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Existence and uniqueness for a semilinear sixth-order ODE

To my dear professor Dan Tiba with infinite admiration

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Abstract. By using variational methods and maximum principles we discuss the existence, uniqueness and multiplicity of solutions for a semilinear sixth-order ODE. The main difference between our work and other related papers is that we treat a general case and we do not impose sign restrictions on the nonlinearity f or on its potential F.

Keywords: ODE, sixth-order, semilinear, variational method.

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1 Introduction

In this paper, we study the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} u^{(6)} + Au^{(4)} + Bu'' - C(x)u + f(x,u) = 0 & \text{in } \Omega \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where A, B are some given constants, $C(x)$ is a given function, f is a continuous function on $[0, L] \times \mathbb{R}$ and $\Omega = (0, L)$.

The treatment of (1.1) is motivated by the study of stationary solutions (which leads to sixth-order ODEs) of the sixth-order parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + f(x,u), \tag{1.2}$$

arising in the formation of spatial periodic patterns in bistable systems and is also a model for describing the behaviour of phase fronts in materials that are undergoing a transition between the liquid and the solid state. The case $f(u) = u - u^3$ was treated by Gardner and Jones [13] as well as by Caginalp and Fife [7].

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We also note that the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by a boundary value of sixth-order (see [1]):

$$\begin{cases} u^{(6)} + 2u^{(4)} + u'' = f(x,u) & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Boundary value problems of sixth-order also arise in sandwich beam deflection under transverse shear [2].

The existence and multiplicity of solutions to (1.1) were obtained in [20], when $f(u) = -u^3$, $A^2 < 4B$ and $C = -1$ in Ω and in [9] when $C < 0$, $f(u) = -b(x)u^3$ where b is an even continuous $2L$ periodic function. A more general existence and multiplicity result was given in [14] by using variational methods and the Brézis and Nirenberg's linking theorems in the case

$$-\frac{F(x,u)}{u^2} \to +\infty, \quad \text{uniformly with respect to } x \text{ as } |u| \to \infty, \tag{1.4}$$

where $F(x,u) = \int_0^u f(x,s)ds \leq 0$.

In [15], the authors studied the existence of positive solutions of the nonlinear boundary value problem

$$\begin{cases} u^{(6)} + f(x,u,u'',u^{(4)}) = 0 & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

using the Krein–Rutman Theorem and the Global Bifurcation Theory under the assumptions (again a sign restriction is assumed)

1). $f : \overline{\Omega} \times [0,\infty) \times (-\infty,0] \times [0,\infty) \to [0,\infty)$ is continuous and there exist functions a, b, c, d, m, n with $a(t) + b(t) + c(t) > 0$ and $d(t) + m(t) + n(t) > 0$ in Ω such that

$$f(t,u,p,q) = a(t)u - b(t)p + c(t)q + o(|(u,p,q)|), as\, |(u,p,q)| \to 0,$$

uniformly for $t \in \overline{\Omega}$, and

$$f(t,u,p,q) = d(t)u - m(t)p + n(t)q + o(|(u,p,q)|), as\, |(u,p,q)| \to 0,$$

uniformly for $t \in \overline{\Omega}$. Here $|(u,p,q)|^2 = u^2 + p^2 + q^2$.

2). $f > 0$ in $\overline{\Omega}$ and $[0,\infty) \times (-\infty,0] \times [0,\infty) \setminus \{(0,0,0)\}$.

3). there exists constants $a_0, b_0, c_0 \geq 0$ satisfying $a_0^2 + b_0^2 + c_0^2 > 0$ and

$$f(t,u,p,q) = a_0 u - b_0 p + c_0 q + o(|(u,p,q)|).$$

It is worth mentioning the new paper of Bonanno and Livrea [4], where the problem

$$\begin{cases} -u^{(6)} + Au^{(4)} - Bu'' + Cu = \lambda f(x,u) & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

is treated.

The authors prove the existence of infinitely many solutions to problem (1.6) under different assumptions on A, B, C and by requiring an oscillation on $f(x,\cdot)$ at infinity. More precisely if

i). $F(x,t) \geq 0$ for every $(x,t) \in ([0,5/12] \cup [7/21,1]) \times \mathbb{R}$.

ii).

$$\liminf_{t\to\infty} \frac{\int_0^1 \max_{|s|<t} F(x,s)dx}{t^2} < \tau \limsup_{t\to\infty} \frac{\int_{5/12}^{7/12} F(x,t)dx}{t^2},$$

then for every

$$\lambda \in \left(\frac{2\delta^4}{\tau} \frac{1}{\limsup_{t\to\infty} \frac{\int_{5/12}^{7/12} F(x,t)dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to\infty} \frac{\int_0^1 \max_{|s|<t} F(x,s)dx}{t^2}} \right)$$

the problem (1.6) admits an unbounded sequence of classical solutions. Here τ and δ are technical constants depending on A, B and C.

Using variational methods we present here some new existence results (Section 3.1). The main difference between our work and the above mentioned papers is that we treat a general case and we do not impose sign restrictions on f or F. We note that we cover nonlinearities that are not treated elsewhere, e.g., the cases $f(u) = \ln(|u|+1) + \frac{|u|}{|u|+1} + u$ and $f(x,u) = a(x)\cos(u^n + C)u^{n-1}$, where a is a bounded function, C is a constant and n is a natural number. We see that these cases are not covered in [14] since the assumption (H1) in [14], i.e. (1.4) is not satisfied. In particular, since (2.10) holds (here $A = 2, B = 1, C = 0, L = 1$), our results apply to (1.3).

We obtain our main existence results under the restriction

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{1.7}$$

where $K_1, K_2, r > 0$.

In Section 3.2 we will briefly present some uniqueness results for the corresponding non-homogeneous linear equation.

The last section is devoted to a multiplicity result. As we mentioned above, the available multiplicity results (see [20, Theorem 3] and [9, Theorem B]) are stated under the restriction $F \le 0$. Here we strengthen relation (1.7), more precisely we impose

$$-K|s|^p \le F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{1.8}$$

where $K, K_1, K_2 > 0, 0 < r < 2, p > 2$ and obtain for sufficiently large L a multiplicity result that holds without a sign restriction on F. We also note that the multiplicity result holds if (1.4) is satisfied without the sign restriction on F.

2 Variational settings and auxiliaries

We consider the Hilbert space $H(\Omega) = \{u \in H^3(\Omega) \,|\, u = u'' = 0 \text{ on } \partial\Omega\}$, endowed with the standard inner product

$$(u,v)_{H^3(\Omega)} = \int_\Omega \Big(u'''v''' + u''v'' + u'v' + uv\Big)dx$$

and standard norm

$$\|u\|_{H^3(\Omega)} = (u,u)^{\frac{1}{2}}_{H^3(\Omega)}.$$

Definition 2.1. A weak solution of (1.1) is a function $u \in H(\Omega)$ such that

$$\int_\Omega \Big(u'''v''' - Au''v'' + Bu'v' + C(x)uv - f(x,u)v\Big)dx = 0, \qquad \forall v \in H(\Omega).$$

A classical solution of (1.1) is a function $u \in C^6(\overline{\Omega})$ that satisfies (1.1).

We note that if f is a continuous function on $[0, L] \times \mathbb{R}$, then a weak solution is a classical solution (for a proof see [20]).

The problem (1.1) has a variational structure and the weak solutions in the space $H(\Omega)$ can be found as critical points of the functional

$$J : H(\Omega) \to \mathbb{R}$$

$$J(u) = \frac{1}{2}\int_\Omega \Big((u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\Big)dx - \int_\Omega F(x,u)dx,$$

which is Fréchet differentiable and its Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int_\Omega \Big(u'''v''' - Au''v'' + Bu'v' + C(x)uv - f(x,u)v\Big)dx, \qquad \forall v \in H(\Omega).$$

Throughout the paper C denotes a universal positive constant depending on the indicated quantities, unless otherwise specified.

The following results will be useful.

Lemma 2.2. *The following relations hold true for any $u \in H(\Omega)$.*

$$\int_\Omega u^2 dx \le \left(\frac{L}{\pi}\right)^{2k} \int_\Omega (u^{(k)})^2 dx, \qquad k = 1, 2, 3. \tag{2.1}$$

$$\int_\Omega (u')^2 dx \le \left(\frac{L}{\pi}\right)^{2} \int_\Omega (u'')^2 dx. \tag{2.2}$$

$$\int_\Omega (u'')^2 dx \le \left(\frac{L}{\pi}\right)^{2} \int_\Omega (u''')^2 dx, \tag{2.3}$$

where L represents the length of Ω.

Lemma 2.2 is proved in [4, Proposition 2.1] in the case when $\Omega = (0, 1)$. By similar calculations we can get the result for the case $\Omega = (0, L)$.

From Lemma 2.2 it follows that the scalar product

$$(u, v)_{H(\Omega)} = \int_\Omega u'''v''' dx$$

induces a norm equivalent to the norm $\|u\|_{H^3(\Omega)}$ in the space $H(\Omega)$.

The next key result is more general version for bounded domains of the result presented in [20], Lemma 5 and will be used to handle the existence in the case $r > 2$ as well the multiplicity result.

Lemma 2.3. *Let $u \in H(\Omega)$. Suppose that* *a).*

$$A > 0, \qquad \frac{A^2}{C_1} < 4B, \qquad C_1 = 1 - C_m\left(\frac{L}{\pi}\right)^6 > 0, \qquad C \geq -C_m, \qquad C_m > 0. \tag{2.4}$$

Then there exists a constant k_1 such that

$$\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\right] dx \geq k_1 \|u\|^2_{H^3(\Omega)}. \tag{2.5}$$

If $C \geq 0$ then $\frac{A^2}{C_1} < 4B$ may be replaced by $A^2 < 4B$.

A similar estimate holds if we assume that

$$A, B > 0, \qquad A^2 < 4C \qquad B, \geq C, \qquad \frac{A^2}{4C} \leq C - 1. \tag{2.6}$$

b).

$$A = 0, \qquad B < 0, \qquad B^2 < 2C_m, \qquad \frac{B^2}{2C_m} \leq \frac{C_m}{2} - 1, \tag{2.7}$$

where $C_m = \inf_\Omega C(x) > 0$.

Then there exists a constant $k_2 > 0$ such that

$$\int_\Omega \left[(u''')^2 + B(u')^2 + C(x)u^2\right] dx \geq k_2 \|u\|^2_{H^3(\Omega)}. \tag{2.8}$$

The inequality (2.8) also holds if

$$A = 0, \qquad B < 0, \qquad C - 1 \geq \left(\frac{-2B}{3}\right)^{4/3}. \tag{2.9}$$

c).

$$C = 0, \qquad A > 0, \qquad B \geq 0, \qquad 1 - \frac{AL^2}{\pi^2} > 0. \tag{2.10}$$

Then there exists a constant $k_3 > 0$ such that

$$\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2\right] dx \geq k_3 \|u\|^2_{H^3(\Omega)}. \tag{2.11}$$

Remark 2.4. Of course if $A \leq 0$, $B, C \geq 0$, then Lemma 2.3 is always true, i.e., there is nothing to prove.

Proof. a). We borrow some ideas from the paper of Bonheure (see [5, Lemma 5]).

It is easy to see that for any real α

$$\int_\Omega \left(u''' + \alpha u'\right)^2 dx = \int_\Omega \left((u''')^2 - 2\alpha(u'')^2 + \alpha^2(u')^2\right) dx.$$

Hence for any α the quantity

$$Q_\alpha = \int_\Omega \left((u''')^2 - 2\alpha(u'')^2 + \alpha^2(u')^2\right) dx$$

is positive.

For arbitrary $\varepsilon > 0$ we have by Lemma 2.2

$$
\begin{aligned}
&\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\right] dx \\
&\geq C_1 \left[\int_\Omega (u''')^2 - \frac{A}{C_1}(u'')^2 + \frac{B}{C_1}(u')^2\right] dx \\
&= C_1 \Bigg\{ \varepsilon \int_\Omega \left[(u''')^2 + (u'')^2 + (u')^2\right] dx \\
&\quad + (1-\varepsilon) \int_\Omega \left[(u''')^2 - \frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}(u'')^2 + \frac{1}{4}\left(\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}\right)^2 (u')^2\right] dx \\
&\quad + \left[\frac{B}{C_1} - \varepsilon - \frac{1}{4}\frac{\left(\frac{A}{C_1}+\varepsilon\right)^2}{1-\varepsilon}\right] \int_\Omega (u')^2 dx \Bigg\} \\
&\geq \varepsilon C_1 \int_\Omega (u''')^2 dx + (1-\varepsilon) C_1 Q_{\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}} + C_1 \left[\frac{B}{C_1} - \varepsilon - \frac{1}{4}\frac{\left(\frac{A}{C_1}+\varepsilon\right)^2}{1-\varepsilon}\right] \int_\Omega (u')^2 dx.
\end{aligned}
$$

Choosing ε sufficiently small, using that $Q_{\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}} \geq 0$ and the equivalence of norms $\|\cdot\|_{H^3(\Omega)}$ and $\|\cdot\|_{H(\Omega)}$ we get the desired result.

b). The proof of inequality (2.8) under the assumption (2.7) is deduced by different means, namely by using the Fourier transform.

We first note that if one of the inequalities (2.5), (2.8) or (2.11) holds for $u \in H^3(\mathbb{R})$, then it follows that the inequalities are also true for $u \in H(\Omega)$.

Indeed, for $u \in H(\Omega)$, we have

$$\int_\Omega [\,*\,] dx = \int_{\mathbb{R}} [\,*\,] dx \geq k\|u\|^2_{H^3(\mathbb{R})} \geq k\|u\|^2_{H^3(\Omega)}. \tag{2.12}$$

Here $*$ stands for one of the expressions in the inequalities (2.5), (2.8) or (2.11) that is inside the square brackets.

We now prove the required inequalities for $u \in H^3(\mathbb{R})$.

We note that the proof of (2.5) under the conditions (2.6) is similar to the proof of (2.8) under the hypothesis (2.7) and hence is omitted.

To prove inequality (2.8) we see that for all $\xi \in \mathbb{R}$

$$-B\xi^2 \leq \frac{B^2}{2C_m}\xi^4 + \frac{C_m}{2} \leq \frac{B^2}{2C_m}\xi^6 + \frac{C_m}{2} + \frac{B^2}{2C_m} \leq \frac{B^2}{2C_m}\xi^6 + C_m - 1. \tag{2.13}$$

Hence

$$\xi^6 + B\xi^2 + C_m \geq \xi^6 - \frac{B^2}{2C_m}\xi^6 - C_m + 1 + C_m \geq \left(1 - \frac{B^2}{2C_m}\right)\left(\xi^6 + 1\right). \tag{2.14}$$

As a consequence, we get

$$\xi^6 + B\xi^2 + C_m \geq \frac{1}{3}\left(1 - \frac{B^2}{2C_m}\right)\left(1 + \xi^2 + \xi^4 + \xi^6\right), \qquad \forall\, \xi \in \mathbb{R}. \tag{2.15}$$

Let $\hat{u}(\xi)$ be the Fourier transform of $u(x) \in H^3(\mathbb{R})$.
By Parseval's identity and (2.15) we get

$$\int_{\mathbb{R}} \Big((u''')^2 + B(u')^2 + C(x)u^2\Big)dx \tag{2.16}$$

$$\geq \int_{\mathbb{R}} \Big((u''')^2 + B(u')^2 + C_m u^2\Big)dx = \int_{\mathbb{R}} \Big(\xi^6 + B\xi^2 + C_m\Big)\|\hat{u}(\xi)\|^2 d\xi$$

$$\geq \frac{1}{3}\Big(1 - \frac{B^2}{2C_m}\Big)\int_{\mathbb{R}} \Big(1 + \xi^2 + \xi^4 + \xi^6\Big)\|\hat{u}(\xi)\|^2 d\xi$$

$$= \frac{1}{3}\Big(1 - \frac{B^2}{2C_m}\Big)\int_{\mathbb{R}} \Big(u^2 + (u')^2 + (u'')^2 + (u''')^2\Big)dx \tag{2.17}$$

$$= \frac{1}{3}\Big(1 - \frac{B^2}{2C_m}\Big)\|u\|^2_{H^3(\mathbb{R})},$$

which is the desired result.

If (2.9) holds then we can achieve the proof in a similar way by showing that

$$-B\xi^2 \leq \frac{1}{2}\xi^6 + C - 1, \qquad \forall\, \xi \in \mathbb{R}. \tag{2.18}$$

To prove (2.18) we easily see that the function $\varphi(t) = \frac{1}{2}t^3 + Bt + C - 1$, $t \geq 0$ has a global minimum at $(\frac{-2B}{3})^{1/2}$.

To prove the estimate (2.11) we use inequality (2.3)

$$\int_{\Omega} \Big[(u''')^2 - A(u'')^2 + B(u')^2\Big]dx \geq \Big(1 - \frac{AL^2}{\pi^2}\Big)\int_{\Omega}(u''')^2 dx \geq k_3\|u\|^2_{H^3(\Omega)}. \qquad \square$$

Lemma 2.5. *Let $u \in H(\Omega)$. Then we have the estimates*

a).

$$\int_{\Omega} u^2 dx \leq \Big(\frac{L}{\pi}\Big)^6 \|u\|^2_{H(\Omega)}, \tag{2.19}$$

b).

$$\int_{\Omega} u^r dx \leq C(L,r)\mathcal{S}^{r-2}\|u\|^r_{H(\Omega)}, \qquad r > 2, \tag{2.20}$$

where C is a positive constant depending only on the indicated quantities and $\mathcal{S}$ is the best constant in the imbedding $H^3(\Omega) \subset C^0(\overline{\Omega})$.

Proof. a). Follows from inequality (2.1).

b). By the Sobolev imbedding and Lemma 2.2 we get

$$\int_{\Omega} u^r dx \leq \|u\|^{r-2}_{C^0(\overline{\Omega})}\int_{\Omega} u^2 dx$$

$$\leq \mathcal{S}^{r-2}\Big(\|u\|^2_{H^3(\Omega)}\Big)^{(r-2)/2}\int_{\Omega} u^2 dx \tag{2.21}$$

$$\leq \mathcal{S}^{r-2}C(L)^{(r-2)/2}\Big(\|u\|^2_{H(\Omega)}\Big)^{(r-2)/2}\int_{\Omega} u^2 dx$$

$$\leq \mathcal{S}^{r-2}C(L,r)\|u\|^r_{H(\Omega)}. \qquad \square$$

3 Main results

3.1 Existence

We split the study of existence into three cases:

Case $0 \le r < 2$

Lemma 3.1. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \le r < 2$ and $A \le 0, B, C \ge 0, C \in C^0(\Omega)$. Then the boundary value problem (1.1) has at least one solution.

Proof. The result is a consequence of the Weierstrass theorem, which tells us that if the functional J is coercive and weakly lower semicontinuous on $H(\Omega)$, then J has a global minimum.

We first establish that $J(u)$ is coercive.

By Young's inequality

$$\int_\Omega F(x,u)dx \le \varepsilon \int_\Omega u^2 dx + \int_\Omega \left(C(r,\varepsilon) K_1^{\frac{2}{2-r}} + K_2 \right) dx. \tag{3.1}$$

Using Lemma 2.2 it follows that

$$\begin{aligned} J(u) &\ge \frac{1}{2}\int_\Omega \left((u''')^2 + C(x)u^2\right) dx - \varepsilon \int_\Omega u^2 dx - \int_\Omega \left(C(r,\varepsilon) K_1^{\frac{2}{2-r}} + K_2 \right) dx \\ &\ge \int_\Omega (u''')^2 \left(\frac{1}{2} - \varepsilon \left(\frac{L}{\pi}\right)^6 \right) dx - C(K_1, K_2, r, L, \varepsilon) \\ &\ge \|u\|^2_{H(\Omega)} \left(\frac{1}{2} - \varepsilon \left(\frac{L}{\pi}\right)^6 \right) - C(K_1, K_2, r, L, \varepsilon). \end{aligned}$$

If we choose now $\varepsilon > 0$ sufficiently small we get that $J(u)$ is coercive on $H(\Omega)$.

We now show that $J(u)$ is weakly lower semicontinuous on the reflexive space $H(\Omega)$. Since $A \le 0$ and $B \ge 0$ we get that

$$J_1(u) = \frac{1}{2}\int_\Omega \left((u''')^2 - A(u'')^2 + B(u')^2 \right) dx$$

is convex.

Hence $J(u)$ can be represented as the sum $J(u) = J_1(u) + J_2(u)$, where $J_1(u)$ is convex and

$$J_2(u) = \frac{1}{2}\int_\Omega \left(C(x)u^2 - 2F(x,u) \right) dx$$

is sequentially weakly continuous.

Therefore, $J(u)$ is weakly lower semicontinuous by the result in [3, Criterion 6.1.3, p. 30], and the proof follows. □

Remark 3.2. From the proof of Lemma 3.1 it can easily be seen that Lemma 3.1 still works if C takes negative values. More precisely, if $C \geq -C_m$, where $C_m > 0$ and

$$C_1 = 1 - C_m \left(\frac{L}{\pi}\right)^6 > 0. \tag{3.2}$$

The next lemma ensures that the solution we have found is nontrivial.

Lemma 3.3. *Suppose that the following condition holds:*

$$\lim_{s\to\infty} \frac{f(x,s)}{s^\alpha} = q(x) \quad \textit{and} \quad \lim_{s\to\infty} F(x,s) = \infty \quad \textit{uniformly in } \overline{\Omega}, \tag{3.3}$$

where $q(x) \geq 0$, $\|q\|_{L^\infty(\Omega)} > 0$, $\alpha > 1$.

Then there exists $e \in H(\Omega)$ *such that* $J(e) < 0$.

Proof. We can find a function $\varphi > 0$ in Ω such that $\varphi \in H(\Omega)$ and $\int_\Omega q(x)\varphi^{\alpha+1}(x)dx \geq \delta$, where δ is a positive constant.

A candidate for φ is

$$\varphi(x) = \sin\frac{\pi x}{L}.$$

We note that by the first part of relation (3.3) and by the fact that $f(x,s)/s^\alpha$ is continuous in $\overline{\Omega} \times (0,\infty)$ we get that there exists a strictly positive function $Q(x) \in L^1(\Omega)$ such that

$$f(x,s) \leq Q(x)s^\alpha \quad \text{in } \Omega \times [N,\infty),$$

where N is a positive constant.

Integrating with respect to s the last inequality, we obtain that $F(x,s)/s^{\alpha+1}$ is "dominated" by the L^1 function $Q(x)/(\alpha+1)$ in $\Omega \times [N,\infty)$.

Hence by the dominated convergence theorem and (3.3)

$$\begin{aligned}
\lim_{s\to\infty} \frac{J(s\varphi)}{s^{\alpha+1}} &= \frac{1}{2}\lim_{s\to\infty} \frac{s^2\int_\Omega \Big((\varphi''')^2 - A(\varphi'')^2 + B(\varphi')^2 + C(x)\varphi^2\Big)dx}{s^{\alpha+1}} \\
&\quad - \lim_{s\to\infty}\int_\Omega \frac{F(x,s\varphi)}{s^{\alpha+1}}dx \\
&= -\int_\Omega \lim_{s\to\infty}\frac{F(x,s\varphi)}{s^{\alpha+1}}dx = -\int_\Omega \lim_{s\to\infty}\frac{f(x,s\varphi)\varphi}{(\alpha+1)s^\alpha}dx \\
&= -\frac{1}{\alpha+1}\int_\Omega q(x)\varphi^{\alpha+1}(x)dx < 0,
\end{aligned}$$

which is the desired result.

Hence there exists $e = s\varphi \in H(\Omega)$ such that $J(e) < 0$. ☐

Our first main existence result reads.

Theorem 3.4. *Suppose that* F *satisfies*

$$F(x,s) \leq K_1|s|^r + K_2, \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \leq r < 2$ *and* $A \leq 0, B, C \geq 0, C \in C^0(\Omega)$. *If in addition* (3.3) *holds, then the boundary value problem* (1.1) *has at least one nontrivial solution.*

Case $r = 2$

Lemma 3.5. *Suppose that F satisfies*

$$F(x,s) \leq K_1|s|^2 + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{3.4}$$

where K_1, K_2, $A \leq 0$, $B, C \geq 0, C \in C^0(\Omega)$. If in addition we assume that

$$1 - 2K_1\left(\frac{L}{\pi}\right)^6 > 0, \tag{3.5}$$

then the boundary value problem (1.1) has at least one solution.

Proof. Since relations and (3.4) and (3.5) ensure the coercivity of $J(u)$, we can imitate the proof of Lemma 3.1. □

Similarly, we get the corresponding existence result in the case $r = 2$.

Theorem 3.6. *Suppose that F satisfies*

$$F(x,s) \leq K_1|s|^2 + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where K_1, K_2, $A \leq 0$, $B, C \geq 0$ in Ω. If in addition we assume that

$$1 - 2K_1\left(\frac{L}{\pi}\right)^6 > 0,$$

and that (3.3) holds, then the boundary value problem (1.1) has at least one nontrivial solution.

Proof. Follows from Lemma 3.3 and Lemma 3.5. □

Case $r > 2$, $K_2 = 0$

The existence for the case $r > 2$ will be treated differently. We shall see that $J(u)$ has a mountain-pass structure and the nontrivial critical points of $J(u)$ will be found by using the Mountain-Pass theorem of Brézis and Nirenberg.

The following two lemmas show when $J(u)$ has a mountain-pass structure.

Lemma 3.7. *Let F satisfy*

$$F(x,s) \leq K_1|s|^r, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{3.6}$$

where $K_1 > 0, r > 2$.

If $A \leq 0$, $B, C \geq 0$, or if one of the relations (2.4), (2.6), (2.7), (2.9) or (2.10) is satisfied, then there exist two positive constants ρ and η such that

$$J(u)_{|_{\|u\|_i = \rho}} \geq \eta, \qquad i = 1,2,3. \tag{3.7}$$

Here $\|u\|_i$ denotes one of the following norms

$$\|u\|_1^2 = \int_\Omega \left((u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\right)dx,$$

when $A \leq 0$, $B, C \geq 0$ or when one of the relations (2.4) or (2.6) is satisfied;

$$\|u\|_2^2 = \int_\Omega \Big((u''')^2 + B(u')^2 + C(x)u^2\Big)dx,$$

when one of the relations (2.7) *or* (2.9) *is satisfied;*

$$\|u\|_3^2 = \int_\Omega \Big((u''')^2 - A(u'')^2 + B(u')^2\Big)dx,$$

when the relation (2.10) *is satisfied.*

Proof. By virtue of Lemma 2.3 we see that $H(\Omega)$ endowed with one of the scalar products $(u,v)_i$, $i = 1,2,3$, becomes a Hilbert space.

We give the proof in the case when (2.4) is satisfied. The cases when relations (2.7), (2.9) or (2.10) hold can be treated similarly.

We note that (2.5) reads

$$\|u\|_1 \ge k_1 \|u\|^2_{H^3(\Omega)}. \tag{3.8}$$

$J(u)$ becomes

$$J(u) = \frac{1}{2}\|u\|_1^2 - \int_\Omega F(x,u)dx.$$

Since $r > 2$, we can choose $q > 1$ such that $r = 1 + q$.

From (3.8), (1.7) and Young's inequality it follows that

$$\begin{aligned}\int_\Omega F(x,u)dx &\le \varepsilon K_1 \int_\Omega u^2 dx + \frac{K_1}{4\varepsilon}\int_\Omega u^{2q}dx \\ &\le \varepsilon K_1 C(L)\|u\|^2_{H(\Omega)} + \frac{K_1}{4\varepsilon}\mathcal{S}^{2q-2}C(L)^{\frac{q}{2}}\|u\|^{2q}_{H(\Omega)} \\ &\le \varepsilon C(L,k_1,K_1)\|u\|_1^2 + \frac{1}{4\varepsilon}C(L,\mathcal{S},k_1,K_1,q)\|u\|_1^{2q}.\end{aligned}$$

Hence

$$J(u) \ge \|u\|_1^2\left(\frac{1}{2} - \varepsilon C(L,k_1,K_1) - \frac{1}{4\varepsilon}C(L,\mathcal{S},k_1,K_1,q)\|u\|_1^{2q-2}\right).$$

We choose now ε small such that $\frac{1}{2} - \varepsilon C(L,k_1,K_1) > 0$.

If we choose ρ sufficiently small we see that the required inequality holds. □

Lemma 3.8. *Let F satisfy*

$$F(x,s) \le K_1|s|^r, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1 > 0, r > 2$.

Suppose that $A \le 0$, $B, C \ge 0$, or one of the relations (2.4), (2.6), (2.7), (2.9) *or* (2.10) *is satisfied. Suppose in addition that the condition* (3.3) *is satisfied and let ρ be as in Lemma 3.7. Then there exists $e \in H(\Omega)$ with $||e||_i > \rho$, $i = 1,2,3$ such that $J(e) < 0$.*

Proof. The condition $A \le 0$, $B, C \ge 0$, or one of the relations (2.4), (2.6), (2.7), (2.9) or (2.10) assures in view of Lemma 3.7 the existence of ρ, while relation (3.3) assures the existence of e with $J(e) < 0$. Since $e = s\varphi$ where s is large we get that $\|e\|_i > \rho$, $i = 1,2,3$. □

The following celebrated result is useful.

Theorem 3.9 (Mountain Pass Theorem [6])**.** *Let E be a real Banach space with its dual E^* and suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} J(u),$$

for some constants $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\lambda \geq \eta$ be characterized by

$$\lambda = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} J(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C^0([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e.

Then there exists a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to \lambda \geq \eta \quad \text{and} \quad \|J'(u_n)\|_{E^*} \to 0, \quad \text{as } n \to \infty.$$

We can now apply the Mountain Pass Theorem (Theorem 3.9) in $H(\Omega)$ to find a Cerami type sequence, i.e.,

$$\text{there exists } \{u_n\} \subset H(\Omega) \text{ such that } J(u_n) \to \lambda \text{ and } \|J'(u_n)\|_{H^*(\Omega)} \to 0. \tag{3.9}$$

Lemma 3.10. *Suppose that we are under the hypotheses of Lemma 3.8. Let $\alpha \in (0,2)$. If in addition there exist the constants $\beta > 0$, $\gamma > 0$, $\theta \geq 2$ such that*

$$F(x,s) - \frac{1}{\theta} f(x,s)s \leq \gamma |s|^{\alpha-1} s, \qquad \forall x \in \Omega,\ s \in \mathbb{R},\ s \neq 0, \tag{3.10}$$

then the sequence $\{u_n\}$ defined by (3.9) is bounded in $H(\Omega)$.

Proof. We give the proof in the case when (2.4) is satisfied.

By the Mountain Pass Theorem 3.9 there exists

$$\{u_n\} \subset H(\Omega) \text{ such that } J(u_n) \to \lambda \text{ and } \langle J'(u_n), u_n \rangle \to 0. \tag{3.11}$$

Hence for sufficiently large n we have,

$$\lambda + 1 \geq J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle. \tag{3.12}$$

Since

$$\langle J'(u_n), u_n \rangle = \|u_n\|_1^2 - \int_\Omega f(x, u_n) u_n dx, \tag{3.13}$$

we get that

$$J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|_1^2 - \int_\Omega \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) dx. \tag{3.14}$$

By (3.10)

$$\begin{aligned} - \int_\Omega \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) dx &\geq -\gamma \int_\Omega |u_n|^{\alpha-1} u_n dx \\ &\geq -\gamma \int_\Omega |u_n|^\alpha dx. \end{aligned} \tag{3.15}$$

Now by Young's inequality we get

$$\int_\Omega |u_n|^\alpha dx \le \varepsilon \int_\Omega (u_n)^2 dx + C(\varepsilon,\alpha,L),$$

by Lemma 2.2 we get

$$\int_\Omega (u_n)^2 dx \le \left(\frac{L}{\pi}\right)^6 \int_\Omega (u_n''')^2 dx = \left(\frac{L}{\pi}\right)^6 \|u_n\|^2_{H(\Omega)}, \tag{3.16}$$

and since (2.5) reads

$$\|u_n\|_1^2 \ge k\|u_n\|^2_{H^3(\Omega)} \ge k\|u_n\|^2_{H(\Omega)}, \qquad k = \frac{1}{3}\left(1 - \frac{A^2}{4B}\right),$$

we obtain from (3.15) that

$$\begin{aligned} -\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right) dx &\ge -\varepsilon\gamma \int_\Omega (u_n)^2 dx - \gamma C(\varepsilon,\alpha,L) \\ &\ge -\varepsilon\gamma \mathcal{C}(\mathcal{L})\|u_n\|^2_{H^3(\Omega)} - C(\varepsilon,\alpha,\gamma,L) \\ &\ge -\varepsilon\frac{\gamma}{k}\mathcal{C}(\mathcal{L})\|u_n\|_1^2 - C(\varepsilon,\alpha,\gamma,L), \end{aligned} \tag{3.17}$$

where $\mathcal{C}(\mathcal{L}) = \left(\frac{L}{\pi}\right)^6$.

Combining relations (3.12), (3.14), (3.17) we have the estimate

$$\lambda + 1 \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_1^2 - \varepsilon\frac{\gamma}{k}\mathcal{C}(\mathcal{L})\|u_n\|_1^2 - C(\varepsilon,\alpha,\gamma,L). \tag{3.18}$$

We can choose $\delta > 0$ such that $\theta = 2 + 2\delta$.

If we now choose

$$\varepsilon = \frac{\delta k}{2\gamma(2+2\delta)\mathcal{C}(\mathcal{L})},$$

it follows that

$$\begin{aligned} \lambda + 1 &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_1^2 - \frac{\delta}{2(2+2\delta)}\|u_n\|_1^2 - C(k,\alpha,\gamma,\delta,L) \\ &\ge \frac{\delta}{2(2+2\delta)}\|u_n\|_1^2 - C(k,\alpha,\gamma,\delta,L), \end{aligned} \tag{3.19}$$

which shows that $\{u_n\}$ is bounded. □

Remark 3.11. Instead of (3.10) we could have imposed the following hypotheses:

Let $\alpha \in (0,2)$ and suppose that there exist the constants $\beta > 0$, $\gamma > 0$, $\theta \ge 2$ such that

$$F(x,s) - \frac{1}{\theta} f(x,s)s \le \gamma s^\alpha, \qquad \forall x \in \Omega,\ s > 0, \tag{3.20}$$

$$F(x,s) - \frac{1}{\theta} f(x,s)s \le \beta, \qquad \forall x \in \Omega,\ s \le 0. \tag{3.21}$$

As (3.10) requires that $F(x,s) - \frac{1}{\theta} f(x,s)s$ is negative for $s < 0$ we see that (3.21) is less restrictive that (3.10).

Sketch of proof. For each fixed n we define $\Omega^+ = \{x \in \Omega \mid u_n(x) > 0\}$ and $\Omega^- = \{x \in \Omega \mid u_n(x) \le 0\}$.

By (3.20) and (3.21)

$$\begin{aligned}-\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right)dx &\ge -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta \int_{\Omega^-} dx \\ &= -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta \operatorname{meas}(\Omega^-) \\ &\ge -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta L.\end{aligned}$$

By Young's inequality we get

$$\int_{\Omega^+} (u_n)^\alpha dx \le \varepsilon \int_{\Omega^+} (u_n)^2 dx + C(\varepsilon,\alpha,L) \le \varepsilon \int_\Omega (u_n)^2 dx + C(\varepsilon,\alpha,L).$$

Hence

$$-\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right)dx \ge -\varepsilon\gamma \int_\Omega (u_n)^2 dx - C(\gamma,\varepsilon,\alpha,L,\beta),$$

which is similar to the first inequality in (3.17). Now the proof follows exactly as the proof of Lemma 3.10. □

Remark 3.12. Lemma 3.10 still holds when $\alpha = 2$, if we impose the restriction

$$\frac{1}{2} - \frac{1}{\theta} - \frac{\gamma}{k}\left(\frac{L}{\pi}\right)^6 > 0. \tag{3.22}$$

Lemma 3.13. *Under the hypotheses of Lemma 3.10, there exists a sequence $\{u_n\}$ such that $u_n \to u_0$ strongly in $H(\Omega)$.*

Proof. By Lemma 3.10 there is a bounded Cerami type sequence $\{u_n\}$. Hence we can extract a subsequence, still denoted $\{u_n\}$, such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{weakly in } H(\Omega), \\ u_n &\to u_0 \quad \text{strongly in } C^2(\overline{\Omega}). \end{aligned}$$

Let $v_n = u_n - u_0$.

Using (3.13) with u_n replaced by v_n and the fact that

$$\langle J'(v_n), v_n\rangle \to 0 \quad \text{as } n \to \infty$$

we can find a sequence $\{\alpha_n\}$, $\alpha_n > 0$, $\alpha_n \to 0$ as $n \to \infty$ such that (for sufficiently large n)

$$\alpha_n \ge \|v_n\|_1^2 - \int_\Omega f(x,v_n)v_n dx.$$

Hence

$$\alpha_n \ge \|v_n\|_1^2 - K_1 \int_\Omega |v_n|^r dx. \tag{3.23}$$

By (2.20) we have the estimate

$$\begin{aligned}
\int_\Omega |v_n(x)|^r dx &= \left(\int_\Omega |v_n(x)|^r dx\right)^{\frac{r-2}{r}} \left(\int_\Omega |v_n(x)|^r dx\right)^{\frac{2}{r}} \\
&\le \|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})} L^{\frac{r-2}{2}} \left(\frac{C(L,r)}{k} \mathcal{S}^{r-2} \|v_n(x)\|_1^r\right)^{\frac{2}{r}} \\
&= C(L,r,k,\mathcal{S}) \|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})} \|v_n(x)\|_1^2.
\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24)

$$\alpha_n \ge \|v_n\|_1^2 \left(1 - C(L,r,k,K_1,\mathcal{S}) \|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})}\right) > 0.$$

Thus $v_n \to 0$ strongly in $H(\Omega)$. This completes the proof. □

We can now conclude the existence result in the case $r > 2$, $K_2 = 0$.

Theorem 3.14. *Let F satisfy*

$$F(x,s) \le K_1 |s|^r, \qquad \forall\, (x,s) \in \Omega \times \mathbb{R},$$

where $K_1 > 0, r > 2$. Suppose that one of the conditions of Lemma 2.3 is satisfied and that (3.10) holds. If the condition (3.3) is satisfied, then problem (1.1) has a nontrivial solution in $H(\Omega)$.

We end this section by giving the following examples as an application of the results.

Example 3.15. We see that the theory presented includes the typical example

$$f(x,s) = b(x) s\, |s|^{p-2}, \qquad p > 2,$$

where b is a bounded function which is either strictly positive or strictly negative in Ω (no sign changing is allowed).

For the sake of simplicity we take p even. We can check that

$$F(x,s) = b(x) \frac{s^p}{p}$$

satisfies (1.7) and relation (3.20) becomes

$$b(x) s^p \left(\frac{1}{p} - \frac{1}{\theta}\right) \le \gamma s, \qquad x \in \Omega,\ s > 0. \tag{3.25}$$

If $b > 0$ then we can choose $2 < \theta < p$ and see that the left hand side of (3.25) becomes negative and hence (3.25) is satisfied. Due to the negativity of the left hand side of (3.25) for $s \le 0$ it is also obvious that (3.21) is satisfied.

We can argue similarly if $b < 0$ by choosing $\theta > p$.

Also since (2.10) holds with $A = 2, B = 1, C = 0, L = 1$ we get by Theorem 3.14 that the boundary value problem that describes the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported (see [1])

$$\begin{cases} u^{(6)} + 2u^{(4)} + u'' = b(x) u\, |u|^{p-2} & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.26}$$

has at least one nontrivial solution.

Example 3.16. We consider the following function

$$g(x,s) = a(x)\cos(s^n + C)s^{n-1},$$

where a is a bounded function, C is a constant and n is a natural number.

Since the potential of g is

$$G(x,s) = \frac{a(x)}{n}\left(\sin(s^n + C) - \sin(C)\right)$$

and satisfies the requirements of Lemma 3.1, we get that problem (1.1) with f replaced by g has a nontrivial solution in $H(\Omega)$ if $A \le 0, B, C \ge 0, C \in C^0(\Omega)$.

Example 3.17. If we consider

$$h(s) = \ln(|s|+1) + \frac{|s|}{|s|+1} + s$$

we see that its potential is $H(s) = s\ln(|s|+1) + s^2/2$. Due to the inequality $\ln(|s|+1) \le |s|$, we see that H satisfies the requirements of Lemma 3.3. Hence problem (1.1) with f replaced by h has a nontrivial solution in $H(\Omega)$ if $A \le 0, B, C \ge 0, C \in C^0(\Omega)$.

3.2 Uniqueness

Our first uniqueness result reads

Theorem 3.18. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \le r < 2$ and $A \le 0, B, C \ge 0, C \in C^0(\Omega)$ and that relation (3.3) holds. If in addition

$$\frac{\partial f(x,s)}{\partial s} < 0 \quad \text{in } \Omega \times \mathbb{R} \tag{3.27}$$

holds, then problem (1.1) has a unique nontrivial solution in $H(\Omega)$.

Proof. By the proof of Lemma 3.1, $J(u)$ can be represented as the sum $J(u) = J_1(u) + J_2(u)$, where $J_1(u)$ is convex and

$$J_2(u) = \frac{1}{2}\int_\Omega \Big(C(x)u^2 - 2F(x,u)\Big)dx.$$

Condition (3.27) assures that the function $s \to F(x,s)$ is strictly convex and hence $J_2(u)$ is strictly convex. The last statement implies that $J(u)$ is strictly convex and the uniqueness follows. □

The next uniqueness result is a consequence of the following one dimensional generalized maximum principle (for results concerning the generalized maximum principle see [18, p. 73]) and collects several author's uniqueness results in the case when the coefficients A, B, C are nonconstant or have arbitrary sign and $f = f(x)$.

Theorem 3.19. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality $\mathrm{L}u \equiv u'' + \gamma(x)u \geq 0$ in Ω, where $\gamma \geq 0$ in Ω.*

Suppose that

$$\sup_{\Omega} \gamma < \frac{\pi^2}{L^2} \tag{3.28}$$

holds.

Then, there exists a function $w > 0$ in $\overline{\Omega}$, $w \in C^\infty(\overline{\Omega})$ such that u/w satisfies a generalized maximum principle in Ω, i.e., there exists a constant $k \in \mathbb{R}$ such that $u/w \equiv k$ in Ω or u/w does not attain a nonnegative maximum in Ω.

Proof. The proof follows directly from [11, Theorem 2.1] (which holds for all dimensions $n \geq 1$). □

The interested reader may consult the paper [16] for a different kind of one dimensional maximum principle for sixth order operators. The authors prove (Theorem 3.1) the positivity of the solution u that satisfies a sixth order differential inequality assuming that u, u' are positive on the boundary of the domain $\Omega = (a, b)$ and (in particular) $u'''(a) \leq 0, u'''(b) \geq 0$.

Theorem 3.20. *The boundary value problem*

$$\begin{cases} u^{(6)} + A(x)u^{(4)} + B(x)u'' - C(x)u = f(x) & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u^{(4)} = g_3 & \text{on } \partial\Omega, \end{cases} \tag{3.29}$$

has at most one solution if one of the following conditions is satisfied (here $g_i, i = 1, 2, 3$ are arbitrary constants)

1).

$$\sup_{\Omega} \frac{A(B+C)^2}{2B^2(A+1)} < \frac{\pi^2}{L^2} \quad \text{in } \Omega. \tag{3.30}$$

Here $A < -1, B > 0$ are constants and $C > 0$ in Ω is a function.

2). Suppose that the functions A, C satisfy $-A = C > 0$ in Ω and that the function B satisfies

$$B > 1 \text{ in } \overline{\Omega}, \qquad \Big(1/(B-1)\Big)'' \leq 0 \quad \text{in } \Omega. \tag{3.31}$$

3). Suppose that the functions $A < 0, B, C > 0$ in Ω satisfy

$$\sup_{\Omega} \frac{-(C-A)^2}{2A(B-1)} < \frac{\pi^2}{L^2} \quad \text{in } \Omega \tag{3.32}$$

and also (3.31) holds.

4).

$$\sup_{\Omega} \frac{-2C}{A+C+1} < \frac{\pi^2}{L^2} \text{ in } \Omega, \tag{3.33}$$

where the functions $A < 0, C > 0$ satisfy $A + C + 1 < 0$ in $\overline{\Omega}$ and $\big(1/(A+C+1)\big)'' \geq 0$ in Ω.

Proof. 1). The proof uses the P-function method introduced by L. E. Payne [17]. Many results concerning the P-function method and its applications can be found in the book [19].

We give the proof when (3.30) holds.

We define $u = u_1 - u_2$, where u_1 and u_2 are solutions of (3.29). Then u satisfies (3.29), where $f = 0$ and with zero boundary data $u = u'' = u^{(4)} = 0$ on $\partial\Omega$.

According to [11], Lemma 3.1, i), the function

$$P = (-Au^{(4)} + Bu)^2 + AB(A+1)(u'')^2 - B^2(A+1)u^2$$

satisfies the inequality

$$P'' + \frac{A(B+C)^2}{2B^2(A+1)}P \geq 0 \quad \text{in } \Omega.$$

Hence by Theorem 3.19 there exists $w > 0$ in $\overline{\Omega}$ such that P/w satisfies a generalized maximum principle in Ω, i.e., either there exists a constant $k \in \mathbb{R}$ such that

$$\frac{P}{w} \equiv k \quad \text{in } \Omega, \tag{3.34}$$

or

$$\frac{P}{w} \quad \text{does not attain a maximum in } \Omega. \tag{3.35}$$

If (3.34) holds then since the function P/w is smooth (3.34) holds in $\overline{\Omega}$. By the zero boundary conditions we have $P = 0$ on $\partial\Omega$, i.e., $k = 0$. It follows that $P = 0$ in Ω. Since P is a sum of squares multiplied by positive constants, $P = 0$ in Ω implies $u \equiv 0$ in Ω. Hence $u_1 = u_2$ in Ω.

Alternatively, if (3.35) holds, then

$$\max_{\overline{\Omega}} \frac{P}{w} = \max_{\partial\Omega} \frac{P}{w} = 0$$

by the zero boundary conditions. It follows that

$$0 \leq \max_{\overline{\Omega}} \frac{P}{w} = 0,$$

i.e., $P = 0$ in Ω. Using the same arguments as above, we get $u \equiv 0$ in Ω, i.e., $u_1 = u_2$ in Ω.

2). If (3.31) holds then, by [11, Lemma 3.1, ii)], the P-function

$$P_1 = (u^{(4)} + u)^2 + (B-1)(u'')^2 + (B-1)u^2$$

satisfies the classical maximum principle, which means that it attains its maximum on the boundary of Ω, i.e., $\max_{\overline{\Omega}} P_1 = \max_{\partial\Omega} P_1 = 0$.

3). If (3.32) holds then [11, Lemma 3.1, ii)] tells that P_1/w satisfies a generalized maximum principle and we can argue as in Case 1).

4). If (3.33) holds then we can use [11, Lemma 3.2, ii)] which shows that P_2/w satisfies a generalized maximum principle and the proof follows. Here

$$P_2 = (u^{(4)} - u'')^2 + C(u'' - u)^2 - (A+C+1)(u'')^2. \qquad \square$$

3.3 Multiplicity

Finally, we present a multiplicity result for (1.1) that is based on the result presented in [20, Theorem 3], Lemma 2.3 and the next result.

Lemma 3.21. *Let A, B, C be real constants such that $C > 0$. The polynomial*

$$P(L) = L^6 - \frac{B}{C}\pi^2 L^4 + \frac{A}{C}\pi^4 L^2 - \frac{\pi^6}{C}$$

has exactly one positive zero ξ_0 if either

$$B \le 0, \qquad A > 0, \tag{3.36}$$

$$A, B > 0, \qquad B^2 \le 3AC, \tag{3.37}$$

$$A, B > 0, \qquad B^2 > 3AC, \qquad \frac{1}{C} \in (0, \gamma_-) \cup (\gamma_+, \infty), \tag{3.38}$$

where

$$\gamma_\pm = \frac{1}{27}\left[\frac{9AB}{C^2} - \frac{2B^3}{C^3} \pm \left(\frac{2B^2}{C^2} - \frac{3A}{C}\right)^{\frac{3}{2}}\right],$$

or

$$A \le 0, \qquad B \in \mathbb{R} \tag{3.39}$$

holds.

Moreover, for $L > \xi_0$ we have $P(L) > 0$.

The proof is a direct consequence of the result presented in [8, Lemma 4.3].

To prove the multiplicity result we need the following

Definition 3.22. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. We say that J satisfies a Palais–Smale condition if any sequence $\{u_n\}_n$ in X for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Theorem 3.23 (Clark [10])**.** *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$ be even, bounded below and satisfy the Palais–Smale condition. Suppose that $J(0) = 0$ and there is a set $Y \subset X$ such that Y is homeomorphic to S^{m-1} by an odd map and $\sup_Y J < 0$. Then J possesses at least m distinct pairs of critical points.*

Our multiplicity result reads

Theorem 3.24. *Let $L > m\xi_0$, for some positive natural number m. Suppose that $F(x, 0) = 0$, $s \to F(x, s)$ is even for all $x \in \Omega$ and A, B, C are constants. If in addition one of the following relations holds*

$$\text{one of the hypotheses of Lemma 3.21, relation (1.4) and } -F(x, s) \le K|s|^p,\ p > 2, \tag{3.40}$$

$$A \le 0,\ B \ge 0, 0 \ge C \ge -C_m \text{ and relation (1.8)}, \tag{3.41}$$

$$A^2 \le 4BC_1,\ B^2 \le -3AC \text{ and relation (1.8)}, \tag{3.42}$$

where $C_1 = 1 - C_m\left(\frac{L}{\pi}\right)^6$, then problem (1.1) has m distinct nontrivial solutions.

Proof. We first note that if relation (1.4) holds, then by [14, Lemma 7], we get that $J(u)$ is bounded from below and satisfies the Palais–Smale condition for any real constants A, B, C.

If one of the relations (3.41) or (3.42) is assumed, in view of Lemma 2.3 and the structure condition (1.8) we can use a similar argument that was used in the proof of Lemma 3.1 to show that $J(u)$ is bounded from below on $H(\Omega)$ by a negative constant. Since f is continuous on $\mathbb{R}^2$ we can follow [20], proof of Theorem 3, to get that $J(u)$ satisfies the Palais–Smale condition.

We now use the same techniques as in [20, Theorem 3] and prove the case when (3.40) holds. We can treat similarly the other cases.

Consider the set $Y \subset H(\Omega)$,

$$Y = \left\{ \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} : \lambda_1^2 + \cdots + \lambda_m^2 = \rho^2 \right\},$$

where ρ is a positive number to be chosen later. Y is a subset of the finite-dimensional space X_m

$$X_m = \operatorname{span}\left\{ \sin \frac{\pi x}{L}, \ldots, \sin \frac{m\pi x}{L} \right\}$$

equipped with the norm

$$\left\| \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} \right\|_m^2 = \lambda_1^2 + \cdots + \lambda_m^2.$$

Since

$$J(v) = \frac{1}{2} \int_\Omega \left((v''')^2 - A(v'')^2 + B(v')^2 + Cv^2 \right) dx - \int_\Omega F(x, v) dx,$$

we get by computation and by (1.8) that for any $v \in Y$

$$J(v) \le \frac{L}{4} \|v\|_m^2 \left[\left(\frac{\pi}{L}\right)^6 - A\left(\frac{\pi}{L}\right)^4 + B\left(\frac{\pi}{L}\right)^2 + C \right] + K \int_\Omega |v|^p dx.$$

Using Hölder's inequality we have

$$\begin{aligned} |v| &= \left| \lambda_1 \sin \frac{\pi x}{L} + \cdots + \lambda_m \sin \frac{m\pi x}{L} \right| \\ &\le \left(\lambda_1^2 + \cdots + \lambda_m^2 \right)^{\frac{1}{2}} \left(\sin^2 \frac{\pi x}{L} + \cdots + \sin^2 \frac{m\pi x}{L} \right)^{\frac{1}{2}} \\ &\le m \|v\|_m. \end{aligned}$$

Hence we get

$$J(v) \le \frac{L}{4} \|v\|_m^2 Q(L) + C(K, m, p, L) \|v\|_m^p,$$

where

$$Q(L) = \left(\frac{\pi}{L}\right)^6 - A\left(\frac{\pi}{L}\right)^4 + B\left(\frac{\pi}{L}\right)^2 - C.$$

It is easy to check that $Q(L) < 0$ iff $P(L) > 0$.

Hence by Lemma 3.21 we see that for $L > \xi_0$ $Q(L) < 0$ and by choosing ρ sufficiently small, we get

$$J(v) \leq \|v\|_m^2 \left(\frac{L}{4} Q(L) + C(K, m, p, L) \|v\|_m^{p-2} \right) < 0,$$

for any $v \in Y$.

Now the proof follows from Clark's theorem. □

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