega_k\cup E. \tag{3.6}$$

with $|E| = 0$. Now, for each $k \in \mathbb{N}$ and $x \in \Omega_k$, we have $\boldsymbol{u}(x) > \frac{1}{k} d_\Omega(x)$ and so, since $\boldsymbol{u}(x) = \lim_{M\to\infty} u_M(x)$, there exists $N_{k,x}$ such that $u_M(x) > \frac{1}{k} d_\Omega(x)$ for any $M > N_{k,x}$. Let $M_{k,x} := \max\left\{N_{k,x}, g\left(x, \frac{1}{k} d_\Omega(x)\right)\right\}$. Since $g(x, \cdot)$ is nonincreasing we have, for $M > M_{k,x}$,

$$g(x, u_M(x)) \leq g\left(x, \frac{1}{k} d_\Omega(x)\right) \leq M_{k,x} < M,$$

and so $g_M(x, u_M(x)) = g(x, u_M(x))$ whenever $M > M_{k,x}$. Thus, for any $x \in \Omega_k$,

$$\lim_{M\to\infty} g_M(x, u_M(x)) = \lim_{M\to\infty} g(x, u_M(x)) = g(x, \boldsymbol{u}(x)),$$

the last equality because g is a Carathéodory function. Then, for each k,

$$\lim_{M\to\infty} g_M(\cdot, u_M) = g(\cdot, \boldsymbol{u}) \quad a.e. \text{ in } \Omega_k,$$

and so, taking into account (3.6) and that $|E| = 0$, we get (3.5).

Let us see that $\{g_M(\cdot, u_M)\}_{M\in\mathbb{N}}$ converges to $g(\cdot, \boldsymbol{u})$ with convergence in $L^q(\Omega)$. From (3.5),

$$\lim_{M\to\infty} |g_M(\cdot, u_M) - g(\cdot, \boldsymbol{u})|^q = 0 \quad a.e. \text{ in } \Omega.$$

Also, since $\tau^* \geq 0$, from (3.3) and (3.2) we have that $\boldsymbol{u} \geq c_0 d_\Omega$ in Ω and that $u_M \geq c_0 d_\Omega$ in Ω for any $M \geq 1$. Then, recalling that g and g_M are nonincreasing in the second variable,

$$\begin{aligned} |g_M(\cdot, u_M) - g(\cdot, \boldsymbol{u})|^q &\leq (g_M(\cdot, u_M) + g(\cdot, \boldsymbol{u}))^q \\ &\leq (2g(\cdot, c_0 d_\Omega))^q \quad a.e. \text{ in } \Omega. \end{aligned}$$

By H3), $(2g(\cdot, c_0 d_\Omega))^q \in L^1(\Omega)$. By Lebesgue's dominated convergence theorem,

$$\begin{gathered} g(\cdot, \boldsymbol{u}) \in L^q(\Omega), \\ \text{and } \lim_{M\to\infty} g_M(\cdot, u_M) = g(\cdot, \boldsymbol{u}) \text{ with convergence in } L^q(\Omega). \end{gathered} \tag{3.7}$$

Let $\boldsymbol{v} = \boldsymbol{u} - \tau^*$. Since $v_M = u_M - \tau^*$, Lemma 2.7 gives

$$\begin{cases} -\Delta v_M = -\Delta u_M = g_M(\cdot, u_M) & \text{in } \Omega, \\ v_M = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e., $v_M = (-\Delta)^{-1} g_M(\cdot, u_M)$; and so, by (3.7),

$$\boldsymbol{v} = \lim_{M\to\infty} v_M = (-\Delta)^{-1} g(\cdot, \boldsymbol{u}) \quad \text{with convergence in } W^{2,q}(\Omega). \tag{3.8}$$

Then $\boldsymbol{u} - \tau^* = \boldsymbol{v} = (-\Delta)^{-1} g(\cdot, \boldsymbol{u})$, which gives that $\boldsymbol{u} \in W^{2,q}(\Omega)$ and that

$$\begin{cases} -\Delta \boldsymbol{u} = g(\cdot, \boldsymbol{u}) & \text{in } \Omega, \\ \boldsymbol{u} = \tau & \text{on } \partial\Omega. \end{cases}$$

□

Remark 3.1. It is a well known fact that, for $\eta \in \mathbb{R}$, $d_\Omega^{-\eta} \in L^1(\Omega)$ if, and only if, $\eta < 1$. Moreover, if $S \subset \Omega$ is a closed C^2 and $n-1$ dimensional surface, and if $\rho_S(x) := \operatorname{dist}(x, S)$, then $\rho_S^{-\eta} \in L^1(\Omega)$ whenever $\eta < 1$. From these facts, and taking into account that $\operatorname{dist}(S, \partial\Omega) > 0$, it follows easily that if $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega \to \mathbb{R}$ are measurable functions such that $ess \sup_\Omega \alpha < \frac{1}{q}$ and $ess \sup_\Omega \beta < \frac{1}{q}$, then $d_\Omega^{-\alpha} s^{-\beta} \in L^q(\Omega)$.

Example 3.2. The conditions H1–H3) of Theorem 1.1 allow some cases where the function $g(x,s)$ is singular at $s=0$, and also at $x \in \partial\Omega$. For instance, consider the case where $g(x,s) := b(x)\, d_\Omega^{-\alpha} s^{-\beta}$, with $\alpha : \Omega \to \mathbb{R}$, and $\beta : \Omega \to [0,\infty)$ measurable functions such that $ess\sup_\Omega \alpha + ess\sup_\Omega \beta < \frac{1}{q}$, and with $b : \Omega \to \mathbb{R}$ such that

$$0 \le b \in L^\infty(\Omega) \quad \text{and} \quad |\{x \in \Omega : b(x) > 0\}| > 0. \tag{3.9}$$

Clearly g satisfies H1) and H2) and, for $q \in \left(1, \frac{1}{\alpha+\beta}\right)$, the first assertion of Remark (3.1), jointly with (3.9), implies that g satisfies also H3).

Example 3.3. A second example of application of Theorem 1.1 is given by the function $g(x,s) := |x-x_0|^{-\gamma}\, b(x)\, s^{-\beta}$, where $x_0 \in \Omega$, $0 < \gamma < n$, $0 < \beta < 1$, $1 < q < \min\left\{\frac{1}{\beta}, \frac{n}{\gamma}\right\}$ and with $b : \Omega \to \mathbb{R}$ satisfying (3.9).

Example 3.4. A third example can be given by taking $g(x,s) := b(x)\, \rho_S^{-\gamma}(x)\, s^{-\beta}$, where $S \subset \Omega$ is a closed C^2 and $n-1$ dimensional surface, $\rho_S(x) := \operatorname{dist}(x,S)$, $0 < \gamma < 1$, $0 < \beta < 1$, $1 < q < \min\left\{\frac{1}{\beta}, \frac{1}{\gamma}\right\}$ and with b satisfying (3.9). Indeed, H1) and H2) clearly hold, and H3) follows easily from the last assertion of Remark 3.1.

If U and V' are domains in $\mathbb{R}^n$, we will write $U \subset\subset V$ to mean that $U \subset \overline{U} \subset V$.

Proof of Theorem 1.3. Let u be a solution of (1.1). By H1) and H2), $g(\cdot,u)$ is nonnegative and nonidentically zero on Ω and, since u is a strong solution of problem (1.1), then $g(\cdot,u) \in L^q(\Omega)$. Let $v := u - \tau^*$. Then $-\Delta v = -\Delta u = g(\cdot,u)$ in Ω and $v = 0$ on $\partial\Omega$, i.e., $v = (-\Delta)^{-1} g(\cdot,u)$. Then, by Remark 2.5 ii), there exists a positive constant c' such that $v \ge c' d_\Omega$ in Ω. On the other hand, $\tau^* \ge 0$ in Ω. Thus, since $u = v + \tau^*$,

$$u \ge \tau^* + c' d_\Omega \text{ in } \Omega. \tag{3.10}$$

Also, since $\tau^* \ge 0$ in Ω, and taking into account that g is nonincreasing in the second variable and that $v \ge c' d_\Omega$ in Ω, we have $g(\cdot, \tau^* + v) \le g(\cdot, c' d_\Omega)$ and so $v = (-\Delta)^{-1} g(\cdot,u) = (-\Delta)^{-1} g(\cdot, \tau^* + v) \le (-\Delta)^{-1} g(\cdot, c' d_\Omega)$. Then

$$u \le \tau^* + (-\Delta)^{-1} g(\cdot, c' d_\Omega) \quad \text{in } \Omega. \tag{3.11}$$

Then

$$\tau^* + c' d_\Omega \le u \le \tau^* + (-\Delta)^{-1} g(\cdot, c' d_\Omega) \quad \text{in } \Omega, \tag{3.12}$$

which, taking into account H5) and that $\tau^* \in C(\overline{\Omega})$, implies that u is continuous at $\partial\Omega$.

Now we prove, by a bootstrap argument, that $u \in W^{2,n}_{loc}(\Omega)$. For $1 \le p \le \infty$ define p^* by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ if $p < n$ and by $p^* = \infty$ if $p \ge n$; and, for $k \in \mathbb{N} \cup \{0\}$, define inductively q_k, by $q_0 = q$, and by $q_{k+1} = q_k^*$. Thus $\frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}$ when $k < \frac{n}{q}$ and $q_k = \infty$ if $k \ge \frac{n}{q}$. Let $j \in \mathbb{N} \cup \{0\}$ be such that $\frac{j}{n} < \frac{1}{q} \le \frac{j+1}{n}$. Then $0 < \frac{1}{q} - \frac{j}{n} < \frac{1}{n}$, and so $n < q_j < \infty$. Given a domain $\widetilde{\Omega} \subset\subset \Omega$, let $\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_j$ be regular domains such that $\widetilde{\Omega} \subset \Omega_j \subset\subset \Omega_{j-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0 = \Omega$. Now, $u \in W^{2,q}(\Omega) = W^{2,q_0}(\Omega_0)$. Suppose that $u \in W^{2,q_k}(\Omega_k)$ for some $k = 0,1,\ldots,j-1$ and let $\widetilde{\Omega}_k$ be a domain such that $\Omega_{k+1} \subset\subset \widetilde{\Omega}_k \subset\subset \Omega_k$. Then $u \in W^{2,q_k}(\widetilde{\Omega}_k)$ and so, by the embedding theorems for Sobolev spaces, $u \in L^{q_k^*}\left(\widetilde{\Omega}_k\right) = L^{q_k+1}(\widetilde{\Omega}_k)$. Also, by H4), g is continuous on $\Omega \times (0,\infty)$, and so, since $0 \le g(\cdot,u) = g(\cdot, \tau^* + v) \le g(\cdot, c' d_\Omega)$, we have $g(\cdot, c' d_\Omega) \in L^\infty(\widetilde{\Omega}_k)$. Thus, by the inner elliptic estimates (as stated, e.g., in [23, Theorem 9.11]), $u \in W^{2,q_{k+1}}(\Omega_{k+1})$. Thus, inductively, we get that $u \in W^{2,q_j}(\Omega_j)$ and so, since $\widetilde{\Omega} \subset \Omega_j$

and $j > n$, we have $u \in W^{2,n}(\widetilde{\Omega})$. Thus (since $\widetilde{\Omega}$ was an arbitrary domain such that $\widetilde{\Omega} \subset\subset \Omega$), $u \in W^{2,n}_{loc}(\Omega)$. Then $u \in C(\Omega)$ and so, since we had already seen that u is continuous at $\partial\Omega$, we conclude that $u \in C\left(\overline{\Omega}\right)$.

Suppose now that u and $\widetilde{u}$ are solutions of problem (1.1). Then u and $\widetilde{u}$ belong to $W^{2,n}_{loc}(\Omega) \cap C\left(\overline{\Omega}\right)$ and

$$\begin{cases} -\Delta\left(u - \widetilde{u}\right) = g\left(\cdot, u\right) - g\left(\cdot, v\right) & \text{in } \Omega, \\ u - \widetilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, by Lemma 2.3, $u = \widetilde{u}$ in Ω. □

Remark 3.5. Assume the hypothesis of Theorem 1.1 and that $\tau = 0$ in problem (1.1). Assume also that $d_\Omega g\left(\cdot, c d_\Omega\right) \in L^2(\Omega)$ for any $c \in (0, \infty)$, and let $u \in W^{2,q}(\Omega)$ be the strong solution of problem (1.1) given by Theorem 1.1. Then $u \in H^1_0(\Omega)$ and u is a weak solution of problem (1.1), i.e., for any $\varphi \in H^1_0(\Omega)$,

$$g\left(\cdot, u\right)\varphi \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega g\left(\cdot, u\right)\varphi. \tag{3.13}$$

Indeed, by Theorem 1.1, we have $u \geq c_u d_\Omega$ for some $c_u \in (0, \infty)$ and so $0 \leq g\left(\cdot, u\right) \leq g\left(\cdot, c_u d_\Omega\right)$. Now, for $\varphi \in H^1_0(\Omega)$, the Holder's inequality and the Hardy's inequality of Remark 2.5 iii) give

$$\begin{aligned} \int_\Omega \left|g\left(\cdot, u\right)\varphi\right| &= \int_\Omega d_\Omega g\left(\cdot, u\right)\left|\frac{\varphi}{d_\Omega}\right| \leq \int_\Omega d_\Omega g\left(\cdot, c_u d_\Omega\right)\left|\frac{\varphi}{d_\Omega}\right| \\ &\leq \left\|d_\Omega g\left(\cdot, c_u d_\Omega\right)\right\|_2 \left\|\frac{\varphi}{d_\Omega}\right\|_2 \leq c\left\|d_\Omega g\left(\cdot, c_u d_\Omega\right)\right\|_2 \left\|\nabla\varphi\right\|_2, \end{aligned}$$

and thus $g(\cdot, u)\varphi \in L^1(\Omega)$. Moreover, the above inequality gives that the map $\varphi \to \int_\Omega g(\cdot, u)\varphi$ is continuous on $H^1_0(\Omega)$. Then, since $H^1_0(\Omega)$ is a Hilbert space with respect to the inner product $(u, v) := \int_\Omega \langle \nabla u, \nabla v\rangle$, it follows that there exists a function $\widetilde{u} \in H^1_0(\Omega)$ such that, for any $\varphi \in H^1_0(\Omega)$,

$$\int_\Omega \langle \nabla\widetilde{u}, \nabla\varphi\rangle = \int_\Omega \langle \nabla u, \nabla\varphi\rangle.$$

Then $\int_\Omega \langle \nabla\left(\widetilde{u} - u\right), \nabla\varphi\rangle = 0$ for any $\varphi \in C^\infty_c(\Omega)$ and so $z := \widetilde{u} - u$ satisfies, in the sense of distributions, $-\Delta z = 0$ in Ω. Also, $z \in W^{1,\overline{q}}_0(\Omega)$ with $\overline{q} := \min(q, 2)$ and so, in the sense of the trace, $z = 0$ on $\partial\Omega$. Then $z = 0$ and thus $u = \widetilde{u}$ in Ω. Therefore $u \in H^1_0(\Omega)$. Since u is a strong solution of problem (1.1) we have

$$\int_\Omega \langle \nabla u, \nabla\psi\rangle = \int_\Omega g\left(\cdot, u\right)\psi \quad \text{for any } \psi \in C^\infty_c(\Omega). \tag{3.14}$$

and then, by density, (3.14) holds also for any $\varphi \in H^1_0(\Omega)$.

For $f : \Omega \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ we will write $f \approx h$ to mean that there exist positive constants c_1 and c_2 such that $c_1 f \leq h \leq c_2 h$ *a.e.* in Ω

Remark 3.6. In order to illustrate the relationship between the existence of classical solutions, strong solutions and weak solutions in $H^1_0(\Omega)$ let us consider the case when Ω is a $C^{2+\alpha}$ domain in $\mathbb{R}^n$ for some $\alpha \in (0, 1)$, $n \geq 3$ and $g(x, s) = a(x)s^{-\gamma}$ with $a \in C^\alpha\left(\overline{\Omega}\right)$ such that $\min_\Omega \alpha > 0$. Assume also that $\tau = 0$ in problem (1.1). In this situation, [26, Theorem 1] states that problem (1.1) has a unique classical solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for any $\gamma > 0$ and that, when $\gamma > 1$, $u \approx d_\Omega^{\frac{2}{1+\gamma}}$ in Ω. In addition, [26, Theorem 2] says that when $\gamma > 3$ no (classical) solution belonging to $H^1(\Omega)$ exists. In the case $\gamma = 1$ [29, Theorem 1] states that $u \approx d_\Omega\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{\frac{1}{2}}$, where ω is any constant such that $\omega > \operatorname{diam}(\Omega)$. On the other hand, as a consequence of [42, Theorems 1 and 2] a weak solution $u \in H_0^1(\Omega)$ exists if and only if $\gamma < 3$. However, for $1 \leq \gamma < 3$ these weak solution are not strong solutions. Indeed, when $\gamma = 1$, $au^{-\gamma} = au^{-1} \approx d_\Omega^{-1}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{1}{2}}$ and it is easy to see that, for all $q \geq 1$, $\int_\Omega d_\Omega^{-q}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{q}{2}} = \infty$ (in fact, $\int_\Omega d_\Omega^{-q}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{q}{2}} < \infty$ if and only if $I(\varepsilon) := \int_0^\varepsilon t^{-q}\big(\ln\big(\frac{\omega}{t}\big)\big)^{-\frac{q}{2}}dt < \infty$ for some $\varepsilon > 0$, but the change of variable $s = \ln\frac{\omega}{t}$ immediately shows that $I(\varepsilon) = \infty$ for all $\varepsilon > 0$). When $1 < \gamma < 3$, we have $u \approx d_\Omega^{\frac{2}{1+\gamma}}$ in Ω, and thus $au^{-\gamma} \approx d_\Omega^{-\frac{2\gamma}{1+\gamma}}$. Then, for $q \geq 1$, $au^{-\gamma} \in L^q(\Omega)$ if and only if $\frac{2\gamma q}{1+\gamma} < 1$, that is $\gamma < \frac{1}{2q-1}$. Since $\frac{1}{2q-1} \leq 1$ we get that $\gamma < 1$, which contradicts our assumption $1 < \gamma < 3$.

4 A related problem in a punctured domain

Let $x_0 \in \Omega$, let $U := \Omega \setminus \{x_0\}$ and let $w \in L^1(U)$. Then $w \in L^1(\Omega)$, and so w can be viewed as a distribution on U and also as a distribution on Ω. For $1 \leq i, j \leq n$, we will denote by $\partial_i^U w$ and $\partial_i^U\partial_j^U w$ (respectively by $\partial_i^\Omega w$ and $\partial_i^\Omega\partial_j^\Omega w$) the first and the second derivatives of w considered as a distribution on U (resp. as a distribution on Ω), and, if $\varphi \in C^\infty(\mathbb{R}^n)$, we will write simply $\partial_i\varphi$ and $\partial_i\partial_j\varphi$ for the first and the second derivatives of φ.

If $w \in W^{2,q}(U)$ for some $q \in (1,\infty)$, then $\partial_i^U w$ and $\partial_i^U\partial_j^U w$ belong to $L^q(U)$ and so they also belong to $L^q(\Omega)$. One may ask if $\partial_i^U w = \partial_i^\Omega w$ and $\partial_i^U\partial_j^U w = \partial_i^\Omega\partial_j^\Omega w$, i.e., if the equalities

$$\left\langle \partial_i^U w, \varphi \right\rangle = -\int_\Omega w\partial_i\varphi \text{ and } \left\langle \partial_i^U\partial_j^U w, \varphi \right\rangle = \int_\Omega w\partial_i\partial_j\varphi,$$

which hold for $\varphi \in C_c^\infty(U)$, hold also for $\varphi \in C_c^\infty(\Omega)$. The next lemma provides a partial answer to this question.

Lemma 4.1. *Let $x_0 \in \Omega$, let $U := \Omega \setminus \{x_0\}$, and, for $\delta > 0$, let A_δ be defined by (1.5) and let $w \in W^{2,q}(U)$. If either $\lim_{x\to x_0} w(x)$ exists and is finite, or if*

$$\limsup_{\delta\to 0^+} \frac{1}{\delta^2}\int_{A_\delta} |w| = 0, \tag{4.1}$$

then $\partial_i^U w = \partial_i^\Omega w$ and $\partial_i^U\partial_j^U w = \partial_i^\Omega\partial_j^\Omega w$ for each i and j, and so, in particular, $w \in W^{2,q}(\Omega)$.

Proof. Observe that, in the case when $\lim_{x\to x_0} w(x)$ exists and is finite, it is enough to prove the lemma under the additional assumption that $\lim_{x\to x_0} w(x) = 0$ (because the functions $w - \lim_{x\to x_0} w(x)$ and w have the same derivatives, either in $D'(U)$ or in $D'(\Omega)$). Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi \leq 1$ in $\mathbb{R}^n$, $\psi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\psi(x) = 0$ for $|x| \geq 1$; and for $\delta > 0$, define $\psi_{1,\delta}$ and $\psi_{2,\delta}$ by $\psi_{1,\delta}(x) := \psi\left(\frac{x-x_0}{\delta}\right)$ and $\psi_{2,\delta}(x) := 1 - \psi_{1,\delta}(x)$. For $\varphi \in C_c^\infty(\Omega)$ and $0 < \delta < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$ we have

$$\int_\Omega \varphi\partial_i^U\partial_j^U w = \int_\Omega \varphi\psi_{1,\delta}\partial_i^U\partial_j^U w + \int_\Omega \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w.$$

Now, $\left|\varphi\psi_{1,\delta}\partial_i^U\partial_j^U w\right| \leq \|\varphi\|_{L^\infty(\Omega)}\left|\partial_i^U\partial_j^U w\right| \in L^q(U) = L^q(\Omega) \subset L^1(\Omega)$. Also $\lim_{\delta\to 0}\varphi\psi_{1,\delta}\partial_i^U\partial_j^U w = 0$ *a.e.* in Ω. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0}\int_\Omega \varphi\psi_{1,\delta}\partial_i^U\partial_j^U w = 0. \tag{4.2}$$

Thus, to prove the assertion of the lemma for the second derivatives, it suffices to show that

$$\lim_{\delta\to 0}\int_\Omega \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_\Omega w\partial_i\partial_j\varphi \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{4.3}$$

Notice that $\partial_i\psi_{2,\delta}(x) = \frac{1}{\delta}\frac{\partial\psi}{\partial x_i}\left(\frac{x-x_0}{\delta}\right)$ and $\partial_i\partial_j\psi_{2,\delta}(x) = \frac{1}{\delta^2}\left(\partial_i\partial_j\psi\right)\left(\frac{x-x_0}{\delta}\right)$, and so there exists a positive constant c, independent of δ, such that

$$|\partial_i\psi_{2,\delta}| \leq \frac{c}{\delta} \quad \text{and} \quad \left|\partial_i\partial_j\psi_{2,\delta}\right| \leq \frac{c}{\delta^2} \quad \text{in } \Omega. \tag{4.4}$$

Now,

$$\int_\Omega \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_U \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_U w\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right),$$

and a computation gives that

$$\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right) = \partial_i\varphi\partial_j\psi_{2,\delta} + \varphi\partial_i\partial_j\psi_{2,\delta} + \partial_i\psi_{2,\delta}\partial_j\varphi + \psi_{2,\delta}\partial_i\partial_j\varphi,$$

and so, $\psi_{2,\delta}$, $\partial_i\psi_{2,\delta}$ and $\partial_i\partial_j\psi_{2,\delta}$ have their supports contained in A_δ, we have

$$\int_U w\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right) = I_{1,\delta} + I_{2,\delta} + I_{3,\delta} + I_{4,\delta}, \tag{4.5}$$

where

$$I_{1,\delta} := \int_{A_\delta} w\partial_i\varphi\partial_j\psi_{2,\delta}, \qquad I_{2,\delta} := \int_{A_\delta} w\varphi\partial_i\partial_j\psi_{2,\delta},$$
$$I_{3,\delta} := \int_{A_\delta} w\partial_i\psi_{2,\delta}\partial_j\varphi, \qquad I_{4,\delta} := \int_U w\psi_{2,\delta}\partial_i\partial_j\varphi.$$

Thus, by (4.4),

$$|I_{1,\delta}| \leq c\,\|\partial_i\varphi\|_{L^\infty(\Omega)}\frac{1}{\delta}\int_{A_\delta}|w|, \tag{4.6}$$

with c a positive constant independent of δ. If (4.1) holds then clearly

$$\lim_{\delta\to 0^+} I_{1,\delta} = 0. \tag{4.7}$$

and, in the case when $\lim_{x\to x_0} w(x) = 0$, we have $\lim_{\delta\to 0^+}\sup_{A_\delta}|w| = 0$, and so, from (4.6), $|I_{1,\delta}| \leq \frac{c}{\delta}\|\partial_i\varphi\|_{L^\infty(\Omega)}|A_\delta|\sup_{A_\delta}|w|$, where $|A_\delta|$ denotes the Lebesgue measure of A_δ. Since $|A_\delta| = \alpha_n\left(1-\frac{1}{2^n}\right)\delta^n$ where α_n is the volume of the unit ball in $\mathbb{R}^n$ and taking into account that $n \geq 2$, we get (4.7) again in this case. Similarly,

$$\lim_{\delta\to 0^+} I_{3,\delta} = 0. \tag{4.8}$$

To estimate $I_{2,\delta}$ observe that, by (4.4),

$$|I_{2,\delta}| \leq \frac{c}{\delta^2}\|\varphi\|_{L^\infty(\Omega)}\int_{A_\delta}|w|, \tag{4.9}$$

and so, proceeding similarly to the estimative of $I_{3,\delta}$ we get, in both cases of the lemma, that

$$\lim_{\delta\to 0^+} I_{2,\delta} = 0. \tag{4.10}$$

Consider now $I_{4,\delta}$. We have $|w\psi_{2,\delta}\partial_i\partial_j\varphi| \leq |w||\partial_i\partial_j\varphi| \in L^1(\Omega)$, and clearly $\lim_{\delta\to 0^+} w\psi_{2,\delta}\partial_i\partial_j\varphi = w\partial_i\partial_j\varphi$ *a.e.* in Ω. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0^+} I_{4,\delta} = \int_\Omega w\partial_i\partial_j\varphi = \left\langle \partial_i^\Omega\partial_j^\Omega w, \varphi \right\rangle. \tag{4.11}$$

From (4.7), (4.8), (4.10), and (4.11), we get (4.3), and so the assertion of the lemma for the second derivatives holds. The proof of the assertion of the lemma for the first derivatives follows similar lines and we omit it. $\square$

Proof of Theorem 1.4. Let $w \in W^{2,q}(U)$ be a strong solution of problem (1.6). If either $w \in C(\Omega)$ or $\limsup_{\delta\to 0^+} \frac{1}{\delta^2}\int_{A_\delta}|w| = 0$, then, by Lemma 4.1, $w \in W^{2,q}(\Omega)$. Since the equality $-\Delta w = h(\cdot, w)$ holds *a.e.* in Ω, and, in the sense of the trace, $w = \tau$ on $\partial\Omega$, we have that w is a strong solution u of problem (1.7). $\square$

Proof of Theorem 1.5. To see i), suppose that $\limsup_{x\to x_0} |x-x_0|^{n-2} w(x) = 0$, and let $\varepsilon > 0$. Then there exist $\delta_0 > 0$ such that $|x-x_0|^{n-2} w(x) \leq \varepsilon$ if $0 < |x-x_0| < \delta_0$. Now, for $\delta \in (0,\delta_0)$,

$$\begin{aligned}
\frac{1}{\delta^2}\int_{A_\delta}|w| &= \frac{1}{\delta^2}\int_{A_\delta}\frac{1}{|x-x_0|^{n-2}}|x-x_0|^{n-2} w(x)\,dx \\
&\leq \frac{1}{\delta^2}\int_{A_\delta}\left(\frac{2}{\delta}\right)^{n-2}|x-x_0|^{n-2} w(x)\,dx \\
&\leq 2^{n-2}\varepsilon\delta^{-n}|A_\delta| = 2^{n-2}\left(1-\frac{1}{2^n}\right)\alpha_n\varepsilon,
\end{aligned}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. Thus $\lim_{\delta\to 0}\frac{1}{\delta^2}\int_{A_\delta}|w| = 0$, and then i) follows from Theorems 1.4 and 1.3.

ii) follows directly from i). If $\|w\|_{L^\infty(U)} = \infty$ and $\limsup_{x\to x_0}|x-x_0|^{n-2} w(x) = 0$, then, by i), after redefining w in a set with zero measure, we would have $C(\overline{\Omega})$, which is impossible when $\|w\|_{L^\infty(U)} = \infty$. $\square$

Remark 4.2. Theorems 1.4 and 1.5 say that if $x_0 \in \Omega$, $U = \Omega \setminus \{x_0\}$, and if w is a nice enough strong solution of problem (1.7) then w is a strong solution of problem 1.1.

On the other hand, it was proved in ([30], Theorem 3.6) that, if μ is a bounded Radon measure in Ω, $\gamma \leq 1$, and $f \in L^1(\Omega)$, then the problem

$$\begin{cases} -\Delta w = fu^{-\gamma} + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases}$$

has a solution in the sense that:

i) $u \in W_0^{1,1}(\Omega)$ and for any compact $K \subset \Omega$ there exists a positive constant c such that $u \geq c$ *a.e.* in K,

ii) $\int_\Omega \langle \nabla w, \nabla\varphi\rangle = \int_\Omega fu^{-\gamma}\varphi + \int_\Omega \varphi d\mu$ for any $\varphi \in C_c^1(\Omega)$.

By taking $\mu = \delta_{x_0}$ (the Dirac's measure concentrated at x_0), and, for instance, $f = 1$, in [30, Theorem 3.6] it is clear that the conclusions of Theorems 1.4 and 1.5 could not hold anymore if the notion of solution is changed and the requirement that w is "nice enough" is dropped.

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