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Mild solutions, variation of constants formula, and linearized stability for delay differential equations

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Abstract. The method and the formula of variation of constants for ordinary differential equations (ODEs) is a fundamental tool to analyze the dynamics of an ODE near an equilibrium. It is natural to expect that such a formula works for delay differential equations (DDEs), however, it is well-known that there is a conceptual difficulty in the formula for DDEs. Here we discuss the variation of constants formula for DDEs by introducing the notion of a *mild solution*, which is a solution under an initial condition having a discontinuous history function. Then the *principal fundamental matrix solution* is defined as a matrix-valued mild solution, and we obtain the variation of constants formula with this function. This is also obtained in the framework of a Volterra convolution integral equation, but the treatment here gives an understanding in its own right. We also apply the formula to show the principle of linearized stability and the Poincaré–Lyapunov theorem for DDEs, where we do not need to assume the uniqueness of a solution.

Keywords: delay differential equations, discontinuous history functions, fundamental matrix solution, variation of constants formula, principle of linearized stability, Poincaré–Lyapunov theorem.

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1 Introduction

Studies concerning with the variation of constants formula for delay differential equations (DDEs) have a long history of over fifty years. Nevertheless, the reason why we try to discuss the variation of constants formula in this paper is that such a consideration gives rise to a conceptual difficulty that is peculiar to the theory of DDEs. Specifically, it is usual to discuss DDEs within the scope of continuous history functions, but a class of discontinuous history functions emerges as initial conditions when we try to obtain the variation of constants formula. In connection with this, a matrix-valued solution having a certain discontinuous matrix-valued function as the initial condition is called the *fundamental matrix solution*. However, it is quite difficult to understand why the solution is called the "fundamental matrix solution" when compared with the theory of ordinary differential equations (ODEs).

This conceptual difficulty has arisen in the theoretical development about the variation of constants formula in the texts [18] and [19] by Jack Hale. In the revised edition [22], the theoretical development is rewritten based on the consideration in [34]. There also exist studies to understand the conceptual difficulty of the variation of constants formula for DDEs within the framework of Functional Analysis (e.g., see [7], [12], and [13]). In this framework, it is essential that the Banach space of continuous functions on closed and bounded interval endowed with the supremum norm is not reflexive, and the theory is constructed by using the so called "sun-star calculus". See [14] for the details. See also [36] for a survey article.

The idea of discussing the variation of constants formula for DDEs in this paper is to define a solution under an initial condition having a discontinuous history function as a *mild solution*. This concept comes from the analogy of the notion of mild solutions of abstract linear evolution equations, and its terminology also originates from this. It can be said that the notion of mild solutions is to elevate the technique to exchange the order of integration to a concept.

The dependence of the derivative $\dot{x}(t)$ of an unknown function x on the past value of x is abstracted to the concept of *retarded functional differential equations* (RFDEs). In this paper, we consider an autonomous linear RFDE

$$\dot{x}(t) = Lx_t \quad (t \geq 0) \tag{1.1}$$

for a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 1$ is an integer, and $r > 0$ is a constant, which are fixed throughout this paper. The derivative of x at 0 is interpreted as the right-hand derivative. We are using the following notations:

- $C([-r,0],\mathbb{K}^n)$ denotes the Banach space of all continuous functions from $[-r,0]$ to $\mathbb{K}^n$ endowed with the supremum norm $\|\cdot\|$. Here a norm $|\cdot|$ on $\mathbb{K}^n$, which is not necessarily the Euclidean norm, is fixed throughout this paper.
- For each $t \geq 0$, $x_t\colon [-r,0] \to \mathbb{K}^n$ is a continuous function defined by
$$x_t(\theta) := x(t+\theta) \quad (\theta \in [-r,0])$$
when $x\colon [-r,\infty) \to \mathbb{K}^n$ is continuous. See also Definition 2.1.

In addition to the linear RFDE (1.1), we also consider a non-homogeneous linear RFDE

$$\dot{x}(t) = Lx_t + g(t) \quad (\text{a.e. } t \geq 0) \tag{1.2}$$

for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$. Here $\mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$ denotes the linear space of all locally Lebesgue integrable functions from $[0,\infty)$ to $\mathbb{K}^n$ defined almost everywhere. See also the notations given below. We refer the reader to [32] and [30] as references of the theory of Lebesgue integration for scalar-valued functions.

To study these differential equations, the following expression of L by a *Riemann–Stieltjes integral*

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta) \tag{1.3}$$

for $\psi \in C([-r,0],\mathbb{K}^n)$ is useful. Here $\eta\colon [-r,0] \to M_n(\mathbb{K})$ is an $n \times n$ matrix-valued function of bounded variation. The above representability is ensured by a corollary of the Riesz representation theorem (see Corollary B.3). It is a useful convention that the domain of definition of η is extended to $(-\infty,0]$ by letting

$$\eta(\theta) := \eta(-r)$$

for $\theta \in (-\infty,-r]$. See Appendix A for the Riemann–Stieltjes integrals with respect to matrix-valued functions. For the use of Riemann–Stieltjes integrals in the context of RFDEs, see [19, Chapters 6 and 7], [34, Chapter 2], [24, Chapter 4], [22, Chapters 6 and 7], and [14, Chapter I], for example.

This paper is organized as follows:

In Section 2, we introduce the notion of a history segment x_t for a discontinuous function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$. By using this, we also introduce the notion of a mild solution to the linear RFDE (1.1) under an initial condition

$$x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n). \tag{1.4}$$

Here $\mathcal{M}^1([-r,0],\mathbb{K}^n)$ consists of elements of $\mathcal{L}^1([-r,0],\mathbb{K}^n)$ that are defined at 0. Roughly speaking, a function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ is said to be a mild solution of (1.1) under the initial condition (1.4) if it satisfies

$$x(t) = \phi(0) + L\int_0^t x_s \,\mathrm{d}s \quad (t \ge 0).$$

Here $\int_0^t x_s \,\mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ is defined by

$$\left(\int_0^t x_s \,\mathrm{d}s\right)(\theta) := \int_0^t x(s+\theta)\,\mathrm{d}s \quad (\theta \in [-r,0]).$$

See Definitions 2.5 and 2.7 for the details. After proving the existence and uniqueness of a mild solution of the linear RFDE (1.1) under the initial condition (1.4), we define the *principal fundamental matrix solution* of (1.1) as a matrix-valued mild solution $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ under the initial condition $X_0^L = \hat{I}$. Here $\hat{I}\colon [-r,0] \to M_n(\mathbb{K})$ is a discontinuous function defined by

$$\hat{I}(\theta) := \begin{cases} O & (\theta \in [-r,0)), \\ I & (\theta = 0). \end{cases} \tag{1.5}$$

In Section 3, we derive a differential equation

$$\dot{X}^L(t) = \int_{-t}^0 \mathrm{d}\eta(\theta)\, X^L(t+\theta)$$

satisfied by the principal fundamental matrix solution X^L of (1.1). In the derivation, it is useful to use the notions of *Volterra operator* and *Riemann–Stieltjes convolution*. See Subsection 3.1 for the definitions and Subsection 3.3 for the fundamental properties. The above differential equation is the key to obtain a variation of constants formula.

In Section 4, we consider the non-homogeneous linear RFDE (1.2). To study a mild solution of (1.2) under the initial condition (1.4), we also consider an integral equation

$$x(t) = \phi(0) + L\int_0^t x_s \,\mathrm{d}s + G(t) \quad (t \ge 0) \tag{1.6}$$

for a continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$. We show that the above integral equation has a unique solution $x^L(\cdot\,;\phi, G)$ under the initial condition (1.4).

In Section 5, we consider a non-homogeneous linear RFDE

$$\dot{x}(t) = Lx_t + f(t) \quad (t \ge 0) \tag{1.7}$$

for a continuous function $f\colon [0,\infty) \to \mathbb{K}^n$ to motivate the use of the convolution for locally Riemann integrable functions. We show that the function $x(\cdot\,; f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot\,; f)_0 = 0$ and

$$x(t; f) := \int_0^t X^L(t-u) f(u)\,\mathrm{d}u \tag{1.8}$$

for $t \geq 0$ is a solution to Eq. (1.7) after developing the results of convolution for locally Riemann integrable functions. See Subsection 5.2 for the developments.

In Section 6, we study the non-homogeneous linear RFDE (1.2) under the initial condition (1.4) and find a variation of constants formula expressed by the principal fundamental matrix solution X^L. For this purpose, we indeed consider the integral equation (1.6) for some continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$. One of the main results of this paper is that the solution $x^L(\cdot\,;\phi, G)$ of (1.6) under the initial condition (1.4) satisfies

$$x^L(t;\phi,G) = X^L(t)\phi(0) + \left[G^L(t;\phi) + G(t)\right] + \int_0^t \dot{X}^L(t-u)\left[G^L(u;\phi) + G(u)\right]\mathrm{d}u \tag{1.9}$$

for all $t \geq 0$. Here $\dot{X}^L(t)$ denotes the derivative of the locally absolutely continuous function $X^L|_{[0,\infty)}$ at $t \geq 0$ (when it exists), and $G^L(\cdot\,;\phi)\colon [0,\infty) \to \mathbb{K}^n$ is a function determined by the initial history function ϕ. See Subsection 6.2 for the detail of the derivation of the function $G^L(\cdot\,;\phi)$. We note that before we obtain the variation of constants formula (1.9), we show that

$$x^L(t;0,G) = G(t) + \int_0^t \dot{X}^L(t-u) G(u)\,\mathrm{d}u \tag{1.10}$$

holds for all $t \geq 0$. Then the derivation of (1.9) is performed by defining a function

$$z^L(\cdot\,;\phi)\colon [-r,\infty) \to \mathbb{K}^n$$

by $z^L(\cdot\,;\phi)_0 = 0$ and

$$z^L(t;\phi) := x^L(t;\phi,0) - X^L(t)\phi(0) \tag{1.11}$$

for $t \geq 0$ and showing that $z := z^L(\cdot\,;\phi)$ satisfies an integral equation

$$z(t) = L\int_0^t z_s\,\mathrm{d}s + G^L(t;\phi) \qquad (t \geq 0), \tag{1.12}$$

because (1.12) shows that

$$z^L(\cdot\,;\phi) = x^L(\cdot\,;0,G^L(\cdot\,;\phi))$$

holds. Here we need to know the regularity of the function $G^L(\cdot\,;\phi)$, which is discussed in Subsection 6.3.

In Section 7, we discuss the exponential stability of the principal fundamental matrix solution X^L of the linear RFDE (1.1) and the uniform exponential stability of the C_0-semigroup $\left(T^L(t)\right)_{t\geq 0}$ on the Banach space $C([-r,0],\mathbb{K}^n)$ defined by

$$T^L(t)\phi := x^L(\cdot\,;\phi,0)_t \tag{1.13}$$

for $(t,\phi) \in [0,\infty) \times C([-r,0],\mathbb{K}^n)$. We show that X^L is α-exponentially stable if and only if $\left(T^L(t)\right)_{t\geq 0}$ is uniformly α-exponentially stable. See Theorems 7.3 and 7.4 for the details.

In Section 8, we apply the obtained variation of constants formulas to a proof of the stability part of the principle of linearized stability and Poincaré–Lyapunov theorem for RFDEs. This is indeed an appropriate modification of the proof for ODEs. However, the given proof makes clear the importance of the principal fundamental matrix solution. In the statement, we do not need to assume the uniqueness of a solution. Therefore, this should be compared with the proof relying on the nonlinear semigroup theory.

We have five appendices. In Appendix A, we collect results on Riemann–Stieltjes integrals for matrix-valued functions that are needed for this paper. In Appendix B, we give a proof of the representability of L by a Riemann–Stieltjes integral (1.3) because there does not seem to be any proof of the representability in the literature. In Appendix C, we discuss Gronwall's inequality and its variants used in the context of RFDEs. In Appendix D, we give lemmas that are used in the fixed point argument in this paper. In Appendix E, we continue to discuss the convolution. The contents of this appendix will not be used in this paper, but it will be useful to share the proofs of results on the convolution for matrix-valued locally Lebesgue integrable functions in the literature of RFDEs.

Notations

Throughout this paper, the following notations will be used.

- Let $E = (E, \|\cdot\|)$ be a Banach space. For each subset $I \subset \mathbb{R}$, let $C(I, E)$ denote the linear space of all continuous functions from I to E. When the subset I is a closed and bounded interval, the linear space $C(I, E)$ is considered as the Banach space of continuous functions endowed with the supremum norm $\|\cdot\|$ given by

$$\|f\| := \sup_{x \in I} \|f(x)\|$$

  for $f \in C(I, E)$.

- For each pair of Banach spaces $E = (E, \|\cdot\|)$ and $F = (F, \|\cdot\|)$, let $\mathcal{B}(E, F)$ denote the linear space of all continuous linear maps (i.e., all bounded linear operators) from E to F. For each $T \in \mathcal{B}(E, F)$, its operator norm is denoted by $\|T\|$. Then $\mathcal{B}(E, F)$ is considered as the Banach space of continuous linear maps endowed with the operator norm. When $F = E$, $\mathcal{B}(E, F)$ is also denoted by $\mathcal{B}(E)$.

- An $n \times n$ matrix $A \in M_n(\mathbb{K})$ is considered as a continuous linear map on the Banach space $\mathbb{K}^n$ endowed with the given norm $|\cdot|$. The operator norm of A is denoted by $|A|$. The linear space $M_n(\mathbb{K})$ of all $n \times n$ matrices is considered as the Banach space of matrices endowed with the operator norm.

- Let $d \geq 1$ be an integer, X be a measurable set of $\mathbb{R}^d$, and $Y = \mathbb{K}^n$ or $M_n(\mathbb{K})$.

  - We say that a function $f \colon X \supset \operatorname{dom}(f) \to Y$ is a *Lebesgue integrable function defined almost everywhere* if (i) $\operatorname{dom}(f)$ is measurable, (ii) $X \setminus \operatorname{dom}(f)$ has measure 0, and (iii) $f|_{\operatorname{dom}(f)} \colon \operatorname{dom}(f) \to Y$ is Lebesgue integrable, i.e., it is measurable and

$$\|f\|_1 := \int_X |f(x)| \,\mathrm{d}x := \int_{\operatorname{dom}(f)} |f(x)| \,\mathrm{d}x$$

    is finite. We note that the function $\operatorname{dom}(f) \ni x \mapsto |f(x)| \in [0, \infty)$ is also measurable by the continuity of the norm $|\cdot|$, and the above integral is the unsigned Lebesgue integral.

– Let $\mathcal{L}^1(X, Y)$ be the set of all Lebesgue integrable functions from X to Y defined almost everywhere. For $f \in \mathcal{L}^1(X, Y)$, let

$$\int_X f(x)\,\mathrm{d}x := \int_{\mathrm{dom}(f)} f(x)\,\mathrm{d}x.$$

Then one can prove that

$$\left| \int_X f(x)\,\mathrm{d}x \right| \le \int_{\mathrm{dom}(f)} |f(x)|\,\mathrm{d}x = \|f\|_1$$

holds.

– For $f, g \in \mathcal{L}^1(X, Y)$, the addition $f + g \colon X \supset \mathrm{dom}(f) \cap \mathrm{dom}(g) \to Y$ is defined by

$$(f + g)(x) := f(x) + g(x)$$

for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Then $f + g \in \mathcal{L}^1(X, Y)$. The scalar multiplication αf for $\alpha \in \mathbb{K}$ is also defined, and it holds that $\alpha f \in \mathcal{L}^1(X, Y)$.

- Let X be an interval of $\mathbb{R}$ and $Y = \mathbb{K}^n$ or $M_n(\mathbb{K})$. Let $\mathcal{L}^1_{\mathrm{loc}}(X, Y)$ be the set of all functions $f \colon X \supset \mathrm{dom}(f) \to Y$ satisfying (i) $\mathrm{dom}(f)$ is measurable, (ii) $X \setminus \mathrm{dom}(f)$ has measure 0, and (iii) for each closed and bounded interval I contained in X, the restriction $f|_I \colon I \supset \mathrm{dom}(f) \cap I \to Y$ belongs to $\mathcal{L}^1(I, Y)$.

2 Mild solutions and fundamental matrix solutions

2.1 Definitions

2.1.1 History segments and memory space

We first make clear the notion of history segments in our setting.

Definition 2.1. Let $x \colon [-r, \infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ be a function. For each $t \ge 0$, we define a function $x_t \colon [-r, 0] \supset \mathrm{dom}(x_t) \to \mathbb{K}^n$ by

$$\begin{aligned} \mathrm{dom}(x_t) &:= \{\theta \in [-r, 0] : t + \theta \in \mathrm{dom}(x)\}, \\ x_t(\theta) &:= x(t + \theta) \qquad (\theta \in \mathrm{dom}(x_t)). \end{aligned}$$

We call x_t the *history segment* of x at t.

We note that $\mathrm{dom}(x_t)$ is expressed by

$$\mathrm{dom}(x_t) = (\mathrm{dom}(x) - t) \cap [-r, 0],$$

where $\mathrm{dom}(x)$ is not necessarily equal to $[-r, \infty)$.

In this paper, we need discontinuous initial history functions. For this purpose, we adopt the following space of history functions.

Definition 2.2 (cf. [10])**.** We define a linear subspace $\mathcal{M}^1([-r, 0], \mathbb{K}^n)$ of $\mathcal{L}^1([-r, 0], \mathbb{K}^n)$ by

$$\mathcal{M}^1([-r, 0], \mathbb{K}^n) := \left\{ \phi \in \mathcal{L}^1([-r, 0], \mathbb{K}^n) : 0 \in \mathrm{dom}(\phi) \right\}$$

and call it the *memory space* of $\mathcal{L}^1$-type. We consider $\mathcal{M}^1([-r, 0], \mathbb{K}^n)$ as a seminormed space endowed with the seminorm $\|\cdot\|_{\mathcal{M}^1} \colon \mathcal{M}^1([-r, 0], \mathbb{K}^n) \to [0, \infty)$ defined by

$$\|\phi\|_{\mathcal{M}^1} := \|\phi\|_1 + |\phi(0)|.$$

Remark 2.3. Let $1 \le p < \infty$ and E be a Banach space. The memory space of $\mathcal{L}^1$-type should be compared with a Banach space $M^p([-r,0],E)$ introduced by Delfour and Mitter [10]. It is isomorphic to the product Banach space

$$L^p([-r,0],E) \oplus E.$$

See also [3], [8], and references therein for the use of the product space.

Definition 2.4. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we will call a function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ a *continuous prolongation* of ϕ if it satisfies the following properties: (i) $x_0 = \phi$, (ii) $[0,\infty) \subset \mathrm{dom}(x)$, and (iii) $x|_{[0,\infty)}$ is continuous.

For a continuous prolongation x of ϕ,

$$\mathrm{dom}(x) = \mathrm{dom}(\phi) \cup [0,\infty)$$

holds.

2.1.2 Mild solutions

The following is the notion of a mild solution, whose introduction is one of the contributions of this paper. We use the expression of L by the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$.

Definition 2.5 (cf. [38])**.** Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. We say that a function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ is a *mild solution* of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ if the following conditions are satisfied: (i) x is a continuous prolongation of ϕ and (ii) for all $t \ge 0$,

$$x(t) = \phi(0) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_0^t x(s+\theta)\,\mathrm{d}s \right) \tag{2.1}$$

holds. Here $\int_0^t x(s+\theta)\,\mathrm{d}s$ is a Lebesgue integral.

Since

$$\int_0^t x(s+\theta)\,\mathrm{d}s = \int_\theta^{t+\theta} x(s)\,\mathrm{d}s,$$

the integrand in Eq. (2.1) is continuous with respect to $\theta \in [-r,0]$. Therefore, the integral in (2.1) is meaningful as a Riemann–Stieltjes integral. Eq. (2.1) is also expressed by

$$x(t) = \phi(0) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_\theta^0 \phi(s)\,\mathrm{d}s \right) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_0^{t+\theta} x(s)\,\mathrm{d}s \right), \tag{2.2}$$

where the third term of the right-hand side may depend on ϕ.

Remark 2.6. Eq. (2.1) appeared at [38, (5.19) in Corollary 5.13] after developing a nonlinear semigroup theory for some class of RFDEs. Compared with this approach, the method of this paper is considered to be taking the notion of mild solutions as a starting point.

2.1.3 Notation $\int_0^t x_s \, \mathrm{d}s$

For ease of notation, we introduce the following.

Definition 2.7. Let $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ be given. For each $t \geq 0$, we define $\int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ by

$$\left(\int_0^t x_s \, \mathrm{d}s\right)(\theta) := \int_0^t x_s(\theta) \, \mathrm{d}s = \int_\theta^{t+\theta} x(s) \, \mathrm{d}s$$

for $\theta \in [-r,0]$.

We note that $\int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ introduced above is not an integral of a vector-valued function

$$[0,t] \ni s \mapsto x_s \in \mathcal{X}$$

for some function space $\mathcal{X}$.

2.2 $\int_0^t x_s \, \mathrm{d}s$ and its properties

We have the following lemma.

Lemma 2.8. *If $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$, then*

$$[0,\infty) \ni t \mapsto \int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n) \tag{2.3}$$

is continuous.

Proof. We define a function $y \colon [-r,\infty) \to \mathbb{K}^n$ by

$$y(t) = \int_{-r}^t x(s) \, \mathrm{d}s$$

for $t \geq -r$. Then y is continuous, and

$$y(t+\theta) = \int_\theta^{t+\theta} x(s) \, \mathrm{d}s + \int_{-r}^\theta x(s) \, \mathrm{d}s$$

holds for all $t \geq 0$ and all $\theta \in [-r,0]$. This shows that the function (2.3) is continuous if and only if

$$[0,\infty) \ni t \mapsto y_t \in C([-r,0],\mathbb{K}^n)$$

is continuous. Since the continuity of this function is ensured by the uniform continuity of y on any closed and bounded interval, the conclusion is obtained. □

When $x \in C([-r,\infty),\mathbb{K}^n)$, the Riemann integral

$$(\mathrm{R}) \int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$$

of the continuous function

$$[0,t] \ni s \mapsto x_s \in C([-r,0],\mathbb{K}^n)$$

exists. See Graves [16, Section 2] for the definition of the Riemann integrability of functions on closed and bounded intervals taking values in normed spaces. We now show that when $x \in C([-r,\infty),\mathbb{K}^n)$, the Riemann integral $(\mathrm{R}) \int_0^t x_s \, \mathrm{d}s$ coincides with $\int_0^t x_s \, \mathrm{d}s$ introduced in Definition 2.7. More generally, one can prove the following result.

Lemma 2.9. *Let E be a Banach space, $[a, b]$ and $[c, d]$ be closed and bounded intervals of $\mathbb{R}$, and $f\colon [a, b] \times [c, d] \to E$ be a continuous function. For each $y \in [c, d]$, let $f(\cdot, y) \in C([a, b], E)$ be defined by*

$$f(\cdot, y)(x) := f(x, y)$$

for $x \in [a, b]$. Then

$$\left(\int_c^d f(\cdot, y)\,\mathrm{d}y\right)(x) = \int_c^d f(x, y)\,\mathrm{d}y$$

holds for all $x \in [a, b]$. Here $\int_c^d f(\cdot, y)\,\mathrm{d}y$ is the Riemann integral of the continuous function $[c, d] \ni y \mapsto f(\cdot, y) \in C([a, b], E)$.

We note that the continuity of $[c, d] \ni y \mapsto f(\cdot, y) \in C([a, b], E)$ is a consequence of the uniform continuity of f.

Proof of Lemma 2.9. We fix $x \in [a, b]$. Let $T\colon C([a, b], E) \to E$ be the evaluation map defined by

$$Tg := g(x)$$

for $g \in C([a, b], E)$. Since T is a bounded linear operator, we have

$$\left(\int_c^d f(\cdot, y)\,\mathrm{d}y\right)(x) = T\int_c^d f(\cdot, y)\,\mathrm{d}y = \int_c^d Tf(\cdot, y)\,\mathrm{d}y,$$

where the last term is equal to $\int_c^d f(x, y)\,\mathrm{d}y$. This completes the proof. □

As an application of Lemma 2.9, the following result can be obtained.

Theorem 2.10. *If $x \in C([-r, \infty), \mathbb{K}^n)$, then*

$$(\mathrm{R})\int_0^t x_s\,\mathrm{d}s = \int_0^t x_s\,\mathrm{d}s$$

holds for all $t \geq 0$.

Proof. Let $t > 0$ be given. We consider a function $f\colon [-r, 0] \times [0, t] \to \mathbb{K}^n$ defined by

$$f(\theta, s) := x(s + \theta).$$

Then the function $f(\cdot, s)$ is equal to x_s. By applying Lemma 2.9 with this f,

$$\left[(\mathrm{R})\int_0^t x_s\,\mathrm{d}s\right](\theta) = \int_0^t x(s + \theta)\,\mathrm{d}s$$

holds for all $\theta \in [-r, 0]$. Since the right-hand side is equal to $\left(\int_0^t x_s\,\mathrm{d}s\right)(\theta)$, this shows the conclusion. □

Remark 2.11. When $x \in C([-r, \infty), \mathbb{K}^n)$, Theorem 2.10 yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x_s\,\mathrm{d}s = x_t \in C([-r, 0], \mathbb{K}^n)$$

holds by the fundamental theorem of calculus for vector-valued functions.

We have the following corollary.

Corollary 2.12. *Let F be a Banach space over $\mathbb{K}$ and $T\colon C([-r,0],\mathbb{K}^n) \to F$ be a bounded linear operator. If $x \in C([-r,\infty),\mathbb{K}^n)$, then*

$$T\int_0^t x_s\,\mathrm{d}s = \int_0^t Tx_s\,\mathrm{d}s \tag{2.4}$$

holds for all $t \geq 0$. Here the right-hand side is the Riemann integral of the continuous function $[0,t] \ni s \mapsto Tx_s \in F$.

Proof. From Theorem 2.10,

$$T\int_0^t x_s\,\mathrm{d}s = T\left[(\mathrm{R})\int_0^t x_s\,\mathrm{d}s\right] = \int_0^t Tx_s\,\mathrm{d}s$$

holds since T is a bounded linear operator. □

Remark 2.13. Corollary 2.12 yields the following: Let $x\colon [-r,\infty) \to \mathbb{K}^n$ be a continuous function satisfying $x_0 = \phi \in C([-r,0],\mathbb{K}^n)$. Since $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ is a bounded linear operator, x is a mild solution of the linear RFDE (1.1) with the initial history function ϕ if and only if it satisfies

$$x(t) = \phi(0) + \int_0^t Lx_s\,\mathrm{d}s$$

for all $t \geq 0$. This shows that a mild solution coincides with a solution in the usual sense when the initial history function ϕ is continuous.

2.3 Existence and uniqueness of a mild solution

By using the contraction mapping principle with an *a priori* estimate, we will prove the unique existence of a mild solution of the linear RFDE (1.1) under an initial condition (1.4)

$$x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n).$$

We note that a solution of (1.1) in the usual sense is also a mild solution (see Remark 2.13).

We will use the following notation.

Notation 1. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, let $\bar{\phi}\colon \mathrm{dom}(\phi) \cup [0,\infty) \to \mathbb{K}^n$ be the function defined by

$$\bar{\phi}(t) := \begin{cases} \phi(t) & (t \in \mathrm{dom}(\phi)), \\ \phi(0) & (t \geq 0). \end{cases} \tag{2.5}$$

$\bar{\phi}$ is a constant prolongation of ϕ.

Theorem 2.14. *For any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, the linear RFDE (1.1) has a unique mild solution under the initial condition $x_0 = \phi$.*

In the following, we give a proof based on an *a priori* estimate. See Chicone [5, Subsection 2.1] for a similar argument.

Proof of Theorem 2.14. We divide the proof into the following steps.

Step 1: Reduction to a continuous unknown function and derivation of an *a priori* estimate. For a continuous prolongation $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ of ϕ, we consider the function $y\colon [-r,\infty) \to \mathbb{K}^n$ defined by

$$y(t) := \begin{cases} x(t) - \bar{\phi}(t) & (t \in \mathrm{dom}(x)), \\ 0 & (t \notin \mathrm{dom}(x)). \end{cases}$$

Then y is a continuous function satisfying $y_0 = 0$. The problem of finding a mild solution $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ is reduced to find a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and

$$y(t) = L\int_0^t (y + \bar{\phi})_s \,\mathrm{d}s = \int_0^t Ly_s \,\mathrm{d}s + L\int_0^t \bar{\phi}_s \,\mathrm{d}s \tag{2.6}$$

for all $t \geq 0$. Here Corollary 2.12 is used. By noticing the following estimate from above

$$\left|\int_0^t \bar{\phi}(s+\theta)\,\mathrm{d}s\right| \leq \|\phi\|_1 + t|\phi(0)|$$

for $t \geq 0$ and $\theta \in [-r,0]$, a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and Eq. (2.6) must satisfy

$$|y(t)| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|) + \int_0^t \|L\|\|y_s\|\,\mathrm{d}s$$

for all $t \geq 0$. By applying Lemma C.4,

$$\|y_t\| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|)\mathrm{e}^{\|L\|t}$$

holds for all $t \geq 0$.

Step 2: Setting of function space. For each $\gamma > \|L\|$, Step 1 indicates that for a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and Eq. (2.6), we have

$$\mathrm{e}^{-\gamma t}\|y_t\| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|)\mathrm{e}^{(\|L\|-\gamma)t}.$$

Here the right-hand side converges to 0 as $t \to \infty$. Therefore,

$$\|y\|_\gamma := \sup_{t\geq 0}(\mathrm{e}^{-\gamma t}\|y_t\|) = \sup_{t\geq 0}(\mathrm{e}^{-\gamma t}|y(t)|) < \infty$$

holds (see Lemma D.1 for the detail). For each $\gamma > \|L\|$, let Y_γ be the linear subspace of $C([-r,\infty),\mathbb{K}^n)$ given by

$$Y_\gamma := \left\{y \in C([-r,\infty),\mathbb{K}^n) : y_0 = 0,\ \|y\|_\gamma < \infty\right\},$$

which is considered as a normed space endowed with the norm $\|\cdot\|_\gamma$. Then Y_γ is a Banach space (see Lemma D.2). We fix $\gamma > \|L\|$ arbitrarily, and let $Y := Y_\gamma$ and $\|\cdot\|_Y := \|\cdot\|_\gamma$.

Step 3: Reduction to fixed point problem. We define a transformation $T\colon Y \to C([-r,\infty),\mathbb{K}^n)$ by $(Ty)_0 = 0$ and

$$(Ty)(t) := \int_0^t Ly_s\,\mathrm{d}s + L\int_0^t \bar{\phi}_s\,\mathrm{d}s \qquad (t \geq 0).$$

We now claim that $T(Y) \subset Y$ holds. Let $y \in Y$ be given. In the same way as in Step 1,

$$|Ty(t)| \le \|L\|(\|\phi\|_1 + t|\phi(0)|) + \|L\| \int_0^t \|y_s\| \, \mathrm{d}s$$

holds for all $t \ge 0$. Since $\mathrm{e}^{-\gamma t}\|L\|(\|\phi\|_1 + t|\phi(0)|) \to 0$ as $t \to \infty$, we only need to show

$$\sup_{t \ge 0} \mathrm{e}^{-\gamma t} \int_0^t \|y_s\| \, \mathrm{d}s < \infty$$

in order to obtain $Ty \in Y$. By the assumption of $y \in Y$, $\|y_t\| \le \|y\|_Y \mathrm{e}^{\gamma t}$ holds for all $t \ge 0$. Therefore, we have

$$\int_0^t \|y_s\| \, \mathrm{d}s \le \|y\|_Y \int_0^t \mathrm{e}^{\gamma s} \, \mathrm{d}s \le \frac{\|y\|_Y}{\gamma} \mathrm{e}^{\gamma t} \qquad (t \ge 0),$$

which implies $\sup_{t \ge 0} \mathrm{e}^{-\gamma t} \int_0^t \|y_s\| \, \mathrm{d}s < \infty$. Thus, $Ty \in Y$ is concluded.

Step 4: Application of contraction mapping principle. We now claim that the mapping $T \colon Y \to Y$ is a contraction. For any $y^1, y^2 \in Y$,

$$\mathrm{e}^{-\gamma t} \left| Ty^1(t) - Ty^2(t) \right| \le \mathrm{e}^{-\gamma t} \|L\| \int_0^t \left\| y^1_s - y^2_s \right\| \mathrm{d}s$$

holds. Since we have

$$\begin{aligned} \left\| y^1_s - y^2_s \right\| &= \mathrm{e}^{\gamma s} \cdot \mathrm{e}^{-\gamma s} \left\| (y^1 - y^2)_s \right\| \\ &\le \mathrm{e}^{\gamma s} \left\| y^1 - y^2 \right\|_Y \end{aligned}$$

for the integrand in the right-hand side,

$$\begin{aligned} \mathrm{e}^{-\gamma t} \|L\| \int_0^t \left\| y^1_s - y^2_s \right\| \mathrm{d}s &\le \frac{\|L\|}{\gamma} (1 - \mathrm{e}^{-\gamma t}) \left\| y^1 - y^2 \right\|_Y \\ &\le \frac{\|L\|}{\gamma} \left\| y^1 - y^2 \right\|_Y \end{aligned}$$

is concluded. Therefore, $T \colon Y \to Y$ is a contraction. By applying the contraction mapping principle, there exists a unique $y_* \in Y$ such that

$$Ty_* = y_*.$$

The function $x_* \colon [-r, \infty) \supset \mathrm{dom}(\phi) \cup [0, \infty) \to \mathbb{K}^n$ defined by

$$x_*(t) := y(t) + \bar{\phi}(t) \qquad (t \in \mathrm{dom}(\phi) \cup [0, \infty))$$

is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi$. The uniqueness follows by the above discussion. □

We hereafter use the following notation.

Notation 2. For each $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$, we denote the unique mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ by $x^L(\cdot\,; \phi) \colon \mathrm{dom}(\phi) \cup [0, \infty) \to \mathbb{K}^n$.

We have the following corollary.

Corollary 2.15. *Let $\alpha, \beta \in \mathbb{K}$ and $\phi, \psi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ be given. Then for all $t \geq 0$,*

$$x^L(t; \alpha\phi + \beta\psi) = \alpha x^L(t; \phi) + \beta x^L(t; \psi) \tag{2.7}$$

holds.

Proof. Let $\chi := \alpha\phi + \beta\psi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ and $x\colon [-r, \infty) \supset \operatorname{dom}(x) \to \mathbb{K}^n$ be the function defined by

$$\operatorname{dom}(x) := \operatorname{dom}(\chi) \cup [0, \infty), \quad x(t) := \alpha x^L(t; \phi) + \beta x^L(t; \psi).$$

Since the map L and the Lebesgue integration are linear, x is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \chi$ by the definition of mild solutions (see Definition 2.5). Therefore, (2.7) is a consequence of Theorem 2.14. □

2.4 Fundamental matrix solutions

Since ODEs are special DDEs, it is natural to expect that the notions of fundamental systems of solutions and fundamental matrix solutions for linear ODEs are meaningful for DDEs in some way. However, the solution space of the linear RFDE (1.1) is infinite-dimensional. Therefore, it is impossible to define these notions to (1.1) as a simple generalization.

A key to this consideration is to focus on a "finite-dimensionality". For this purpose, we consider an "instantaneous input" as an initial history function. We will use the following notation.

Definition 2.16. For each $\xi \in \mathbb{K}^n$, we define a function $\hat{\xi}\colon [-r, 0] \to \mathbb{K}^n$ by

$$\hat{\xi}(\theta) := \begin{cases} 0 & (\theta \in [-r, 0)), \\ \xi & (\theta = 0). \end{cases}$$

$\hat{0}$ is the constant function whose value is identically equal to the zero vector $0 \in \mathbb{K}^n$.

Since $\hat{\xi} \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ for each $\xi \in \mathbb{K}^n$, one can consider the mild solution

$$x^L(\cdot; \hat{\xi})\colon [-r, \infty) \to \mathbb{K}^n$$

of the linear RFDE (1.1) under the initial condition $x_0 = \hat{\xi}$ from Theorem 2.14. Then Corollary 2.15 yields that the subset $\mathcal{S}$ given by

$$\mathcal{S} := \left\{ x^L(\cdot; \hat{\xi})\colon [-r, \infty) \to \mathbb{K}^n : \xi \in \mathbb{K}^n \right\}$$

forms a linear space. We have the following lemma.

Lemma 2.17. *Let $\xi_1, \dots, \xi_m \in \mathbb{K}^n$ be vectors and let $x_j := x^L(\cdot; \hat{\xi}_j)\colon [-r, \infty) \to \mathbb{K}^n$ for each $j \in \{1, \dots, m\}$. Then the following properties are equivalent:*

(a) The system of vectors $\xi_1, \dots, \xi_m$ is linearly independent.

(b) The system of functions $x_1, \dots, x_m$ is linearly independent.

Here the system of functions $x_1, \dots, x_m$ is said to be *linearly independent* if for any scalars $\alpha_1, \dots, \alpha_m$, $\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$ implies $\alpha_1 = \cdots = \alpha_m = 0$.

Proof of Lemma 2.17. (a) $\Rightarrow$ (b): Since $\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$ implies

$$\alpha_1 \xi_1 + \cdots + \alpha_m \xi_m = (\alpha_1 x_1 + \cdots + \alpha_m x_m)(0) = 0,$$

this part follows by the definition of linear independence for functions.

(b) $\Rightarrow$ (a): We suppose $\alpha_1 \xi_1 + \cdots + \alpha_m \xi_m = 0$ for $\alpha_1, \dots, \alpha_m \in \mathbb{K}$. Since this implies

$$\alpha_1 \hat{\xi}_1 + \cdots + \alpha_m \hat{\xi}_m = \hat{0},$$

(2.7) yields

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 0.$$

Therefore, we have $\alpha_1 = \cdots = \alpha_m = 0$ by the assumption (b).

This completes the proof. □

Theorem 2.18. *The linear space $\mathcal{S}$ is n-dimensional.*

Proof. Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ be a basis of $\mathbb{K}^n$. From Lemma 2.17, the system of functions

$$x^L(\cdot; \hat{\boldsymbol{b}}_1), \dots, x^L(\cdot; \hat{\boldsymbol{b}}_n) \in \mathcal{S}$$

is linearly independent. Furthermore, for any $x_1, \dots, x_{n+1} \in \mathcal{S}$, the system of functions is linearly dependent from Lemma 2.17 because the system $x_1(0), \dots, x_{n+1}(0) \in \mathbb{K}^n$ of vectors is linearly dependent. Therefore, the statement holds. □

Theorem 2.18 naturally leads us to the following definition.

Definition 2.19 (cf. [18], [19])**.** We call a basis of the n-dimensional linear space $\mathcal{S}$ a *fundamental system of solutions* to the linear RFDE (1.1). Equivalently, a fundamental system of solutions is the linear independent system

$$x^L(\cdot; \hat{\boldsymbol{b}}_1), \dots, x^L(\cdot; \hat{\boldsymbol{b}}_n) \colon [-r, \infty) \to \mathbb{K}^n$$

for some basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ of $\mathbb{K}^n$. We call a matrix-valued function having a fundamental system of solutions as its column vectors a *fundamental matrix solution*. In particular, we call the fundamental matrix solution

$$X \colon [-r, \infty) \to M_n(\mathbb{K})$$

satisfying $X(0) = I$ the *principal fundamental matrix solution*. Here I denotes the identity matrix.

The above definition is considered as a natural generalization of the corresponding definition for linear ODEs (see [6, Definition 2.12 in Section 2.1 of Chapter 2]). See also [37, Definition 5.10] for a related definition.

We hereafter use the following notation.

Notation 3. Let $X^L \colon [-r, \infty) \to M_n(\mathbb{K})$ denote the principal fundamental matrix solution of the linear RFDE (1.1). By the above definition,

$$X^L(\cdot) = \Big(x^L(\cdot; \hat{\boldsymbol{e}}_1) \ \cdots \ x^L(\cdot; \hat{\boldsymbol{e}}_n) \Big) \tag{2.8}$$

holds. Here $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ denotes the standard basis of $\mathbb{K}^n$.

Remark 2.20. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n$ be given. From (2.8), we have

$$X^L(\cdot)\xi = \xi_1 x^L(\cdot\,;\hat{e}_1) + \cdots + \xi_n x^L(\cdot\,;\hat{e}_n).$$

Here the right-hand side is equal to $x^L(\cdot\,;\xi_1\hat{e}_1 + \cdots + \xi_n\hat{e}_n)$ from (2.7). Therefore,

$$X^L(\cdot)\xi = x^L(\cdot\,;\hat{\xi})$$

holds.

Remark 2.21. We consider an autonomous linear ODE

$$\dot{x} = Ax \tag{2.9}$$

for some $A \in M_n(\mathbb{K})$. For a system of global solutions $y_1, \dots, y_m \colon \mathbb{R} \to \mathbb{K}^n$ to the linear ODE (2.9), the following statements are equivalent:

(a) For any $t \in \mathbb{R}$, $y_1(t), \dots, y_m(t) \in \mathbb{K}^n$ is linearly independent.

(b) For some $t_0 \in \mathbb{R}$, $y_1(t_0), \dots, y_m(t_0) \in \mathbb{K}^n$ is linearly independent.

(c) The system of functions $y_1, \dots, y_m$ is linearly independent.

The nontrivial part is (c) $\Rightarrow$ (a), which is proved by the principle of superposition and by the unique existence of a solution of (2.9) under an initial condition

$$x(t_0) = \xi \in \mathbb{K}^n.$$

Compared with this situation, the linear independence of vectors $x_1(t_0), \dots, x_m(t_0) \in \mathbb{K}^n$ for each $t_0 > 0$ is not necessarily guaranteed for the functions $x_1, \dots, x_m$ in Lemma 2.17 under the assumption that (a) or (b) in Lemma 2.17 holds. This should be compared with an example given by Popov [29], which is a three dimensional system of linear DDEs whose solution values are contained in a hyperplane of $\mathbb{R}^3$ after a certain amount of time has elapsed. See also [19, Section 3.5] and [22, Section 3.5].

2.5 Remarks

2.5.1 Consideration by Delfour

The definition of a mild solution in Definition 2.5 is also related to the consideration by Delfour [8]. In that paper, the author considered a continuous linear map

$$L \colon W^{1,p}((-r,0),\mathbb{R}^n) \to \mathbb{R}^n$$

for some $p \in [1,\infty)$. Here $W^{1,p}((-r,0),\mathbb{R}^n)$ is the Sobolev space (e.g., see Brezis [4, Section 8.2]). The author used the integral representation of L given by

$$L\phi := \int_{-r}^{0} [A_1(\theta)\phi(\theta) + A_2(\theta)\phi'(\theta)]\,\mathrm{d}\theta, \tag{2.10}$$

where $A_1, A_2 \colon (-r,0) \to M_n(\mathbb{R})$ are $n \times n$ real matrix-valued q-integrable functions with $(1/p) + (1/q) = 1$. For the first term of the right-hand side of (2.10), we have

$$\int_0^t \left(\int_{-r}^0 A_1(\theta)x(s+\theta)\,\mathrm{d}\theta \right) \mathrm{d}s = \int_{-r}^0 A_1(\theta) \left(\int_0^t x(s+\theta)\,\mathrm{d}s \right) \mathrm{d}\theta$$

under the exchange of order of integration. Here we have replaced ϕ with x_s and have integrated from 0 to t with respect to s. In view of the above equality, it can be said that the concept of mild solutions in Definition 2.5 is also hidden in [8]. Theorem 2.14 and its proof should be compared with the existence and uniqueness result in [8].

2.5.2 Mild solutions for linear differential difference equations

We consider an autonomous linear *differential difference equation*

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \qquad (t \geq 0) \tag{2.11}$$

for $n \times n$ matrices $A, B_1, \dots, B_m \in M_n(\mathbb{K})$ and $\tau_1, \dots, \tau_m \in (0, r]$. We refer the reader to [2] as a general reference of the theory of differential difference equations.

The linear DDE (2.11) can be expressed in the form of the linear RFDE (1.1) by defining a continuous linear map $L\colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$ by

$$L\psi = A\psi(0) + \sum_{k=1}^{m} B_k \psi(-\tau_k) \tag{2.12}$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$. Let $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ be given and $x := x^L(\cdot\,; \phi)$ for the above continuous linear map L. By the definition of mild solutions (see Definitions 2.5 and 2.7), x satisfies

$$\begin{aligned} x(t) &= \phi(0) + L \int_0^t x_s \, \mathrm{d}s \\ &= \phi(0) + A \int_0^t x(s) \, \mathrm{d}s + \sum_{k=1}^{m} B_k \int_{-\tau_k}^{t-\tau_k} x(s) \, \mathrm{d}s \end{aligned}$$

for all $t \geq 0$. Since the last term is equal to

$$\phi(0) + \int_0^t Ax(s) \, \mathrm{d}s + \sum_{k=1}^{m} \int_{-\tau_k}^{t-\tau_k} B_k x(s) \, \mathrm{d}s,$$

x also satisfies

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \qquad (\text{a.e. } t \geq 0)$$

by the Lebesgue differentiation theorem (see Subsection 3.1).

3 Differential equation satisfied by principal fundamental matrix solution

In this section, we consider the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0)$$

for a continuous linear map $L\colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$. We choose a matrix-valued function $\eta\colon [-r, 0] \to M_n(\mathbb{K})$ of bounded variation so that L is represented as the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta) \, \psi(\theta)$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$. We recall that the domain of definition of η is extended to $(-\infty, 0]$ by letting $\eta(\theta) := \eta(-r)$ for $\theta \in (-\infty, -r]$. We will use the following notation.

Notation 4. Let $\check{\eta}\colon [0,\infty) \to M_n(\mathbb{K})$ be the function given by

$$\check{\eta}(u) := -\eta(-u)$$

for $u \in [0,\infty)$.

In this paper, a function defined on $[0,\infty)$ is said to be of *locally bounded variation* if it is of bounded variation on any closed and bounded interval of $[0,\infty)$. A function of locally bounded variation is also called a *locally BV function*. Then the above function $\check{\eta}$ is a function of locally bounded variation whose value is constant on $[r,\infty)$. It is related to the reversal formula for Riemann–Stieltjes integrals (see Theorem A.9).

It will turn out that the notions of Volterra operator and Riemann–Stieltjes convolution are useful to deduce a differential equation that is satisfied by the principal fundamental matrix solution $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ of the linear RFDE (1.1).

3.1 Definitions

Definition 3.1. For each $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, let $Vf\colon [0,\infty) \to M_n(\mathbb{K})$ be the function defined by

$$(Vf)(t) = \int_0^t f(s)\,\mathrm{d}s. \tag{3.1}$$

Here the right-hand side is a Lebesgue integral. We call V the *Volterra operator*.

For details related to the Volterra operator as a linear operator on $C([0,T],\mathbb{K})$ for each $T > 0$, see [31]. By using the Lebesgue differentiation theorem (e.g., see [32, Theorem 1.3 in Section 1 of Chapter 3]) component-wise, it holds that Vf is *locally absolutely continuous* (i.e., locally absolutely continuous on any closed and bounded interval of $[0,\infty)$), differentiable almost everywhere on $[0,\infty)$, and

$$(Vf)'(t) = f(t)$$

holds for almost all $t \in [0,\infty)$.

Definition 3.2. For each function $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ of locally bounded variation and for each continuous function $f\colon [0,\infty) \to M_n(\mathbb{K})$, we define a function $\mathrm{d}\alpha * f\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(\mathrm{d}\alpha * f)(t) := \int_0^t \mathrm{d}\alpha(u)\, f(t-u).$$

Here the right-hand side is a Riemann–Stieltjes integral. This function is called a *Riemann–Stieltjes convolution*.

See [31, Definition 10.3 in Section 10.3] for the scalar-valued case. The above definition should be compared with the treatment in [34, Eq. (2.13) in Chapter 2] and [14, Corollary 2.5 in Section I.2 of Appendix I], where an appearing integral is not a Riemann–Stieltjes integral but a Lebesgue–Stieltjes integral.

3.2 Motivation

The following lemma motivates the use of Volterra operator and Riemann–Stieltjes convolution.

Lemma 3.3. *If $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ satisfies $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$, then*

$$L\int_0^t x_s\,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right) = \int_0^t \mathrm{d}\check{\eta}(u)\left(\int_0^{t-u} x(s)\,\mathrm{d}s\right)$$

holds for all $t \ge 0$.

Proof. Let $t > 0$ be fixed. By the assumption, $\int_\theta^0 x(s)\,\mathrm{d}s = 0$ holds for all $\theta \in [-r,0]$. Therefore, we have

$$L\int_0^t x_s\,\mathrm{d}s = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^{t+\theta} x(s)\,\mathrm{d}s\right) = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right).$$

We examine the last term by dividing the consideration into the following cases:

- Case: $t \in [0,r)$. In this case, $t+\theta \ge 0$ is equivalent to $\theta \ge -t$ for each $\theta \in [-r,0]$. Since $\int_0^{t+\theta} x(s)\,\mathrm{d}s = 0$ for $\theta \in [-r,-t)$,

  $$L\int_0^t x_s\,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right)$$

  holds by the additivity of Riemann–Stieltjes integrals on sub-intervals.

- Case: $t \ge r$. In this case, $t+\theta \ge 0$ holds for all $\theta \in [-r,0]$. Since η is constant on $[-t,-r]$,

  $$L\int_0^t x_s\,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right)$$

  holds.

Therefore, the expressions of $L\int_0^t x_s\,\mathrm{d}s$ are obtained in combination with the reversal formula for Riemann–Stieltjes integrals (see Theorem A.9). □

3.3 Properties of Volterra operator and Riemann–Stieltjes convolution

Throughout this subsection, let $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation and $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function.

3.3.1 Continuity and local integrability

The following is a simple result about the continuity of Riemann–Stieltjes convolution.

Lemma 3.4. *If $f(0) = O$, then $\mathrm{d}\alpha * f$ is continuous.*

Proof. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0) = O$ for $t \le 0$. Then the obtained function $f\colon \mathbb{R} \to M_n(\mathbb{K})$ is continuous. Let $s,t \in [0,\infty)$ be given so that $s < t$. By the additivity of Riemann–Stieltjes integrals on sub-intervals,

$$\begin{aligned}(\mathrm{d}\alpha * f)(s) &= \int_0^s \mathrm{d}\alpha(u)\,f(s-u)\\ &= \int_0^t \mathrm{d}\alpha(u)\,f(s-u) - \int_s^t \mathrm{d}\alpha(u)\,f(s-u)\end{aligned}$$

holds. Since

$$\int_s^t \mathrm{d}\alpha(u)\,f(s-u) = [\alpha(t)-\alpha(s)]f(0) = O,$$

we have

$$(\mathrm{d}\alpha * f)(t) - (\mathrm{d}\alpha * f)(s) = \int_0^t \mathrm{d}\alpha(u)\,[f(t-u) - f(s-u)].$$

By combining this and the uniform continuity of f on closed and bounded intervals, the continuity of $\mathrm{d}\alpha * f$ is obtained. □

See [31, Lemma 10.4 in Section 10.3] for the corresponding result for scalar-valued functions. In this paper, we say that a function is *locally Riemann integrable* if it is Riemann integrable on any closed and bounded interval.

Theorem 3.5. $\mathrm{d}\alpha * f$ *is a sum of a continuous function and a function of locally bounded variation. Consequently,* $\mathrm{d}\alpha * f$ *is locally Riemann integrable.*

Proof. By using $f = (f - f(0)) + f(0)$, we have

$$\mathrm{d}\alpha * f = \mathrm{d}\alpha * (f - f(0)) + \mathrm{d}\alpha * f(0). \tag{3.2}$$

The first term in the right-hand side is continuous from Lemma 3.4. The second term is of locally bounded variation since

$$(\mathrm{d}\alpha * f(0))(t) = [\alpha(t) - \alpha(0)]f(0)$$

holds for all $t \geq 0$. Therefore, the conclusion holds. □

Remark 3.6. Theorem 3.5 yields that $V(\mathrm{d}\alpha * f)$ makes sense. Furthermore if α is continuous, then (3.2) shows that $\mathrm{d}\alpha * f$ is also continuous.

3.3.2 Riemann–Stieltjes convolution under Volterra operator

The Riemann–Stieltjes convolution and Volterra operator are related in the following way.

Theorem 3.7. *The equality*

$$V(\mathrm{d}\alpha * f) = \mathrm{d}\alpha * Vf \tag{3.3}$$

holds. Consequently, $\mathrm{d}\alpha * Vf$ *is locally absolutely continuous, differentiable almost everywhere, and satisfies*

$$(\mathrm{d}\alpha * Vf)'(t) = (\mathrm{d}\alpha * f)(t)$$

holds for almost all $t \geq 0$.

For the proof, we need the following theorem. It contains the result on iterated Riemann integrals for continuous functions on rectangles as a special case.

Theorem 3.8. *Let* $[a,b]$ *and* $[c,d]$ *be closed and bounded intervals of* $\mathbb{R}$. *If* $f\colon [a,b] \times [c,d] \to M_n(\mathbb{K})$ *is continuous and* $\alpha\colon [a,b] \to M_n(\mathbb{K})$ *is a function of bounded variation, then*

$$\int_a^b \mathrm{d}\alpha(x) \left(\int_c^d f(x,y)\,\mathrm{d}y \right) = \int_c^d \left(\int_a^b \mathrm{d}\alpha(x)\, f(x,y) \right) \mathrm{d}y \tag{3.4}$$

holds.

See also [39, Theorem 15a in Section 15 of Chapter I]. We will give the proof in Appendix A.7.

Proof of Theorem 3.7. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0)$ for $t \le 0$. By the proof of Lemma 3.4, we have

$$V(\mathrm{d}\alpha * f)(t) = \int_0^t \left(\int_0^t \mathrm{d}\alpha(u) f(s-u) \right) \mathrm{d}s - \int_0^t [\alpha(t) - \alpha(s)] f(0) \,\mathrm{d}s$$

for $t \ge 0$, where

$$\int_0^t \left(\int_0^t \mathrm{d}\alpha(u) f(s-u) \right) \mathrm{d}s = \int_0^t \mathrm{d}\alpha(u) \left(\int_0^t f(s-u) \,\mathrm{d}s \right)$$

holds from Theorem 3.8. The last term is expressed by

$$(\mathrm{d}\alpha * Vf)(t) + \int_0^t \mathrm{d}\alpha(u) \int_{-u}^0 f(s) \,\mathrm{d}s$$

by using the Volterra operator and the Riemann–Stieltjes convolution. Since $\int_{-u}^0 f(s)\,\mathrm{d}s = uf(0)$ for $u \in [0,t]$, the proof is complete by showing

$$\int_0^t [\alpha(t) - \alpha(s)] \,\mathrm{d}s = \int_0^t u \,\mathrm{d}\alpha(u).$$

This is indeed true because

$$\begin{aligned} \int_0^t u \,\mathrm{d}\alpha(u) &= [u\alpha(u)]_{u=0}^t - \int_0^t \alpha(u)\,\mathrm{d}u \\ &= t\alpha(t) - \int_0^t \alpha(u)\,\mathrm{d}u \end{aligned}$$

holds by the integration by parts formula for Riemann–Stieltjes integrals. See Theorem A.10 for the detail. □

The following is a corollary of Theorem 3.7. It will not be used in the sequel.

Corollary 3.9. *Furthermore, if f is continuously differentiable, then $\mathrm{d}\alpha * f$ is expressed by*

$$\mathrm{d}\alpha * f = (\alpha - \alpha(0)) f(0) + V\big(\mathrm{d}\alpha * f'\big).$$

Consequently, $\mathrm{d}\alpha * f$ is of locally bounded variation, differentiable almost everywhere, and satisfies

$$(\mathrm{d}\alpha * f)'(t) = \alpha'(t) f(0) + (\mathrm{d}\alpha * f')(t)$$

for almost all $t \ge 0$.

Proof. By the fundamental theorem of calculus, $f = f(0) + Vf'$ holds. By combining this and (3.3), the expression of $\mathrm{d}\alpha * f$ is obtained. Since $V(\mathrm{d}\alpha * f')$ is locally absolutely continuous, it is also of locally bounded variation. Therefore, the expression of $\mathrm{d}\alpha * f$ yields that $\mathrm{d}\alpha * f$ is of locally bounded variation. The remaining properties are consequences of the fact that matrix-valued functions of bounded variation are differentiable almost everywhere. This is obtained by applying the corresponding result[1] for real-valued functions component-wise. □

[1]See [32, Theorem 3.4 in Subsection 3.1 of Chapter 3], for example.

3.4 Differential equation and principal fundamental matrix solution

As an application of Theorem 3.7, one can derive a differential equation that is satisfied by $x^L(\cdot\,;\hat{\xi})$ for each $\xi \in \mathbb{K}^n$.

Theorem 3.10. *Let $x := x^L(\cdot\,;\hat{\xi})$ for some $\xi \in \mathbb{K}^n$. Then x satisfies*

$$x(t) = \xi + \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_0^{t+\theta} x(s)\,\mathrm{d}s \right) = \xi + \left(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\right)(t) \tag{3.5}$$

for all $t \geq 0$. Furthermore, $x|_{[0,\infty)}$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$\dot{x}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, x(t+\theta) = \left(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\right)(t) \tag{3.6}$$

for almost all $t \in [0,\infty)$.

Proof. By definition, x satisfies

$$x(t) = \xi + L \int_0^t x_s\,\mathrm{d}s$$

for all $t \geq 0$, and $x|_{[0,\infty)}$ is continuous. Then Eq. (3.5) is a consequence of Lemma 3.3. Theorem 3.7 and Eq. (3.5) yield that

$$x(t) = \xi + V\left(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\right)(t)$$

holds for all $t \geq 0$. Therefore, it holds that $x|_{[0,\infty)}$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$\dot{x}(t) = \left(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\right)(t)$$

for almost all $t \geq 0$. The remaining expression in Eq. (3.6) is a consequence of the reversal formula for Riemann–Stieltjes integrals. □

The following theorem also holds.

Theorem 3.11. *Let $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ be given so that $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$. If x satisfies (3.5) for all $t \geq 0$, then $x = x^L(\cdot\,;\hat{\xi})$.*

Proof. From Lemma 3.3, x satisfies

$$x(t) = \xi + L \int_0^t x_s\,\mathrm{d}s$$

for all $t \geq 0$. From Lemma 2.8 and by the continuity of L, the right-hand side is continuous with respect to $t \geq 0$. Therefore, x is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \hat{\xi}$. By the uniqueness (see Theorem 2.14), the conclusion is obtained. □

We obtain the following result as a direct consequence of Theorem 3.10 and (2.8). We omit the proof.

Theorem 3.12 (cf. [34])**.** *The principal fundamental matrix solution $X^L \colon [-r,\infty) \to M_n(\mathbb{K})$ of the linear RFDE (1.1) satisfies*

$$X^L(t) = I + \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_0^{t+\theta} X^L(s)\,\mathrm{d}s \right) = I + \left(\mathrm{d}\check{\eta} * VX^L|_{[0,\infty)}\right)(t) \tag{3.7}$$

for all $t \geq 0$. Furthermore, $X^L|_{[0,\infty)}$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$\dot{X}^L(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, X^L(t+\theta) = \big(\mathrm{d}\check{\eta} * X^L|_{[0,\infty)}\big)(t) \tag{3.8}$$

for almost all $t \in [0,\infty)$.

Roughly speaking, Eq. (3.8) is used as the defining equation of the fundamental matrix solution in [34, (9.1) in Chapter 9].

4 Non-homogeneous linear RFDEs

In this section, we study a non-homogeneous linear RFDE (1.2)

$$\dot{x}(t) = Lx_t + g(t) \qquad (\text{a.e. } t \geq 0)$$

for a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ and some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$.

4.1 Non-homogeneous linear RFDE and mild solutions

It is natural to define the notion of mild solutions to Eq. (1.2) in the following way.

Definition 4.1. Let $t_0 \geq 0$ and $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. We say that a function $x\colon [t_0 - r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ is a *mild solution* of Eq. (1.2) under the initial condition $x_{t_0} = \phi$ if the following conditions are satisfied: (i) $x_{t_0} = \phi$, (ii) $[t_0,\infty) \subset \mathrm{dom}(x)$, (iii) $x|_{[t_0,\infty)}$ is continuous, and (iv) for all $t \geq t_0$,

$$x(t) = \phi(0) + L\int_{t_0}^{t} x_s \,\mathrm{d}s + \int_{t_0}^{t} g(s)\,\mathrm{d}s$$

holds.

We note that $\int_{t_0}^{t} x_s\,\mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ is defined by

$$\left(\int_{t_0}^{t} x_s\,\mathrm{d}s\right)(\theta) := \int_{t_0}^{t} x(s+\theta)\,\mathrm{d}s = \int_{t_0+\theta}^{t+\theta} x(s)\,\mathrm{d}s$$

for $\theta \in [-r,0]$, and

$$\mathrm{dom}(x) = (t_0 + \mathrm{dom}(\phi)) \cup [t_0,\infty) = t_0 + (\mathrm{dom}(\phi) \cup [0,\infty))$$

holds for a mild solution of Eq. (1.2) under the initial condition $x_{t_0} = \phi$.

Lemma 4.2. *Let $t_0 \geq 0$ and $\phi \in C([-r,0],\mathbb{K}^n)$ be given. If $x\colon [t_0 - r,\infty) \to \mathbb{K}^n$ is a mild solution of Eq. (1.2) under the initial condition $x_{t_0} = \phi$, then x satisfies*

$$\dot{x}(t) = Lx_t + g(t)$$

for almost all $t \geq t_0$.

Proof. By the translation, we may assume $t_0 = 0$. Since $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ is a bounded linear operator,

$$x(t) = \phi(0) + \int_0^t Lx_s\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}s$$

holds for all $t \geq 0$ from Corollary 2.12. Then the fundamental theorem of calculus and the Lebesgue differentiation theorem yield that $x|_{[0,\infty)}$ is differentiable almost everywhere and

$$\dot{x}(t) = Lx_t + g(t)$$

holds for almost all $t \geq 0$. $\square$

Remark 4.3. Let $\mathbb{K} = \mathbb{R}$. We assume that $\operatorname{dom}(g) = [0,\infty)$ and consider the function $F\colon [0,\infty) \times C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$F(t,\phi) := L\phi + g(t).$$

Then F satisfies the Carathéodory condition. See [19, Section 2.6 of Chapter 2] and [22, Section 2.6 of Chapter 2] for the detail of the Carathéodory condition for RFDEs. Lemma 4.2 shows that a mild solution $x\colon [t_0 - r,\infty) \to \mathbb{R}^n$ of Eq. (1.2) under the initial condition $x_{t_0} = \phi \in C([-r,0],\mathbb{R}^n)$ is a solution (in the Carathéodory sense).

4.2 Integral equation with a general forcing term

More generally, for a given $t_0 \geq 0$ and a given continuous function $G\colon [t_0,\infty) \to \mathbb{K}^n$ with $G(t_0) = 0$, we can discuss a solution of the following integral equation

$$x(t) = \phi(0) + L\int_{t_0}^{t} x_s \,\mathrm{d}s + G(t) \qquad (t \geq t_0) \tag{4.1}$$

under an initial condition $x_{t_0} = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$. Here the assumption $G(t_0) = 0$ is natural because the right-hand side of (4.1) is equal to

$$\phi(0) + G(t_0)$$

at $t = t_0$. The notion of a solution of (4.1) can be defined in a similar way as in Definition 4.1. The following theorem holds.

Theorem 4.4. *Let $t_0 \geq 0$ be given. Suppose that $G\colon [t_0,\infty) \to \mathbb{K}^n$ is a continuous function with $G(t_0) = 0$. Then for any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, Eq. (4.1) has a unique solution under the initial condition $x_{t_0} = \phi$.*

The following proof should be compared with the proof of Theorem 2.14.

Proof of Theorem 4.4. By the translation, it is sufficient to consider the case $t_0 = 0$. We will solve the integral equation locally and will connect the obtained local solutions. For this purpose, we need to consider an integral equation under the initial condition $x_\sigma = \psi$ for each $\sigma \geq 0$ and each $\psi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$. Here an appropriate forcing term is given by

$$G(t,\sigma) := G(t) - G(\sigma)$$

for $t \geq \sigma$. Then we are going to consider an integral equation

$$x(t) = \psi(0) + L\int_{\sigma}^{t} x_s \,\mathrm{d}s + G(t,\sigma) \qquad (t \geq \sigma) \tag{4.2}$$

under the initial condition $x_\sigma = \psi$. The remainder of the proof is divided into the following steps.

Step 1: Existence and uniqueness of a local solution. We fix the above σ and ψ. By defining a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ by

$$y(s) := \begin{cases} x(\sigma+s) - \bar{\psi}(s) & (\sigma + s \in \mathrm{dom}(x)), \\ 0 & (\sigma+s \notin \mathrm{dom}(x)), \end{cases}$$

Eq. (4.2) is transformed into

$$y(s) = \int_0^s Ly_u \,\mathrm{d}u + L\int_0^s \bar{\psi}_u \,\mathrm{d}u + G(\sigma+s,\sigma) \quad (s \geq 0),$$

which is an integral equation under the initial condition $y_0 = 0$. We choose a constant $a > 0$ so that

$$\|L\|a < 1$$

and consider a closed subset Y of the Banach space $C([-r,a],\mathbb{K}^n)$ given by

$$Y := \{y \in C([-r,a],\mathbb{K}^n) : y_0 = 0\}.$$

Furthermore, we define a transformation $T\colon Y \to Y$ by $(Ty)_0 = 0$ and

$$(Ty)(s) := \int_0^s Ly_u \,\mathrm{d}u + L\int_0^s \bar{\psi}_u \,\mathrm{d}u + G(\sigma+s,\sigma) \quad (s \geq 0).$$

Then it holds that T is contractive, and the application of the contraction mapping principle yields the unique existence of a fixed point y_* of T. By defining a function $x_*\colon [\sigma-r,\sigma+a] \to \mathbb{K}^n$ by

$$x_*(\sigma+s) := y_*(s) + \bar{\psi}(s) \quad (s \in \mathrm{dom}(\psi) \cup [0,a]),$$

it is concluded that x_* is a solution of Eq. (4.2). We note that such a local solution is unique by the choice of the above a.

Step 2: Existence and uniqueness of a (global) solution. We note that the time $a > 0$ of existence of a local solution to Eq. (4.2) in Step 1 does not depend on the considered integral equation (4.2) and the specified initial condition $x_\sigma = \psi$. In this step, we will show that by connecting these local solutions, we obtain a global solution. For this purpose, for each $\sigma \geq 0$ and each $\psi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, let

$$x(\,\cdot\,;\sigma,\psi)\colon [\sigma-r,\sigma+a] \to \mathbb{K}^n$$

be the obtained unique solution of Eq. (4.2) under an initial condition $x_\sigma = \psi$. We fix σ and ψ. Let

$$x := x(\,\cdot\,;\sigma,\psi) \quad \text{and} \quad y := x(\,\cdot\,;\sigma+a,x_{\sigma+a}).$$

We now claim that the function $z\colon [\sigma-r,\sigma+2a] \to \mathbb{K}^n$ defined by

$$z(t) := \begin{cases} x(t) & (t \in [\sigma-r,\sigma+a]) \\ y(t) & (t \in [\sigma+a-r,\sigma+2a]) \end{cases}$$

is a solution to Eq. (4.2). We note that this definition makes sense because $y_{\sigma+a} = x_{\sigma+a}$. To show the claim, it is sufficient to consider the case $t \in [\sigma+a,\sigma+2a]$. In this case, we have

$$z(t) = y(t) = x_{\sigma+a}(0) + L\int_{\sigma+a}^t y_s \,\mathrm{d}s + G(t,\sigma+a),$$

where

$$x_{\sigma+a}(0) = x(\sigma+a) = \phi(0) + L\int_{\sigma}^{\sigma+a} x_s \, \mathrm{d}s + G(\sigma+a,\sigma).$$

In the above equations, one can replace y_s and x_s with z_s. Therefore, in view of

$$G(t,\sigma+a) + G(\sigma+a,\sigma) = G(t,\sigma),$$

it holds that z is a solution of Eq. (4.2) under the initial condition $x_{t_0} = \phi$.

By repeating the above procedure, a global solution of the original integral equation (4.1) is obtained. By the uniqueness of each local solution, such a global solution is unique. □

Remark 4.5. Let $\mathbb{K} = \mathbb{R}$. In [21], Hale and Meyer studied the following equation

$$x(t) = \phi(0) + g(t,x_t) - g(t_0,\phi) + \int_{t_0}^{t} f(s,x_s)\,\mathrm{d}s + \int_{t_0}^{t} h(s)\,\mathrm{d}s$$

under an initial condition $x_{t_0} = \phi \in C([-r,0],\mathbb{R}^n)$ for each $t_0 \in \mathbb{R}$. Here

$$f,g \colon \mathbb{R} \times C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$$

are continuous maps with the properties that

$$C([-r,0],\mathbb{R}^n) \ni \phi \mapsto f(t,\phi) \in \mathbb{R}^n \quad \text{and} \quad C([-r,0],\mathbb{R}^n) \ni \phi \mapsto g(t,\phi) \in \mathbb{R}^n$$

are linear for each $t \in \mathbb{R}$, and $h\colon \mathbb{R} \to \mathbb{R}^n$ is a locally Lebesgue integrable function. In [21, Theorem 1 in Chapter II], it is shown that the above problem has a unique solution under an additional assumption of the non-atomicity of g at 0. See [21, Chapter I] for the detail of this condition. The proof of Theorem 4.4 should be compared with [21, Proof of Theorem 1 in Chapter II].

We hereafter use the following notation.

Notation 5. Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and

$$\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$$

be given. The unique solution of Eq. (1.6)

$$x(t) = \phi(0) + L\int_0^t x_s \,\mathrm{d}s + G(t) \qquad (t \ge 0)$$

is denoted by $x^L(\cdot;\phi,G)\colon [-r,\infty) \to \mathbb{K}^n$. Then $x^L(\cdot;\phi,0) = x^L(\cdot;\phi)$.

We obtain the following corollary. It will be a basics of considering a variation of constants formula for Eq. (1.6).

Corollary 4.6. *For any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ and any continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$,*

$$x^L(\cdot;\phi,G) = x^L(\cdot;\phi,0) + x^L(\cdot;0,G)$$

holds.

Proof. Let $x := x^L(\cdot;\phi,0) + x^L(\cdot;0,G)$. Then x satisfies $x_0 = \phi$. Furthermore, we have

$$\begin{aligned} x(t) &= x^L(t;\phi,0) + x^L(t;0,G) \\ &= \phi(0) + L\int_0^t x^L(\cdot;\phi,0)_s \,\mathrm{d}s + L\int_0^t x^L(\cdot;0,G)_s \,\mathrm{d}s + G(t) \end{aligned}$$

for all $t \ge 0$. Since the last term is equal to

$$\phi(0) + L\int_0^t x_s \,\mathrm{d}s + G(t)$$

by the linearity of L, Theorem 4.4 yields $x = x^L(\cdot;\phi,G)$. □

In the same way as in Theorems 3.10 and 3.11 under Theorem 4.4, we obtain the following theorems. The proof can be omitted.

Theorem 4.7. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and $x := x^L(\cdot;\hat{\xi},G)$ for some $\xi \in \mathbb{K}^n$. Then x satisfies*

$$x(t) = \xi + \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right) + G(t) = \xi + \left(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\right)(t) + G(t) \tag{4.3}$$

for all $t \ge 0$.

Theorem 4.8. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and*

$$x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$$

be given so that $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$. If x satisfies (4.3) for all $t \ge 0$, then $x = x^L(\cdot;\hat{\xi},G)$.

5 Convolution and Volterra operator

5.1 A motivation to introduce convolution

5.1.1 Variation of constants formula for non-homogeneous linear ODEs

As a motivation to introduce convolution for locally Riemann integrable functions on $[0,\infty)$, we first recall the variation of constants formula for a non-homogeneous linear ODE

$$\dot{x} = Ax + f(t) \tag{5.1}$$

for an $n \times n$ matrix $A \in M_n(\mathbb{K})$ and a continuous function $f\colon \mathbb{R} \to \mathbb{K}^n$. The unique global solution $x^A(\cdot;t_0,\xi,f)\colon \mathbb{R} \to \mathbb{K}^n$ of Eq. (5.1) satisfying an initial condition $x(t_0) = \xi \in \mathbb{K}^n$ is expressed by

$$x^A(t;t_0,\xi,f) = \mathrm{e}^{tA}\left[\mathrm{e}^{-t_0A}\xi + \int_{t_0}^t \mathrm{e}^{-uA} f(u)\,\mathrm{d}u\right] \quad (t \in \mathbb{R}) \tag{5.2}$$

with the matrix exponential. This is the *variation of constants formula* for (5.1), which is obtained by finding an equation of $y = y(t)$ under the change of variable $x(t) = \mathrm{e}^{tA}y(t)$. Indeed, the function y must satisfy an initial condition $y(t_0) = \mathrm{e}^{-t_0A}\xi$ and

$$\dot{y}(t) = \mathrm{e}^{-tA}f(t) \quad (t \in \mathbb{R}).$$

This procedure to derive the formula (5.2) corresponds to replacing a constant vector $v \in \mathbb{K}^n$ in the general solution

$$x(t) = \mathrm{e}^{tA}v$$

for the linear ODE (2.9) with a vector-valued function $y = y(t)$. This is the reason for the terminology of the variation of constants formula.

The above method to derive (5.2) should be called the *method of variation of constants*. Unfortunately, this method does not exist for a non-homogeneous linear RFDE (1.2) because the solution space of the linear RFDE (1.1) is infinite-dimensional and (1.1) does not have a general solution. Even if the method itself does not exist for (1.2), a formula similar to (5.2) if it exists will be useful to analyze the dynamics of RFDEs near equilibria. For this purpose, a form

$$x^A(t;\xi,f) = \mathrm{e}^{tA}\xi + \int_0^t \mathrm{e}^{(t-u)A} f(u)\,\mathrm{d}u, \tag{5.3}$$

which is equivalent to (5.2) is helpful. Here the initial time t_0 is set to 0, and it has been omitted in $x^A(t;\xi,f)$. The first term of the right-hand side of (5.3) is the solution of the linear ODE (2.9) under the initial condition $x(0) = \xi$. Therefore, the second term of the right-hand side of (5.3) is the solution of (5.1) under the initial condition $x(0) = 0$. This can be checked directly by differentiating the second term as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-u)A} f(u)\,\mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{tA}\int_0^t \mathrm{e}^{-uA} f(u)\,\mathrm{d}u\right] \\ &= f(t) + A\mathrm{e}^{tA}\int_0^t \mathrm{e}^{-uA} f(u)\,\mathrm{d}u. \end{aligned}$$

We note that this gives another proof of (5.3).

5.1.2 Convolution and non-homogeneous linear RFDEs

For a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ and a continuous function $f\colon [0,\infty) \to \mathbb{K}^n$, we consider the non-homogeneous linear RFDE (1.7)

$$\dot{x}(t) = Lx_t + f(t) \qquad (t \geq 0).$$

Since $\mathbb{R} \ni t \mapsto \mathrm{e}^{tA} \in M_n(\mathbb{K})$ is the *principal fundamental matrix solution* of the linear ODE (2.9) in the sense that it is a matrix solution to (2.9) and $\mathrm{e}^{tA}|_{t=0}$ is the identity matrix, it is natural to ask whether the function $x(\cdot\,;f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot\,;f)_0 = 0$ and (1.8)

$$x(t;f) := \int_0^t X^L(t-u)f(u)\,\mathrm{d}u$$

for $t \geq 0$ is a solution to Eq. (1.7). Here $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ is the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0).$$

In Theorem 3.12, we obtained the differential equation that is satisfied by X^L. However, one can not directly prove that the function $x(\cdot\,;f)$ is a solution to (1.7) by differentiating the right-hand side of (1.8) as in the case of the non-homogeneous linear ODE (5.1) because one cannot take the term $X^L(t)$ out of the integral. This comes from the property that initial value problems of RFDEs cannot be solved backward in general. Therefore, one needs to treat the integral of the right-hand side of (1.8) as it is.

Such an integral is a convolution for locally (Riemann) integrable functions, which should be distinguished from the convolution for integrable functions. The convolution for locally integrable functions has been used in the literature of DDEs. For example, see [2, Chapter 1] with the context of the Laplace transform. The convolution is also used in [34] and [14], however, the detail has been omitted there.

5.2 Convolution and Riemann–Stieltjes convolution

In this subsection, we study a convolution of the following type.

Definition 5.1. For each pair of locally Riemann integrable functions $f, g\colon [0,\infty) \to M_n(\mathbb{K})$, we define a function $g * f\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(g * f)(t) := \int_0^t g(t-u)f(u)\,\mathrm{d}u = \int_0^t g(u)f(t-u)\,\mathrm{d}u$$

for $t \geq 0$. Here the above integrals are Riemann integrals. We call the function $g * f$ the *convolution* of g and f.

See [31, Section 5.3] for the convolution of continuous functions. We note that when f is a constant function,

$$(g * f)(t) = \int_0^t g(u)f(0)\,\mathrm{d}u = (Vg)(t)f(0) \tag{5.4}$$

holds for all $t \geq 0$. In the same way, $g * f = g(0)Vf$ holds when g is constant.

Lemma 5.2 (cf. [31]). *Let $f, g\colon [0,\infty) \to M_n(\mathbb{K})$ be locally Riemann integrable functions. If f is continuous, then $g * f$ is a sum of a continuous function and a locally absolutely continuous function.*

Proof. By using (5.4),

$$g * f = g * (f - f(0)) + (Vg)f(0)$$

holds. Therefore, the conclusion is obtained by showing that $g * f$ is continuous when $f(0) = O$. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0) = O$ for $t \leq 0$. Let $s, t \in [0,\infty)$ be given so that $s < t$. By the same reasoning as in the proof of Lemma 3.4, we have

$$(g * f)(t) - (g * f)(s) = \int_0^t g(u)[f(t-u) - f(s-u)]\,\mathrm{d}u.$$

By combining this and the uniform continuity of f on closed and bounded intervals, the continuity of $g * f$ is obtained. □

5.2.1 Convolution of locally BV functions and continuous functions

By using Theorem 3.7, one can obtain the following result on the regularity of convolution.

Theorem 5.3 (cf. [33]). *If $f\colon [0,\infty) \to M_n(\mathbb{K})$ is continuous and $g\colon [0,\infty) \to M_n(\mathbb{K})$ is of locally bounded variation, then*

$$g * f = g(0)Vf + \mathrm{d}g * (Vf) = V(g(0)f + \mathrm{d}g * f) \tag{5.5}$$

holds. Consequently, the convolution $g * f$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$(g * f)'(t) = g(0)f(t) + (\mathrm{d}g * f)(t)$$

for almost all $t \geq 0$.

The above result is considered as the finite-dimensional version of [33, Theorem 3.2] (i.e., the case that the Banach space X in [33, Theorem 3.2] is finite-dimensional) except for the equality

$$g * f = g(0)Vf + \mathrm{d}g * (Vf).$$

In the following, we give a simpler proof of Theorem 5.3 based on Theorem 3.7.

Proof of Theorem 5.3. Since Vf is continuously differentiable and $(Vf)(0) = O$,

$$\begin{aligned}[\mathrm{d}g * (Vf)](t) &= [g(u)(Vf)(t-u)]_{u=0}^{t} + \int_0^t g(u)f(t-u)\,\mathrm{d}u \\ &= -g(0)(Vf)(t) + (g * f)(t)\end{aligned}$$

holds for all $t \geq 0$ by the integration by parts formula for Riemann–Stieltjes integrals and from Theorem A.19. By combining the obtained equality

$$g * f = g(0)(Vf) + \mathrm{d}g * (Vf)$$

and Theorem 3.7, the equality (5.5) is obtained. □

Remark 5.4. Let $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. By defining a function $V(\mathrm{d}g)\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$V(\mathrm{d}g)(t) := g(t) - g(0)$$

for $t \geq 0$, we have

$$V(\mathrm{d}g * f) = \mathrm{d}g * (Vf) = V(\mathrm{d}g) * f$$

from Theorems 3.7 and 5.3. This formula is easy to remember. We note that the above definition of $V(\mathrm{d}g)$ is reasonable because

$$\int_0^t \mathrm{d}g(u) = g(t) - g(0)$$

holds for all $t \geq 0$.

We have the following corollaries.

Corollary 5.5. *If $f\colon [0,\infty) \to M_n(\mathbb{K})$ is continuous and $g\colon [0,\infty) \to M_n(\mathbb{K})$ is of locally bounded variation, then*

$$V(g * f) = g * (Vf)$$

holds.

Proof. From Theorems 5.3 and 3.7, we have

$$V(g * f) = g(0)\big(V^2 f\big) + \mathrm{d}g * \big(V^2 f\big),$$

where $V^2 f := V(Vf)$. Since the right-hand side is equal to $g * (Vf)$ from Theorem 5.3, the equality is obtained. □

Corollary 5.6. *Let $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. Then the following statements hold:*

1. *If g is continuous or $f(0) = O$, then $g * f$ is continuously differentiable and*
$$(g * f)' = g(0)f + \mathrm{d}g * f$$
holds.

2. *If g is locally absolutely continuous, then $g * f$ is continuously differentiable and*
$$(g * f)' = g(0)f + g' * f$$
holds. Here $g' * f\colon [0,\infty) \to M_n(\mathbb{K})$ is the function defined by
$$(g' * f)(t) := \int_0^t g'(t-u)f(u)\,\mathrm{d}u = \int_0^t g'(u)f(t-u)\,\mathrm{d}u$$
for $t \geq 0$, where the integrals are Lebesgue integrals.

Proof. 1. Under the assumption, $\mathrm{d}g * f$ is continuous from Lemma 3.4 and Remark 3.6. Therefore, the conclusion follows by the formula (5.5).

2. The continuous differentiability of $g * f$ follows by the statement 1. When g is locally absolutely continuous,
$$\mathrm{d}g * f = g' * f$$
holds from Theorem A.20. □

5.2.2 Associativity of Riemann–Stieltjes convolution

For the proof of Theorem 5.9 below, we need the following result.

Theorem 5.7 (refs. [17], [31])**.** *Let $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. Then for any continuous functions $f, g\colon [0,\infty) \to M_n(\mathbb{K})$,*
$$\mathrm{d}\alpha * (g * f) = (\mathrm{d}\alpha * g) * f \tag{5.6}$$
holds.

Remark 5.8. Both sides of Eq. (5.6) are meaningful from Lemma 5.2 and Theorem 3.5. Eq. (3.3) is a special case of (5.6) since we have
$$Vf = f * \mathcal{I} = \mathcal{I} * f$$
for any $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Here $\mathcal{I}\colon [0,\infty) \to M_n(\mathbb{K})$ denotes the constant function whose value is equal to the identity matrix I.

The above is a result on the associativity for Riemann–Stieltjes convolutions. The corresponding statements in a more general setting are given in [17, Section 6 in Chapter 3]. See also [31, Proposition D.9 in Appendix D] for a similar result to Theorem 5.7.

One can prove Theorem 5.7 by the same reasoning as in the proof of Theorem 3.7, however, we give an outline of the proof for reader's convenience.

Outline of the proof of Theorem 5.7. We extend the domain of definition of g to $\mathbb{R}$ by defining $g(t) := g(0)$ for $t \le 0$. Then the obtained function $g\colon \mathbb{R} \to M_n(\mathbb{K})$ is continuous. Let $t > 0$ be fixed. By the proof of Lemma 3.4, we have

$$(\mathrm{d}\alpha * g)(u) = \int_0^t \mathrm{d}\alpha(v)\, g(u-v) - [\alpha(t) - \alpha(u)]g(0)$$

for $u \in [0,t]$. Therefore, $[(\mathrm{d}\alpha * g) * f](t)$ is expressed as

$$[(\mathrm{d}\alpha * g) * f](t) = \int_0^t \left(\int_0^t \mathrm{d}\alpha(v)\, g(u-v) \right) f(t-u)\,\mathrm{d}u - \int_0^t [\alpha(t) - \alpha(u)]g(0) f(t-u)\,\mathrm{d}u.$$

Since

$$[0,t] \times [0,t] \ni (u,v) \mapsto g(u-v)f(t-u) \in M_n(\mathbb{K})$$

is continuous, the first term of the right-hand side becomes

$$\int_0^t \mathrm{d}\alpha(v) \left(\int_0^t g(u-v) f(t-u)\,\mathrm{d}u \right)$$

from Theorem 3.8. Here the integrand also becomes

$$\int_{-v}^{t-v} g(u) f(t-u-v)\,\mathrm{d}u = (g * f)(t-v) + \int_{-v}^{0} g(0) f(t-u-v)\,\mathrm{d}u.$$

Then the proof is complete by showing

$$\int_0^t [\alpha(t) - \alpha(u)]g(0) f(t-u)\,\mathrm{d}u = \int_0^t \mathrm{d}\alpha(v) \left(\int_{-v}^{0} g(0) f(t-u-v)\,\mathrm{d}u \right).$$

One can prove this by using the integration by parts formula for Riemann–Stieltjes integrals. □

5.3 A formula for non-homogeneous equations with trivial initial history

Let $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ be a continuous linear map. We recall that for a continuous map $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$, the function $x^L(\cdot\,;0,G)\colon [-r,\infty) \to \mathbb{K}^n$ denotes the unique solution of an integral equation

$$x(t) = L\int_0^t x_s\,\mathrm{d}s + G(t) \qquad (t \ge 0) \tag{5.7}$$

under the initial condition $x_0 = 0$.

In this subsection, as an application of the results in Subsection 5.2, we show that the function $x(\cdot\,;f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot\,;f)_0 = 0$ and (1.8) is a solution to the non-homogeneous linear RFDE (1.7).

Theorem 5.9 (cf. [35]). *Let $f\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function. Then*

$$x^L(t;0,Vf) = \int_0^t X^L(t-u) f(u)\,\mathrm{d}u \tag{5.8}$$

holds for all $t \ge 0$.

We note that $x^L(\cdot\,;0,Vf)$ is a solution to Eq. (1.7) (see Lemma 4.2).

Proof of Theorem 5.9. Let $x := x(\cdot\,; f)|_{[0,\infty)}$ and $X := X^L|_{[0,\infty)}$. Since X is locally absolutely continuous (see Theorem 3.12) and $X(0) = I$,

$$x = X * f = Vf + \dot{X} * (Vf)$$

holds from Corollary 5.6. For the term $\dot{X} * (Vf)$, we have

$$\begin{aligned} \dot{X} * (Vf) &= (\mathrm{d}\check{\eta} * X) * (Vf) \\ &= \mathrm{d}\check{\eta} * [X * (Vf)] \\ &= \mathrm{d}\check{\eta} * V(X * f) \end{aligned}$$

from Theorems 3.12, 5.7, and Corollary 5.5. This shows that $x(\cdot\,; f)$ satisfies

$$x(t; f) = \big(\mathrm{d}\check{\eta} * Vx(\cdot\,; f)|_{[0,\infty)}\big)(t) + (Vf)(t)$$

for all $t \geq 0$. Therefore, the equality (5.8) is obtained from Theorem 4.8. □

The above proof of Theorem 5.9 is different from the proofs in the literature (e.g., see [35, Section 4]).

6 Variation of constants formula

Let $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ be a continuous linear map and $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ be the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0).$$

In this section, we obtain a "variation of constants formula" for the non-homogeneous linear RFDE (1.2)

$$\dot{x}(t) = Lx_t + g(t) \qquad (\text{a.e. } t \geq 0)$$

for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$ expressed by X^L. In view of Corollary 4.6, we will divide our consideration into the following steps:

- Step 1: To find a formula for the mild solution $x^L(\cdot\,;0,Vg)$ of Eq. (1.2) under the initial condition $x_0 = 0$.
- Step 2: To find a formula for the mild solution $x^L(\cdot\,;\phi,0)$ of Eq. (1.1) under the initial condition $x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$.

Then the full formula for the mild solution of (1.2) under the initial condition $x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ is obtained by combining the above formulas. In Step 1, for a given continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$, we indeed consider the integral equation (5.7)

$$x(t) = L\int_0^t x_s \,\mathrm{d}s + G(t) \qquad (t \geq 0)$$

under the initial condition $x_0 = 0$ and try to find a formula for the solution $x^L(\cdot\,;0,G)$ expressed by X^L.

Remark 6.1. Since $x_0 = 0 = \hat{0}$, Eq. (5.7) is equivalent to

$$x(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_0^{t+\theta} x(s)\,\mathrm{d}s \right) + G(t) \quad (t \geq 0)$$

from Lemma 3.3.

The following is the main result of this section.

Theorem 6.2. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. Then the solution $x^L(\cdot;\phi,G)$ of the integral equation* (1.6)

$$x(t) = \phi(0) + L\int_0^t x_s\,\mathrm{d}s + G(t) \quad (t \geq 0)$$

under the initial condition $x_0 = \phi$ satisfies (1.9)

$$x^L(t;\phi,G) = X^L(t)\phi(0) + \left[G^L(t;\phi) + G(t)\right] + \int_0^t \dot{X}^L(t-u)\left[G^L(u;\phi) + G(u)\right]\mathrm{d}u$$

for all $t \geq 0$.

We will call the formula (1.9) the *variation of constants formula* for Eq. (1.6). The definition of the function $G^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$ for $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ will be given later. For this definition, the expression of L by the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$ is a key tool.

6.1 Motivation: Naito's consideration

We first concentrate our consideration to the case that $g \in C([0,\infty),\mathbb{K}^n)$ and $\phi \in C([-r,0],\mathbb{K}^n)$. From Theorem 5.9, we only need to find a formula for $x^L(\cdot;\phi,0)$ in this case.

Naito [26, Theorem 6.5] has discussed an expression of the form

$$x(t) = \phi(0) + \int_0^t X(t-u)L\bar{\phi}_u\,\mathrm{d}u \quad (t \geq 0).$$

In the above formula, $x\colon [-r,\infty) \to \mathbb{K}^n$ is the solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi \in C([-r,0],\mathbb{K}^n)$, and $\bar{\phi}\colon [-r,\infty) \to \mathbb{K}^n$ is the function defined by

$$\bar{\phi}(t) := \begin{cases} \phi(t) & (t \in [-r,0]), \\ \phi(0) & (t \geq 0). \end{cases}$$

See also Notation 1. Although the study of [26] is in the setting of infinite retardation, we are now interpreting this in the setting of finite retardation (i.e., the history function space is $C([-r,0],\mathbb{K}^n)$). We note that the matrix-valued function $X\colon [0,\infty) \to M_n(\mathbb{K})$ is defined by using the inverse Laplace transform. See [26] for the detail. See also [27], where an interpretation of the matrix-valued function X is given.

In our setting, a formula expressed by the principal fundamental matrix solution X^L

$$x^L(t;\phi,0) = \phi(0) + \int_0^t X^L(t-u)L\bar{\phi}_u\,\mathrm{d}u \quad (t \geq 0) \tag{6.1}$$

is true. To see this, let $y(t) := x^L(t;\phi,0) - \bar{\phi}(t)$ for $t \in [-r,\infty)$. Then the function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfies $y_0 = 0$ and

$$\dot{y}(t) = Ly_t + L\bar{\phi}_t \qquad (t \geq 0).$$

See also the proof of Theorem 2.14. Since the function $[0,\infty) \ni t \mapsto L\bar{\phi}_t \in \mathbb{K}^n$ is continuous, we obtain

$$y(t) = \int_0^t X^L(t-u)L\bar{\phi}_u \,\mathrm{d}u \qquad (t \geq 0)$$

by applying Theorem 5.9.

6.2 Derivation of a general forcing term

The formula (6.1) is not sufficient for the application to the linearized stability. See Section 8 for the detail of the application of the variation of constants formula to the linearized stability. We now introduce the following function.

Notation 6. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we define a function $z^L(\,\cdot\,;\phi)\colon [-r,\infty) \to \mathbb{K}^n$ by $z^L(\,\cdot\,;\phi)_0 = 0$ and (1.11)

$$z^L(t;\phi) := x^L(t;\phi,0) - X^L(t)\phi(0)$$

for $t \geq 0$.

Remark 6.3. Since

$$z^L(0;\phi) = \phi(0) - X^L(0)\phi(0) = 0,$$

the function $z^L(\,\cdot\,;\phi)$ is continuous. In view of $X^L(\,\cdot\,)\phi(0) = x^L\Big(\,\cdot\,;\widehat{\phi(0)},0\Big)$, we also have

$$z^L(t;\phi) = x^L\Big(t;\phi - \widehat{\phi(0)},0\Big) \qquad (t \geq 0)$$

from Corollary 2.15. We note that this equality is not valid for $t \in [-r,0)$ because $z^L(\,\cdot\,;\phi)_0 = 0$.

From the expression (2.2) for a mild solution, the function $z^L(\,\cdot\,;\phi)$ satisfies

$$z^L(t;\phi) = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \phi(s)\,\mathrm{d}s\right) + \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x^L\Big(s;\phi - \widehat{\phi(0)},0\Big)\,\mathrm{d}s\right)$$

for all $t \geq 0$. The second term of the right-hand side is further calculated as follows:

- When $t \in [0,r)$, $\theta \in [-r,0]$ satisfies $t+\theta \geq 0$ if and only if $\theta \in [-t,0]$. Since

$$x^L\Big(s;\phi - \widehat{\phi(0)},0\Big) = \phi(s)$$

  for $s \in \mathrm{dom}(\phi) \setminus \{0\}$, the second term is decomposed by

$$\int_{-r}^{-t} \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right) + \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} z^L(s;\phi)\,\mathrm{d}s\right)$$

  by the additivity of Riemann–Stieltjes integrals on sub-intervals.

- When $t \geq r$, the second term is equal to $\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} z^L(s;\phi)\,\mathrm{d}s\right)$.

This leads to the following definition.

Definition 6.4. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we define a function $G^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$ by

$$G^L(t;\phi) := \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right) + \int_{-r}^{-t} \mathrm{d}\eta(\theta) \left(\int_{0}^{t+\theta} \phi(s)\,\mathrm{d}s \right)$$

for $t \in [0,r)$ and

$$G^L(t;\phi) := \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right)$$

for $t \in [r,\infty)$.

By definition, $G^L(0;\phi) = 0$ holds. Summarizing the above discussion, we obtain the following lemma.

Lemma 6.5. *For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, the function $z := z^L(\cdot;\phi)$ is a solution of an integral equation* (1.12)

$$z(t) = L\int_0^t z_s\,\mathrm{d}s + G^L(t;\phi) \qquad (t \geq 0)$$

under the initial condition $z_0 = 0$.

6.3 Regularity of the general forcing term

To study Eq. (1.12), it is important to reveal the regularity of the function $G^L(\cdot;\phi)$ for each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$.

6.3.1 Forcing terms for continuous initial histories

Before we tackle this problem, we find a differential equation satisfied by $z := z^L(\cdot;\phi)$ for $\phi \in C([-r,0],\mathbb{K}^n)$. It should be noted that this is not straightforward because (1.11) is only valid for $t \geq 0$.

Let $x := x^L(\cdot;\phi,0)$ and $\tilde{x} := x^L\left(\cdot;\widehat{\phi(0)},0\right)$. In view of

$$Lz_t = \int_{-r}^{0} \mathrm{d}\eta(\theta)\, z(t+\theta) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, z(t+\theta)$$

for each $t \geq 0$, we express the linear RFDE (1.1) as

$$\dot{x}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, x(t+\theta) + \int_{-r}^{-t} \mathrm{d}\eta(\theta)\, \phi(t+\theta)$$

by using the additivity of Riemann–Stieltjes integrals on sub-intervals. Here we are interpreting that the second term of the right-hand side is equal to 0 when $t \geq r$. More precisely, we introduce the following.

Definition 6.6 (cf. [3], [9], [25])**.** For each $\phi \in C([-r,0],\mathbb{K}^n)$, we define a function

$$g^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$$

by

$$g^L(t;\phi) := \int_{-r}^{-t} \mathrm{d}\eta(\theta)\, \phi(t+\theta)$$

for $t \in [0,r)$ and $g^L(t;\phi) = 0$ for $t \geq r$. Here the right-hand side is a Riemann–Stieltjes integral.

We note that similar concepts have appeared in the literature. See [3, (3.1) and (3.2)], [9, (2.7) and (2.13)], and [25, Lemma 1.10], for example.

From Theorem 3.10, $\tilde{x}$ satisfies

$$\dot{\tilde{x}}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, \tilde{x}(t+\theta)$$

for almost all $t \geq 0$. In combination with the above consideration, z satisfies

$$\begin{aligned}\dot{z}(t) &= \dot{x}(t) - \dot{\tilde{x}}(t) \\ &= \int_{-t}^{0} \mathrm{d}\eta(\theta)\, z(t+\theta) + g^L(t;\phi)\end{aligned}$$

for almost all $t \geq 0$. Here the property that $t + \theta \geq 0$ for all $\theta \in [-t, 0]$ is used.

In summary, we have the following statement.

Lemma 6.7. *For each $\phi \in C([-r,0],\mathbb{K}^n)$, $z := z^L(\cdot;\phi)$ is locally absolutely continuous, differentiable almost everywhere, and*

$$\dot{z}(t) = Lz_t + g^L(t;\phi) \tag{6.2}$$

holds for almost all $t \geq 0$.

We note that since $g^L(\cdot;\phi)$ is not necessarily continuous, Theorem 5.9 is not sufficient to obtain an expression of $z = z^L(\cdot;\phi)$ by X^L.

6.3.2 Relationship with the forcing terms

Comparing (1.12) and (6.2), it is natural to expect that

$$G^L(t;\phi) = \int_0^t g^L(s;\phi)\,\mathrm{d}s \tag{6.3}$$

holds for all $t \geq 0$ when $\phi \in C([-r,0],\mathbb{K}^n)$. We now justify this relationship.

Lemma 6.8. *Suppose $\phi \in C([-r,0],\mathbb{K}^n)$. Then*

$$g^L(t;\phi) = L\bar{\phi}_t - [\eta(0) - \eta(-t)]\phi(0) \tag{6.4}$$

holds for all $t \geq 0$. Consequently, $g^L(\cdot;\phi)$ is a locally Riemann integrable function vanishing at $[r,\infty)$.

Proof. When $t \geq r$,

$$L\bar{\phi}_t = \int_{-r}^{0} \mathrm{d}\eta(\theta)\, \phi(0) = [\eta(0) - \eta(-r)]\phi(0)$$

holds. Therefore, the right-hand side of (6.4) is equal to 0 for all $t \geq r$. We next consider the case $t \in [0,r)$. In this case, we have

$$\begin{aligned}g^L(t;\phi) &= \int_{-r}^{-t} \mathrm{d}\eta(\theta)\, \bar{\phi}(t+\theta) \\ &= \int_{-r}^{0} \mathrm{d}\eta(\theta)\, \bar{\phi}(t+\theta) - \int_{-t}^{0} \mathrm{d}\eta(\theta)\, \bar{\phi}(t+\theta)\end{aligned}$$

by the additivity of Riemann–Stieltjes integrals on sub-intervals. Since

$$\int_{-t}^{0} \mathrm{d}\eta(\theta)\, \bar{\phi}(t+\theta) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, \phi(0) = [\eta(0) - \eta(-t)]\phi(0),$$

the expression (6.4) is obtained. Since $[0,\infty) \ni t \mapsto L\bar{\phi}_t \in \mathbb{K}^n$ is continuous and $[0,\infty) \ni t \mapsto \eta(-t)\phi(0)$ is of locally bounded variation, the local Riemann integrability of $g^L(\cdot;\phi)$ follows by the expression (6.4). □

Remark 6.9. The expression (6.4) also shows that $g^L(\cdot;\phi)$ is continuous if $\phi(0)=0$. This should be compared with [9, Theorem 2.1(ii) and Remark 2.1].

The following theorem reveals a connection between $G^L(\cdot;\phi)$ and $g^L(\cdot;\phi)$.

Theorem 6.10. *Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. Then for all $t \geq 0$,*

$$G^L(t;\phi) = L\int_0^t \bar{\phi}_s \,\mathrm{d}s - \int_0^t [\eta(0)-\eta(-s)]\phi(0)\,\mathrm{d}s \tag{6.5}$$

holds.

Proof. For the first term of the definition of $G^L(t;\phi)$, we have

$$\begin{aligned}\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \phi(s)\,\mathrm{d}s\right) &= \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \bar{\phi}(s)\,\mathrm{d}s\right)\\ &= \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right) - \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right),\end{aligned}$$

where the first term of the last equation is equal to $L\int_0^t \bar{\phi}_s\,\mathrm{d}s$. The remainder of the proof is divided into the cases $t\in[0,r]$ and $t\in(r,\infty)$ in order to study the term $\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta}\bar{\phi}(s)\,\mathrm{d}s\right)$.

Case 1: $t\in[0,r]$**.** When $t=r$, we have

$$\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right) = \int_{-r}^0 \mathrm{d}\eta(\theta)\,(r+\theta)\phi(0)$$

because $r+\theta\geq 0$ for all $\theta\in[-r,0]$. We next consider the case $t\in[0,r)$. In this case,

$$\begin{aligned}&\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)\\ &= \int_{-r}^{-t} \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right) + \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)\end{aligned}$$

holds by the additivity of Riemann–Stieltjes integrals on sub-intervals and by the property that $t+\theta\leq 0$ for all $\theta\in[-r,-t]$. Here the second term of the right-hand side is equal to

$$\int_{-t}^0 \mathrm{d}\eta(\theta)\,(t+\theta)\phi(0).$$

Therefore, the definition of $G^L(t;\phi)$ yields

$$\begin{aligned}G^L(t;\phi) &= L\int_0^t \bar{\phi}_s\,\mathrm{d}s - \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta}\bar{\phi}(s)\,\mathrm{d}s\right)\\ &= L\int_0^t \bar{\phi}_s\,\mathrm{d}s - \int_{-t}^0 \mathrm{d}\eta(\theta)\,(t+\theta)\phi(0)\end{aligned}$$

including the case $t=r$. The proof is complete in view of

$$\begin{aligned}\int_{-t}^0 (t+\theta)\,\mathrm{d}\eta(\theta) &= [(t+\theta)\eta(\theta)]_{\theta=-t}^0 - \int_{-t}^0 \eta(\theta)\,\mathrm{d}\theta\\ &= t\eta(0) - \int_0^t \eta(-s)\,\mathrm{d}s,\end{aligned}$$

where the integration by parts formula for Riemann–Stieltjes integrals is used.

Case 2: $t \in (r, \infty)$**.** Since we have shown that (6.5) holds for $t = r$,

$$G^L(t;\phi) = L\int_0^r \bar{\phi}_s \, \mathrm{d}s - \int_0^r [\eta(0) - \eta(-s)]\phi(0) \, \mathrm{d}s$$

holds for all $t \geq r$. Here the property that $G^L(\cdot;\phi)$ is constant on $[r, \infty)$ is used. Then the proof is complete by showing that the right-hand side of (6.5) is constant on $[r, \infty)$. For this purpose, we calculate

$$L\int_0^t \bar{\phi}_s \, \mathrm{d}s - L\int_0^r \bar{\phi}_s \, \mathrm{d}s.$$

By the linearity of L, it is calculated as

$$\begin{aligned}\int_{-r}^0 \mathrm{d}\eta(\theta) \left(\int_{r+\theta}^{t+\theta} \bar{\phi}(s) \, \mathrm{d}s \right) &= \int_{-r}^0 \mathrm{d}\eta(\theta)\, (t-r)\phi(0) \\ &= (t-r)[\eta(0) - \eta(-r)]\phi(0).\end{aligned}$$

Since η is constant on $(-\infty, -r]$, the last value is expressed as

$$\int_r^t [\eta(0) - \eta(-s)]\phi(0) \, \mathrm{d}s.$$

This shows that

$$L\int_0^t \bar{\phi}_s \, \mathrm{d}s = L\int_0^r \bar{\phi}_s \, \mathrm{d}s + \int_r^t [\eta(0) - \eta(-s)]\phi(0) \, \mathrm{d}s,$$

which also implies that the right-hand side of (6.5) is equal to

$$L\int_0^r \bar{\phi}_s \, \mathrm{d}s - \int_0^r [\eta(0) - \eta(-s)]\phi(0) \, \mathrm{d}s$$

for all $t \geq r$. □

Remark 6.11. $G^L(t;\phi)$ is also expressed as

$$G^L(t;\phi) = \int_{-r}^0 [\eta(\theta) - \eta(\theta - t)]\phi(\theta) \, \mathrm{d}\theta.$$

See [14, Section I.2 of Chapter I] for the detail. See also [34, Remark 2.10(iii) in Chapter 2]. In this paper, we do not need the above expression.

By combining the obtained results, we obtain the following result on the regularity of $G^L(\cdot;\phi)$. See also [34, Remark 2.10(ii) in Chapter 2].

Theorem 6.12. *For any $\phi \in \mathcal{M}^1([-r,0], \mathbb{K}^n)$, the function $G^L(\cdot;\phi)$ is continuous with $G^L(0;\phi) = 0$. Furthermore, if $\phi \in C([-r,0], \mathbb{K}^n)$, then it is locally absolutely continuous, differentiable almost everywhere, and*

$$\dot{G}^L(t;\phi) = g^L(t;\phi)$$

holds for almost all $t \in [0, \infty)$. Here $\dot{G}^L(t;\phi)$ denotes the derivative of $G^L(\cdot;\phi)$ at t.

Proof. Let $\phi \in \mathcal{M}^1([-r,0], \mathbb{K}^n)$ be given. Then (6.5) yields the continuity of $G^L(\cdot;\phi)$ from Lemma 2.8 and by the continuity of L. The property $G^L(0;\phi) = 0$ follows by definition. We next assume $\phi \in C([-r,0], \mathbb{K}^n)$. Since $\bar{\phi} \colon [-r, \infty) \to \mathbb{K}^n$ is continuous, Theorem 6.10, Corollary 2.12, and Lemma 6.8 show that (6.3)

$$G^L(t;\phi) = \left(Vg^L(\cdot;\phi)\right)(t)$$

holds for all $t \geq 0$. This yields the properties of $G^L(\cdot;\phi)$. □

6.4 Derivation of the variation of constants formula

6.4.1 Formulas for trivial initial histories

For the derivation of the variation of constants formula, we use the following result.

Theorem 6.13. *Let $[a, b]$ be a closed and bounded interval of $\mathbb{R}$. If $F, G\colon [a, b] \to M_n(\mathbb{K})$ are absolutely continuous, then*

$$\int_a^b F'(x)G(x)\,\mathrm{d}x = [F(x)G(x)]_{x=a}^b - \int_a^b F(x)G'(x)\,\mathrm{d}x$$

holds.

This should be called the integration by parts formula for matrix-valued absolutely continuous functions. We note that the above result also holds when $G\colon [a, b] \to \mathbb{K}^n$ is an absolutely continuous function. Since the Lebesgue integral of a matrix-valued function is defined component-wise, Theorem 6.13 can be obtained by the corresponding result for scalar-valued functions in combination with the linearity of Lebesgue integration. We note that the result for scalar-valued functions is mentioned in [30, Exercise 14 of Chapter 7]. One can also give a direct proof based on the matrix product.

By using the local absolute continuity of $X^L|_{[0,\infty)}$ (see Theorem 3.12), Theorem 6.13 shows that

$$\begin{aligned}\int_0^t X^L(t-u)g(u)\,\mathrm{d}u &= [X^L(t-u)(Vg)(u)]_{u=0}^t + \int_0^t \dot{X}^L(t-u)(Vg)(u)\,\mathrm{d}u \\ &= (Vg)(t) + \int_0^t \dot{X}^L(t-u)(Vg)(u)\,\mathrm{d}u\end{aligned}$$

holds for any $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), \mathbb{K}^n)$. Here $X^L(0) = I$ and $(Vg)(0) = 0$ are also used. The following theorem is motivated by this.

Theorem 6.14. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$. Then* (1.10)

$$x^L(t; 0, G) = G(t) + \int_0^t \dot{X}^L(t-u)G(u)\,\mathrm{d}u$$

holds for all $t \geq 0$.

Proof. Let $X := X^L|_{[0,\infty)}$. We define a function $x\colon [-r,\infty) \to \mathbb{K}^n$ by $x_0 = 0$ and

$$x(t) := G(t) + \big(\dot{X} * G\big)(t)$$

for $t \geq 0$. By applying Corollary 5.6 in combination with the fundamental theorem of calculus, we have $Vx|_{[0,\infty)} = X * G$. Here $(X * G)(0) = 0$ is also used. Furthermore, we have

$$x(t) = G(t) + [\mathrm{d}\check{\eta} * (X * G)](t) \quad (t \geq 0)$$

from Theorems 3.12 and 5.7. Therefore, x satisfies

$$x(t) = G(t) + \big(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\big)(t)$$

for all $t \geq 0$. This implies that (1.10) holds by applying Theorem 4.8. □

The following corollary is obtained from Theorem 6.14 by using the discussion before Theorem 6.14. It is an extension of Theorem 5.9.

Corollary 6.15 (cf. [18], [19])**.** *Let $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), \mathbb{K}^n)$. Then*

$$x^L(t; 0, Vg) = \int_0^t X^L(t-u)g(u)\,\mathrm{d}u \tag{6.6}$$

holds for all $t \geq 0$.

6.4.2 Formulas for homogeneous equations

We next find an expression of $x^L(\cdot;\phi,0)$ by X^L as an application of Theorem 6.14.

Theorem 6.16. *Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$. Then*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + G^L(t;\phi) + \int_0^t \dot{X}^L(t-u)G^L(u;\phi)\,\mathrm{d}u \tag{6.7}$$

holds for all $t \geq 0$.

Proof. From Lemma 6.5 and Theorem 6.14 together with Theorem 6.12,

$$z^L(t;\phi) = G^L(t;\phi) + \int_0^t \dot{X}^L(t-u)G^L(u;\phi)\,\mathrm{d}u$$

holds for all $t \geq 0$. Then the formula (6.7) is obtained in view of

$$z^L(t;\phi) = x^L(t;\phi,0) - X^L(t)\phi(0)$$

for $t \geq 0$. □

Remark 6.17. The above proof of Theorem 6.16 is considered to be a reorganization of [14, Section I.2 of Chapter I]. It leads us to the understanding of the variation of constants formula for non-homogeneous linear RFDEs that does not rely on the theory of Volterra convolution integral equations.

We have the following corollary.

Corollary 6.18 (cf. [25])**.** *Let $\phi \in C([-r,0],\mathbb{K}^n)$. Then*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \int_0^t X^L(t-u)g^L(u;\phi)\,\mathrm{d}u \tag{6.8}$$

holds for all $t \geq 0$.

Proof. From Theorem 6.12,

$$G^L(\cdot;\phi) = V\big(g^L(\cdot;\phi)\big)$$

holds. Therefore, the formula (6.8) is obtained from (6.7) by using the integration by parts formula for matrix-valued absolutely continuous functions. □

Corollary 6.18 should be compared with [25, Theorem 1.11], where the inverse Laplace transform is used to obtain a formula.

6.4.3 Derivation of the main result of this section

Theorem 6.2 is a combination of Theorems 6.14 and 6.16 in view of Corollary 4.6. Therefore, the proof can be omitted. The following is a corollary of Theorem 6.2, which is a combination of Corollaries 6.15 and 6.18 in view of Corollary 4.6. The proof can be omitted.

Corollary 6.19. *If $\phi \in C([-r,0],\mathbb{K}^n)$ and $G = Vg$ for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$, then*

$$x^L(t;\phi,G) = X^L(t)\phi(0) + \int_0^t X^L(t-u)[g^L(u;\phi) + g(u)]\,\mathrm{d}u$$

holds for all $t \geq 0$.

6.5 Variation of constants formula for linear differential difference equations

We apply Theorem 6.16 to an autonomous linear differential difference equation (2.11)

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \qquad (t \geq 0)$$

for $n \times n$ matrices $A, B_1, \dots, B_m \in M_n(\mathbb{K})$ and $\tau_1, \dots, \tau_m \in (0, r]$. We recall that the linear DDE (2.11) can be expressed in the form of the linear RFDE (1.1) by defining a continuous linear map $L\colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$ by (2.12)

$$L\psi = A\psi(0) + \sum_{k=1}^{m} B_k \psi(-\tau_k)$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$.

For the above mentioned application, we need to calculate the function $G^L(\cdot; \phi)$ for each $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ based on Definition 6.4. By the linearity of $L \mapsto G^L(\cdot; \phi)$, this can be reduced to the calculation of $G^{L_k}(\cdot; \phi)$ for each $k \in \{0, \dots, m\}$, where $L_k\colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$ is the continuous linear map given by

$$L_0\psi := A\psi(0),$$

and

$$L_k\psi := B_k\psi(-\tau_k)$$

for $k \in \{1, \dots, m\}$. We have the following lemma.

Lemma 6.20. *Let $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ be given. Then the following statements hold:*

1. $G^{L_0}(\cdot; \phi) = 0$.
2. *For each $k \in \{1, \dots, m\}$, $G^{L_k}(0; \phi) = 0$ and*

$$G^{L_k}(t; \phi) = \begin{cases} B_k \int_{-\tau_k}^{t-\tau_k} \phi(s)\,\mathrm{d}s & (t \in (0, \tau_k]), \\ B_k \int_{-\tau_k}^{0} \phi(s)\,\mathrm{d}s & (t \in (\tau_k, \infty)). \end{cases}$$

holds.

Proof. 1. Let $\eta_0\colon [-r, 0] \to M_n(\mathbb{K})$ be the matrix-valued function given by

$$\eta_0(\theta) := \begin{cases} O & (-r \leq \theta < 0), \\ A & (\theta = 0). \end{cases}$$

Then L_0 is expressed as

$$L_0\psi = \int_{-r}^{0} \mathrm{d}\eta_0(\theta)\,\psi(\theta)$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$. Therefore, the definition of $G^L(\cdot; \phi)$ yields the conclusion.

2. Let $k \in \{1, \dots, m\}$ be fixed and $\eta_k\colon [-r, 0] \to M_n(\mathbb{K})$ be the matrix-valued function given by

$$\eta_k(\theta) := \begin{cases} O & (-r \leq \theta \leq -\tau_k), \\ B_k & (-\tau_k < \theta \leq 0). \end{cases}$$

Then L_k is expressed as

$$L_k\psi = \int_{-r}^{0} \mathrm{d}\eta_k(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$. By the definition of L_k, we have

$$\int_{-r}^{0} \mathrm{d}\eta_k(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right) = B_k \int_{-\tau_k}^{0} \phi(s)\,\mathrm{d}s.$$

Furthermore, the integral $\int_{-r}^{-t} \mathrm{d}\eta_k(\theta) \left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s \right)$ is calculated as

$$\int_{-r}^{-t} \mathrm{d}\eta_k(\theta) \left(\int_{0}^{t+\theta} \phi(s)\,\mathrm{d}s \right) = \begin{cases} B_k \int_0^{t-\tau_k} \phi(s)\,\mathrm{d}s & (t \in [0,\tau_k]), \\ 0 & (t \in (\tau_k,\infty)). \end{cases}$$

By combining the above expressions, the conclusion is obtained. □

Theorem 6.21 (cf. [19], [22]). *Let $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ be the continuous linear map given by (2.12). Then for any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$,*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \sum_{k=1}^{m} \int_{-\tau_k}^{0} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta \tag{6.9}$$

holds for all $t \geq 0$.

Proof. Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. From Lemma 6.20,

$$G^L(\cdot;\phi) = \sum_{k=1}^{m} G^{L_k}(\cdot;\phi)$$

is locally absolutely continuous. Therefore, Theorem 6.16 and the integration by parts formula for absolutely continuous functions yield that

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \sum_{k=1}^{m} \int_0^t X^L(t-u)\dot{G}^{L_k}(u;\phi)\,\mathrm{d}u$$

holds for all $t \geq 0$. We now fix $k \in \{1,\dots,m\}$ and find an expression of the integral

$$\int_0^t X^L(t-u)\dot{G}^{L_k}(u;\phi)\,\mathrm{d}u.$$

Lemma 6.20 shows that $\dot{G}^{L_k}(t;\phi) = B_k\phi(t-\tau_k)$ holds for almost all $t \in [0,\tau_k]$, and $\dot{G}^{L_k}(t;\phi) = 0$ holds for all $t \in (\tau_k,\infty)$. Then the integral is expressed as follows:

- When $t \in [0,\tau_k]$, the integral becomes

$$\begin{aligned} \int_0^t X^L(t-u)B_k\phi(u-\tau_k)\,\mathrm{d}u &= \int_{-\tau_k}^{t-\tau_k} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta \\ &= \int_{-\tau_k}^{0} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta \end{aligned}$$

  because $t - \tau_k \leq 0$.

- When $t \in (\tau_k, \infty)$, the integral becomes

$$\int_0^{\tau_k} X^L(t-u) B_k \phi(u-\tau_k)\,\mathrm{d}u = \int_{-\tau_k}^0 X^L(t-\tau_k-\theta) B_k \phi(\theta)\,\mathrm{d}\theta.$$

This completes the proof. □

Remark 6.22. Suppose $\phi \in C([-r,0],\mathbb{R})$ and $m = 1$. In [19, Theorem 6.1 in Section 1.6] and [22, Theorem 6.1 in Section 1.6], (6.9) is obtained by using the Laplace transform. See also [28, Theorem 4.2], where (6.9) is obtained under different assumptions for a linear evolution equation with commensurate delays.

6.6 Remarks on definitions of "fundamental matrix"

6.6.1 Definition by Hale

Let $\mathbb{K} = \mathbb{R}$. In [18, Theorem 16.3 and Corollary 16.1] and [19, Theorem 2.1 and Corollary 2.1 in Chapter 6], a matrix-valued function $X\colon [0,\infty) \to M_n(\mathbb{R})$ is defined by using the property that for every $t \geq 0$,

$$\mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) \ni g \mapsto x^L(t;0,Vg) \in \mathbb{R}^n$$

is a bounded linear operator to show

$$x^L(t;0,Vg) = \int_0^t X(t-u)g(u)\,\mathrm{d}u \qquad (t \geq 0).$$

Furthermore, by the formal exchange of order of integration, the function X is interpreted as a "matrix-valued solution" to the linear RFDE (1.1). Indeed, Hale argued that X satisfies (i) $X_0 = \hat{I}$, (ii) $X|_{[0,\infty)}$ is locally absolutely continuous, and (iii) X satisfies

$$\dot{X}(t) = \int_{-r}^0 \mathrm{d}\eta(\theta)\, X(t+\theta)$$

for almost all $t \in [0,\infty)$. Here $\hat{I}\colon [-r,0] \to M_n(\mathbb{R})$ is defined by (1.5)

$$\hat{I}(\theta) := \begin{cases} O & (\theta \in [-r,0)), \\ I & (\theta = 0). \end{cases}$$

However, the above integral does not make sense in general because X is not continuous.

6.6.2 Volterra convolution integral equations and fundamental matrix solutions

Let $x := x^L(\,\cdot\,;\hat{\xi})|_{[0,\infty)}$ for some $\xi \in \mathbb{K}^n$ and suppose $\eta(0) = O$. By using the integration by parts formula for Riemann–Stieltjes integrals and Theorem A.19 in (3.5)

$$x = \xi + \mathrm{d}\check{\eta} * Vx,$$

we have

$$\begin{aligned} x(t) &= \xi + [\check{\eta}(u)(Vx)(t-u)]_{u=0}^t + \int_0^t \check{\eta}(u)x(t-u)\,\mathrm{d}u \\ &= \xi + (\check{\eta} * x)(t) \end{aligned}$$

for all $t \geq 0$. Here $(Vx)(0) = 0$ is also used. The above calculation shows that the function $x\colon [0,\infty) \to \mathbb{K}^n$ satisfies

$$x = \check{\eta} * x + \xi,$$

which is a Volterra convolution integral equation with the kernel function $\check{\eta}$ and with the constant forcing term ξ. Therefore, $X := X^L|_{[0,\infty)}$ satisfies

$$X = \check{\eta} * X + I.$$

This means that the restriction $X = X^L|_{[0,\infty)}$ is the *fundamental matrix solution* for the Volterra convolution integral equation with the kernel function $\check{\eta}$ under the assumption that $\eta(0) = O$. For an approach by the Volterra convolution integral equation, see [14, Section I.2 of Chapter I].

7 Exponential stability of principal fundamental matrix solution

For a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$, we consider a linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0).$$

Let $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ be the principal fundamental matrix solution. We use the following terminology.

Definition 7.1. We say that the principal fundamental matrix solution X^L is *exponentially stable* if there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$\left|X^L(t)\right| \leq M\mathrm{e}^{-\alpha t} \tag{7.1}$$

holds for all $t \geq 0$. We also say that X^L is *α-exponentially stable*.

In the following calculations, it is useful to extend the domain of definition of X^L to $\mathbb{R}$ by letting $X^L(t) := O$ for $t \in (-\infty,-r)$.

Lemma 7.2. *If X^L is α-exponentially stable for some $\alpha > 0$, then there exists a constant $M \geq 1$ such that*

$$\sup_{\theta\in[-r,0]} \left|X^L(t+\theta)\right| \leq M\mathrm{e}^{-\alpha t}$$

holds for all $t \in \mathbb{R}$.

Proof. By the assumption, one can choose a constant $M_0 \geq 1$ so that

$$\left|X^L(t)\right| \leq M_0\mathrm{e}^{-\alpha t}$$

holds for all $t \geq 0$. Since the statement is trivial when $t \leq 0$, we only have to consider the case $t > 0$. Let $\theta \in [-r,0]$. When $t+\theta \geq 0$, we have

$$\left|X^L(t+\theta)\right| \leq M_0\mathrm{e}^{-\alpha(t+\theta)} \leq M_0\mathrm{e}^{\alpha r}\mathrm{e}^{-\alpha t}.$$

The above estimate also holds when $t+\theta < 0$ because $X^L(t+\theta) = O$ in this case. Therefore, the conclusion is obtained. □

Theorem 7.3 (cf. [19], [22]). *If X^L is α-exponentially stable for some $\alpha > 0$, then the C_0-semigroup $\left(T^L(t)\right)_{t\geq 0}$ on $C([-r,0],\mathbb{K}^n)$ defined by* (1.13)

$$T^L(t)\phi := x^L(\,\cdot\,;\phi,0)_t$$

for $(t,\phi) \in [0,\infty) \times C([-r,0],\mathbb{K}^n)$ is uniformly α-exponentially stable, i.e., there exists a constant $M \geq 1$ such that for all $t \geq 0$,

$$\left\|T^L(t)\right\| \leq M\mathrm{e}^{-\alpha t}$$

holds.

Proof. By applying Lemma 7.2, we choose a constant $M_0 \geq 1$ so that

$$\sup_{\theta\in[-r,0]} \left|X^L(t+\theta)\right| \leq M_0\mathrm{e}^{-\alpha t}$$

holds for all $t \in \mathbb{R}$. Since the statement is trivial when $t = 0$, we only have to consider the case $t > 0$. Let $\theta \in [-r,0]$ and $\phi \in C([-r,0],\mathbb{K}^n)$ be given. Then

$$\left[T^L(t)\phi\right](\theta) = \begin{cases} X^L(t+\theta)\phi(0) + \int_0^{t+\theta} X^L(t+\theta-u)g^L(u;\phi)\,\mathrm{d}u & (t+\theta \geq 0), \\ \phi(t+\theta) & (t+\theta \in [-r,0]) \end{cases}$$

holds from Corollary 6.18 (see Definition 6.6 for the definition of $g^L(t;\phi)$). We divide the consideration into the following cases.

Case 1: $t+\theta \geq 0$. For the first term of the right-hand side,

$$\left|X^L(t+\theta)\phi(0)\right| \leq M_0\mathrm{e}^{-\alpha t}|\phi(0)| \leq M_0\mathrm{e}^{-\alpha t}\|\phi\|$$

holds. For the second term,

$$\begin{aligned}\left|\int_0^{t+\theta} X^L(t+\theta-u)g^L(u;\phi)\,\mathrm{d}u\right| &\leq \int_0^{t+\theta} \left|X^L(t-u+\theta)\right|\left|g^L(u;\phi)\right|\mathrm{d}u \\ &\leq \int_0^{t+\theta} M_0\mathrm{e}^{-\alpha(t-u)}\left|g^L(u;\phi)\right|\mathrm{d}u\end{aligned}$$

holds from Lemma 7.2. Since

$$\left|g^L(t;\phi)\right| = \left|\int_{-r}^{-t} \mathrm{d}\eta(\theta)\,\phi(t+\theta)\right| \leq \operatorname{Var}(\eta)\|\phi\|$$

holds for all $t \in [0,r)$ (see Lemma A.4) and $g^L(t;\phi) = 0$ for all $t \geq r$, we have

$$\begin{aligned}\int_0^{t+\theta} M_0\mathrm{e}^{-\alpha(t-u)}\left|g^L(u;\phi)\right|\mathrm{d}u &\leq \int_0^r M_0\mathrm{e}^{-\alpha(t-u)}\operatorname{Var}(\eta)\|\phi\|\,\mathrm{d}u \\ &= M_0\left(\int_0^r \mathrm{e}^{\alpha u}\,\mathrm{d}u\right)\operatorname{Var}(\eta)\mathrm{e}^{-\alpha t}\|\phi\|.\end{aligned}$$

We note that

$$\int_0^r \mathrm{e}^{\alpha u}\,\mathrm{d}u = \frac{1}{\alpha}(\mathrm{e}^{\alpha r} - 1)$$

holds.

Case 2: $t + \theta < 0$**.** In this case, we have

$$|\phi(t+\theta)| \leq \mathrm{e}^{-\alpha(t+\theta)}|\phi(t+\theta)| \leq \mathrm{e}^{\alpha r}\mathrm{e}^{-\alpha t}\|\phi\|.$$

By combining the estimates obtained in Cases 1 and 2, one can choose a constant $M \geq 1$ so that

$$\sup_{\theta \in [-r,0]} \left| \left[T^L(t)\phi \right](\theta) \right| \leq M\mathrm{e}^{-\alpha t}\|\phi\|$$

holds for all $(t, \phi) \in [0, \infty) \times C([-r, 0], \mathbb{K}^n)$. This completes the proof. □

The converse of Theorem 7.3 also holds.

Theorem 7.4 (cf. [19], [22]). *If $\left(T^L(t)\right)_{t \geq 0}$ is uniformly α-exponentially stable for some $\alpha > 0$, then X^L is α-exponentially stable.*

Proof. By the assumption, we choose a constant $M_0 \geq 1$ so that

$$\left\| T^L(t) \right\| \leq M_0\mathrm{e}^{-\alpha t}$$

holds for all $t \geq 0$. We fix $\xi \in \mathbb{K}^n$ and let

$$\phi_\xi := x^L(\cdot\,; \hat{\xi})_r \in C([-r, 0], \mathbb{K}^n).$$

Then the map $\mathbb{K}^n \ni \xi \mapsto \phi_\xi \in C([-r, 0], \mathbb{K}^n)$ is linear from Corollary 2.7. Since $X^L(\cdot)\xi = x^L(\cdot\,; \xi)$, we have

$$\|\phi_\xi\| = \sup_{t \in [0,r]} \left| x^L(t; \hat{\xi}) \right| \leq \left(\sup_{t \in [0,r]} \left| X^L(t) \right| \right) \cdot |\xi|.$$

This yields that the linear operator $\mathbb{K}^n \ni \xi \mapsto \phi_\xi \in C([-r, 0], \mathbb{K}^n)$ is bounded.

We now show that X^L is α-exponentially stable by dividing into the following cases.

Case 1: $t \geq r$**.** From Theorem 2.14, we have

$$x^L(t; \hat{\xi}) = x^L(t - r; \phi_\xi),$$

where the right-hand side is equal to $[T^L(t - r)\phi_\xi](0)$. Therefore,

$$\left| X^L(t)\xi \right| \leq \left\| T^L(t - r) \right\| \|\phi_\xi\|$$

holds. Since $\left\| T^L(t - r) \right\| \leq M_0\mathrm{e}^{\alpha r}\mathrm{e}^{-\alpha t}$, we obtain

$$\left| X^L(t) \right| \leq \left(M_0\mathrm{e}^{\alpha r} \sup_{t \in [0,r]} \left| X^L(t) \right| \right) \cdot \mathrm{e}^{-\alpha t}$$

by combining the above estimate on $\|\phi_\xi\|$.

Case 2: $t \in [0, r]$**.** In this case, $\left| X^L(t) \right|$ is estimated by

$$\left| X^L(t) \right| \leq \left(\mathrm{e}^{\alpha r} \sup_{t \in [0,r]} \left| X^L(t) \right| \right) \cdot \mathrm{e}^{-\alpha t}.$$

Here $1 = \mathrm{e}^{-\alpha t}\mathrm{e}^{\alpha t}$ is used.

By combining the above estimates, the conclusion is obtained. □

See [19, Lemmas 6.1, 6.2, and 6.3 in Chapter 6] and [22, Lemmas 5.1, 5.2, and 5.3 in Chapter 6] for related results. We note that the statements of Theorems 7.3 and 7.4 are included in these results, where the detailed proofs are not given.

8 Principle of linearized stability and Poincaré–Lyapunov theorem

Throughout this section, let $L\colon C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous linear map. We consider a non-autonomous RFDE

$$\dot{x}(t) = Lx_t + f(t,x_t) \tag{8.1}$$

for some continuous map

$$f\colon \mathbb{R} \times C([-r,0],\mathbb{R}^n) \supset \operatorname{dom}(f) \to \mathbb{R}^n.$$

Let $X^L\colon [-r,\infty) \to M_n(\mathbb{R})$ be the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \quad (t \ge 0)$$

and $\big(T^L(t)\big)_{t\ge 0}$ be the C_0-semigroup on $C([-r,0],\mathbb{R}^n)$ generated by (1.1). See also Section 7.

We recall the definition of a solution to the RFDE (8.1). For each $(t_0,\phi) \in \operatorname{dom}(f)$ and each $T > 0$, a continuous function

$$x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$$

is called a *solution* of (8.1) under an initial condition $x_{t_0} = \phi$ if the following conditions are satisfied: (i) $x_{t_0} = \phi$, (ii) $(t,x_t) \in \operatorname{dom}(f)$ for all $t \in [t_0, t_0+T]$, and (iii) $x|_{[t_0,t_0+T]}$ is differentiable and satisfies the RFDE (8.1) for all $t \in [t_0,t_0+T]$. Here the derivative of x at t_0 and t_0+T are understood as the right-hand derivative at t_0 and the left-hand derivative at t_0+T, respectively.

8.1 Variation of constants formula and nonlinear equations

Theorem 8.1. *Let $(t_0,\phi) \in \operatorname{dom}(f)$ and $T > 0$ be given. Then for a continuous function $x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$ with the properties (i) $x_{t_0} = \phi$ and (ii) $(t,x_t) \in \operatorname{dom}(f)$ for all $t \in [t_0,t_0+T]$, x is a solution of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$ if and only if x satisfies*

$$x(t) = x^L(t - t_0;\phi,0) + \int_{t_0}^{t} X^L(t-u) f(u,x_u)\,\mathrm{d}u$$

for all $t \in [t_0,t_0+T]$.

We note that the above statement is not a simple application of Corollaries 4.6 and 6.15 because there is no method of variation of constants for RFDEs (see Subsection 5.1).

Proof of Theorem 8.1. Let $x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$ be a continuous function with the properties (i) and (ii) in Theorem 8.1. Then it is a solution of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$ if and only if

$$x(t) = \phi(0) + \int_{t_0}^{t} [Lx_s + f(s,x_s)]\,\mathrm{d}s$$

holds for all $t \in [t_0,t_0+T]$. Let $z\colon [-r,T] \to \mathbb{R}^n$ be the function defined by

$$z(s) := x(t_0+s) - x^L(s;\phi)$$

for $s \in [-r,T]$. Then z satisfies $z_0 = 0$ and an integral equation

$$z(s) = \int_0^s Lz_u\,\mathrm{d}u + \int_0^s f(t_0+u, x_{t_0+u})\,\mathrm{d}u$$

for $s \in [0,T]$. Since $[0,T] \ni u \mapsto f(t_0+u, x_{t_0+u}) \in \mathbb{R}^n$ is continuous, $z|_{[0,T]}$ is expressed by

$$z(s) = \int_0^s X^L(s-u) f(t_0+u, x_{t_0+u}) \, \mathrm{d}u \quad (s \in [0,T])$$

from Theorem 5.9 or Corollary 6.15. Therefore, we have

$$x(t_0+s) = x^L(s;\phi) + \int_0^s X^L(s-u) f(t_0+u, x_{t_0+u}) \, \mathrm{d}u$$

for $s \in [0,T]$. The expression of x is obtained by the change of variable $t_0 + s = t$. □

8.2 Stability part of principle of linearized stability

In this subsection, we consider a continuous map

$$h \colon \mathbb{R} \times C([-r,0],\mathbb{R}^n) \supset \mathbb{R} \times U_0 \to \mathbb{R}^n$$

for some open neighborhood U_0 of 0 in $C([-r,0],\mathbb{R}^n)$ with the property that $h(t,\phi) = o(\|\phi\|)$ as $\|\phi\| \to 0$ uniformly in t. This means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(t,\phi) \in \mathbb{R} \times U_0$, $\|\phi\| < \delta$ implies

$$|h(t,\phi)| \le \varepsilon \|\phi\|.$$

In the following theorem, we suppose that $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = h(t,\phi)$ holds for all $(t,\phi) \in \mathbb{R} \times U_0$ in the RFDE (8.1). Then (8.1) is considered as a perturbation of the linear RFDE (1.1). Since $f(t,0) = 0$ holds for all $t \in \mathbb{R}$, (8.1) has the zero solution. The statement in the following theorem is the stability part of the *principle of linearized stability* for RFDEs.

Theorem 8.2 (cf. [14])**.** *If X^L is exponentially stable, then there exist $M \ge 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r,0],\mathbb{R}^n)$ such that for every $t_0 \in \mathbb{R}$, every $\phi \in U$, and every non-continuable solution x of* (8.1) *under the initial condition $x_{t_0} = \phi$, x is defined for all $t \ge t_0$ and satisfies*

$$\|x_t\| \le M\mathrm{e}^{-\beta(t-t_0)} \|\phi\|$$

for all $t \ge t_0$.

Remark 8.3. See [2, Chapter 11] for the corresponding result for differential difference equations. See [11] for the general result of the principle of linearized stability in the context of nonlinear semigroups. See also [14, Chapter VII] for a general treatment of the principle of linearized stability and its application to RFDEs under the local Lipschitz continuity of h.

In the proof of Theorem 8.2 given below, the Peano existence theorem and the continuation of solutions for RFDEs play key roles. See [19, Chapter 2] and [22, Chapter 2] for the fundamental theory of RFDEs.

Proof of Theorem 8.2. From Lemma 7.2 and Theorem 7.3, we choose constants $M \ge 1$ and $\alpha > 0$ so that

$$\sup_{\theta \in [-r,0]} \left| X^L(t+\theta) \right| \le M\mathrm{e}^{-\alpha t} \quad (t \in \mathbb{R})$$

and

$$\left\| T^L(t) \right\| \le M\mathrm{e}^{-\alpha t} \quad (t \ge 0)$$

hold. We also choose an $\varepsilon > 0$ so that

$$-\beta := M\varepsilon - \alpha < 0.$$

We divide the proof into the following steps.

Step 1: Choice of a neighborhood of 0 and a non-continuable solution. Since $f(t, \phi) = o(\|\phi\|)$ as $\|\phi\| \to 0$ uniformly in t, there exists a $\widetilde{\delta} > 0$ for this $\varepsilon > 0$ with the following properties:

(i) For all $\phi \in C([-r, 0], \mathbb{R}^n)$, $\|\phi\| < \widetilde{\delta}$ implies $\phi \in U_0$.

(ii) $\|\phi\| < \widetilde{\delta}$ implies $|f(t, \phi)| \le \varepsilon\|\phi\|$ for all $t \in \mathbb{R}$.

Let $\delta := \widetilde{\delta}/M$. We define open sets U and $\widetilde{U}$ by

$$\begin{aligned} U &:= \{\phi \in C([-r, 0], \mathbb{R}^n) : \|\phi\| < \delta\}, \\ \widetilde{U} &:= \left\{\phi \in C([-r, 0], \mathbb{R}^n) : \|\phi\| < \widetilde{\delta}\right\}. \end{aligned}$$

Then

$$U \subset \widetilde{U} \subset U_0$$

holds. From now on, we fix $t_0 \in \mathbb{R}$ and $\phi \in U$ and proceed with the discussion. By applying the Peano existence theorem for RFDEs, the RFDE

$$\dot{x}(t) = L|_{\widetilde{U}}(x_t) + f|_{\mathbb{R}\times\widetilde{U}}(t, x_t) \tag{8.2}$$

has a solution under the initial condition $x_{t_0} = \phi$. Let x be a non-continuable solution of the RFDE (8.2) under this initial condition. Then its domain of definition is written as $[t_0 - r, t_0 + T)$ for some $T \in (0, \infty]$.

Step 2: Estimate by Gronwall's inequality. Let $t \in [t_0, t_0 + T)$ and $\theta \in [-r, 0]$. By applying Theorem 8.1,

$$x(t) = x^L(t - t_0; \phi, 0) + \int_{t_0}^{t} X^L(t - u) f(u, x_u)\,\mathrm{d}u \qquad (t \in [t_0, t_0 + T))$$

holds for this non-continuable solution $x \colon [t_0 - r, t_0 + T) \to \mathbb{R}^n$. When $t + \theta \ge t_0$, we have

$$\begin{aligned} |x(t+\theta)| &\le \left|x^L(t + \theta - t_0; \phi, 0)\right| + \int_{t_0}^{t+\theta} \left|X^L(t - u + \theta)\right| |f(u, x_u)|\,\mathrm{d}u \\ &\le \left\|T^L(t - t_0)\phi\right\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-u)} |f(u, x_u)|\,\mathrm{d}u \\ &\le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha t}\mathrm{e}^{\alpha u}\varepsilon\|x_u\|\,\mathrm{d}u. \end{aligned}$$

When $t + \theta < t_0$, the estimate

$$|x(t+\theta)| \le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha t}\mathrm{e}^{\alpha u}\varepsilon\|x_u\|\,\mathrm{d}u$$

also holds in view of

$$|x(t+\theta)| = |\phi(t - t_0 + \theta)| = \left|[T^L(t - t_0)\phi](\theta)\right| \le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\|.$$

These estimates yield

$$\mathrm{e}^{\alpha(t-t_0)}\|x_t\| \leq M\|\phi\| + \int_{t_0}^{t} M\varepsilon \mathrm{e}^{\alpha(u-t_0)}\|x_u\|\,\mathrm{d}u,$$

and we obtain

$$\mathrm{e}^{\alpha(t-t_0)}\|x_t\| \leq M\|\phi\|\mathrm{e}^{M\varepsilon(t-t_0)}$$

by applying Gronwall's inequality (see Lemma C.1). This means that

$$\|x_t\| \leq M\|\phi\|\mathrm{e}^{-\beta(t-t_0)} \tag{8.3}$$

holds for all $t \in [t_0, t_0 + T)$.

Step 3: Proof by contradiction. We next show that T is equal to ∞, i.e., the non-continuable solution x is defined on $[t_0 - r, \infty)$. We suppose $T < \infty$ and derive a contradiction. Since $\|x_t\| < \widetilde{\delta}$ holds for all $t \in [t_0, t_0 + T)$, we have

$$\begin{aligned}|\dot{x}(t)| &\leq \|L\|\|x_t\| + |f(t, x_t)| \\ &\leq (\|L\| + \varepsilon)\widetilde{\delta} \\ &< \infty.\end{aligned}$$

This shows that $x|_{[t_0,t_0+T)}$ is Lipschitz continuous. In particular, $x|_{[t_0,t_0+T)}$ is uniformly continuous, and therefore, the limit $\lim_{t\uparrow t_0+T} x(t)$ exists. Since this yields the existence of the limit

$$\lim_{t\uparrow t_0+T} x_t =: \psi \in C([-r, 0], \mathbb{R}^n),$$

we have

$$\|\psi\| \leq M\|\phi\|\mathrm{e}^{-\beta T} < M\delta = \widetilde{\delta},$$

i.e., $\psi \in \widetilde{U}$, by taking the limit as $t \uparrow t_0 + T$ in the inequality (8.3). Then the RFDE (8.2) has a solution under the initial condition $x_{t_0+T} = \psi$ by the Peano existence theorem for RFDEs, and one can construct a continuation of x. It contradicts the property that x is non-continuable. Therefore, T should be infinity.

The above steps yield the conclusion. □

The above proof of Theorem 8.2 is an appropriate modification of the stability part of the principle of linearized stability for ODEs (e.g., see [6, Section 2.3]). It also should be compared with [35, Theorem 2 and its proof]. We note that the continuity of the higher-order term f in the RFDE (8.1) is sufficient for the proof.

8.3 Poincaré–Lyapunov theorem for RFDEs

In this subsection, we consider the continuous map $h\colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \supset \mathbb{R} \times U_0 \to \mathbb{R}^n$ used in Subsection 8.2 and a map

$$N\colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$$

with the following properties:

- For each $t \in \mathbb{R}$, the map $N(t) \colon C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ defined by
$$N(t)\phi := N(t,\phi)$$
for $\phi \in C([-r,0],\mathbb{R}^n)$ is a bounded linear operator.
- $\mathbb{R} \ni t \mapsto N(t) \in \mathcal{B}(C([-r,0],\mathbb{R}^n),\mathbb{R}^n)$ is continuous.
- $\lim_{t\to\infty} \|N(t)\| = 0$ holds.

We note that the map N with the above properties is continuous.

Lemma 8.4. *Suppose that f satisfies $\operatorname{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = N(t)\phi + h(t,\phi)$ for all $(t,\phi) \in \mathbb{R} \times U_0$. Then for every $\varepsilon > 0$, there exist $a \in \mathbb{R}$ and $\widetilde{\delta} > 0$ such that*
$$|f(t,\phi)| \leq \varepsilon\|\phi\|$$
holds for all $t \geq a$ and all $\|\phi\| < \widetilde{\delta}$.

Proof. Let $\varepsilon > 0$ be given. Then we can choose $a \in \mathbb{R}$ and $\widetilde{\delta} > 0$ with the following properties:

- $\|N(t)\| < \varepsilon/2$ holds for all $t \geq a$.
- For all $\phi \in C([-r,0],\mathbb{R}^n)$, $\|\phi\| < \widetilde{\delta}$ implies $\phi \in U_0$.
- $|h(t,\phi)| \leq (\varepsilon/2)\|\phi\|$ holds for all $t \in \mathbb{R}$ and all $\|\phi\| < \widetilde{\delta}$.

Then for all $t \geq a$ and all $\|\phi\| < \widetilde{\delta}$,
$$|f(t,\phi)| \leq \|N(t)\|\|\phi\| + |h(t,\phi)| \leq \varepsilon\|\phi\|$$
holds. □

Lemma 8.5. *Suppose that f satisfies $\operatorname{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = N(t)\phi + h(t,\phi)$ for all $(t,\phi) \in \mathbb{R} \times U_0$. Let $\sigma \in \mathbb{R}$ be given. Then for every $\varepsilon > 0$, there exist $a \in \mathbb{R}$, $\widetilde{\delta} > 0$, and a continuous function $R \colon [\sigma,\infty) \to (0,\infty)$ with the following properties: (i) $R(t) \leq \varepsilon$ for all $t \geq a$, (ii) there exists an $R_0 > \varepsilon$ such that $R(t) \leq R_0$ holds for all $t \in [\sigma,a]$, and (iii)*
$$|f(t,\phi)| \leq R(t)\|\phi\|$$
holds for all $t \geq \sigma$ and all $\|\phi\| < \widetilde{\delta}$.

Proof. Let $\varepsilon > 0$ be given. In the same way as in the proof of Lemma 8.4, we choose $a > 0$ and $\widetilde{\delta} > 0$. When $\sigma \geq a$, the condition (ii) in Lemma 8.5 is vacuous, and one can choose the constant function whose value is equal to ε as a function R. When $\sigma < a$, we choose $R_0 > \varepsilon$ so that
$$\sup_{t\in[\sigma,\infty)} \|N(t)\| + \frac{\varepsilon}{2} < R_0.$$
We note that $\sup_{t\in[\sigma,\infty)} \|N(t)\| < \infty$ holds because $\|N(t)\| < \varepsilon/2$ for all $t \geq a$ and $t \mapsto \|N(t)\|$ is continuous. Then the continuous function $R \colon [\sigma,\infty) \to (0,\infty)$ given by
$$R(t) := \|N(t)\| + \frac{\varepsilon}{2}$$
satisfies the properties (i), (ii), and (iii). □

Theorem 8.6. *Suppose that the map f in the RFDE (8.1) satisfies $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and the conclusion of Lemma 8.5. If X^L is exponentially stable, then for each given $\sigma \in \mathbb{R}$, there exist $M \geq 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r,0],\mathbb{R}^n)$ with the following property: for every $t_0 \geq \sigma$, every $\phi \in U$, and every non-continuable solution x of (8.1) under the initial condition $x_{t_0} = \phi$, x is defined for all $t \geq t_0$ and satisfies*

$$\|x_t\| \leq M\mathrm{e}^{-\beta(t-t_0)}\|\phi\|$$

for all $t \geq t_0$.

Proof. From Lemma 7.2 and Theorem 7.3, we choose constants $M_0 \geq 1$ and $\alpha > 0$ so that

$$\sup_{\theta\in[-r,0]} \left|X^L(t+\theta)\right| \leq M_0\mathrm{e}^{-\alpha t} \qquad (t \in \mathbb{R})$$

and

$$\left\|T^L(t)\right\| \leq M_0\mathrm{e}^{-\alpha t} \qquad (t \geq 0)$$

hold. We also choose an $\varepsilon > 0$ so that

$$-\beta := M_0\varepsilon - \alpha < 0.$$

For this $\varepsilon > 0$, we choose the $a \in \mathbb{R}$, $\widetilde{\delta} > 0$, and the continuous function $R\colon [\sigma,\infty) \to (0,\infty)$ in Lemma 8.5. We divide the proof into the following cases: (I) $\sigma \geq a$, (II) $\sigma < a$.

When (I) $\sigma \geq a$, the completely same argument as in the proof of Theorem 8.2 is valid by choosing $M_0 := M$. See also Lemma 8.4. Therefore, we only have to consider the case (II) $\sigma < a$. In this case, we further divide the proof into the following steps.

Step 1: Choice of a neighborhood of 0 and a non-continuable solution. Let

$$M := M_0\mathrm{e}^{M_0(R_0-\varepsilon)(a-\sigma)} \quad \text{and} \quad \delta := \frac{\widetilde{\delta}}{M}.$$

We define open sets U and $\widetilde{U}$ by

$$U := \{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \delta\},$$
$$\widetilde{U} := \left\{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \widetilde{\delta}\right\}.$$

Since $M > M_0 \geq 1$,

$$U \subset \widetilde{U} \subset U_0$$

holds. We now fix $t_0 \geq \sigma$ and $\phi \in U$, and let $x\colon [t_0 - r, t_0 + T) \to \mathbb{R}^n$ be a non-continuable solution of the RFDE (8.2)

$$\dot{x}(t) = L|_{\widetilde{U}}(x_t) + f|_{\mathbb{R}\times\widetilde{U}}(t,x_t)$$

under the initial condition $x_{t_0} = \phi$.

Step 2: Estimate by Gronwall's inequality. A similar argument as in Step 2 of the proof of Theorem 8.2 yields that

$$\mathrm{e}^{\alpha(t-t_0)}\|x_t\| \leq M_0\|\phi\| \exp\left(\int_{t_0}^t M_0R(u)\,\mathrm{d}u\right)$$

holds for all $t \in [t_0, t_0 + T)$ by Gronwall's inequality (see Lemma C.1). This is just obtained by replacing M and ε with M_0 and $R(u)$, respectively. The above inequality means that

$$\|x_t\| \le M_0\|\phi\| \exp\left(\int_{t_0}^{t} [M_0 R(u) - \alpha]\, \mathrm{d}u\right)$$

holds for all $t \in [t_0, t_0 + T)$. We now estimate

$$C(t) := \exp\left(\int_{t_0}^{t} [M_0 R(u) - \alpha]\, \mathrm{d}u\right)$$

from above for $t \in [t_0, t_0 + T)$ by dividing into the following cases:

- Case: $t_0 + T < a$. Since $t < a$, we have

  $$C(t) \le \mathrm{e}^{(M_0 R_0 - \alpha)(t - t_0)}$$

  by the property (ii) in Lemma 8.5. Here the right-hand side is equal to

  $$\mathrm{e}^{M_0(R_0 - \varepsilon)(t - t_0)} \mathrm{e}^{-\beta(t - t_0)}$$

  by the choice of β. In view of $\sigma \le t_0 \le t < a$, the above is further estimated from above by

  $$\mathrm{e}^{M_0(R_0 - \varepsilon)(a - \sigma)} \mathrm{e}^{-\beta(t - t_0)}.$$

  Therefore, inequality (8.3)

  $$\|x_t\| \le M\|\phi\|\mathrm{e}^{-\beta(t - t_0)}$$

  holds for all $t \in [t_0, t_0 + T)$ with $M = M_0 \mathrm{e}^{M_0(R_0 - \varepsilon)(a - \sigma)}$.

- Case: $t_0 + T \ge a$. The integral in $C(t)$ is estimated from above by

  $$\int_{t_0}^{a} (M_0 R_0 - \alpha)\, \mathrm{d}u + \int_{a}^{t} (M_0 \varepsilon - \alpha)\, \mathrm{d}u = (M_0 R_0 - \alpha)(a - t_0) + (-\beta)(t - a).$$

  Here $-\beta = M_0\varepsilon - \alpha$ is used. In view of $t - a = (t - t_0) + (t_0 - a)$, the above value becomes

  $$(M_0 R_0 - \alpha + \beta)(a - t_0) + (-\beta)(t - t_0) = M_0(R_0 - \varepsilon)(a - t_0) + (-\beta)(t - t_0).$$

  The last term is also estimated from above by

  $$M_0(R_0 - \varepsilon)(a - \sigma) + (-\beta)(t - t_0)$$

  because of $t_0 \ge \sigma$ and $M_0(R_0 - \varepsilon) > 0$. Therefore, inequality (8.3) holds for all $t \in [t_0, t_0 + T)$ with $M = M_0 \mathrm{e}^{M_0(R_0 - \varepsilon)(a - \sigma)}$.

Step 3: Proof by contradiction. We next show that T is equal to ∞, i.e., the non-continuable solution x is defined on $[t_0 - r, \infty)$. We suppose $T < \infty$ and derive a contradiction. Since $\|x_t\| < \widetilde{\delta}$ holds for all $t \in [t_0, t_0 + T)$, we have

$$\begin{aligned} |\dot{x}(t)| &\le \|L\|\|x_t\| + |f(t, x_t)| \\ &\le (\|L\| + R(t))\widetilde{\delta} \\ &< \infty. \end{aligned}$$

We note that the continuous function R is bounded. The remainder of the proof is completely same as in Step 3 of the proof of Theorem 8.2.

The above steps yield the conclusion. □

As a consequence of Theorem 8.6 and Lemma 8.5, the following *Poincaré–Lyapunov theorem* for RFDEs is obtained. See [6, Exercise 2.79] for the theorem for ODEs. In the theorem, we suppose that $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t, \phi) = N(t)\phi + h(t, \phi)$ holds for all $(t, \phi) \in \mathbb{R} \times U_0$ in the RFDE (8.1).

Theorem 8.7. *If X^L is exponentially stable, then for each given $\sigma \in \mathbb{R}$, there exist $M \geq 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r, 0], \mathbb{R}^n)$ with the following property: for every $t_0 \geq \sigma$, every $\phi \in U$, and every non-continuable solution x of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$, x is defined for all $t \geq t_0$ and satisfies*

$$\|x_t\| \leq M\mathrm{e}^{-\beta(t-t_0)}\|\phi\|$$

for all $t \geq t_0$.

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A Riemann–Stieltjes integrals with respect to matrix-valued functions

Throughout this appendix, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 1$ be an integer, and $[a, b]$ be a closed and bounded interval of $\mathbb{R}$. In this appendix, we study Riemann–Stieltjes integrals with respect to matrix-valued functions. We refer the reader to [39, Chapter 1] and [31, Appendix D] as references of Riemann–Stieltjes integrals for scalar-valued functions. See also [24, Section 3.1] and [14, Section I.1 in Appendix I].

A.1 Definitions

Definition A.1. A finite sequence $(x_k)_{k=0}^m$ for some integer $m \geq 1$ satisfying

$$a = x_0 < x_1 < \cdots < x_m = b$$

is called a *partition* of $[a, b]$. This is also denoted by a symbol $P : a = x_0 < x_1 < \cdots < x_m = b$. For a finite sequence $\xi := (\xi_k)_{k=1}^m$ satisfying

$$x_{k-1} \leq \xi_k \leq x_k \quad (k \in \{1, \dots, m\}),$$

we call a pair (P, ξ) a *tagged partition* of $[a, b]$. For the tagged partition (P, ξ), let

$$|(P, \xi)| := |P| := \max_{1 \leq k \leq m}(x_k - x_{k-1}),$$

which is called the *norm* of (P, ξ).

The above terminology of tagged partition comes from [15].

Definition A.2. Let $f, \alpha \colon [a, b] \to M_n(\mathbb{K})$ be functions. For a tagged partition (P, ξ) of $[a, b]$ given in Definition A.1, let

$$S(f; \alpha, (P, \xi)) := \sum_{k=1}^{m} [\alpha(x_k) - \alpha(x_{k-1})] f(\xi_k).$$

We call $S(f; \alpha, (P, \xi))$ the *Riemann–Stieltjes sum* of f with respect to α under the tagged partition (P, ξ).

Definition A.3. Let $f, \alpha \colon [a, b] \to M_n(\mathbb{K})$ be functions. We say that f is *Riemann–Stieltjes integrable with respect to* α if there exists a $J \in M_n(\mathbb{K})$ with the following property: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all tagged partition (P, ξ) of $[a, b]$, $|(P, \xi)| < \delta$ implies

$$|S(f; \alpha, (P, \xi)) - J| < \varepsilon.$$

We note that such a J is unique if it exists. It is called the *Riemann–Stieltjes integral of f with respect to α* and is denoted by $\int_a^b \mathrm{d}\alpha(x) f(x)$.

A.1.1 Remarks

Remark A.4. One can also consider a sum

$$\sum_{k=1}^{m} f(\xi_k)[\alpha(x_k) - \alpha(x_{k-1})],$$

which is different from $S(f; \alpha, (P, \xi))$ in general. If a limit of the above sum as $|(P, \xi)| \to 0$ exists in the sense of Definition A.3, we will write the limit as $\int_a^b f(x) \, \mathrm{d}\alpha(x)$. By taking the transpose,

$$\left(\int_a^b \mathrm{d}\alpha(x) f(x) \right)^{\mathrm{T}} = \int_a^b f(x)^{\mathrm{T}} \, \mathrm{d}\alpha(x)^{\mathrm{T}}$$

holds. Here A^{T} denotes the transpose of a matrix $A \in M_n(\mathbb{K})$. When $n = 1$,

$$\int_a^b \mathrm{d}\alpha(x) f(x) = \int_a^b f(x) \, \mathrm{d}\alpha(x)$$

holds.

Remark A.5. The notions of the Riemann–Stieltjes sum $S(f; \alpha, (P, \xi))$ and the Riemann–Stieltjes integrability of f with respect to α are also defined for functions

$$f \colon [a, b] \to \mathbb{K}^n \quad \text{and} \quad \alpha \colon [a, b] \to M_n(\mathbb{K}).$$

In this case, the sum $S(f; \alpha, (P, \xi))$ and the integral $\int_a^b \mathrm{d}\alpha(x) f(x)$ belong to $\mathbb{K}^n$.

A.2 Reduction to scalar-valued case

Since the linear space $M_n(\mathbb{K})$ is finite-dimensional, the operator norm $|\cdot|$ on $M_n(\mathbb{K})$ is equivalent to the norm $|\cdot|_2$ on $M_n(\mathbb{K})$ defined by

$$|A|_2 := \sqrt{\sum_{i,j \in \{1,\dots,n\}} |a_{i,j}|^2}, \tag{A.1}$$

where $a_{i,j}$ is the (i, j)-component of the matrix $A \in M_n(\mathbb{K})$. This means that the notion of convergence in $M_n(\mathbb{K})$ can be treated component-wise.

Lemma A.6. *Let $f, \alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions. Then the following properties are equivalent:*

(a) f is Riemann–Stieltjes integrable with respect to α.

(b) For each column vector $f_j \colon [a,b] \to \mathbb{K}^n$ of $f = (f_1 \ \cdots \ f_n)$, it is Riemann–Stieltjes integrable with respect to α.

Furthermore,

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \left(\int_a^b \mathrm{d}\alpha(x)\, f_1(x) \ \cdots \ \int_a^b \mathrm{d}\alpha(x)\, f_n(x) \right)$$

holds when one of the above properties are satisfied.

The proof is based on the definition of the matrix product and on the property that the operator norm $|\cdot|$ is equivalent to the norm $|\cdot|_2$ given in (A.1). Therefore, we omit the proof.

Lemma A.7. *Let $f \colon [a,b] \to \mathbb{K}^n$ and $\alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions with $f = (f_1, \dots, f_n)$ and $\alpha = (\alpha_{i,j})_{i,j \in \{1,\dots,n\}}$. If $f_j \colon [a,b] \to \mathbb{K}$ is Riemann–Stieltjes integrable with respect to $\alpha_{i,j} \colon [a,b] \to \mathbb{K}$ for every $i, j \in \{1, \dots, n\}$, then so is f with respect to α. Furthermore,*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \left(\sum_{j=1}^n \int_a^b f_j(x)\, \mathrm{d}\alpha_{i,j}(x) \right)_{i=1}^n$$

holds.

Proof. By the definition of the product of a matrix and a vector, the i-th component of

$$S(f; \alpha, (P, \xi)) \in \mathbb{K}^n$$

is equal to

$$\sum_{j=1}^n S(f_j; \alpha_{i,j}, (P, \xi)).$$

Therefore, the conclusion is obtained by the triangle inequality. □

The converse of Lemma A.7 does not necessarily hold as the following example shows.

Example A.8. Let $n = 2$ and $g, \beta \colon [a,b] \to \mathbb{K}$ be given functions. Let

$$f := (g, -g) \colon [a,b] \to \mathbb{K}^2 \quad \text{and} \quad \alpha := (\beta)_{i,j \in \{1,2\}} \colon [a,b] \to M_2(\mathbb{K}),$$

i.e., $f_1 = f$, $f_2 = -f$, and $\alpha_{i,j} = \beta$. Then the Riemann–Stieltjes sum of f with respect to α is equal to 0 under any tagged partition of $[a,b]$. This means that f is Riemann–Stieltjes integrable with respect to α for any pair (g, β) of functions.

In view of the above example, the Riemann–Stieltjes integration of vector-valued functions with respect to matrix-valued functions is not completely reduced to that for scalar-valued functions. However, it is often useful to reduce the integration to scalar-valued case in view of Lemma A.7.

A.3 Fundamental results

The following are fundamental results on Riemann–Stieltjes integrals for matrix-valued functions.

A.3.1 Reversal formula

Theorem A.9. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. We define functions $\bar{f}, \bar{\alpha}\colon [-b,-a] \to M_n(\mathbb{K})$ by*

$$\bar{f}(y) := f(-y), \quad \bar{\alpha}(y) := \alpha(-y)$$

for $y \in [-b,-a]$. If f is Riemann–Stieltjes integrable with respect to α, then so is $\bar{f}$ with respect to $\bar{\alpha}$. Furthermore,

$$\int_{-b}^{-a} \mathrm{d}\bar{\alpha}(y)\, \bar{f}(y) = -\int_a^b \mathrm{d}\alpha(x)\, f(x) \tag{A.2}$$

holds.

We call Eq. (A.2) the *reversal formula* for Riemann–Stieltjes integrals. The proof is obtained by returning to the definition of Riemann–Stieltjes integrals. Therefore, it can be omitted.

A.3.2 Integration by parts formula

The following is the *integration by parts formula* for Riemann–Stieltjes integrals with respect to matrix-valued functions.

Theorem A.10. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is Riemann–Stieltjes integrable with respect to α, then so is α with respect to f. Furthermore,*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = [\alpha(x) f(x)]_{x=a}^b - \int_a^b \alpha(x)\, \mathrm{d}f(x)$$

holds. Here $[\alpha(x) f(x)]_{x=a}^b := \alpha(b) f(b) - \alpha(a) f(a)$.

The proof is basically same as the proof for the case $n = 1$ (i.e., the scalar-valued case). See [31, Proposition D.3] for the proof of this case. See also [39, Theorems 4a and 4b in Chapter 1].

A.4 Integrability

A.4.1 Matrix-valued functions of bounded variation

We first recall the definition of matrix-valued functions of bounded variation.

Definition A.11. Let $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a function. For each partition $P : a = x_0 < x_1 < \cdots < x_m = b$ of $[a,b]$, let

$$\mathrm{Var}(\alpha; P) := \sum_{k=1}^m |\alpha(x_k) - \alpha(x_{k-1})|,$$

which is called the *variation* of α under the partition P. The value

$$\mathrm{Var}(\alpha) := \sup \{ \mathrm{Var}(\alpha; P) : P \text{ is a partition of } [a,b] \}$$

is called the *total variation* of α. α is said to be *of bounded variation* if $\mathrm{Var}(\alpha) < \infty$.

Since the operator norm $|\cdot|$ on $M_n(\mathbb{K})$ and the norm $|\cdot|_2$ on $M_n(\mathbb{K})$ given in (A.1) are equivalent, a matrix-valued function $\alpha\colon [a,b] \to M_n(\mathbb{K})$ is of bounded variation if and only if each component function $\alpha_{i,j}\colon [a,b] \to \mathbb{K}$ is of bounded variation.

Remark A.12. Let $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a function. Then for any $c \in (a,b)$,

$$\operatorname{Var}\bigl(\alpha|_{[a,c]}\bigr) + \operatorname{Var}\bigl(\alpha|_{[c,b]}\bigr) = \operatorname{Var}(\alpha) \tag{A.3}$$

holds. This equality is obtained from

$$\operatorname{Var}\bigl(\alpha|_{[a,c]}; P_1\bigr) + \operatorname{Var}\bigl(\alpha|_{[c,b]}; P_2\bigr) = \operatorname{Var}(\alpha; P),$$

where P_1 is a partition of $[a,c]$, P_2 is a partition of $[c,b]$, and P is the partition of $[a,b]$ obtained by joining P_1 and P_2.

Lemma A.13. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is Riemann–Stieltjes integrable with respect to α, then*

$$\left|\int_a^b \mathrm{d}\alpha(x) f(x)\right| \le \operatorname{Var}(\alpha) \cdot \sup_{x\in[a,b]} |f(x)| \tag{A.4}$$

holds.

Proof. Let (P,ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Since $|AB| \le |A||B|$ holds for any $A, B \in M_n(\mathbb{K})$, we have

$$|S(f;\alpha,(P,\xi))| \le \sum_{k=1}^{m} |\alpha(x_k) - \alpha(x_{k-1})||f(\xi_k)| \le \operatorname{Var}(\alpha) \cdot \sup_{x\in[a,b]} |f(x)|.$$

Then the remaining proof is essentially same as the scalar-valued case. □

Remark A.14. In the completely similar way, (A.4) also holds for any function $f\colon [a,b] \to \mathbb{K}^n$ which is Riemann–Stieltjes integrable with respect to α. This can also be seen from Lemma A.13 because for any $A \in M_n(\mathbb{K})$ of the form

$$A = (a\ 0\ \cdots\ 0) \quad (a \in \mathbb{K}^n, 0 \in \mathbb{K}^n),$$

$|A| = |a|$ holds.

A.4.2 Integrability of matrix-valued functions

The following is a fundamental theorem on the Riemann–Stieltjes integrability for scalar-valued functions.

Theorem A.15. *Let $f, \alpha\colon [a,b] \to \mathbb{K}$ be functions. If f is continuous and α is of bounded variation, then f is Riemann–Stieltjes integrable with respect to α.*

See [31, Theorem D.1] for a proof, which is valid for the case $\mathbb{K} = \mathbb{C}$ because it does not use the order structure. By using Theorem A.15, one can obtain the following.

Theorem A.16. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is continuous and α is of bounded variation, then f is Riemann–Stieltjes integrable with respect to α.*

Proof. From Lemma A.6, the problem is reduced to the Riemann–Stieltjes integrability of each column vector of f with respect to α. From Lemma A.7, it is sufficient to show that each component $f_{i,j}\colon [a,b] \to \mathbb{K}$ of f is Riemann–Stieltjes integrable with respect to each component $\alpha_{i,j}\colon [a,b] \to \mathbb{K}$ of α. Since each $f_{i,j}$ is continuous and each $\alpha_{i,j}$ is of bounded variation, the conclusion is obtained from Theorem A.15. □

The following is the result on additivity of Riemann–Stieltjes integrals with respect to matrix-valued functions on sub-intervals.

Theorem A.17. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a continuous function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a function of bounded variation. Then for any $c \in (a,b)$,*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^c \mathrm{d}\alpha(x)\, f(x) + \int_c^b \mathrm{d}\alpha(x)\, f(x)$$

holds.

The proof is same as that for the case $n = 1$. See [31, Proposition D.2] for the proof. We note that the statement can be proved by considering partitions of $[a,b]$ with $c \in (a,b)$ as an intermediate point.

Remark A.18. In Theorem A.17, the assumptions that f is continuous and α is of bounded variation are essential because these assumptions ensure the existence of three integrals (see (A.3) and Theorem A.16). Without these assumptions, the integral in the left-hand side does not necessarily exist even if the integrals in the right-hand side exist. Such a situation will occur when the functions f and α share a discontinuity at c. See [39, Section 5 in Chapter I] for the detail.

A.5 Integration with respect to continuously differentiable functions

The following theorem shows a relationship between Riemann–Stieltjes integrals and Riemann integrals.

Theorem A.19. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a Riemann integrable function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a continuously differentiable function. Then f is Riemann–Stieltjes integrable with respect to α, and*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^b \alpha'(x) f(x)\, \mathrm{d}x$$

holds. Here the right-hand side is a Riemann integral.

Since the above statement is not mentioned in [39] and [31] even for the case $n = 1$, we now give an outline of the proof.

Outline of the proof of Theorem A.19. Let (P,ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Let

$$S(\alpha' f; (P,\xi)) := \sum_{k=1}^{m} (x_k - x_{k-1}) \alpha'(\xi_k) f(\xi_k).$$

Since

$$\alpha(x_k) - \alpha(x_{k-1}) = \int_{x_{k-1}}^{x_k} \alpha'(t)\, \mathrm{d}t$$

holds for each $k \in \{1, \dots, m\}$ by the fundamental theorem of calculus, we have

$$S(f; \alpha, (P,\xi)) - S(\alpha' f; (P,\xi)) = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} [\alpha'(t) - \alpha'(\xi_k)]\, \mathrm{d}t \cdot f(\xi_k).$$

From this, we also have

$$\left| S(f; \alpha, (P,\xi)) - S(\alpha' f; (P,\xi)) \right| \le \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \left| \alpha'(t) - \alpha'(\xi_k) \right| \mathrm{d}t \cdot |f(\xi_k)|.$$

By combining this and the uniform continuity of α', one can obtain the conclusion. □

When $n = 1$ and $\mathbb{K} = \mathbb{R}$, one can use the mean value theorem for the proof of Theorem A.19.

A.6 Integration with respect to absolutely continuous functions

The following theorem should be compared with Theorem A.19.

Theorem A.20. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a continuous function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be an absolutely continuous function. Then*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^b \alpha'(x) f(x)\, \mathrm{d}x$$

holds. Here the right-hand side is a Lebesgue integral.

See [39, Theorem 6a in Chapter I] for the proof of the scalar-valued case. We note that the existence of the Riemann–Stieltjes integral in the left-hand side is ensured by Theorem A.16 because the absolutely continuous function α is of bounded variation. We also note that the function $[a,b] \ni x \mapsto \alpha'(x) f(x) \in M_n(\mathbb{K})$ is Lebesgue integrable because it is measurable and

$$\int_a^b |\alpha'(x) f(x)|\, \mathrm{d}x \le \int_a^b |\alpha'(x)||f(x)|\, \mathrm{d}x \le \|\alpha'\|_1 \|f\| < \infty$$

holds.

Since it is interesting to compare the proof of Theorem A.19 and the proof of Theorem A.20, we now give an outline of the proof.

Outline of the proof of Theorem A.20. Let (P, ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Since $\alpha = \alpha(0) + V\alpha'$,

$$S(f; \alpha, (P, \xi)) = \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \alpha'(t)\, \mathrm{d}t \cdot f(\xi_k)$$

holds. Therefore, we have

$$\int_a^b \alpha'(x) f(x)\, \mathrm{d}x - S(f; \alpha, (P, \xi)) = \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \alpha'(t)[f(t) - f(\xi_k)]\, \mathrm{d}t.$$

In combination with the uniform continuity of f, the conclusion is obtained by taking the limit as $|(P, \xi)| \to 0$. □

A.7 Proof of the theorem on iterated integrals

In this subsection, we give a proof of Theorem 3.8.

Proof of Theorem 3.8. We define a bounded linear operator $T\colon C([a,b], M_n(\mathbb{K})) \to M_n(\mathbb{K})$ by

$$Tg := \int_a^b \mathrm{d}\alpha(x)\, g(x)$$

for $g \in C([a,b], M_n(\mathbb{K}))$. From Lemma 2.9, the left-hand side of (3.4) is equal to

$$T \int_c^d f(\cdot, y)\, \mathrm{d}y,$$

which is also equal to $\int_c^d Tf(\cdot, y)\, \mathrm{d}y$ since T is a bounded linear operator. By the definition of T, this integral is equal to the right-hand side of (3.4). This completes the proof. □

B Riesz representation theorem

Throughout this appendix, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $[a, b]$ be a closed and bounded interval of $\mathbb{R}$.

The following is the cerebrated *Riesz representation theorem*.

Theorem B.1. *For any continuous linear functional $A \colon C([a, b], \mathbb{K}) \to \mathbb{K}$, there exists a function $\alpha \colon [a, b] \to \mathbb{K}$ with the following properties: (i) $\operatorname{Var}(\alpha) = \|A\|$, (ii) every $f \in C([a, b], \mathbb{K})$ is Riemann–Stieltjes integrable with respect to α, and (iii)*

$$A(f) = \int_a^b f(x)\,\mathrm{d}\alpha(x)$$

holds for all $f \in C([a, b], \mathbb{K})$.

In a proof of Theorem B.1 (e.g., see discussions on [31, Chapter 9]), we construct such a function α by using a continuous linear extension

$$\bar{A} \colon B([a, b], \mathbb{K}) \to \mathbb{K}$$

of A with $\|\bar{A}\| = \|A\|$. Here $B([a, b], \mathbb{K})$ denotes the linear space of all bounded functions from $[a, b]$ to $\mathbb{K}$ endowed with the supremum norm. Its existence is ensured by the Hahn–Banach extension theorem in normed spaces (see [40, Theorem 1 in Section 5 of Chapter IV]). See also [1, Section 4 of Chapter IV].

Remark B.2. The Riemann-Stieltjes integrability of any $f \in C([a, b], \mathbb{K})$ with respect to the constructed function α is also obtained in the proof. This should be compared with Theorem A.15.

The following is a corollary of Theorem B.1.

Corollary B.3. *For any integer $n \geq 1$ and any continuous linear map $A \colon C([a, b], \mathbb{K}^n) \to \mathbb{K}^n$, there exists a function $\alpha \colon [a, b] \to M_n(\mathbb{K})$ of bounded variation such that*

$$A(f) = \int_a^b \mathrm{d}\alpha(x)\, f(x)$$

holds for all $f \in C([a, b], \mathbb{K}^n)$.

Corollary B.3 has been used in the literature of RFDEs (e.g., see [18], [19], [22], and [14]). We now give the proof of Corollary B.3 because it is not given in these references.

Proof of Corollary B.3. Let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ be the standard basis of $\mathbb{K}^n$. For each $g \in C([a, b], \mathbb{K})$ and each $j \in \{1, \dots, n\}$, let $g\boldsymbol{e}_j \in C([a, b], \mathbb{K}^n)$ be defined by

$$(g\boldsymbol{e}_j)(x) := g(x)\boldsymbol{e}_j$$

for $x \in [a, b]$. For each $i, j \in \{1, \dots, n\}$, we define a functional $A_{i,j} \colon C([a, b], \mathbb{K}) \to \mathbb{K}$ by

$$A_{i,j}(g) := A(g\boldsymbol{e}_j)_i.$$

Here y_i denotes the i-th component of $y \in \mathbb{K}^n$. Since $A_{i,j}$ is a continuous linear functional, one can choose a function $\alpha_{i,j} \colon [a, b] \to \mathbb{K}$ of bounded variation so that

$$A_{i,j}(g) = \int_a^b g(x)\,\mathrm{d}\alpha_{i,j}(x)$$

holds for all $g \in C([a,b],\mathbb{K})$ from Theorem B.1. By using $f = \sum_{j=1}^{n} f_j \boldsymbol{e}_j$ for $f = (f_1,\dots,f_n)$, we have

$$A(f)_i = \sum_{j=1}^{n} A(f_j \boldsymbol{e}_j)_i = \sum_{j=1}^{n} A_{i,j}(f_j) = \sum_{j=1}^{n} \int_a^b f_j(x)\,\mathrm{d}\alpha_{i,j}(x).$$

From Lemma A.7, this yields that

$$A(f) = \int_a^b \mathrm{d}\alpha(x)\, f(x)$$

holds for all $f \in C([a,b],\mathbb{K}^n)$ by defining a matrix-valued function $\alpha\colon [a,b] \to M_n(\mathbb{K})$ of bounded variation by $\alpha := (\alpha_{i,j})_{i,j}$. This completes the proof. □

C Variants of Gronwall's inequality

Throughout this appendix, let $[a,b]$ be a closed and bounded interval of $\mathbb{R}$.

C.1 Gronwall's inequality and its generalization

The following is known as Gronwall's inequality.

Lemma C.1 (ref. [20]). *Let $\alpha \in \mathbb{R}$ be a constant and $\beta\colon [a,b] \to [0,\infty)$ be a continuous function. If a continuous function $u\colon [a,b] \to \mathbb{R}$ satisfies*

$$u(t) \le \alpha + \int_a^t \beta(s)u(s)\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$u(t) \le \alpha \exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

Outline of the proof. To use a technique for scalar homogeneous linear ODEs, let

$$v(t) := \int_a^t \beta(s)u(s)\,\mathrm{d}s.$$

Then the given inequality becomes

$$\dot{v}(t) \le \beta(t)[v(t) + \alpha] \qquad (t \in [a,b]),$$

where the non-negativity of β is used. Since the left-hand side is the derivative of the function $t \mapsto v(t) + \alpha$, it is natural to consider the derivative of

$$t \mapsto \exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)[v(t) + \alpha].$$

Then it holds that this function is monotonically decreasing, which yields the conclusion. □

The following is a generalized version of Gronwall's inequality.

Lemma C.2 (refs. [19], [20], [22])**.** *Let $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to [0,\infty)$ be given continuous functions. If a continuous function $u\colon [a,b] \to \mathbb{R}$ satisfies*

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$

holds for all $t \in [a,b]$. Furthermore, if α is monotonically increasing, then

$$u(t) \le \alpha(t)\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds.

By letting $v(t) := \int_a^t \beta(s)u(s)\,\mathrm{d}s$, one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)v(t) \le \exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)\beta(t)\alpha(t).$$

Then the first inequality is obtained by integrating both sides in combination with $u(t) \le \alpha(t) + v(t)$. See [20, Section I.6], [19, Lemma 3.1 in Section 1.3], and [22, Lemma 3.1 in Section 1.3] for the detail of the proof.

C.2 Gronwall's inequality and RFDEs

In this subsection, let $r > 0$ and $E = (E, \|\cdot\|)$ be a normed space. For each continuous function $u\colon [a-r,b] \to E$ and each $t \in [a,b]$, let $u_t \in C([-r,0],E)$ be defined by

$$u_t(\theta) := u(t+\theta) \qquad (\theta \in [-r,0]).$$

It holds that the function $[a,b] \ni t \mapsto u_t \in C([-r,0],E)$ is continuous.

In the context of RFDEs, it is often convenient to use the following result rather than to use Gronwall's inequality directly.

Lemma C.3 (cf. [23])**.** *Let $\alpha \in \mathbb{R}$ be a constant and $\beta\colon [a,b] \to [0,\infty)$ be a given continuous function. If a continuous function $u\colon [a-r,b] \to E$ satisfies*

$$\|u(t)\| \le \alpha + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$\|u_t\| \le \max\{\|u_a\|, \alpha\}\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

This should be compared with [23, Lemma 2.1]. We note that the argument of the proof has appeared in [19, Theorem 1.1 in Chapter 6] and [22, Theorem 1.1 in Chapter 6].

A generalization of Lemma C.3 is possible by using Lemma C.2.

Lemma C.4. *Let $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to [0,\infty)$ be given continuous functions. If a continuous function $u\colon [a-r,b] \to E$ satisfies*

$$\|u(t)\| \le \alpha(t) + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s$$

for all $t \in [a,b]$ and α is monotonically increasing, then

$$\|u_t\| \le \max\{\|u_a\|, \alpha(t)\} \exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

Proof. Let $t \in [a,b]$ be fixed and $\theta \in [-r,0]$ be given. When $t+\theta \ge a$, we have

$$\begin{aligned}\|u(t+\theta)\| &\le \alpha(t+\theta) + \int_a^{t+\theta} \beta(s)\|u_s\|\,\mathrm{d}s \\ &\le \alpha(t) + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s.\end{aligned}$$

Here the property that α is monotonically increasing and the non-negativity of β are used. When $t+\theta \le a$, we have

$$\|u(t+\theta)\| \le \|u_a\|.$$

By combining the above inequalities, we obtain

$$\|u_t\| \le \max\{\|u_a\|, \alpha(t)\} + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s.$$

Since the functions $[a,b] \ni t \mapsto \|u_t\| \in [0,\infty)$ and $[a,b] \ni t \mapsto \max\{\|u_a\|, \alpha(t)\} \in \mathbb{R}$ are continuous, the conclusion is obtained by applying Lemma C.2. □

D Lemmas on fixed point argument

Let $E = (E, \|\cdot\|)$ be a normed space and $r > 0$ be a constant. For each $\gamma > 0$, let

$$Y_\gamma := \left\{y \in C([-r,\infty), E) : y_0 = 0,\ \|y\|_\gamma < \infty\right\}$$

be a normed space endowed with the norm $\|\cdot\|_\gamma$ given by

$$\|y\|_\gamma := \sup_{t \ge 0}\,(\mathrm{e}^{-\gamma t}\|y_t\|) < \infty.$$

For the notation $\|y_t\|$, see Subsection C.2.

Lemma D.1. *For any continuous function $y\colon [-r,\infty) \to E$ with $y_0 = 0$,*

$$\|y\|_\gamma = \sup_{t \ge 0}\,(\mathrm{e}^{-\gamma t}\|y(t)\|)$$

holds.

Proof. Since $\|y(t)\| \le \|y_t\|$ holds for all $t \ge 0$,

$$\sup_{t\ge 0} \left(\mathrm{e}^{-\gamma t}\|y(t)\|\right) \le \|y\|_\gamma$$

holds. The reverse inequality also follows in view of

$$\mathrm{e}^{-\gamma t}\|y(t+\theta)\| = \mathrm{e}^{-\gamma(t+\theta)}\|y(t+\theta)\| \cdot \mathrm{e}^{\gamma\theta} \le \sup_{t\ge 0} \left(\mathrm{e}^{-\gamma t}\|y(t)\|\right)$$

for $t \ge 0$ and $\theta \in [-r, 0]$. Here $y_0 = 0$ and $\mathrm{e}^{\gamma\theta} \le 1$ are used. □

Lemma D.2. *If E is a Banach space, then Y_γ is also a Banach space.*

Proof. Let $(y^k)_{k=1}^\infty$ be a Cauchy sequence in Y_γ. We choose $\varepsilon > 0$. Then for all sufficiently large $k, \ell \ge 1$, we have $\left\|y^k - y^\ell\right\|_\gamma \le \varepsilon$. From Lemma D.1, this means that for all sufficiently large $k, \ell \ge 1$,

$$\left\|y^k(t) - y^\ell(t)\right\| \le \varepsilon \mathrm{e}^{\gamma t}$$

holds for all $t \ge 0$. This implies that $(y^k(t))_{k=1}^\infty$ is a Cauchy sequence for each $t \ge 0$, and therefore, $(y^k)_{k=1}^\infty$ has the limit function $y\colon [-r, \infty) \to E$ with $y_0 = 0$. Since the above relation shows that the convergence of $(y^k)_{k=1}^\infty$ to y is uniform on each closed and bounded interval of $\mathbb{R}$ by taking the limit as $\ell \to \infty$, the limit function y is continuous. Then it is concluded that

$$\left\|y^k - y\right\|_\gamma \le \varepsilon$$

holds for all sufficiently large $k \ge 1$, which implies that $(y^k)_{k=1}^\infty$ converges to y in Y_γ. □

E Convolution continued

In this appendix, we discuss the convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. The purpose is to share results on the convolution and their proofs in the literature of RFDEs. The results discussed here extend the results in Subsection 5.2, but they will not be used in this paper. See also [25, Proposition A.4, Theorems A.5, A.6, A.7 in Appendix A].

E.1 Convolution for locally essentially bounded functions and locally Lebesgue integrable functions

We first recall that a function $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ is said to be *locally essentially bounded* if

$$\operatorname*{ess\,sup}_{t\in[0,T]} |g(t)| := \inf\left\{M > 0 : |g(t)| \le M \text{ holds for almost all } t \in [0, T]\right\}$$

is finite for all $T > 0$. Let

$$\mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K})) := \left\{g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K})) : g \text{ is locally essentially bounded}\right\},$$

which is a linear subspace of $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. As in Definition 5.1, we introduce the following.

Definition E.1. For each $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and each $g \in \mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, we define a function $g * f\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(g * f)(t) := \int_0^t g(t-u) f(u) \, \mathrm{d}u = \int_0^t g(u) f(t-u) \, \mathrm{d}u$$

for $t \geq 0$. Here the integrals are Lebesgue integrals. The function $g * f$ is called the *convolution* of g and f.

We note that

$$|(g * f)(t)| \leq \operatorname*{ess\,sup}_{u \in [0,t]} |g(u)| \cdot \int_0^t |f(u)| \, \mathrm{d}u$$

holds for all $t \geq 0$. The following result should be compared with Lemma 5.2.

Lemma E.2. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and $g \in \mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Then $g * f$ is continuous.*

Outline of the proof. We show the continuity of $g * f$ on $[0,T]$ for each fixed $T > 0$. We define a function $\tilde{f}\colon \mathbb{R} \to M_n(\mathbb{K})$ by

$$\tilde{f}(t) := \begin{cases} f(t) & (t \in \operatorname{dom}(f) \cap [0,T]), \\ O & (\text{otherwise}). \end{cases}$$

Then $\tilde{f} \in \mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$, and

$$(g * f)(t) = \int_0^t g(u) \tilde{f}(t-u) \, \mathrm{d}u$$

holds for all $t \in [0,T]$. We fix $t_0 \in [0,T]$. By the reasoning as in the proof of Lemma 3.4, we have

$$(g * f)(t) - (g * f)(t_0) = \int_0^{t_0} g(u) \left[\tilde{f}(t-u) - \tilde{f}(t_0-u) \right] \mathrm{d}u + \int_{t_0}^t g(u) \tilde{f}(t-u) \, \mathrm{d}u$$

for all $t \in [0,T]$. Therefore, the continuity of $g * f$ on $[0,T]$ is obtained by Hölder's inequality, the continuity of the translation in $\mathcal{L}^1$, and the integrability of $\tilde{f}$. □

E.2 Convolution for locally Lebesgue integrable functions

The notion of convolution in Definition E.1 is not satisfactory in the sense that the condition on f and g is not symmetry. To introduce the notion of convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, we need the following.

Theorem E.3. *Let $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ be given. Then the following statements hold:*

1. *For almost all $t > 0$, $u \mapsto g(t-u) f(u)$ belongs to $\mathcal{L}^1([0,t], M_n(\mathbb{K}))$.*
2. *The function $g * f$ defined by*

$$(g * f)(t) := \int_0^t g(t-u) f(u) \, \mathrm{d}u$$

for almost all $t \geq 0$ belongs to $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$.

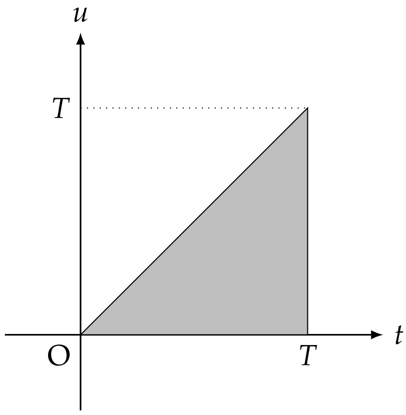

Figure E.1: The light gray region is the subset A.

3. *For all $T \geq 0$,*
$$\int_0^T |(g * f)(t)|\,\mathrm{d}t \leq \left(\int_0^T |g(t)|\,\mathrm{d}t\right) \cdot \left(\int_0^T |f(t)|\,\mathrm{d}t\right)$$
holds.

In the following, we give a direct proof of Theorem E.3 by using Fubini's theorem and Tonelli's theorem for functions on the Euclidean space $\mathbb{R}^d$. See [32, Theorems 3.1 and 3.2 in Section 3 of Chapter 2] for these statements and their proofs.

A direct proof of Theorem E.3. Let $\bar{f} \colon \mathbb{R} \to M_n(\mathbb{K})$ be the function defined by
$$\bar{f}(t) := \begin{cases} f(t) & (t \in \mathrm{dom}(f)), \\ O & (t \in \mathbb{R} \setminus \mathrm{dom}(f)). \end{cases}$$
In the same way, we define the function $\bar{g} \colon \mathbb{R} \to M_n(\mathbb{K})$. Then $\bar{f}, \bar{g} \colon \mathbb{R} \to M_n(\mathbb{K})$ are locally Lebesgue integrable functions.

Let $T > 0$ be fixed. The remainder of the proof is divided into the following steps.

Step 1: Setting of triangle region and function. We consider a closed set A of $\mathbb{R}^2$ given by
$$A := \{\, (t, u) \in \mathbb{R}^2 : t \in [0, T],\, u \in [0, t] \,\}.$$
See Fig. E.1 for the picture of A. Then the characteristic function $\mathbf{1}_A$ is measurable and
$$\mathbf{1}_A(t, u) = \mathbf{1}_{[0,T]}(t)\mathbf{1}_{[0,t]}(u) = \mathbf{1}_{[0,T]}(u)\mathbf{1}_{[u,T]}(t)$$
holds for all $(t, u) \in \mathbb{R}^2$. We define a function $h \colon \mathbb{R}^2 \to M_n(\mathbb{K})$ by
$$h(t, u) := \mathbf{1}_A(t, u)\bar{g}(t - u)\bar{f}(u).$$
Then h is measurable because
$$\mathbb{R}^2 \ni (t, u) \mapsto \bar{g}(t - u) \in M_n(\mathbb{K}) \quad \text{and} \quad \mathbb{R}^2 \ni (t, u) \mapsto \bar{f}(u) \in M_n(\mathbb{K})$$
are measurable.[2] This implies that the function $\mathbb{R}^2 \ni (t, u) \mapsto |h(t, u)| \in [0, \infty)$ is also measurable.

Step 2: Application of Tonelli's theorem. By applying Tonelli's theorem, the following statements hold:

[2]See [32, Corollary 3.7 and Proposition 3.9 in Section 3 of Chapter 2] for the results of scalar-valued case.

- For almost all $u \in \mathbb{R}$, the function $\mathbb{R} \ni t \mapsto |h(t,u)| \in [0,\infty)$ is measurable.
- For almost all $t \in \mathbb{R}$, the function $\mathbb{R} \ni u \mapsto |h(t,u)| \in [0,\infty)$ is measurable.
- The functions

$$u \mapsto \int_{\mathbb{R}} |h(t,u)| \,\mathrm{d}t \in [0,\infty], \quad t \mapsto \int_{\mathbb{R}} |h(t,u)| \,\mathrm{d}u \in [0,\infty]$$

  are measurable functions defined almost everywhere.
- We have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)| \,\mathrm{d}t \right) \mathrm{d}u = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)| \,\mathrm{d}u \right) \mathrm{d}t = \int_{\mathbb{R}^2} |h(t,u)| \,\mathrm{d}(t,u)$$

  including the possibility that all the unsigned Lebesgue integrals are ∞.

Step 3: Application of Fubini's theorem. By Step 2, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |h(t,u)| \,\mathrm{d}(t,u) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)| \,\mathrm{d}t \right) \mathrm{d}u \\
&\leq \int_0^T \left(\int_u^T |\bar{g}(t-u)| \,\mathrm{d}t \right) |\bar{f}(u)| \,\mathrm{d}u \\
&\leq \left(\int_0^T |g(t)| \,\mathrm{d}t \right) \cdot \left(\int_0^T |f(t)| \,\mathrm{d}t \right).
\end{aligned}$$

Since the last term is finite, it holds that h is integrable. By applying Fubini's theorem component-wise, the following statements hold:

- For almost all $u \in \mathbb{R}$, the function $\mathbb{R} \ni t \mapsto h(t,u) \in M_n(\mathbb{K})$ is Lebesgue integrable.
- For almost all $t \in \mathbb{R}$, the function $\mathbb{R} \ni u \mapsto h(t,u) \in M_n(\mathbb{K})$ is Lebesgue integrable.
- The functions

$$u \mapsto \int_{\mathbb{R}} h(t,u) \,\mathrm{d}t \in M_n(\mathbb{K}), \quad t \mapsto \int_{\mathbb{R}} h(t,u) \,\mathrm{d}u \in M_n(\mathbb{K})$$

  belong to $\mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$.
- The equalities

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(t,u) \,\mathrm{d}t \right) \mathrm{d}u = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(t,u) \,\mathrm{d}u \right) \mathrm{d}t = \int_{\mathbb{R}^2} h(t,u) \,\mathrm{d}(t,u)$$

  hold.

Step 4: Conclusion. For each $t \in [0,T]$,

$$h(t,u) = g(t-u)f(u)$$

holds for almost all $u \in [0,t]$. Therefore, for almost all $t \in [0,T]$, the function $u \mapsto g(t-u)f(u)$ belongs to $\mathcal{L}^1([0,t], M_n(\mathbb{K}))$. Furthermore, we have

$$\int_{\mathbb{R}} h(t,u) \,\mathrm{d}u = \int_0^t g(t-u)f(u) \,\mathrm{d}u$$

for almost all $t \in [0, T]$, and it holds that the function

$$t \mapsto \int_0^t g(t-u)f(u)\,\mathrm{d}u$$

is a Lebesgue integrable function defined almost everywhere on $[0, T]$. Since $T > 0$ is arbitrary, the statements 1 and 2 hold. The statement 3 also holds because we have

$$\begin{aligned}
\int_0^T |(g * f)(t)|\,\mathrm{d}t &\le \int_0^T \left(\int_0^t |g(t-u)f(u)|\,\mathrm{d}u \right) \mathrm{d}t \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}u \right) \mathrm{d}t \\
&\le \left(\int_0^T |g(t)|\,\mathrm{d}t \right) \cdot \left(\int_0^T |f(t)|\,\mathrm{d}t \right),
\end{aligned}$$

where the calculation in Step 3 is used.

This completes the proof. □

Another proof of Theorem E.3. Let $T > 0$ be fixed. We define $\tilde{f} \colon \mathbb{R} \to M_n(\mathbb{K})$ by

$$\tilde{f}(t) := \begin{cases} f(t) & (t \in \mathrm{dom}(f) \cap [0, T]), \\ O & (\text{otherwise}). \end{cases}$$

In the same way, we define the function $\tilde{g} \colon \mathbb{R} \to M_n(\mathbb{K})$. Since $\tilde{f}, \tilde{g} \colon \mathbb{R} \to M_n(\mathbb{K})$ are Lebesgue integrable functions, one can prove the following statements as in the scalar-valued case:[3]

1′. For almost all $t \in \mathbb{R}$, the function $u \mapsto \tilde{g}(t-u)\tilde{f}(u)$ is a Lebesgue integrable function defined almost everywhere.

2′. The function $\tilde{g} \star \tilde{f}$ defined by

$$(\tilde{g} \star \tilde{f})(t) := \int_{\mathbb{R}} \tilde{g}(t-u)\tilde{f}(u)\,\mathrm{d}u$$

for almost all $t \in \mathbb{R}$ belongs to $\mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$.

3′. An estimate

$$\int_{\mathbb{R}} \left| (\tilde{g} \star \tilde{f})(t) \right| \mathrm{d}t \le \|\tilde{g}\|_1 \cdot \|\tilde{f}\|_1$$

holds.

1. For each $t \in [0, T]$, we have

$$\tilde{g}(t-u)\tilde{f}(u) = g(t-u)f(u)$$

for almost all $u \in [0, t]$. By combining this and the above statement 1′, it holds that for almost all $t \in [0, T]$, $u \mapsto g(t-u)f(u)$ is a Lebesgue integrable function defined almost everywhere on $[0, t]$. Since $T > 0$ is arbitrary, the statement 1 holds.

2. By the definitions of $\tilde{f}$ and $\tilde{g}$,

$$(\tilde{g} \star \tilde{f})(t) = \int_0^t \tilde{g}(t-u)\tilde{f}(u)\,\mathrm{d}u = \int_0^t g(t-u)f(u)\,\mathrm{d}u$$

[3] See [32, Exercise 21 in Chapter 2] and [30, 8.13 and 8.14 of Chapter 8] for the scalar-valued case.

holds for all $t \in \mathrm{dom}(\tilde{g} \star \tilde{f}) \cap [0, T]$. Since $T > 0$ is arbitrary, this shows that

$$t \mapsto \int_0^t g(t-u) f(u) \, \mathrm{d}u$$

is a measurable function defined almost everywhere on $[0, \infty)$ from the statement 2′. Furthermore, we also have

$$\int_0^T \left| \int_0^t g(t-u) f(u) \, \mathrm{d}u \right| \mathrm{d}t = \int_0^T \left| (\tilde{g} \star \tilde{f})(t) \right| \mathrm{d}t < \infty.$$

Since $T > 0$ is arbitrary, the statement 2 holds.

3. By combining the proof of the statement 2 and the inequality in the statement 3′, we have

$$\int_0^T |(g * f)(t)| \, \mathrm{d}t \leq \|\tilde{g}\|_1 \cdot \|\tilde{f}\|_1$$

Here

$$\|\tilde{f}\|_1 = \int_0^T |f(t)| \, \mathrm{d}t, \quad \|\tilde{g}\|_1 = \int_0^T |g(t)| \, \mathrm{d}t$$

holds since $f(t) = g(t) = O$ for $t \in (-\infty, 0) \cup (T, \infty)$. Therefore, the inequality in the statement 3 is obtained. □

The above proof of Theorem E.3 is not given in [34], [14], [25], and [17]. Based on Theorem E.3, we introduce the following.

Definition E.4. Let $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$. We call $g * f \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$ in Theorem E.3 defined by

$$(g * f)(t) := \int_0^t g(t-u) f(u) \, \mathrm{d}u = \int_0^t g(u) f(t-u) \, \mathrm{d}u$$

the *convolution* of f and g.

E.3 Convolution under Volterra operator

The convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$ and the Volterra operator are related in the following way.

Theorem E.5. *For any pair of $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$,*

$$V(g * f) = (Vg) * f = g * (Vf) \tag{E.1}$$

holds.

The above theorem is an extension of Corollary 5.5.

Proof of Theorem E.5. For each $t > 0$,

$$V(g * f)(t) = \int_0^t \left(\int_0^s g(s-u) f(u) \, \mathrm{d}u \right) \mathrm{d}s$$

holds by the definition of convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. By applying Fubini's theorem in a similar way as in the direct proof of Theorem E.3, the right-hand side is calculated as

$$\begin{aligned}\int_0^t \left(\int_0^s g(s-u)f(u)\,\mathrm{d}u\right)\mathrm{d}s &= \int_0^t \left(\int_u^t g(s-u)\,\mathrm{d}s\right)f(u)\,\mathrm{d}u \\ &= \int_0^t (Vg)(t-u)f(u)\,\mathrm{d}u,\end{aligned}$$

where the last term is equal to $[(Vg)*f](t)$. Therefore, the integration by parts formula for matrix-valued absolutely continuous functions (see Theorem 6.13) yields

$$\begin{aligned}[(Vg)*f](t) &= [(Vg)(t-u)(Vf)(u)]_{u=0}^t + \int_0^t g(t-u)(Vf)(u)\,\mathrm{d}u \\ &= [g*(Vf)](t),\end{aligned}$$

where $(Vg)(0) = (Vf)(0) = O$ is used. This completes the proof. □

Remark E.6. Eq. (E.1) is a special case of the *associativity of convolution*

$$(h*g)*f = h*(g*f) \tag{E.2}$$

for $f, g, h \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ because

$$(f*\mathcal{I})(t) = (\mathcal{I}*f)(t) = \int_0^t f(s)\,\mathrm{d}s = (Vf)(t) \quad (t \geq 0)$$

holds for any $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Here $\mathcal{I}\colon [0,\infty) \to M_n(\mathbb{K})$ denote the constant function whose value is equal to the identity matrix.

The following is a result on the regularity of convolution. It should be compared with Theorem 5.3.

Theorem E.7. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a locally absolutely continuous function. Then $g*f$ is expressed by*

$$g*f = V\big(g(0)f + g'*f\big). \tag{E.3}$$

Consequently, $g*f$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$(g*f)'(t) = g(0)f(t) + (g'*f)(t)$$

for almost all $t \geq 0$.

We note that for a locally absolutely continuous function $g\colon [0,\infty) \to M_n(\mathbb{K})$, the derivative g' belongs to $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Therefore, the convolution $g'*f$ makes sense from Theorem E.3.

Proof of Theorem E.7. Since $g = g(0) + Vg'$, we obtain

$$g*f = g(0)Vf + (Vg')*f = g(0)Vf + V(g'*f)$$

by using Theorem E.5. This yields the expression (E.3) because the Volterra operator is linear. The remaining properties of $g*f$ are derived by the properties of Volterra operator. □

See also [17, 7.4 Corollary in Chapter 3] for related results.

Remark E.8. From Theorem E.7, we have

$$(Vg) * f = V(g * f)$$

for $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. We also have

$$g * (Vf) = V(g * f)$$

in a similar way.

We note that the statement 2 of Corollary 5.6 also follows by Lemma E.2 and Theorem E.7.

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